

Two-loop QCD corrections to $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$

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We present a next-to-next-to-leading-order (NNLO) calculation within the nonrelativistic QCD framework for the process $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$. We find that the NNLO contribution is two to three times larger than the next-to-leading-order contribution, which itself is already three to five times larger than the leading-order result. In the high-energy limit, we provide analytic expressions for the leading-power coefficients in the asymptotic expansion of the two-loop amplitudes. Our results are directly applicable to the bottomonium process $Z \rightarrow \Upsilon + \Upsilon$. Using the obtained hadronic amplitudes, we predict the decay width of the Z boson into these rare dicharmonium and dibottomonium final states.

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I. INTRODUCTION

The study of exclusive double charmonium production in electron-positron annihilation has been a topic of intense theoretical and experimental interest since its first observation at the B factories in the early 2000s [1–4]. Among these processes, $e^+e^- \rightarrow J/\psi + \eta_c$ became particularly notable for exhibiting one of the most significant discrepancies between experimental measurements and Standard Model predictions within the nonrelativistic QCD (NRQCD) framework [5], spurring extensive theoretical investigations [6–13]. Such hard exclusive reactions provide a unique testing ground for probing the interplay between perturbative and nonperturbative QCD dynamics in heavy quarkonium physics.

A closely related process of considerable interest is the production of J/ψ pairs. At B-factory energies, this configuration cannot proceed through a single virtual photon due to C-parity conservation. Instead, the dominant

mechanism is the double-photon channel, whose leading-order (LO) cross section is sizable [14–17]. However, next-to-leading order (NLO) QCD corrections were found to be large and negative, with a K factor ($\sigma_{\text{NLO}}/\sigma_{\text{LO}}$) ranging from -0.31 to 0.25 [18], providing a natural explanation for the nonobservation of double J/ψ production at these facilities. More recently, theoretical calculations have been extended to next-to-next-to-leading order (NNLO) within the framework of the vector dominance model [16,17], highlighting the dominance of the fragmentation-type contributions.

The experimental landscape has since evolved dramatically. The Belle II collaboration is ushering in a new era of precision in heavy flavor physics [19], while the LHC experiments have opened a new frontier by probing quarkonium production in Z -boson decays. Following a CMS proposal to search for Z decays into double quarkonia [20], and motivated by the LHCb observation of novel structures in the J/ψ -pair mass spectrum [21], the CMS collaboration has performed dedicated searches for the rare decay $Z \rightarrow J/\psi + J/\psi$. Using the full Run 2 dataset, they established a stringent upper limit on the branching fraction [22],

$$\mathcal{B}(Z \rightarrow J/\psi + J/\psi) < 1.4 \times 10^{-6}. \quad (1)$$

This new level of experimental precision demands equally precise theoretical predictions. It is well established that NNLO QCD corrections within the NRQCD

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framework are often crucial for achieving accurate descriptions of quarkonium production processes [23,24]. In this paper, we present a state-of-the-art calculation of the $Z \rightarrow J/\psi + J/\psi$ decay rate at NNLO in α_s . Specifically, we provide the first complete computation of the two-loop amplitudes for this process. The paper is organized as follows. We begin by outlining the theoretical framework and establishing our notation. We then detail the calculation of the two-loop amplitudes, including the treatment of ultraviolet and infrared divergences through renormalization and factorization. The PSLQ integer-relation algorithm [25] is employed to reconstruct analytical expressions from high-precision numerical results. Finally, we provide a semianalytical analysis of the decay width and conclude with a discussion of our findings and their implications.

II. FRAMEWORK AND METHODS

We consider the two-loop QCD corrections to the following process

$$e^+(k_1) + e^-(k_2) \rightarrow Z^* \rightarrow J/\psi(P_1) + J/\psi(P_2), \quad (2)$$

where the external momentum satisfies the on-shell conditions,

$$\begin{aligned} k_1^2 = k_2^2 &= 0, \\ P_1^2 = P_2^2 &= 4m_c^2, \\ s = q^2 &= (k_1 + k_2)^2 = (P_1 + P_2)^2. \end{aligned} \quad (3)$$

Figure 1 presents the four tree-level diagrams.

Within the NRQCD formalism, the differential cross section for the process can be written as

$$d\sigma = d\hat{\sigma}_{e^+e^- \rightarrow c\bar{c}[n_1] + c\bar{c}[n_1]} \langle \mathcal{O}^{J/\psi}(n_1) \rangle^2, \quad (4)$$

where the $d\hat{\sigma}$ denotes the short-distance partonic cross section for producing two $c\bar{c}$ pairs in intermediate states labeled by n_1 , and the NRQCD long-distance matrix element $\langle \mathcal{O}^{J/\psi}(n_1) \rangle$ describes the probability for these pairs to hadronize into physical J/ψ mesons. For the

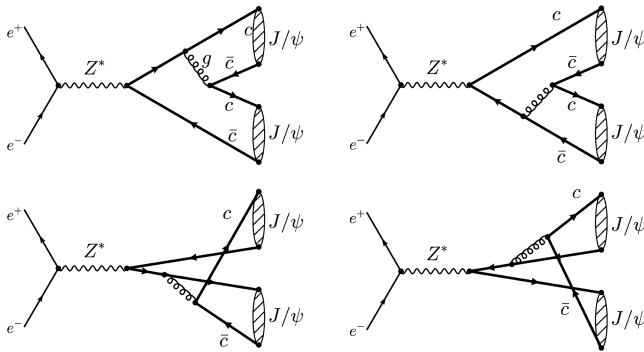


FIG. 1. Tree-level diagrams for $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$.

J/ψ production, we project the $c\bar{c}$ pair onto the color-singlet state $n_1 = {}^3S_1^{[1]}$, which corresponds to the quantum numbers of the physical meson.

