

Toward precise ξ gauge fixing for lattice QCD

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(Received 16 October 2025; revised 3 April 2026; accepted 26 May 2026; published 17 June 2026)

Lattice QCD provides a first-principles framework for solving quantum chromodynamics. However, its application to off-shell partons has been largely restricted to the Landau gauge, as achieving high-precision ξ gauge fixing on the lattice poses significant challenges. Motivated by a universal power-law dependence of off-shell parton matrix elements on gauge-fixing precision in the Landau gauge, we propose an empirical precision-extrapolation method to approximate high-precision ξ gauge fixing. By properly defining the bare gauge coupling and then the effective ξ , we validate our ξ gauge-fixing procedure by successfully reproducing the ξ -dependent regularization-independent momentum-subtraction renormalization constants for local quark bilinear operators at 0.3% level, up to $\xi \sim 1$.

DOI: [10.1103/n19w-y4lj](https://doi.org/10.1103/n19w-y4lj)

I. INTRODUCTION

The gauge invariance is a crucial property of gauge theories. One of the most fundamental principles in constructing a gauge theory is that the Lagrangian should remain gauge invariant. However, the usual quantization and also perturbative calculation requires one to introduce an additional gauge-fixing term, e.g., $(\partial_\mu A^\mu)^2/(2\xi)$, while the choice of gauge can be rather arbitrary and may vary among different physicists. For example, the Feynman gauge with $\xi = 1$ simplifies the form of the gauge boson propagator and then all the perturbative calculations, while the Landau gauge with $\xi = 0$ can minimize the loop correction of the quark field at the one-loop level. Nevertheless, physical observables derived from different

gauge-fixing choices are also gauge invariant as the experimental value is evidently unique.

The scenario in lattice QCD differs slightly from the framework described above. In lattice QCD, the fundamental gauge degree of freedom—the gauge potential $A_\mu(x)$ at a given spacetime point x —is replaced by the gauge link $U_\mu(x + \hat{\mu}a/2) = e^{iag \int_x^{x+\hat{\mu}a} dy^\mu A_\mu(y)}$ connecting x and $x + \hat{\mu}a$. Gauge invariance is then automatically ensured as long as the gauge links terminate at quark fields or form closed loops, as in hadronic correlation functions.

On the other hand, gauge-dependent quantities—such as off-shell parton (quark and gluon) propagators and interaction vertices—vanish entirely unless gauge fixing is imposed. Those quantities are important ingredients of nonperturbative functional QCD, such as the Dyson-Schwinger equations [1–10] and the functional renormalization group [11–22]. Usually, the gauge-dependent propagators and vertices are computed in functional QCD with the Landau gauge $\xi = 0$, from which benchmark comparison between the functional QCD and lattice QCD can be made; see, e.g., Ref. [19] for a recent study. Lattice QCD simulations at low energy scales with the Landau gauge have shown kinds of highly nontrivial features at the hadron scale, such as the emergent masses

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of the quark and gluon [10,23,24], nondegenerate gauge coupling from the gluon ghost [25], triple gluon vertices [26], and so on, which are consistent with the results of functional QCD (cf., e.g., Refs. [11–15,18,19]). However, most of those lattice calculations are restricted to the Landau gauge, despite the fact that lattice implementations of general ξ gauge fixing [27] were proposed years ago [28–30]. The primary challenge lies in severe convergence issues that arise at large ξ and/or strong gauge coupling g , making the application of ξ gauge fixing to realistic configurations numerically demanding, to understand how those inferred features of partons are sensitive to the specific Landau gauge fixing.

Recently, the dependence of gauge links and nonlocal operators on gauge-fixing precision has been investigated at multiple lattice spacings and for varying gauge link lengths, in both Landau and ξ gauges [31]. The values of these quantities follow an empirical power law in terms of gauge-fixing precision, regardless of the gauge link length. In this work, we further validate that this power law also holds for local operators with different γ matrices and off-shell momenta. Based on this, we propose a precision-extrapolation method to approximate high-precision ξ gauge fixing with controllable systematic uncertainty.

The paper is organized as follows: In Sec. II, we briefly review the gauge-fixing procedure on the lattice for Landau and ξ gauge and also the idea of the precision extrapolation based on the empirical power law observed in Ref. [31]. Further numerical evidence of the precision extrapolation in the Landau gauge is presented in Sec. III. Section IV presents our results on the nonperturbative ξ gauge dependence of quark bilinear operators, including a detailed comparison with perturbative calculations. Finally, Sec. V provides a concise summary of our findings.

II. METHODOLOGY

In the path integral formalism, gauge fixing with the additional Lagrangian term $(\partial_\mu A^\mu)^2/(2\xi)$ can be equivalently implemented by introducing $3^2 - 1$ random variables Λ^a which follow the distribution $P(\Lambda^a(x)) = \frac{1}{\sqrt{2\pi\xi}} \exp\{-\frac{1}{2\xi}[\Lambda^a(x)]^2\}$. The gauge-fixing condition is then enforced by integrating over $\Lambda \equiv \Lambda^a t^a$ with the δ function constraint $\delta(\partial_\mu A^\mu(x) - \Lambda(x))$, where t^a are the generators of the adjoint representation of SU(3).

On the lattice, the δ function constraint is discretized into the gauge-fixing condition,

$$\Delta(x) \equiv \sum_{\mu, \eta=\pm} \eta \left[\frac{U_\mu(x + \eta \frac{\hat{\mu}}{2} a) - U_\mu^\dagger(x + \eta \frac{\hat{\mu}}{2} a)}{2ig_0} \right]_{\text{Traceless}} - \Lambda(x)a^2 = 0, \quad (1)$$

and the integration over Λ can be efficiently performed by averaging over different gauge configurations with

independent Λ . The bare gauge coupling g_0 in Eq. (1) can be defined in multiple ways, differing at next-to-leading order in α_s . As an example, consider the tadpole-improved tree-level Symanzik gauge action S_g , defined as

$$S_g = \frac{1}{3} \text{Re} \sum_{x, \mu < \nu} \text{Tr} \left[1 - \beta \left(\frac{5}{3} \mathcal{P}_{\mu\nu}^U(x) - \frac{\mathcal{R}_{\mu\nu}^U(x)}{12u_0^2} \right) \right],$$

where

$$\begin{aligned} \mathcal{P}_{\mu\nu}^U(x) &= U_\mu \left(x + \frac{a}{2} \hat{\mu} \right) U_\nu \left(x + a\hat{\mu} + \frac{a}{2} \hat{\nu} \right) \\ &\quad \times U_\mu^\dagger \left(x + \frac{a}{2} \hat{\mu} + a\hat{\nu} \right) U_\nu^\dagger \left(x + \frac{a}{2} \hat{\nu} \right), \\ \mathcal{R}_{\mu\nu}^U(x) &= U_\mu \left(x + \frac{a}{2} \hat{\mu} \right) U_\mu \left(x + \frac{3a}{2} \hat{\mu} \right) \\ &\quad \times U_\nu \left(x + 2a\hat{\mu} + \frac{a}{2} \hat{\nu} \right) U_\mu^\dagger \left(x + \frac{3a}{2} \hat{\mu} + a\hat{\nu} \right) \\ &\quad \times U_\mu^\dagger \left(x + \frac{a}{2} \hat{\mu} + a\hat{\nu} \right) U_\nu^\dagger \left(x + \frac{a}{2} \hat{\mu} \right), \end{aligned} \quad (2)$$

and the tadpole improvement factor u_0 is given by $u_0 = \langle \text{ReTr} \sum_{x, \mu < \nu} \mathcal{P}_{\mu\nu}^U(x) / (6N_c V) \rangle^{1/4}$.

Then we can have three definitions of g_0 in Eq. (1):

- (1) Naive definition: $g_0^{(a)} = \sqrt{6/\beta}$.
- (2) Full tadpole improvement: including u_0 in both the action and also the gauge link in the gauge-fixing condition, then Eq. (1) should be rewritten into

$$\begin{aligned} \Lambda(x)a^2 &= \sum_{\mu, \eta=\pm} \eta \left[\frac{\frac{U_\mu(x + \eta \frac{\hat{\mu}}{2} a)}{u_0} - \frac{U_\mu^\dagger(x + \eta \frac{\hat{\mu}}{2} a)}{u_0}}{2i\sqrt{6/\beta}/u_0^4} \right]_{\text{Traceless}} \\ &= \sum_{\mu, \eta=\pm} \eta \left[\frac{U_\mu(x + \eta \frac{\hat{\mu}}{2} a) - U_\mu^\dagger(x + \eta \frac{\hat{\mu}}{2} a)}{2i\sqrt{6/\beta}/u_0} \right]_{\text{Traceless}}. \end{aligned} \quad (3)$$

Thus, it leads to a effective gauge coupling $g_0^{(b)} = \sqrt{6/\beta}/u_0$.