We define the amplitude of the process $Z^* \rightarrow J/\psi + J/\psi$ as

$$\begin{aligned} \mathcal{A}^\mu(s) &= \langle J/\psi(P_1) + J/\psi(P_2) | J_Z^\mu | 0 \rangle \\ &= \mathcal{A}^{\alpha_1\alpha_2\mu}(P_1, P_2) \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \\ &= \sum_{i=0}^6 \sum_{j=1}^6 g_j^{(il)} \frac{e}{c_W s_W} \frac{1}{192\pi m_c^4} g_s^{2i+2} \\ &\quad \times b_j^{\alpha_1\alpha_2\mu} \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} M_{J/\psi} R_{J/\psi}^\mu(0). \end{aligned} \quad (5)$$

It is expressed as a sum over Lorentz structures $b_j^{\alpha_1\alpha_2\mu}$, each multiplied by a scalar form factor $g_j^{(il)}(r)$, where i denotes the loop order. The ε_{α_1} and ε_{α_2} denote the polarization vectors of the mesons. The factors $M_{J/\psi}$ and $R_{J/\psi}(0)$ represent the meson mass and the radial wave function at the origin, respectively. The meson mass can be approximated by $M_{J/\psi} = 2m_c$. The $R_{J/\psi}(0)$ encodes the non-perturbative transition amplitude of a color-singlet state ${}^3S_1^{[1]}$ $c\bar{c}$ pair into a physical J/ψ , and its square is the corresponding NRQCD long-distance matrix element, as, $\langle \mathcal{O}^{J/\psi}({}^3S_1^{[1]}) \rangle = |R_{J/\psi}(0)|^2$.

The interaction of the Z boson with quarks involves an axial-vector current, with the presence of a γ^5 matrix. In dimensional regularization, γ^5 cannot be consistently defined with all its four-dimensional properties. In this work, we adopt the Larin scheme [26] to define γ^5 ,

$$\gamma^\mu \gamma^5 \rightarrow \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma. \quad (6)$$

As a cross-check, we also use the cyclic scheme, without commuting γ^5 with any other gamma matrices [27,28]. Both schemes lead to identical results after completing the Lorentz tensor reduction.

There are eight possible Lorentz structures in total,

$$\begin{aligned} b_1^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\alpha_2\mu} P_1, \\ b_2^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\alpha_2\mu} P_2, \\ b_3^{\alpha_1\alpha_2\mu} &= e^{\alpha_2\mu} P_1 P_2 P_1^{\alpha_1} / m_c^2, \\ b_4^{\alpha_1\alpha_2\mu} &= e^{\alpha_2\mu} P_1 P_2 P_2^{\alpha_1} / m_c^2, \\ b_5^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\mu} P_1 P_2 P_1^{\alpha_2} / m_c^2, \\ b_6^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\mu} P_1 P_2 P_2^{\alpha_2} / m_c^2, \\ b_7^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\alpha_2\mu} P_1 P_2 P_1^\mu / m_c^2, \\ b_8^{\alpha_1\alpha_2\mu} &= e^{\alpha_1\alpha_2\mu} P_1 P_2 P_2^\mu / m_c^2. \end{aligned} \quad (7)$$

In four dimensions, the above Lorentz structures involving the Levi-Civita tensor are not all independent due to the Schouten identity.¹ Due to the identical nature of the two J/ψ , the amplitude has the relation $\mathcal{A}^{\alpha_1\alpha_2\mu}(P_1, P_2) = \mathcal{A}^{\alpha_2\alpha_1\mu}(P_2, P_1)$, leading to

$$\begin{aligned} g_1^{(il)} &\equiv -g_2^{(il)}, \\ g_3^{(il)} &\equiv -g_6^{(il)}, \\ g_4^{(il)} &\equiv -g_5^{(il)}, \\ g_7^{(il)} &\equiv g_8^{(il)}. \end{aligned} \quad (10)$$

The definition of a $^3S_1^{[1]} c\bar{c}$ state requires $P_{1\alpha_1} \mathcal{A}^{\alpha_1\alpha_2\mu} = 0$, which is known as the transversality condition and follows from $P_1 \cdot \varepsilon(P_1) = 0$ for vector meson polarization [29–31], resulting in

$$g_1^{(il)} = \frac{P_1^2}{m_c^2} g_3^{(il)} + \frac{P_1 \cdot P_2}{m_c^2} g_4^{(il)}. \quad (11)$$

Furthermore, we find that, within the γ^5 scheme adopted in this work, the coefficient $g_7^{(il)}$ vanishes,

$$g_7^{(il)} = 0. \quad (12)$$

Note that Eq. (12) is a finding of our calculation rather than a condition imposed on it. In fact, Eq. (12) does not necessarily hold in other γ^5 schemes. Nevertheless, it can be easily proved that the structures $b_7^{\alpha_1\alpha_2\mu}$ and $b_8^{\alpha_1\alpha_2\mu}$ do not contribute to the squared amplitudes in Eqs. (13) and (14) below.

The Eqs. (10)–(12) reduce the amplitudes to only two independent coefficients at each order in α_s ; we choose $g_1^{(il)}$ and $g_3^{(il)}$ as the basis.

With the amplitude at hand, the total cross section for $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$ can be written as

¹For example,

$$\begin{aligned} \epsilon^{\mu\nu_1\nu_2\alpha_1} P_1^{\alpha_2} + \epsilon^{\nu_1\nu_2\alpha_1\alpha_2} P_1^\mu + \epsilon^{\nu_2\alpha_1\alpha_2\mu} P_1^{\nu_1} \\ + \epsilon^{\alpha_1\alpha_2\mu\nu_1} P_1^{\nu_2} + \epsilon^{\alpha_2\mu\nu_1\nu_2} P_1^{\alpha_1} = 0. \end{aligned} \quad (8)$$

Contracting both sides with $P_{1,\nu_1} P_{2,\nu_2}$, we obtain

$$\begin{aligned} (P_1 \cdot P_2) \epsilon^{\mu\alpha_1\alpha_2 P_1} + P_1^\mu \epsilon^{\alpha_1\alpha_2 P_1 P_2} + P_1^{\alpha_1} \epsilon^{\alpha_2\mu P_1 P_2} \\ + P_1^{\alpha_2} \epsilon^{\mu\alpha_1 P_1 P_2} - P_1^2 \epsilon^{\mu\alpha_1\alpha_2 P_2} = 0. \end{aligned} \quad (9)$$

As a result, some of the tensor structures appearing in Eq. (7) become linearly dependent in four dimensions, although they remain independent in general D dimensions within dimensional regularization.

$$\begin{aligned} \sigma = \frac{\alpha(8s_W^4 - 4s_W^2 + 1)}{384c_W^2 s_W^2 (m_Z^4 + m_Z^2(\Gamma_Z^2 - 2s) + s^2)} \sqrt{1 - \frac{16m_c^2}{s}} \\ \times \left(\frac{q^\mu q^{\mu'}}{q^2} - g^{\mu\mu'} \right) \mathcal{A}_\mu \mathcal{A}_\mu^*. \end{aligned} \quad (13)$$

We also study the partial decay width for Z boson in this channel, which is

$$\Gamma = \frac{1}{2} \frac{1}{3} \frac{1}{16\pi m_Z} \sqrt{1 - \frac{16m_c^2}{s}} \left(\frac{q^\mu q^{\mu'}}{m_Z^2} - g^{\mu\mu'} \right) \mathcal{A}_\mu \mathcal{A}_\mu^*. \quad (14)$$

We generate the two-loop Feynman diagrams using FeynArts [32] and QGRAF [33], with sample diagrams shown in Fig. 2. Then we use the CalcLoop package to deal with the Dirac and $SU(N_c)$ color algebra [34]. The Feynman integrals are in general the Lorentz-tensor integrals, which are reduced to scalar integrals for subsequent processing. More than 10^4 integrals are categorized into 153 families according to their propagators. The coefficients of these integrals are rational functions of the center-of-mass energy s , the charm quark mass m_c , as well as the dimensional regulator $\epsilon = (4 - D)/2$, multiplied by the Lorentz tensors. Next, we utilize the package Blade [35], which is based on the method of block-triangular form [36,37], to perform integration-by-part (IBP) reduction [38]. This allows us to express integrals in each family as linear combinations of so-called master integrals (MIs). The finite field method [39–44] is employed in the reduction step.

Furthermore, the master integrals are computed using the method of differential equations [45] based on power series expansion [46–51]. The differential equations of master integrals with respect to $r = m_c^2/s$ are constructed using the aforementioned IBP reduction. The boundary conditions at $r = 1/100$ are obtained using the auxiliary mass flow method [52–55] implemented in AMFlow package [56].

The DESolver module in AMFlow solves the differential equations for each family, expressing all MIs as asymptotic expansions at $r = 0$ (equivalently $s = \infty$). By combining the solutions of master integrals and the IBP reduction results, the amplitude can be expressed as an asymptotic expansion at $r = 0$,

$$\mathcal{A}(\epsilon, r) = \sum_{\lambda, k, n} c_{\lambda, k, n}(\epsilon) r^{\lambda(\epsilon)} \ln^k(r) r^n. \quad (15)$$

The leading power $\lambda(\epsilon)$ in Eq. (15) was determined by solving the eigenvalue problem of the residue matrix of the differential equation for the master integrals at $r = 0$ for each family. Hereafter, the Lorentz index μ on the amplitude has been omitted for brevity. In the phenomenological analysis presented in this work, the asymptotic expansion of the amplitude is truncated at $\mathcal{O}(r^{35})$, exhibiting a convergence behavior analogous to that found in Ref. [57].

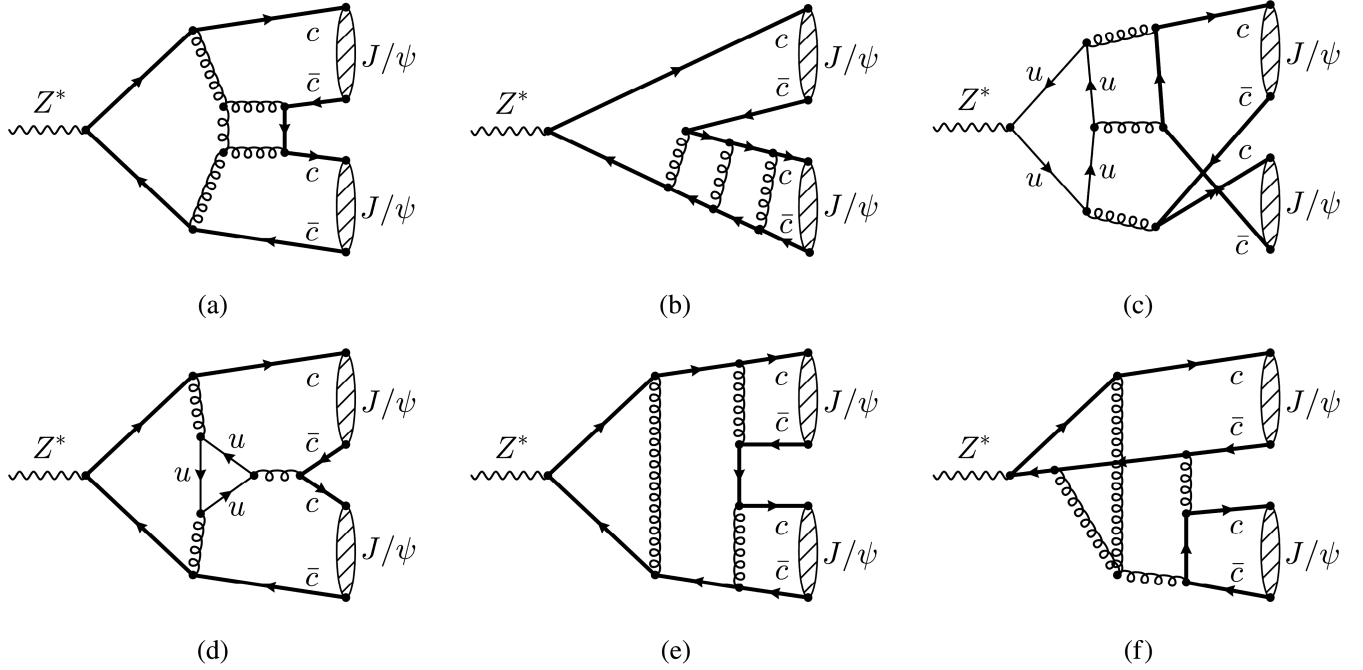


FIG. 2. Some typical two-loop diagrams for $Z \rightarrow J/\psi + J/\psi$. (a) Diagram with three-gluon vertices; (b) Ladder diagram; (c) Diagram with a fermion loop connected to Z boson; (d) Diagram with a fermion loop disconnected to Z boson; (e) Triangle-hexagon diagram; (f) Diagram with a J/ψ meson indirectly attached to Z boson via gluons exchange.

Finally, the cross section or decay width is obtained by combining the two-loop, one-loop, and Born amplitudes.

III. RENORMALIZATION AND IR SUBTRACTION

The bare amplitudes are renormalized according to

$$\mathcal{A}(\alpha_s, m_Q) = Z_{2,Q}^{-2} Z_A Z_5 \mathcal{A}_0(\alpha_{s,0}, m_{Q,0}), \quad (16)$$

where $Z_{2,Q}$ is the quark field renormalization constant and Z_A and Z_5 will be explained later. The bare mass and the strong coupling constants are renormalized as follows:

$$\begin{aligned} m_{Q,0} &= Z_{m,Q} m_Q, \\ \alpha_{s,0} &= \left(\frac{e^{\gamma_E}}{4\pi} \right)^{\epsilon} \mu_R^{2\epsilon} Z_{\alpha_s}^{\overline{\text{MS}}} \alpha_s(\mu_R). \end{aligned} \quad (17)$$

The field strength renormalization constant for a gluon or light quark shows up at the intermediate stage in the two-loop calculation, but they completely cancel out in the results. The renormalization constants to $\mathcal{O}(\alpha_s^2)$ are taken from Ref. [58].