- (3) Approximation from u_0 only: Using u_0 only in gauge fixing while approximating α_s via $\alpha_s \approx -\frac{4 \ln u_0}{3.0684}$ [32,33] which avoids to define g_0 and $\alpha_s = g_0^2/(4\pi)$ from the action, and a similar procedure gives $g_0^{(c)} = \sqrt{-\frac{16\pi \ln u_0}{3.0684}} u_0$.

Different definitions can differ at $\mathcal{O}(\alpha_s^2)$ according to perturbative lattice QCD [34]. Because tadpole

improvement is essential for ensuring good convergence of the lattice perturbative series, the naive definition—though seemingly natural—is generally not suitable. For the MILC ensemble a06m310 at $a = 0.0566$ fm with $m_\pi = 310$ MeV and $\hat{\beta} = 5/3\beta = 6.72$, three definitions yield: $g_0^{(a)} = 1.2199$, $g_0^{(b)} = 1.3768$, $g_0^{(c)} = 1.2476$, respectively. In practice, we generate the random distribution $\tilde{P}(\tilde{\Lambda}^a(x)) = \frac{1}{\sqrt{2\pi\tilde{\xi}}} \exp\{-\frac{1}{2\tilde{\xi}}[\tilde{\Lambda}^a(x)]^2\}$ for the dimensionless quantity $\tilde{\Lambda} \equiv g_0\Lambda a^2$, meaning different g_0 definitions correspond to different effective gauge-fixing parameters $\tilde{\xi} = \frac{1}{g_0^2}\xi$.

In this work, we use $g_0^{(a)}$ to define the $\tilde{\xi}$ needed by $\xi = 0, 0.2, 0.4, 0.8, 1.0$, and the effective ξ with the other g_0 definition can be obtained with the rescale factor $(g_0^a/g_0)^2$.

The gauge-fixing algorithm we employ is the “over-relaxation” method described in Refs. [30,35–37], with its main procedure described as follows:

- (1) Separate all sites into two even-odd subsets, and start from the unitary gauge transformation matrix $G = 1$.
- (2) Calculate the gauge-fixing criterion

$$\theta = \frac{1}{N_c V} \sum_x \text{Tr}[\Delta(x)\Delta^\dagger(x)], \quad (4)$$

where V is the lattice volume.

- (3) For all the sites in the first parity subset:
 - (a) Compute

$$K[x; G] = \sum_{\mu=1}^4 \left[U_\mu \left(x + \hat{\mu} \frac{a}{2} \right) + U_\mu \left(x - \hat{\mu} \frac{a}{2} \right)^\dagger \right] - i\tilde{\Lambda}(x). \quad (5)$$

- (b) Decompose K into three SU(2) subgroups K_{sub} , construct

$$\begin{aligned} \tilde{G}(x) &= \left(A_0 \cdot I - \sum_{i=1}^3 A_i \cdot \sigma_i \right) \Big|_{\text{renormalize}} \\ &= K_{\text{sub}}(x)^\dagger / \sqrt{\det(K_{\text{sub}}(x)^\dagger)} \end{aligned} \quad (6)$$

with four parameters A_μ extracted from K_{sub} . One may apply over-relaxation by updating the parameters as $A'_0 = \cos(\omega \cdot \arccos A_0)$, $A'_i = A_i \frac{\sin(\omega \cdot \arccos A_0)}{\sin(\arccos A_0)}$. While the over-relaxation parameter ω is normally chosen between 1 and 2, we found that, in the ξ gauge, values of $\omega > 1$ lead to poorer convergence in θ . For this reason, we fix $\omega = 1$ in our simulations, which corre-

sponds to performing no over-relaxation. Then update the gauge link and rotation

$$\begin{aligned} U_\mu(x + \hat{\mu}a/2) &\rightarrow \tilde{G}(x)U_\mu(x + \hat{\mu}a/2)\tilde{G}(x + \hat{\mu}a)^\dagger, \\ G(x) &\rightarrow \tilde{G}(x)G(x). \end{aligned} \quad (7)$$

- (c) Repeat the update for the other two SU(2) subgroups.

- (4) Repeat step 3 for the other parity subset.

- (5) Repeat steps 2–4 until θ is smaller than the target precision.

There is another widely used gauge-fixing precision criteria for the Landau gauge by requiring

$$\delta(n) = F[G(n)] - F[G(n-1)] \quad (8)$$

to be smaller than given δ , where

$$\begin{aligned} F[G] &= -\sum_x \text{ReTr} \left\{ \sum_{\mu=1}^4 \left[G(x)U_\mu \left(x + \hat{\mu} \frac{a}{2} \right) G(x + \hat{\mu}a)^\dagger \right. \right. \\ &\quad \left. \left. + G(x)U_\mu \left(x - \hat{\mu} \frac{a}{2} \right)^\dagger G(x - \hat{\mu}a)^\dagger \right] \right\}. \end{aligned} \quad (9)$$

$G(n)$ represents the gauge rotation at the n th step. The gauge fixing will stop at the m th step once $\delta^F(m)$ is smaller than the preassigned value δ^F , and the previous study on the MILC ensembles in the lattice spacing range $a \in [0.03, 0.12]$ fm suggests that $\theta \sim 24\delta^F$ in all the cases [31]. Landau gauge fixing in this work was performed with precisions $\delta^F = 10^{-n}$ for integer n and then correspond to $\theta \sim 2.4 \times 10^{-n+1}$.

To verify the implementation of the ξ gauge fixing on the lattice, one must compare lattice-computed observables with known continuum results. A suitable class of observables are the ξ -dependent, off-shell quark matrix elements for bilinear operators, $\mathcal{O}_\Gamma \equiv \bar{\psi}\Gamma\psi$, specifically, $\langle q|\mathcal{O}_\Gamma|q \rangle$.

In the continuum, the vector and axial-vector currents are protected from renormalization in the continuum as a consequence of current conservation and partially conserved axial current relations, respectively. Furthermore, chiral symmetry ensures that the renormalization of the pseudoscalar current is identical to that of the scalar current. For a Euclidean momentum p^2 , the scalar ($S \equiv \mathcal{O}_\mathcal{T}$) and tensor ($T_{\mu\nu} \equiv \mathcal{O}_{\sigma_{\mu\nu}}$) matrix elements are known to three-loop order in dimensional regularization [38],

$$\begin{aligned}
 \langle q^R(p) | S | q^R(p) \rangle &= 1 + \frac{\alpha_s C_F}{4\pi} \left[3 \left(\frac{1}{\bar{\epsilon}} + \ln \frac{\mu^2}{p^2} \right) + 4 + \xi \right] \\
 &\quad + \mathcal{O}(\alpha_s^2), \\
 \langle q^R(p) | T_{\mu\nu} | q^R(p) \rangle &= \sigma_{\mu\nu} \left\{ 1 + \frac{\alpha_s C_F}{4\pi} \left[- \left(\frac{1}{\bar{\epsilon}} + \ln \frac{\mu^2}{p^2} \right) - \xi \right] \right. \\
 &\quad \left. + \mathcal{O}(\alpha_s^2) \right\}. \quad (10)
 \end{aligned}$$

Note that, while the operator itself is gauge invariant, the off-shell quark state used in its evaluation is defined in a specific gauge.

The logarithmic term $\frac{1}{\bar{\epsilon}} + \ln(\mu^2/p^2)$ is gauge independent. On the lattice, this term is replaced by $-\ln(a^2 p^2) + c_{\mathcal{O}}$, where the constant $c_{\mathcal{O}}$ depends on the specific operator and the discretization of the fermion and gauge actions. In contrast, the finite $\mathcal{O}(\alpha_s)$ coefficient of ξ is nonzero and operator dependent due to the off-shell external state. Notably, for the scalar and tensor operators, this coefficient has opposite signs. Crucially, this specific gauge dependence is a property of the currents themselves in the off-shell quark state; it is therefore independent of the ultraviolet regularization and must match between the lattice and continuum formulations.

This fundamental feature is leveraged in the regularization-independent momentum-subtraction (RI/MOM) scheme [39], where a condition such as $Z_{\mathcal{O}_r}^{\text{RI}} \langle q^R(p) | \mathcal{O}_r | q^R(p) \rangle \equiv \Gamma$ is imposed at a scale $p^2 = \mu_0^2$. The subsequent perturbative matching from the gauge-dependent RI/MOM scheme to the gauge-independent $\overline{\text{MS}}$ scheme is designed to cancel this ξ dependence. However, a direct verification that the final $\overline{\text{MS}}$ renormalization constants are indeed gauge independent has, until now, been absent.

A. Numerical challenge in the ξ gauge fixing

As demonstrated in the upper panel of Fig. 1 using the MILC ensemble a06m310 at $a = 0.06$ fm with the information detailed in the next section, the primary reason for this absence is that convergence to a small $\theta(\xi)$ becomes unattainable at large ξ , even after 100,000 iterations. The mean residual $\bar{\theta}$, measured between steps 100,000 and 200,000, exhibits an exponential scaling with ξ ($\bar{\theta}(\xi) \approx 10^{2.86\xi - 7.31}$), as illustrated in the lower panel. Nevertheless, the Landau gauge with $\xi = 0$ is free of this convergence problem since $\Lambda = 0$, and one can reach much higher precisions like 10^{-15} .