In addition to the usual field, mass, and coupling constant renormalization, the axial current operator must be renormalized multiplicatively by a factor Z_A , and the correct Ward identities must be restored through a finite renormalization factor Z_5 . Moreover, both Z_A and Z_5 have distinct nonsinglet and singlet versions. Here, “nonsinglet” refers to the case where the Z boson couples to an external

fermion line. In our calculation, the tree-level amplitudes involve only nonsinglet contributions, whereas the one-loop amplitudes contain both nonsinglet and singlet components. Consequently, we require $Z_{A,NS}$ and $Z_{5,NS}$ to two-loop order, and $Z_{A,S}$ and $Z_{5,S}$ to one-loop order. The explicit results are [59],

$$\begin{aligned} Z_{A,NS} &= 1 + \left(\frac{\alpha_s}{\pi} \right)^2 \frac{1}{\epsilon} \left(\frac{11}{24} C_A C_F - \frac{1}{6} C_F T_F n_f \right), \\ Z_{5,NS} &= 1 - \frac{\alpha_s}{\pi} C_F \\ &\quad + \left(\frac{\alpha_s}{\pi} \right)^2 \left(-\frac{107}{144} C_A C_F + \frac{11}{8} C_F^2 + \frac{1}{36} C_F T_F n_f \right), \\ Z_{A,S} &= 1, \\ Z_{5,S} &= 1 - \frac{\alpha_s}{\pi} C_F. \end{aligned} \quad (18)$$

After ultraviolet renormalization, a residual infrared pole remains. The infrared divergence can be factorized into the NRQCD matrix element $\langle \mathcal{O}^{J/\psi} \rangle$ under the $\overline{\text{MS}}$ prescription. As a result, the two-loop amplitudes acquire a dependence on the NRQCD factorization scale μ_Λ ,

$$\tilde{\mathcal{A}}^{(2l)} = \mathcal{A}^{(2l)} + \mathcal{A}^{(0l)} \alpha_s^2(\mu_R) \frac{\gamma_{J/\psi}}{\epsilon} \left(\frac{\mu_\Lambda^2}{\mu_R^2} \right)^{-2\epsilon}, \quad (19)$$

where the anomalous dimension for the J/ψ NRQCD current is given by [60,61],

TABLE I. Reduced Born amplitudes.

Lorentz structures $b_j^{\alpha_1\alpha_2\mu}$	$\epsilon^{\alpha_1\alpha_2\mu P_1}$	$\epsilon^{\alpha_1\alpha_2\mu P_2}$	$\epsilon^{\alpha_2\mu P_1 P_2} P_1^{\alpha_1}/m_c^2$	$\epsilon^{\alpha_2\mu P_1 P_2} P_2^{\alpha_1}/m_c^2$	$\epsilon^{\alpha_1\mu P_1 P_2} P_1^{\alpha_2}/m_c^2$	$\epsilon^{\alpha_1\mu P_1 P_2} P_2^{\alpha_2}/m_c^2$
Form factors $g_j^{(0l)}(r)$	$512ir^2$	$-512ir^2$	$128ir^2$	0	0	$-128ir^2$

$$\gamma_{J/\psi} = -\frac{\pi^2}{12} C_F (2C_F + 3C_A). \quad (20)$$

After performing all the subtractions described above, we find that the two-loop amplitudes $\tilde{\mathcal{A}}_j^{(2l)}$ ($j = 1, \dots, 6$ as in Table I) for different Lorentz structures are individually free of divergence. The fact that each structure is finite provides a strong consistency check of our calculation. For brevity, we will omit the tilde symbol in what follows.

IV. ANALYTICAL RESULTS

For subsequent discussions, it is more convenient to work directly with the dimensionless functions $g_j(r)$, defined in Eq. (5). The leading-order results for g_j ($j = 1, \dots, 6$) are summarized in Table I, where we have explicitly verified Eq. (11) at this order. Although g_4 and g_5 vanish at this order, they become nonzero starting from NLO, with their leading contributions scaling as $\mathcal{O}(r^3)$. Owing to Eq. (11), only two form factors, g_1 and g_3 , are independent. Therefore, in the following, we present analytical results only for these two.

The functions $g_j(r)$ admit the following asymptotic expansion in the variable r :

$$\begin{aligned} g_j^{(0l)}(r) &= r^2 G_{j,(2,0)}^{(0l)}, \\ g_j^{(1l)}(r) &= r^2 \left[G_{j,(2,2)}^{(1l)} \ln^2(r) + G_{j,(2,1)}^{(1l)} \ln(r) + G_{j,(2,0)}^{(1l)} \right] \\ &\quad + \mathcal{O}(r^3), \\ g_j^{(2l)}(r) &= r^2 \left[G_{j,(2,4)}^{(2l)} \ln^4(r) + G_{j,(2,3)}^{(2l)} \ln^3(r) + G_{j,(2,2)}^{(2l)} \ln^2(r) \right. \\ &\quad \left. + G_{j,(2,1)}^{(2l)} \ln(r) + G_{j,(2,0)}^{(2l)} \right] + \mathcal{O}(r^3). \end{aligned} \quad (21)$$

The analytical expressions for $G_{j,(2,k)}^{(il)}$ constitute the principal findings of this work, which are reconstructed from high-precision numerical coefficients using the PSLQ algorithm, with the basis of irrational numbers detailed in Ref. [57]. The NLO results are given by

$$\begin{aligned} g_1^{(1l)}(r)/r^2 &= \frac{32i \ln^2(r)}{\pi^2} - \frac{32i(-33 - 18i\pi - 120 \ln(2)) \ln(r)}{9\pi^2} \\ &\quad - \frac{32i}{9\pi^2} (-75 \ln(\bar{\mu}^2) - 120 \ln^2(2) - 120i\pi \ln(2) \\ &\quad - 306 \ln(2) + 11\pi^2 - 33i\pi + 19) + \mathcal{O}(r), \end{aligned} \quad (22)$$

$$\begin{aligned} g_3^{(1l)}(r)/r^2 &= \frac{24i \ln^2(r)}{\pi^2} + \frac{8(-63i - 54\pi + 330i \ln(2)) \ln(r)}{9\pi^2} \\ &\quad + \frac{8}{9\pi^2} (75i \ln(\bar{\mu}^2) + 495i \ln^2(2) - 330\pi \ln(2) \\ &\quad + 602i \ln(2) - 28i\pi^2 + 63\pi - 69i) + \mathcal{O}(r). \end{aligned} \quad (23)$$

In the above equations, we defined a dimensionless variable $\bar{\mu} = \mu_R/m_Q$.