To ensure consistency across different precision levels, we generated a random Λ field ($\Lambda = 0$ in the Landau gauge) for each gauge configuration and recorded the gauge-fixed links dynamically as the minimization process crossed each target accuracy θ . This approach is applied

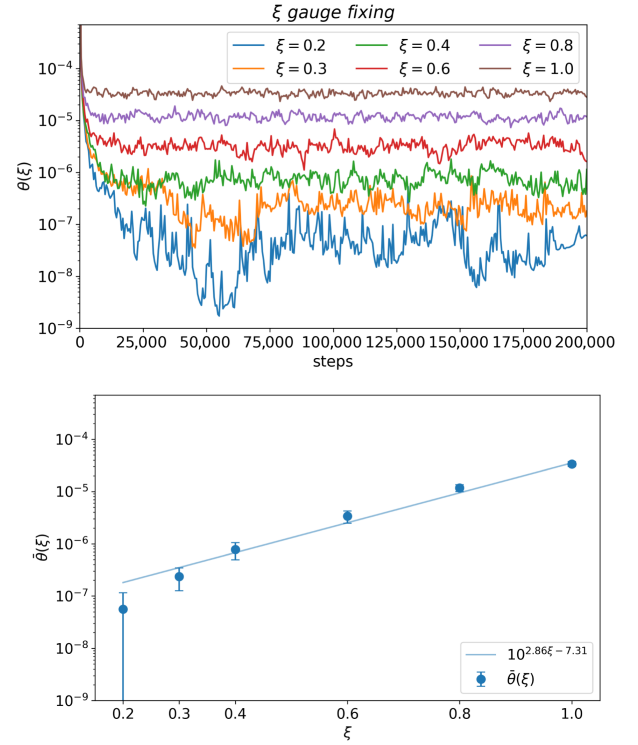


FIG. 1. Upper: gauge-fixing criteria θ as a function of iteration steps, for different ξ using $g_0^{(a)}$ on a typical configuration of the a06m310 ensemble. Lower: exponential growth of the gauge-fixing residual $\bar{\theta}$ with ξ , based on over the iteration window 100,001–200,000. [Please change θ_ξ^G into $\bar{\theta}(\xi)$].

throughout all precision dependence and extrapolation studies in this work.

We further verify that the convergence behavior is insensitive to both the choice of Λ and the specific gauge configuration:

- (1) Robustness against Λ choice: For a fixed gauge configuration, we tested multiple random initializations of the Λ field. As shown in the upper panel of Fig. 2, all sampled Λ configurations for $\xi = 1$ exhibit similar convergence trajectories. The fluctuations of θ ensure that the residual $10^{-4.5}$ (the lowest one used in the work, dashed line in Fig. 2) can eventually be reached for all sampled Λ , even though it lies slightly below the “plateau” of θ . This indicates that the attainable precision is unaffected by the choice of Λ .
- (2) Ensemble-wide consistency: As illustrated in the lower panel of Fig. 2, we repeated the same test for different gauge configurations, and the conclusion remains unchanged.

Consequently, the resulting distribution of Λ functions should remain purely Gaussian, as no selection bias was introduced into the analysis.

In principle, θ cannot be made arbitrarily small for sufficiently large ξ due to the gauge-fixing condition

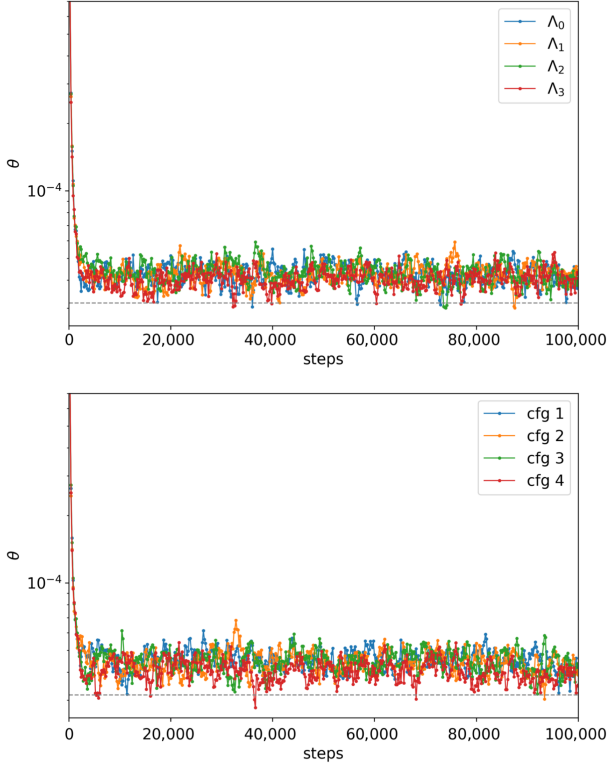


FIG. 2. Convergence trajectories of the residual θ with $\xi = 1$ for different Λ on a given configuration (upper) and different configurations with respective random Λ (lower).

in Eq. (1). If we define $\tilde{\Delta}(x) \equiv g_0 \Delta(x)$ and separate it as

$$\begin{aligned} \tilde{\Delta}(x) &= \tilde{\Delta}_U(x) - \tilde{\Lambda}(x), \\ \tilde{\Delta}_U(x) &\equiv \sum_{\mu, \eta=\pm} \eta \left[\frac{U_\mu(x + \eta \frac{\hat{\mu}}{2} a) - U_\mu^\dagger(x + \eta \frac{\hat{\mu}}{2} a)}{2i} \right]_{\text{Traceless}}, \\ \tilde{\Lambda}(x) &\equiv g_0 \Lambda a^2, \end{aligned} \quad (11)$$

then $\tilde{\Delta}_U$ is bounded but $\tilde{\Lambda} \propto \sqrt{\xi}$ is not. Thus, it is crucial to verify whether the plateau of θ originates from this concept issue.

As shown in Fig. 3, the distribution of $\tilde{\Delta}_U^a \equiv 2\text{Tr}[t^a \tilde{\Delta}_U]$ without the gauge fixing after projected to the SU(3) generators can be roughly described by a Gaussian distribution with variance $\sigma(\tilde{\Delta}_U^a) = 1.63$. Since $\sigma(\tilde{\Delta}_U^a)$ is larger than that of $\tilde{\Lambda}^a \equiv 2\text{Tr}[t^a \tilde{\Lambda}]$ with the largest $\xi = 1.0$, $\sigma(\tilde{\Lambda}^a) = g_0 \sqrt{\xi} = 1.25$, the gauge fixing is a procedure to reduce the variance of $\tilde{\Delta}_U^a$ to match that of the $\tilde{\Lambda}^a$, like the Landau gauge fixing. The practical calculation suggests that the largest $|\tilde{\Lambda}^a|$ that we generated in the $\xi = 1.0$ case is around 7.2, which is around 5.8σ of its distribution, where that of $\tilde{\Delta}_U^a$ is around 8.8, which is larger. It means that practically we have not brought the theoretical concept issue into real calculation.

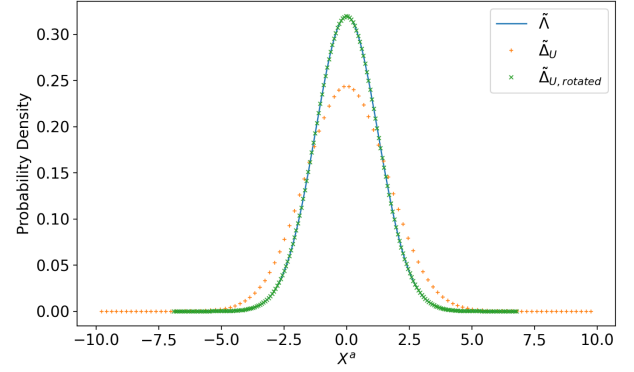


FIG. 3. Probability density of SU(3) components of the $\tilde{\Delta}_U^a$ before (orange upright crosses) and after (green saltire crosses) gauge fixing and that of $\tilde{\Lambda}^a$ (blue curve), in the case of $\xi = 1.0$, $\theta = 10^{-4.5}$. For each spacetime point, eight SU(3) components X^a are extracted through $X^a = 2\text{tr}(X t^a)$.

We further illustrate the distribution of the residual $\tilde{\Delta}^a$ for different θ values in the Landau gauge, shown in Fig. 4. Taking the case $\theta = 10^{-3}$ as an example, the distribution $P(\tilde{\Delta}^a)$ is flat for $\tilde{\Delta}^a \lesssim 10^{-3.5}$, decreases as $\sim (\tilde{\Delta}^a)^{-1}$ for $\tilde{\Delta}^a \lesssim 10^{-1}$, and then falls more rapidly as $\sim (\tilde{\Delta}^a)^{-4}$ for larger $\tilde{\Delta}^a$. As θ decreases, the region of rapid suppression shifts toward smaller $\tilde{\Delta}^a$, accompanied by a higher plateau value near $\tilde{\Delta}^a \sim 0$. This behavior suggests that imprecise Landau gauge fixing affects physical quantities differently than the ξ gauge with small but nonzero ξ , where deviations from the Landau gauge are expected to follow a Gaussian distribution that decays exponentially at relatively large $\tilde{\Delta}^a$.

Based on similar calculations for $\xi \simeq 0.2, 0.6$, and 1.0 , shown in Fig. 5, the distributions of $\tilde{\Delta}^a$ for nonzero ξ exhibit similar patterns in both their $\tilde{\Delta}^a$ and θ dependencies. This observation suggests that the empirical formula established for the Landau gauge may also describe the θ dependence of physical quantities for $\xi \leq 1$, although we emphasize that this remains an assumption given the ill-posed nature of ξ gauge fixing and the absence of a rigorous proof.

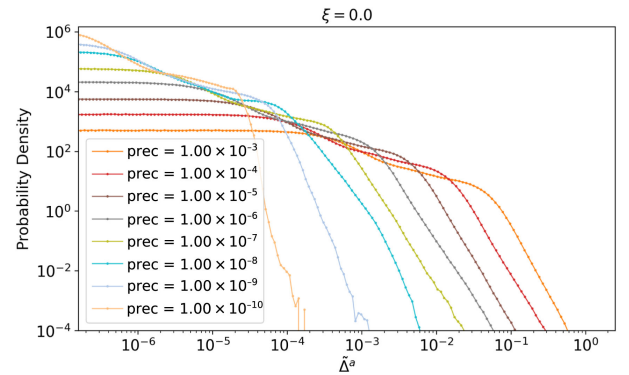


FIG. 4. Probability density of the residual $\tilde{\Delta}^a$ in Landau gauge.

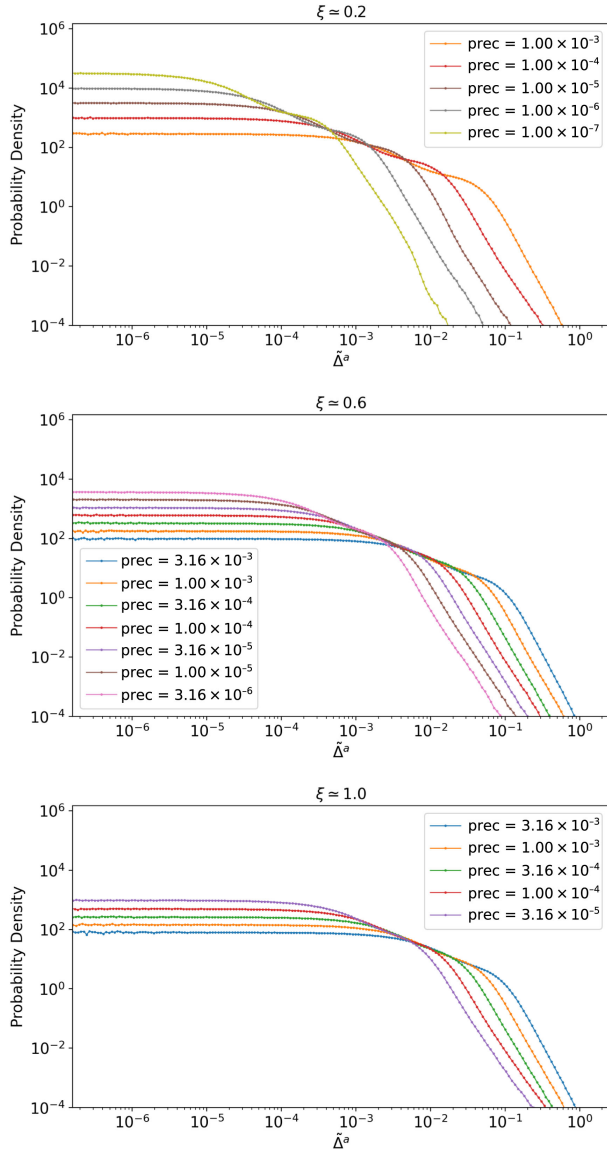


FIG. 5. Probability density of the residual $\tilde{\Delta}^a$ with different precision (prec), for $\xi \approx 0.2, 0.6$, and 1.0 .

Reference [31] demonstrated that the bias introduced by imprecise Landau gauge fixing on nonlocal operators can be captured by an empirical formula,

$$X(\theta) = X^{\text{fit}} e^{-c(X)\theta^{n(X)}}, \quad (12)$$

where X^{fit} , c , and n are fit parameters, and the fitting result of X^{fit} denotes the extrapolated exact result under perfect gauge fixing. For both the gauge link $W(z) \equiv U_z(0, z)$ and the nonlocal quark bilinear $\bar{\psi}(0)\gamma_t U_z(0, z)\psi(z)$, Ref. [31] found $n \sim 0.5$ and $c \propto (z/a)^2$, with c reaching $\mathcal{O}(30)$ at $z \sim 1$ and $a \sim 0.06$ fm. The empirical form in Eq. (12) remains valid up to $\theta \sim 0.01$.

As shown in Fig. 6, fitting the parameters $W(z; 0)$, c_W , and n_W using data at finite precisions

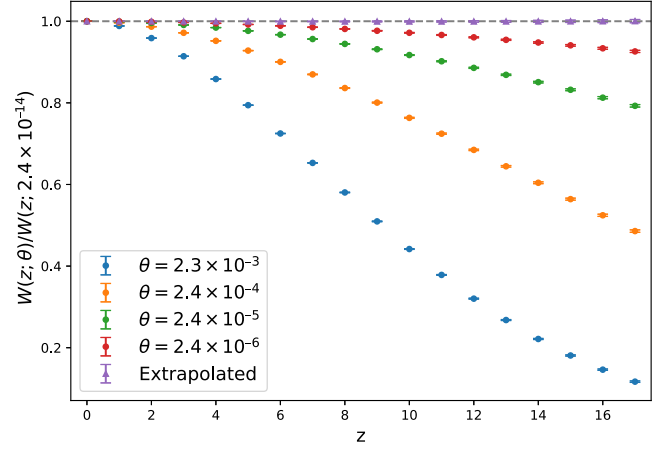


FIG. 6. Ratio of the gauge link $W(z) \equiv U_z(0, z)$ at various Landau gauge-fixing precisions $\theta = \{2.3 \times 10^{-3}, 2.4 \times 10^{-4}, 2.4 \times 10^{-5}, 2.4 \times 10^{-6}\}$ to its high-precision value $W(z; 2.4 \times 10^{-14})$. The ratio for the precision-extrapolated result (purple triangles, using the empirical formula) is also shown.

$\theta = \{2.3 \times 10^{-3}, 2.4 \times 10^{-4}, 2.4 \times 10^{-5}, 2.4 \times 10^{-6}\}$ yields a value of $W(z; 0)$ that agrees with the result obtained at the much higher precision $\theta = 2.4 \times 10^{-14}$ within statistical uncertainties, despite the large deviations present in the data at the coarser precisions.

Although this power-divergent bias vanishes for local operators ($z = 0$), the success of the precision extrapolation using only relatively poor gauge-fixing data suggests that a similar strategy could be highly beneficial in the ξ gauge, where achieving high-precision gauge fixing is currently impractical.

III. PRECISION EXTRAPOLATION UNDER LANDAU GAUGE ON LOCAL OPERATORS

We first verify Eq. (12) and the precision extrapolation in the Landau gauge as a prerequisite for the ξ gauge study. This verification uses the RI/MOM constants computed across a range of gauge-fixing precisions. In this study, we employ configurations generated by the MILC Collaboration [40–42], utilizing the $2+1+1$ highly improved staggered quark (HISQ) fermion action and the one-loop Symanzik-improved gauge action. The specifics of these configurations are detailed in Table I. For the valence quarks, we use both clover and overlap fermion actions across these ensembles with the pion mass tuned to the same as that of the light sea quark. Further details will be elaborated upon later in this section.

A. Renormalization constants of $Z_{S,T}$ on various momentum

With point-source quark propagators, one can define the bare Green's function as

TABLE I. Information of the $2 + 1 + 1$ flavor MILC ensembles [40–42] used in this study. The symbol $S^{(1)}$ denotes the Symanzik gauge action with full one-loop improvement, while the sea quark action employs the HISQ discretization.