The complete expressions for two-loop amplitudes are rather lengthy so that we present only the leading and next-to-leading logarithmic terms in Eq. (21). For $j = 1$,

$$\begin{aligned} G_{1,(2,4)}^{(2l)} &= \frac{4i}{9\pi^4}, \\ G_{1,(2,3)}^{(2l)} &= -\frac{4(-69i + 12\pi - 134i \ln(2))}{27\pi^4}. \end{aligned} \quad (24)$$

For $j = 3$,

$$\begin{aligned} G_{3,(2,4)}^{(2l)} &= \frac{7i}{27\pi^4}, \\ G_{3,(2,3)}^{(2l)} &= -\frac{-90i + 56\pi - 991i \ln(2)}{54\pi^4}. \end{aligned} \quad (25)$$

The full analytical results at $\mathcal{O}(r^2)$ including all logarithmic terms are provided in Supplemental Material [62]. The complete analytic coefficients of the series expansion up to $\mathcal{O}(r^3)$ have also been obtained; they are too extensive to be included and are available from the authors upon reasonable request.

V. NUMERICAL RESULTS

In this Section, we will discuss the phenomenological prediction. The parameters we used in the computation are outlined in Table II. The fine structure constant is fixed in the Thomson limit $\alpha = 1/137$. The strong coupling constant is evaluated to two-loop accuracy with our in-house

TABLE II. Numerical values for parameters.

$m_c = 1.5$ GeV	$R_{J/\psi}^2 = 1.1$ GeV ³
$m_b = 4.8$ GeV	$R_Y^2 = 6.477$ GeV ³
$m_Z = 91.2$ GeV	$\Gamma_Z = 2.5$ GeV
$\sin \theta_w = 38/79$	$\cos \theta_w = 64/73$

program, wherein $n_f = 5$ for $\mu_R \geq 4.8$ GeV while $n_f = 4$ for $\mu_R < 4.8$ GeV.

In our calculation, we adopt the standard QCD color factors with $N_c = 3$, and the numbers of active fermions: $n_{\text{light}} = 3, n_{\text{heavy}} = 1$ for the charmonium case, and $n_{\text{light}} = 4, n_{\text{heavy}} = 1$ for bottomonium. Note that the treatment of active flavors for the two specific processes considered in this work and that used for the running of α_s are handled separately.

The perturbative expansion is truncated at the cross-section level. Specifically, we expand the squared amplitude as

$$\begin{aligned}\sigma &= \sigma_{\text{LO}} + \sigma_{\text{NLO}} + \sigma_{\text{NNLO}} + \cdots \\ &\propto |\mathcal{A}^{(0l)} + \mathcal{A}^{(1l)} + \mathcal{A}^{(2l)} + \mathcal{O}(\alpha_s^3)|^2 \\ &= \mathcal{A}^{(0l)} \mathcal{A}^{(0l)*} + 2\text{Re}[\mathcal{A}^{(1l)} \mathcal{A}^{(0l)*}] \\ &\quad + (2\text{Re}[\mathcal{A}^{(2l)} \mathcal{A}^{(0l)*}] + \mathcal{A}^{(1l)} \mathcal{A}^{(1l)*}) + \mathcal{O}(\alpha_s^3),\end{aligned}\quad (26)$$

and retain terms up to order α_s^2 in the cross section. The same scheme is applied to the decay width.

A. Cross section of $J/\psi + J/\psi$ production

In Table III, we enumerate our predictions at various perturbative accuracies for several colliding energies of interest, including $\sqrt{s} = m_Z/4, m_Z/2, m_Z, 2m_Z$ as well as $\sqrt{s} = 10.52$ GeV, 10.58 GeV, 250 GeV. The central values correspond to $\mu_R = \sqrt{s}$, with the first uncertainty estimated by varying μ_R between $\sqrt{s}/2$ and $2\sqrt{s}$, and the second uncertainty obtained by varying m_c within 1.5 ± 0.2 GeV. An overall comparison indicates that the $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$ production cross section is relatively small compared to those of other channels, such as $\sigma(e^+e^- \rightarrow \gamma\gamma \rightarrow J/\psi + J/\psi)$ or $\sigma(e^+e^- \rightarrow \gamma \rightarrow J/\psi + \eta_c)$, etc. [57,63]. The uncertainty from μ_R dependence can be estimated. Taking $\sqrt{s} = 45.6$ GeV as an example, the differences of finite cross section at $\mu_R = (\sqrt{s}/2, 2\sqrt{s})$ from $\mu_R = \sqrt{s}$ are,

$$\begin{aligned}&(+28.9\%, -20.0\%) \quad \text{for } \sigma_{\text{LO}}, \\ &(+29.9\%, -20.5\%) \quad \text{for } \sigma_{\text{NLO}}, \\ &(+34.8\%, -22.6\%) \quad \text{for } \sigma_{\text{NNLO}}.\end{aligned}$$

That is to say, the μ_R dependence becomes slightly heavier in higher α_s order, as illustrated soon in Fig. 3.