Action	Symbol	$6/g^2$	$L^3 \times T$	a (fm)	$m_{\pi,ss}$ (MeV)
HISQ + $S^{(1)}$	a12m310	3.60	$24^3 \times 64$	0.1222	310
HISQ + $S^{(1)}$	a09m310	3.78	$32^3 \times 96$	0.0879	310
HISQ + $S^{(1)}$	a06m310	4.03	$48^3 \times 144$	0.0566	310

$$G_{\mathcal{O}}(p_1, p_2) = \sum_{x,y} e^{-i(p_1 \cdot x - p_2 \cdot y)} \langle \psi(x) \mathcal{O}(0) \bar{\psi}(y) \rangle, \quad (13)$$

and then the amputated Green's function is generally defined as

$$\Lambda_{\mathcal{O}}(p_1, p_2) = S^{-1}(p_1) G_{\mathcal{O}}(p_1, p_2) S^{-1}(p_2), \quad (14)$$

where $S^{-1}(p)$ represents the inverse of the quark propagator $S(p) \equiv \sum_x e^{-ip \cdot x} \langle \psi(x) \bar{\psi}(0) \rangle$ with momentum p . Following the Lehmann-Symanzik-Zimmermann reduction formalism, the RI/MOM renormalization constant is given by

$$Z_{\mathcal{O}_r}(\mu) \equiv \frac{Z_q(\mu)}{\frac{1}{12} \text{Tr}[\Lambda_{\mathcal{O}}(p, p) \Gamma]} \Big|_{\mu^2=p^2},$$

$$Z_q(\mu) \equiv \frac{\text{Tr}[\not{p} S^{-1}(p)]}{12 p^2} \Big|_{\mu^2=p^2}. \quad (15)$$

Rather than directly computing z_o/z_q , we evaluate z_o/z_v instead to circumvent the explicit use of z_q , which is subject to significant discretization errors, and extract z_v from the vector current conservation condition of the pseudoscalar meson. Further details on these operators are available in Ref. [43].

For this analysis, we utilize valence overlap fermions [44] on two ensembles, a09m310 ($a \sim 0.09$ fm) and a06m310 ($a \sim 0.06$ fm). To investigate the momentum dependence, we employ point-source propagators and compute Z_S/Z_V and Z_T/Z_V at different $a^2 p^2$. The propagators are solved using the deflated Clebsch-Gordon algorithm [45] with a residual tolerance of 1×10^{-7} , a precision demonstrated to be sufficient in previous renormalization studies. These quantities are derived by applying gauge rotations of varying precision $\theta \in (2.4 \times 10^{-14}, 2.2 \times 10^{-1})$ to the same underlying quark propagators and fitted with Eq. (12) to verify this fit ansatz. The Q values shown in the upper panel of Fig. 7 suggest that fit qualities are acceptable except the lowest two momenta of Z_S/Z_V at $a = 0.0566$ fm. The correlation between the data with different θ has been taken into account through the correlation matrix in the precision extrapolation.

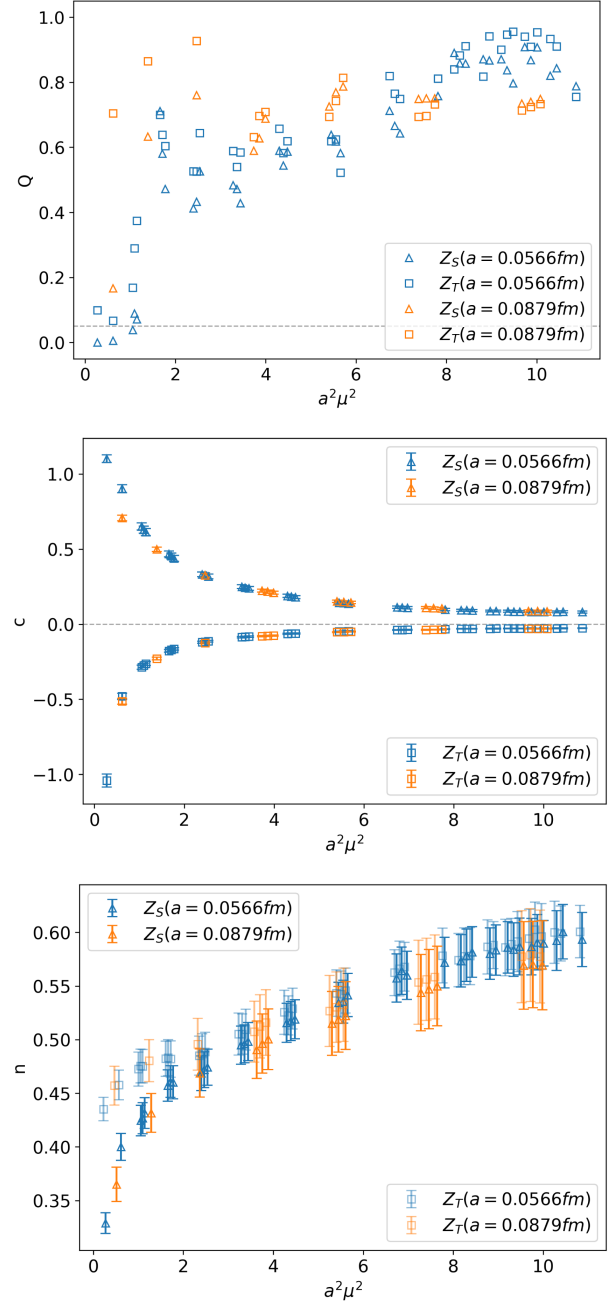


FIG. 7. Q values (upper), fit parameters c (middle), and n (lower) for the ratio $X(\theta)/X^{\text{fit}} = e^{-c \cdot \theta^n}$ of Z_S/Z_V and Z_T/Z_V across various momenta p and lattice spacing a . Original data are computed with point-source propagators rotated by gauge rotation matrices corresponding to gauge-fixing precisions $\theta = \{2.2 \times 10^{-1}, 2.3 \times 10^{-3}, 2.4 \times 10^{-4}, 2.4 \times 10^{-5}, 2.4 \times 10^{-6}, 2.4 \times 10^{-7}, 2.5 \times 10^{-9}, 2.4 \times 10^{-11}, 2.4 \times 10^{-14}\}$. For a given operator (scalar or tensor), both c and n lie on the same $a^2 p^2$ curve and are insensitive to the lattice spacing a .

The results of the fitted c and n are presented in the other two panels of Fig. 7. For a given operator (scalar or tensor), both c and n lie on the same $a^2 p^2$ curve and are insensitive to the lattice spacing a . The approximation $n \approx 0.5$ [31]

seems to be held in large range of momenta, while n decreases to roughly 0.4 at lower momenta. At the same time, the absolute value of c increases rapidly at small $a^2 p^2$. Both features require higher gauge-fixing precision at small momenta for a given relative deviation, and a smaller lattice spacing further tightens this requirement at fixed momentum. For instance, with gauge-fixing precision $\theta \sim 2 \times 10^{-6}$, Z_S at $\mu = 1.75$ GeV deviates by 1.5% for $a = 0.0566$ fm ($a^2 \mu^2 \simeq 0.25$), whereas the deviation drops to 0.6% for $a = 0.0879$ fm at the same scale and precision.

The precision extrapolation allows us to correct for these deviations using data obtained even at relatively coarse gauge-fixing precisions. For the scalar and tensor operators, we fit the renormalization constants $Z(a^2 p^2, \theta)$ using the four largest values of θ ($\theta = 2.2 \times 10^{-1}$, 2.3×10^{-3} , 2.4×10^{-4} , 2.4×10^{-5}) based on the ansatz in Eq. (12), with $\bar{Z}^{\text{fit}}(a^2 p^2)$, $\bar{c}(a^2 p^2)$, and $\bar{n}(a^2 p^2)$ as the renamed free parameters. The quality of the extrapolation is assessed by comparing the extrapolated values $\bar{Z}^{\text{fit}}(a^2 p^2)$ to the high-precision results computed at $\theta = 2.4 \times 10^{-14}$. As shown in Fig. 8, the ratio of these values is consistent with unity within 0.2% or even smaller statistical uncertainties.

In Fig. 9, we explicitly show the data for $Z(a^2 p^2, \theta)$ as a function of θ for three representative momenta spanning the infrared, intermediate, and ultraviolet regions. The data points marked with circles correspond to the data used in the extrapolation, while the solid curves represent the central values of the fits using Eq. (12). The data points indicated with crosses, which lie in the gray-shaded region excluded from the fit, are also shown and agree well with the extrapolated curves, further demonstrating that the fitting procedure captures the θ dependence reliably.

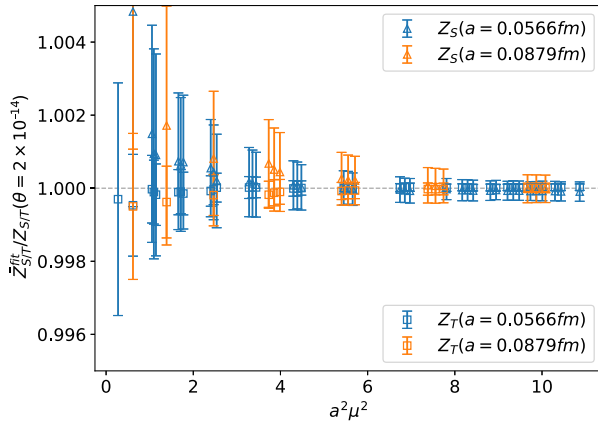


FIG. 8. Ratio of precision-extrapolated $\bar{Z}_{S,T}^{\text{fit}}$ at various $a^2 p^2$ to its high-precision value $Z_{S,T}(2.4 \times 10^{-14})$. Only the results with relatively poor gauge-fixing precision $\theta = \{2.2 \times 10^{-1}, 2.3 \times 10^{-3}, 2.4 \times 10^{-4}, 2.4 \times 10^{-5}\}$ are used for the extrapolation.