Figures 3(a)–3(d) display the cross section as a function of μ_R for $\sqrt{s} = m_Z/4, m_Z/2, m_Z, 2m_Z$ respectively. Here, we adopt a symmetric interval $\mu_R = \sqrt{s} \pm \sqrt{s}/4$, so that the central value is located at the mid point of the horizontal axis. It is evident from Fig. 3 that the absolute magnitudes vary significantly among these energy points, up to several orders. Nevertheless, these figures show similar patterns: The NLO cross section is three times larger than LO, and the NNLO cross section is two times larger than NLO. In general, the relative size can be summarized as follows:

$$\frac{\sigma_{\text{NNLO}}}{\sigma_{\text{NLO}}} \approx 2 \sim 3. \quad (27)$$

We can also estimate the K factor, which is consistent to the Ref. [64],

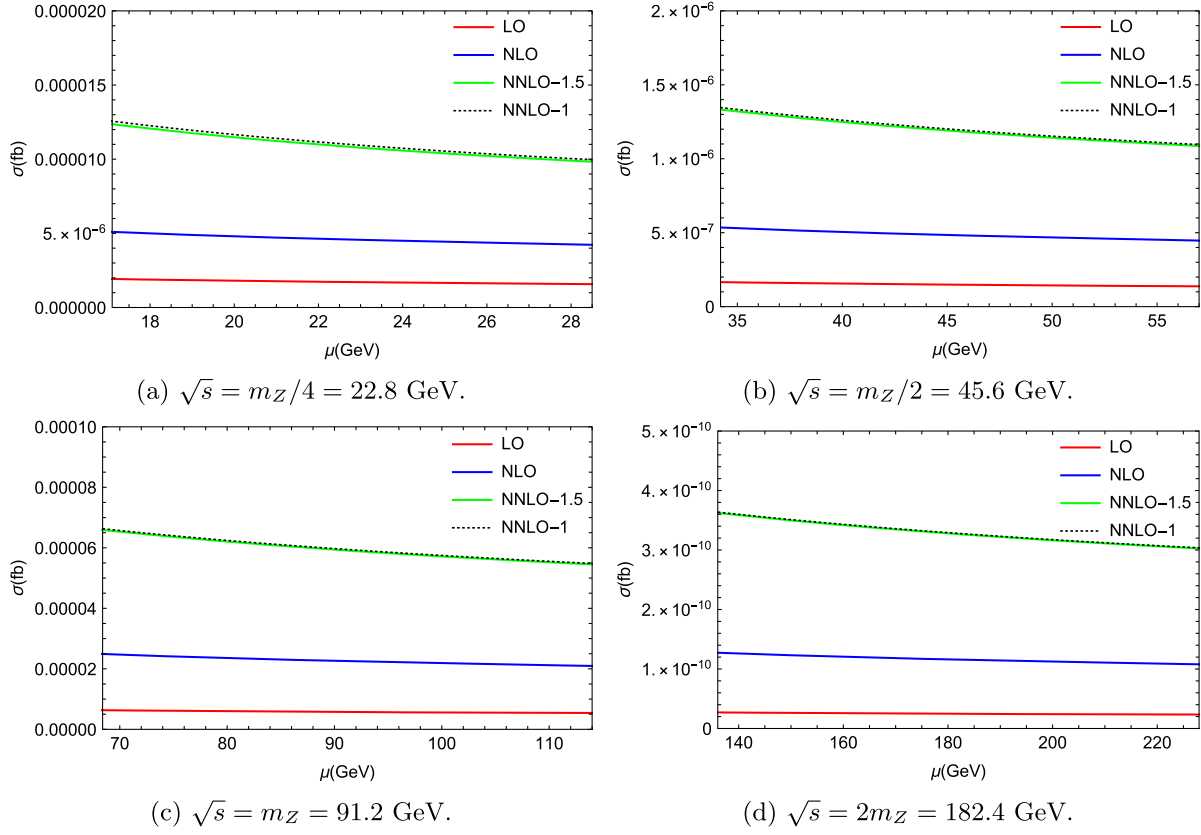
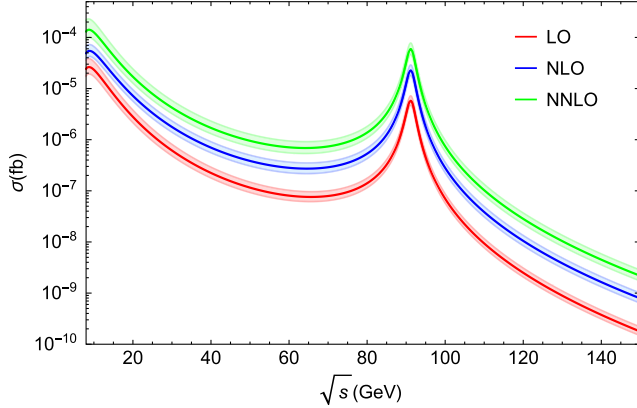
$$\frac{\sigma_{\text{NLO}}}{\sigma_{\text{LO}}} \approx 3 \sim 5. \quad (28)$$

Finally, we check the dependence on the factorization scale μ_Λ . For example, at the colliding energy $\sqrt{s} = 45.6$ GeV, taking the renormalization scale $\mu_R = 45.6$ GeV and the factorization scale, as in Eq. (19), $\mu_\Lambda = 1.5$ GeV, we have $\sigma_{\text{NNLO}} = 1.184 \times 10^{-6}$ fb. If we change μ_Λ to 1.0 GeV, we have $\sigma_{\text{NNLO}} = 1.195 \times 10^{-6}$ fb. In Fig. 3, the solid line “NNLO-1.5” and the dotted line “NNLO-1” correspond to $\mu_\Lambda = 1.5$ GeV and $\mu_\Lambda = 1$ GeV, respectively. It is evident that the μ_Λ dependence is very modest compared to the μ_R dependence. For this reason, we will fix $\mu_\Lambda = 1.5$ GeV in the following discussions.

In Fig. 4, the cross sections for $J/\psi + J/\psi$ production are depicted at the range of $\sqrt{s} = [8, 150]$ GeV, with the

TABLE III. Cross sections for $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$ at different colliding energies. The central values are evaluated at $\mu_R = \sqrt{s}$ and $m_c = 1.5$ GeV. The first uncertainty is estimated by varying μ_R from $\sqrt{s}/2$ to $2\sqrt{s}$, and the second estimated by varying m_c from 1.3 to 1.7 GeV. The factorization scale $\mu_\Lambda = m_c = 1.5$ GeV.