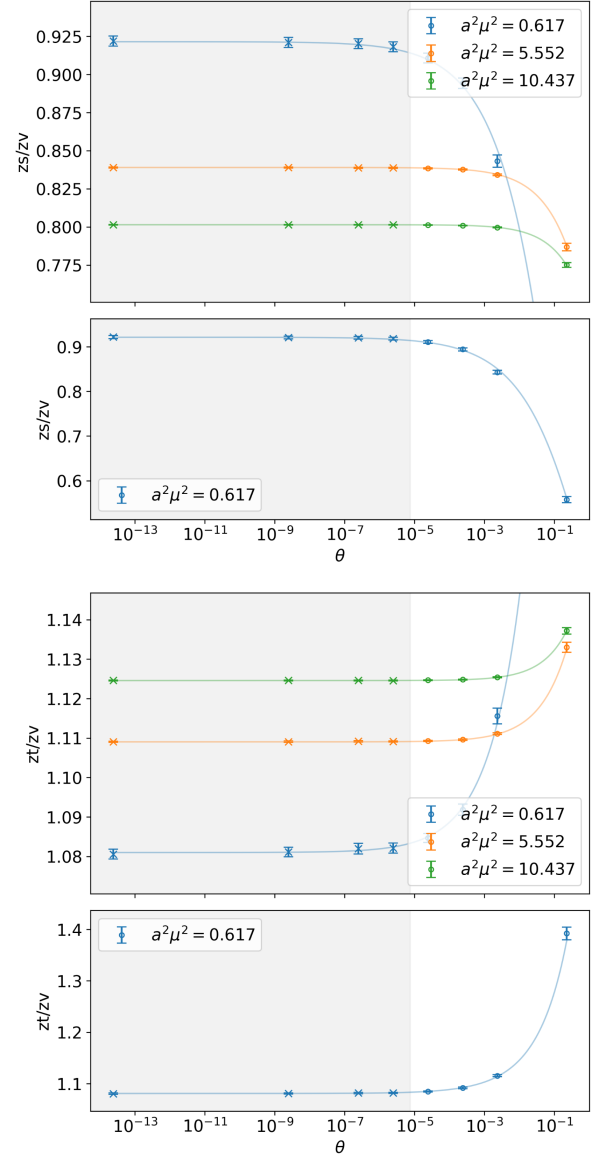


FIG. 9. Gauge-fixing residual θ dependencies of Z_S/Z_V (upper) and Z_T/Z_V (lower) for the scale μ in the infrared ($a^2 \mu^2 = 0.617$), intermediate ($a^2 \mu^2 = 5.552$), and ultraviolet ($a^2 \mu^2 = 10.437$) regions. The gray-shaded region of θ is excluded from the precision extrapolation, yet the data points within it agree well with the extrapolated curves (shown in color).

B. Further check with valence clover volume-source propagators

For a more accurate check on the deviation of imprecise gauge fixing for different operators, we generate volume-source propagators with dimensionless momentum (5, 5, 0, 0) (corresponds to $\mu \simeq 3$ GeV) and gauge-fixing precisions $\theta \in (2.4 \times 10^{-14}, 2.5 \times 10^{-2})$, using valence clover fermions on two ensembles, a06m310 ($a \sim 0.06$ fm and then $a^2 p^2 = 0.85$) used above and also a12m310 ($a \sim 0.12$ fm and then $a^2 p^2 = 3.43$). The volume-source propagators are computed using the multigrid algorithm

TABLE II. Gauge-fixing precision parameters c , n , and Z^{fit} for $X(\theta) = X^{\text{fit}} e^{-c\theta^2}$ of Z_T , $Z_{X_{2a}}$, and $Z_{X_{2b}}$ at two lattice spacings and compared with the values $Z(\epsilon_0)$ with the highest available precision $\epsilon_0 = 2.4 \times 10^{-14}$.

		c	n	Z^{fit}	$Z(\epsilon_0)$
a06m310	Z_T	-0.80(4)	0.522(9)	1.0916(2)	1.0915(1)
	$Z_{X_{2a}}$	-1.60(5)	0.481(6)	1.2376(4)	1.2380(4)
	$Z_{X_{2b}}$	-1.64(5)	0.481(6)	1.2249(4)	1.2253(4)
a12m310	Z_T	-0.185(6)	0.513(9)	1.0578(1)	1.0575(1)
	$Z_{X_{2a}}$	-0.508(9)	0.482(5)	1.1539(3)	1.1542(2)
	$Z_{X_{2b}}$	-0.467(9)	0.487(6)	1.1338(3)	1.1340(2)

[46] to a similar tolerance of $\sim 1 \times 10^{-7}$, employing a setup consistent with Ref. [31]. Those propagators allow us to compute Z_T/Z_V and also those of the quark energy moment tensor operators,

$$X_{2a} \equiv \bar{\psi} \gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}} \psi|_{\mu \neq \nu},$$

$$X_{2b} \equiv \frac{1}{2} \bar{\psi} \left(\gamma_1 \overleftrightarrow{D}_1 + \gamma_2 \overleftrightarrow{D}_2 - \gamma_3 \overleftrightarrow{D}_3 - \gamma_4 \overleftrightarrow{D}_4 \right) \psi, \quad (16)$$

where the symmetric covariant derivative is given by $\overleftrightarrow{D}_\nu = \overrightarrow{D}_\nu - \overleftarrow{D}_\nu$. The use of volume-source propagators enables the efficient calculation of $Z_{X_{2a(b)}}$, as all necessary derivative directions can be obtained at the sink of a single propagator. However, a full investigation of the $a^2 p^2$ dependence requires repeated calculations at multiple momenta; such an analysis is beyond the scope of this work and is therefore omitted.

The values of Z^{fit} , c , and n of different operators using the lowest four gauge-fixing precisions ($\theta \geq 2.4 \times 10^{-6}$ for a06m310 and $\theta \geq 2.4 \times 10^{-5}$ for a12m310) at two lattice spacings are collected in Table II and compared with those using the highest $\theta = 2.4 \times 10^{-14}$. Even with the volume source, the c and n for the scalar current using the clover fermion still have very large uncertainty and are not shown here.

Even with the volume source, the $c(Z_S)$ and $n(Z_S)$ for the scalar current using the clover fermion still have very large uncertainty and are not shown here. The value $c(Z_T) = -0.7(1)$ obtained with clover fermions differs from the overlap fermion result $c(Z_T) = -0.47(2)$ on the identical a06m310 ensemble and renormalization scale μ . These discrepancies imply that the dependence on gauge-fixing precision could depend on the fermion discretization. Based on the comparison of the results at two lattice spacings with the same μ , n is always around 0.5, while $|c|$ becomes larger at smaller lattice spacing, as we found in the previous subsection using the overlap fermion.

As shown in Table II, the extrapolated values Z^{fit} agree with the results from high-precision gauge fixing to within at most 0.03%. This deviation is smaller than the typical

statistical uncertainty in hadron matrix elements. Compared to the value obtained at the highest precision used in the fit, the extrapolation reduces the deviation by at least an order of magnitude. Given $n \sim 0.5$ in the empirical formula, this corresponds to an improvement of 2 orders of magnitude in the effective gauge-fixing precision θ .

IV. APPLICATIONS ON ξ GAUGE

The success of the precision-extrapolation method in the Landau gauge suggests that a similar approach would resolve precision issues in the ξ gauge. Unlike the Landau gauge that can be fixed to the machine precision, the minimal attainable θ in the ξ gauge is inherently limited by the current gauge-fixing algorithm, especially when ξ is large.

As the ξ gauge dependence of $Z_{S,T}^{\text{RI}}$ is known perturbatively to three loops, comparing with nonperturbative determinations of $Z_{S,T}^{\text{RI}}(\xi)$ offers a powerful consistency check. This comparison tests both the sufficiency of ξ gauge-fixing precision and the validity of the precision-extrapolation approach.

We execute our calculations on the ensemble a06m310 using the same valence overlap fermion point-source propagators generated for the Landau gauge calculation, and perform the precision extrapolation for all the combinations of ξ and μ . As Fig. 1 shows, the increasing lower band of gauge-fixing precision θ at larger ξ reduces the number of available data points, as quantified in Table III.

In principle, the ξ gauge dependence of $Z^{\text{RI}}(\xi, \mu; a)$ should match the perturbatively calculated result under dimensional regularization at three loops [38], up to discretization errors,

$$R_X(\xi, \mu; a) = \frac{Z_X^{\text{RI}}(\xi, \mu; a) Z_X^{\text{RI, pert}}(0, \mu)}{Z_X^{\text{RI}}(0, \mu; a) Z_X^{\text{RI, pert}}(\xi, \mu)} = 1 + \mathcal{O}(a^2 \mu^2). \quad (17)$$

However, in practice, the value of ξ in Eq. (17) is sensitive to the definition of the bare coupling g_0 , as discussed earlier.

In Fig. 10, we plot the ratio $R_T((g_0^{(a)}/g_0)^2, a, \mu)$ of the tensor operator for three definitions of g_0 in the $\theta \rightarrow 0$ limit: (1) $g_0 = g_0^{(a)}$ from the naive definition (red dots), (2) $g_0^{(b)}$ with full tadpole improvement (blue crosses), and (3) $g_0^{(c)}$ from the u_0 approximation (black triangles). We can see

TABLE III. Number of gauge-fixing precision θ for different ξ . The largest θ is 10^{-2} and decreases progressively by a factor of $\sqrt{10}$, yielding a precision sequence like $\{10^{-2}, 10^{-2.5}, 10^{-3}, \dots\}$.

ξ	0.0	0.2	0.4	0.6	0.8	1.0
No. of θ	21	11	9	8	7	6

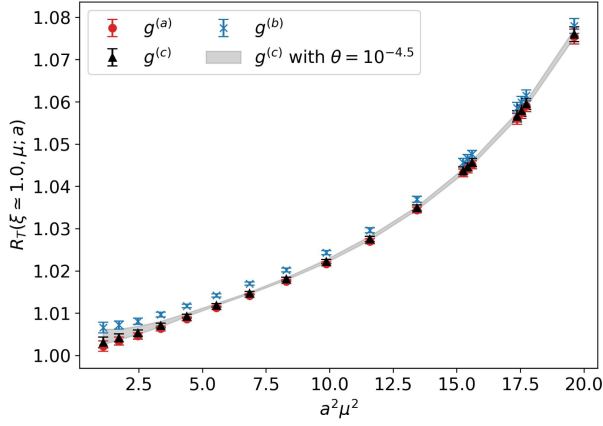


FIG. 10. Ratio of Z_T defined in Eq. (17) which should be unity up to the $a^2\mu^2$ error, with $\xi = (g_0^{(a)}/g_0)^2$ (in the $\theta \rightarrow 0$ limit) with the naive definition $g_0 = g_0^{(a)}$ (red dots), tadpole-improved one $g_0^{(b)} = 1.13g_0^{(a)}$ (blue crosses), and also u_0 approximation $g_0^{(c)} = 1.02g_0^{(a)}$ (black triangles). The gray band using the $g_0^{(c)}$ with $\theta = 10^{-4.5}$ is also shown for comparison.

that the $a^2 p^2$ extrapolated value using either $g_0^{(a)}$ or $g_0^{(c)}$ is closer to 1 than that using $g_0^{(b)}$ and then can be considered as a good choice of g_0 . We adopt $g_0^{(c)}$ to define the effective ξ for two reasons: first, it can be extracted directly from gauge configurations without requiring knowledge of the discretized action; second, unlike the naive definition $g_0^{(a)}$, it implicitly incorporates the tadpole improvement necessary for a well-convergent perturbative series.

For comparison, we also show the $R_T((g_0^{(a)}/g_0^{(c)})^2, a, \mu)$ with finite gauge-fixing precision $\theta = 10^{-4.5}$ as a gray band for comparison. We observe that the precision-extrapolated values align with those obtained using $\theta = 10^{-4.5}$, albeit with slightly smaller statistical uncertainty. This suggests that systematic uncertainties arising from precision extrapolation are well controlled in this case.

Figure 11 shows the fitting results of c and n using the empirical formula in Eq. (12), with good $\chi^2/\text{d.o.f.}$ in all the cases. We can see that n is also around 0.5 regardless of ξ and μ , while $|c|$ becomes larger at both ends of the $a^2\mu^2$ range. Since the empirical formula describes the data well for all values of ξ and $a^2\mu^2$ we studied, the precision extrapolation is expected to reduce the gauge-fixing deviation. Based on the established results in the Landau gauge, this suppression factor is typically $\sim 10^2$. Consequently, for the largest value $\xi \sim 1$, the method reduces the deviation from the practically achievable tolerance of $\theta \sim 10^{-4.5}$ to an effective level of $\theta \sim 10^{-6.5}$.

The ratios R_S (upper panel) and R_T (lower panel) are shown in Fig. 12 as the function of $a^2\mu^2$ with different ξ . Then we use the following polynomial ansatz to fit the data of the operator X using the momentum range of $a^2 p^2 \in (2, 20)$,

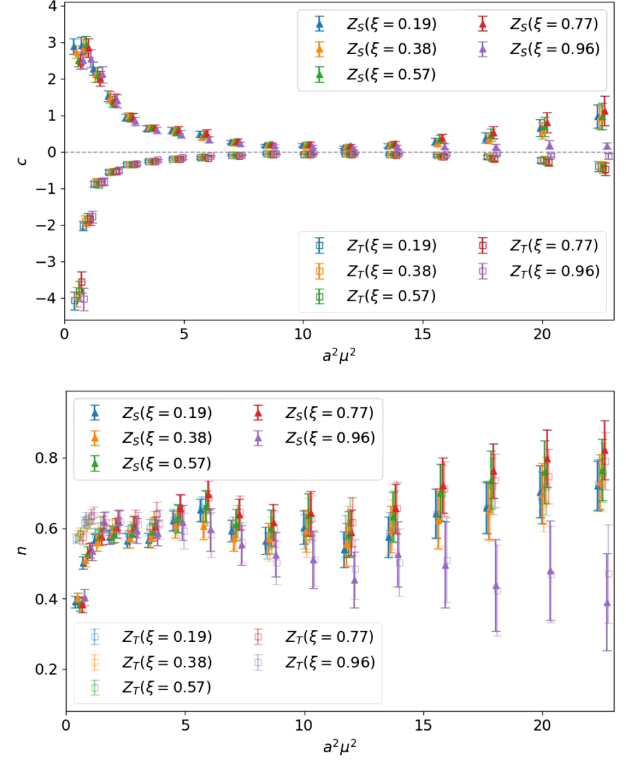


FIG. 11. Gauge-fixing precision parameters c (upper) and n (lower) for the ratio $X(\theta)/X(0) = e^{-c\theta^n}$ of Z_S and Z_T across various RI/MOM scale μ and gauge parameter ξ . Results c and n of $\xi \neq 0$ show good consistency with different ξ , while sensitive to $a^2\mu^2$ and current operator.

$$R_X(\xi, \mu; a) = c_{0,X}(\xi) \left(1 + \sum_{i=1}^3 c_{i,X}(\xi) (a^2\mu^2)^i \right), \quad (18)$$

where $c_0 - 1$ represents the deviation of the numerical ξ gauge fixing after the $a^2\mu^2 \rightarrow 0$ extrapolation.

As illustrated in Fig. 13, the fitted c_0 for different ξ 's are consistent with 1 up to 2σ with no more than 0.3% statistical uncertainty. All the fitting parameters obtained through full jackknife resampled fitting are collected in Table IV, and we can see that $c_{1,S/T}$ can be described by $c_{1,S} = -0.00803(51)\xi$ and $c_{1,T} = 0.00326(24)\xi$ within the statistical uncertainty.

While the Symanzik-improved gauge action is accurate to $\mathcal{O}(a^4)$, the renormalization constants for quark bilinear operators inherit an $\mathcal{O}(a^2)$ error from the fermion actions. Therefore, the overall deviation cannot be expected to be smaller than $\mathcal{O}(a^2)$. The $a^2\mu^2$ error in R_X would originate from the discretized gauge-fixing condition in Eq. (1), which is equivalent to using a μ -dependent ξ parameter. By defining an effective $\tilde{\xi} = \xi a^2\mu^2 / (4 \sin^2(a\mu/2)) = \xi(1 + \frac{1}{12}a^2\mu^2 + \mathcal{O}(a^4\mu^4))$ and using $\tilde{\xi}$ in the perturbative renormalization constant $Z_X^{\text{RI,pert}}$ used in Eq. (17), we suppress the $a^2\mu^2$ dependence in $R_{X=S,T}$ to the 1% level

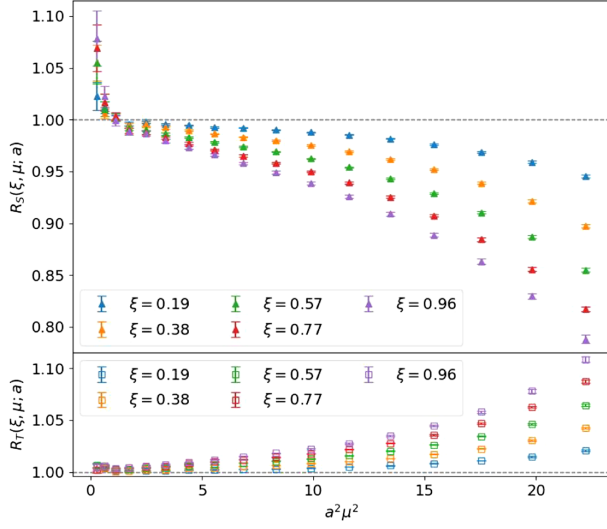


FIG. 12. The ratios R_S (upper) and R_T (lower) as the function of $a^2\mu^2$ with different ξ .

or less for $a^2\mu^2 \leq 10$, as shown in Fig. 14. This result suggests that an improved gauge-fixing condition, like the one proposed in Ref. [47], would be highly effective in suppressing this discretization error.