$\sqrt{s}/\text{GeV}$	LO/fb	NLO/fb	NNLO/fb
10.52	$2.128^{+0.551+0.656}_{-0.919-0.697} \times 10^{-5}$	$4.620^{+1.01+1.57}_{-1.51-1.86} \times 10^{-5}$	$11.50^{+3.28+3.91}_{-6.45-4.72} \times 10^{-5}$
10.58	$2.103^{+0.544+0.639}_{-0.906-0.676} \times 10^{-5}$	$4.574^{+0.998+1.54}_{-1.49-1.82} \times 10^{-5}$	$11.37^{+3.24+3.82}_{-6.36-4.60} \times 10^{-5}$
22.8	$1.714^{+0.384+0.0893}_{-0.586-0.0805} \times 10^{-6}$	$4.578^{+0.965+0.449}_{-1.43-0.500} \times 10^{-6}$	$10.82^{+2.65+1.26}_{-4.40-1.50} \times 10^{-6}$
45.6	$1.487^{+0.298+0.0186}_{-0.429-0.0164} \times 10^{-7}$	$4.824^{+0.988+0.291}_{-1.44-0.338} \times 10^{-7}$	$11.84^{+2.68+1.01}_{-0.12-1.23} \times 10^{-7}$
91.2	$0.5759^{+0.104+0.00178}_{-0.144-0.00156} \times 10^{-5}$	$2.256^{+0.446+0.111}_{-0.637-0.131} \times 10^{-5}$	$5.911^{+1.26+0.443}_{-1.86-0.543} \times 10^{-5}$
182.4	$0.2481^{+0.0410+0.00019}_{-0.0548-0.00017} \times 10^{-10}$	$1.157^{+0.220+0.0504}_{-0.306-0.0594} \times 10^{-10}$	$3.265^{+0.665+0.226}_{-0.947-0.275} \times 10^{-10}$
250	$1.372^{+0.218+0.000562}_{-0.288-0.000492} \times 10^{-11}$	$6.892^{+1.28+0.287}_{-1.77-0.338} \times 10^{-11}$	$20.12^{+4.01+1.34}_{-5.65-1.64} \times 10^{-11}$

FIG. 3. Cross section with respect to renormalization scale by varying μ_R for $\pm\sqrt{s}/4$.FIG. 4. Cross section of $J/\psi + J/\psi$ production as a function of center-of-mass energy. The green, blue, and red bands correspond to the NNLO, NLO, and LO predictions, respectively. Widths of the bands represent theoretical uncertainties by varying μ_R from $\sqrt{s}/2$ to $2\sqrt{s}$. The factorization scale is fixed at $\mu_\Lambda = 1.5$ GeV. The vertical axis uses a logarithmic scale.

red, blue, and green bands stand for the LO, NLO, and NNLO cross sections, respectively. Each curve is represented by three lines and the band width represents the theoretical uncertainty from the renormalization scale variation: the upper one stands for $\mu_R = \sqrt{s}/2$, the centered

one for $\mu_R = \sqrt{s}$, and the lower one for $\mu_R = 2\sqrt{s}$. The cross section displays a prominent peak at the Z -boson pole ($\sqrt{s} = 91.2$ GeV), which is of particular interest for future lepton colliders. Away from the resonance, the cross section decreases rapidly.

At the collision energy $\sqrt{s} = 91.2$ GeV, there is a significant enhancement in the cross section. Alternatively, one may interpret this as $\sigma(\sqrt{s})$ being suppressed by the large mass of the Z boson when $\sqrt{s} < m_Z$, while the cross section at $\sqrt{s} = 91.2$ GeV is not subject to such suppression. Specifically, we find that

$$\sigma(91.2 \text{ GeV}) = 5.911^{+1.26+0.443}_{-1.86-0.543} \times 10^{-5} \text{ fb}, \quad (29)$$

which is sufficiently large to be observed in forthcoming Z factories [65,66].

In summary, our findings are crucial for elucidating the complexities associated with the perturbative convergence in QCD and its effective theory [57].

B. Cross section of $\Upsilon + \Upsilon$ production

As a nontrivial byproduct, our result can be easily utilized for the study of double bottomonium production. Here we simply provide our numerical prediction in Table IV and Fig. 5.

TABLE IV. Cross sections for $e^+e^- \rightarrow Z^* \rightarrow \Upsilon + \Upsilon$ at different colliding energies. The central values are evaluated at $\mu_R = \sqrt{s}$ and $m_b = 4.8$ GeV. The first uncertainty is estimated by varying μ_R from $\sqrt{s}/2$ to $2\sqrt{s}$, and the second estimated by varying m_b from 4.6 to 5.0 GeV. The factorization scale $\mu_\Lambda = m_b = 4.8$ GeV.

$\sqrt{s}/\text{GeV}$	LO/fb	NLO/fb	NNLO/fb
22.8	$0.594^{+0.133+0.0309}_{-0.203-0.0279} \times 10^{-4}$	$1.587^{+0.335+0.156}_{-0.497-0.173} \times 10^{-4}$	$3.751^{+0.92+0.438}_{-1.53-0.518} \times 10^{-4}$
45.6	$0.516^{+0.103+0.00644}_{-0.149-0.00567} \times 10^{-5}$	$1.673^{+0.343+0.101}_{-0.500-0.117} \times 10^{-5}$	$4.107^{+0.928+0.349}_{-1.43-0.426} \times 10^{-5}$
91.2	$1.997^{+0.361+0.00617}_{-0.500-0.00540} \times 10^{-4}$	$7.821^{+1.55+0.384}_{-2.21-0.453} \times 10^{-4}$	$20.49^{+4.38+1.54}_{-6.43-1.88} \times 10^{-4}$
182.4	$0.860^{+0.142+0.000662}_{-0.190-0.000580} \times 10^{-9}$	$4.012^{+0.761+0.175}_{-1.06-0.206} \times 10^{-9}$	$11.32^{+2.30+0.782}_{-3.28-0.955} \times 10^{-9}$
250	$0.476^{+0.0757+0.000195}_{-0.0999-0.000171} \times 10^{-10}$	$2.389^{+0.444+0.095}_{-0.613-0.117} \times 10^{-10}$	$6.977^{+1.39+0.466}_{-1.96-0.567} \times 10^{-10}$

TABLE V. Partial decay width for Z boson. The central values are evaluated at $\mu_R = m_Z = 91.2$ GeV and $m_c = 1.5$ GeV or $m_b = 4.8$ GeV. The first uncertainty is estimated by varying μ_R from $m_Z/2$ to $2m_Z$, and the second from varying $m_{c,b}$ for ± 0.2 GeV. The factorization scale is set as $\mu_\Lambda = m_{c,b}$, respectively.

	LO/GeV	NLO/GeV	NNLO/GeV
$\Gamma_{J/\psi+J/\psi}$	$0.520^{+0.094+0.0016}_{-0.130-0.0014} \times 10^{-12}$	$2.037^{+0.403+0.100}_{-0.575-0.118} \times 10^{-12}$	$5.338^{+1.14+0.400}_{-1.68-0.491} \times 10^{-12}$
$\Gamma_{\Upsilon+\Upsilon}$	$1.628^{+0.295+0.0160}_{-0.407-0.0154} \times 10^{-11}$	$4.094^{+0.670+0.101}_{-0.892-0.105} \times 10^{-11}$	$8.413^{+1.50+0.270}_{-2.12-0.285} \times 10^{-11}$

In Fig. 5, the cross sections for $\Upsilon + \Upsilon$ production are depicted at the range of $\sqrt{s} = [20, 150]$ GeV. The plot style is similar to Fig. 4. From Table IV, we can read the peaking value at 91.2 GeV is

$$\sigma(91.2 \text{ GeV}) = 2.049^{+0.438+0.154}_{-0.643-0.188} \times 10^{-3} \text{ fb}, \quad (30)$$

which is one order larger than its charmonium counterpart.

In summary, the cross section behavior results from the interplay of several mechanisms: (1) a global $1/s^4$ falloff; (2) a resonant enhancement near $\sqrt{s} \approx m_Z$ due to the

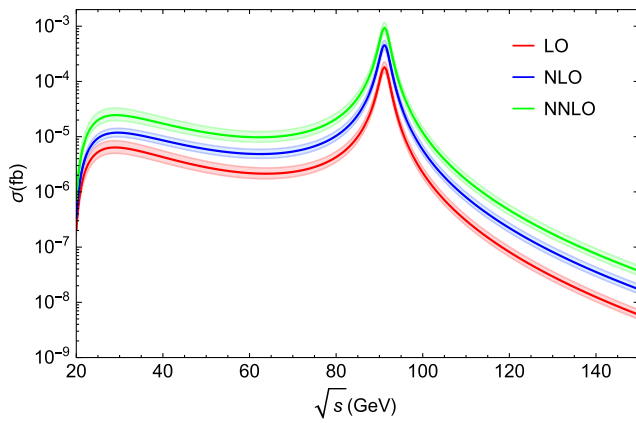


FIG. 5. Cross section of $\Upsilon + \Upsilon$ production as a function of center-of-mass energy. The green, blue, and red bands correspond to the NNLO, NLO, and LO prediction, respectively. Widths of the bands represent theoretical uncertainties by varying μ_R from $\sqrt{s}/2$ to $2\sqrt{s}$. The factorization scale is fixed at $\mu_\Lambda = 4.8$ GeV. The vertical axis uses a logarithmic scale.

$1/[(s - m_Z^2)^2 + A]$ factor; (3) a threshold effect for di-quarkonium production above $4m_Q$; and (4) double-logarithmic enhancements in the asymptotic expansion, moderating the falloff in a high energy region. Comparing Fig. 4 and Fig. 5, the di- J/ψ production is primarily governed by mechanisms (1) and (2), whereas the di- Υ is additionally affected by (3). As a result, the cross section of di- Υ at low energies is suppressed compared to the $\sqrt{s} \approx m_Z$ region.

C. Rare decay of Z boson

In the upcoming high luminosity LHC and the future super Z factory, significant amounts of Z bosons will be produced, providing a good chance to detect many rare decay channels [67]. This offers a good opportunity to investigate the Z property. In this Subsection, we utilize Eq. (14) to compute the decay width. Our results are shown in Table V. Again, we use the numerical value in Table II, select $\mu_R = m_Z$, $\mu_\Lambda = m_Q$, and vary μ_R from $m_Z/2$ to $2m_Z$ while we vary m_Q by ± 0.2 GeV. In conclusion, we have

$$\begin{aligned} \mathcal{B}_{\text{QCD}}^{\text{NNLO}}(J/\psi + J/\psi) &= (2.139^{+0.457+0.160}_{-0.672-0.197}) \times 10^{-12}, \\ \mathcal{B}_{\text{QCD}}^{\text{NNLO}}(\Upsilon + \Upsilon) &= (3.371^{+0.600+0.108}_{-0.850-0.114}) \times 10^{-11}. \end{aligned} \quad (31)$$

Thus, our NNLO prediction for the branching fraction of di- J/ψ channel is well below the current upper bound reported by the CMS experiment [22].

In Refs. [68,69], the authors found those QED diagrams with a photon directly fragmenting to a J/ψ contribute the most to the $Z \rightarrow J/\psi + J/\psi$. Recently, a theoretical advancement has been made regarding the NLO-QCD

correction for both QCD-tree and QED-tree diagrams [69]. The branching fraction of the Z decay to double J/ψ by their calculation is,

$$\mathcal{B}_{\text{QED,QCD}}^{\text{NLO}} = (1.110^{+0.334+0.054}_{-0.241-0.001}) \times 10^{-10}. \quad (32)$$

Applying analogous reasoning as Eq. (27), it seems to be a justifiable inference that the NNLO correction to both QED-tree and QCD-tree diagrams for $Z \rightarrow J/\psi + J/\psi$ would suggest,

$$\mathcal{B}_{\text{QED,QCD}}^{\text{NNLO}} \sim 3 \times 10^{-10}. \quad (33)$$

We believe that considering the full NNLO-QCD correction to photon-fragmenting QED tree diagrams would be important for future precision study of the $Z \rightarrow J/\psi + J/\psi$ channels.

VI. SUMMARY

We investigated the exclusive double charmonium production in electron-positron annihilation, focusing on the process $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$. Our study provided new insights into rare Z boson decays. We carried out extensive calculations at NLO and NNLO, employing various methods to ensure the consistency and reliability of our results.

To validate the treatment of γ^5 in the calculation, we employed both the Larin scheme and the cyclic scheme, finding excellent agreement between the two. The Feynman integral reductions were cross-checked against an independent code that combines `FiniteFlow` [41] and `LiteRed` [70]. All master integrals, obtained by solving differential equations with respect to r with boundary conditions at $r = 1/100$, were verified by `AMFlow` at another phase space point $r = 1.5^2/91.2^2$, achieving agreement up to ten significant digits. Furthermore, the

ultraviolet and infrared divergences were systematically subtracted, providing strong consistency checks.

We have obtained, for the first time, the NNLO QCD corrections to $e^+e^- \rightarrow Z^* \rightarrow J/\psi + J/\psi$ and found that the NNLO correction is two to three times larger than the NLO correction. This result poses significant challenges to the perturbative convergence of QCD and underscores the phenomenological relevance of higher-order effects. Furthermore, the analytical asymptotic expansions derived in this work have direct implications for studies of the four-lepton decay channel of the Z boson conducted in LHC [20–22], once combined with the leptonic decays of the J/ψ . Importantly, our results are independent of the Z boson production mechanism, thereby offering broader applicability to various experimental settings.

The Feynman diagrams in this paper are drawn with the aid of `FeynGame` [71].

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DATA AVAILABILITY

The data underlying this study are available in Supplemental Material at [62].

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