For illustration, we define the $a^2\mu^2$ -extrapolated Z^{RI} as

$$\tilde{Z}^{\text{RI,latt}}(\xi, \mu) \equiv \frac{Z^{\text{RI,latt}}(\xi, \mu; a)}{f(a^2\mu^2, \xi)f_0(a^2\mu^2)}, \quad (19)$$

where $f(a^2\mu^2, \xi) = 1 + \sum_{i=1}^3 c_i(\xi)(a^2\mu^2)^i$ represents the discretization error obtained by fitting Eq. (18), and $f_0(x) \equiv 1 + \sum_{i=1}^3 d_i(a^2\mu^2)^i$ quantifies additional discretization errors in $Z^{\text{MS}}(2 \text{ GeV})$ obtained through $Z^{\text{RI,latt}}(0, \mu; a)$ under the Landau gauge. Specifically, the f_0 term is extracted through the polynomial fit of the following combination:

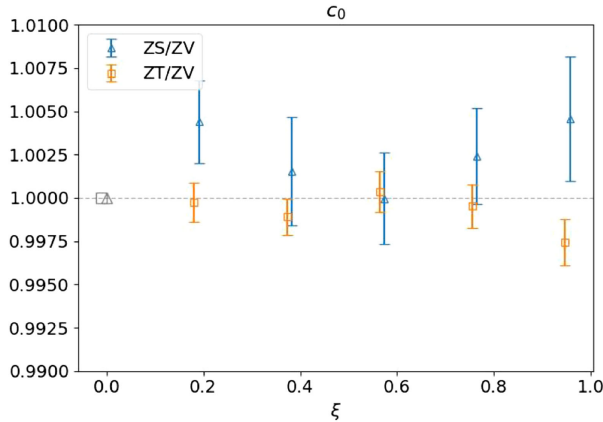


FIG. 13. Extrapolated R_S (blue triangles) and R_T (orange boxes) after the $a^2\mu^2 \rightarrow 0$ extrapolation, with different ξ .

TABLE IV. Fitting parameters of Eq. (18) for different ξ .

	ξ	c_0	$c_1(\times 10^{-3})$	$c_2(\times 10^{-4})$	$c_3(\times 10^{-5})$
Z_S/Z_V	0.191	1.0044(24)	-2.71(61)	1.85(49)	-0.81(13)
	0.383	1.0015(31)	-3.14(77)	1.45(63)	-0.96(16)
	0.574	1.0000(27)	-4.23(61)	1.71(54)	-1.22(15)
	0.765	1.0024(28)	-6.21(70)	2.28(66)	-1.45(20)
	0.957	0.9973(25)	-6.12(74)	1.25(69)	-1.24(19)
Z_T/Z_V	0.191	0.9997(11)	0.57(32)	-0.40(27)	0.24(07)
	0.383	0.9989(11)	1.37(29)	-0.96(26)	0.54(07)
	0.574	1.0004(12)	1.56(31)	-1.01(27)	0.69(08)
	0.765	0.9995(12)	2.33(35)	-1.40(30)	0.91(09)
	0.957	0.9991(14)	3.14(36)	-2.04(30)	1.27(09)

$$\begin{aligned} Z^{\text{RI,latt}}(0, \mu; a) \frac{Z^{\text{MS,pert}}(2 \text{ GeV})}{Z^{\text{RI,pert}}(0, \mu)} \\ = Z^{\text{MS,latt}}(2 \text{ GeV}) f_1(a^2\mu^2), \end{aligned} \quad (20)$$

with $Z^{\text{MS,latt}}(2 \text{ GeV})$ and $d_{i=1,2,3}$ as fit parameters. Here, the perturbative ratio $\frac{Z^{\text{MS,pert}}(2 \text{ GeV})}{Z^{\text{RI,pert}}(0, \mu)}$ is derived by matching $Z^{\text{RI,pert}}$ to Z^{MS} at scale μ first, then evolving to 2 GeV using renormalization group equations. We further define the perturbative ξ and μ dependence of $Z^{\text{RI,pert}}(\xi, \mu)$ as

$$\begin{aligned} \tilde{Z}^{\text{RI,pert}}(\xi, \mu) &\equiv \frac{Z^{\text{RI,pert}}(\xi, \mu)}{Z^{\text{MS,pert}}(2 \text{ GeV})} Z^{\text{MS,latt}}(2 \text{ GeV}) \\ &= \frac{Z^{\text{RI,pert}}(\xi, \mu)}{Z^{\text{MS,pert}}(\mu)} \frac{Z^{\text{MS,pert}}(\mu)}{Z^{\text{MS,pert}}(2 \text{ GeV})} Z^{\text{MS,latt}}(2 \text{ GeV}), \end{aligned} \quad (21)$$

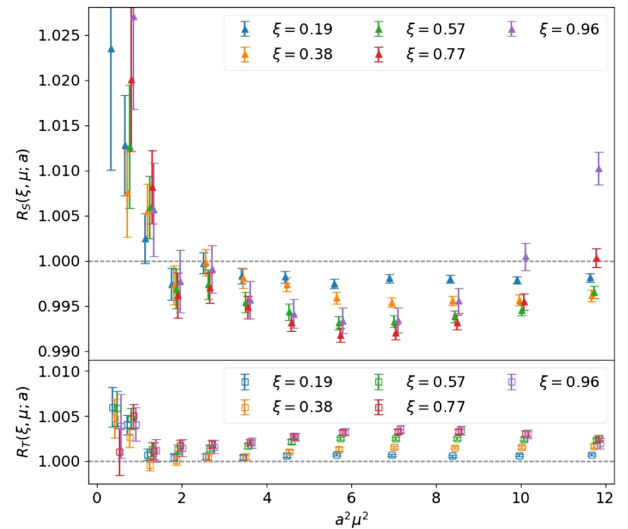


FIG. 14. R_S (upper) and R_T (lower) for using $\tilde{\xi} = \xi a^2\mu^2 / (4 \sin^2(a\mu/2))$ in the perturbative matching.

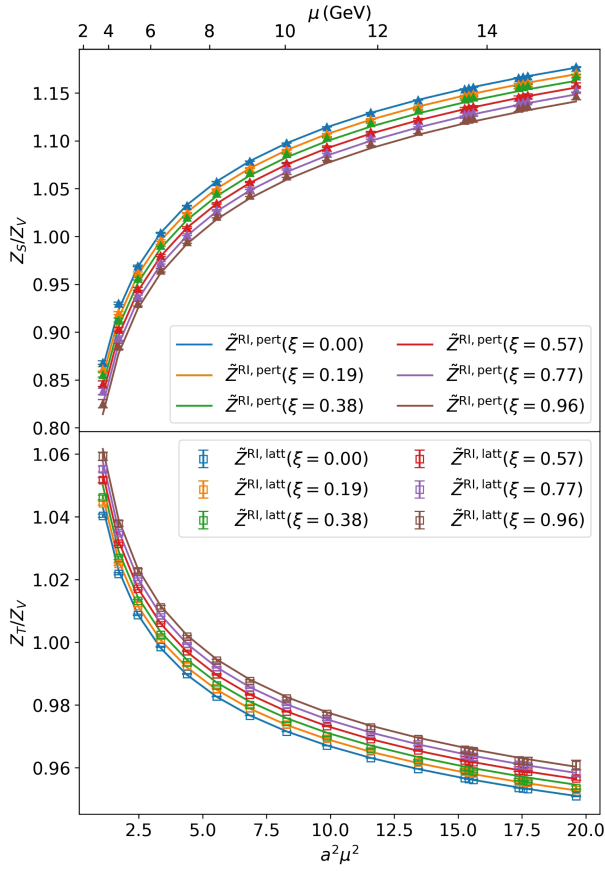


FIG. 15. Comparison of renormalization constants under RI/MOM regenerated from perturbation theory with that of lattice computation. Lines are perturbation results and data points are from lattice. The consistency of the line and points exhibits the success of taking the ξ dependence and discretization error apart.

where both the ratios $\frac{Z^{\text{RI,pert}}(\xi, \mu)}{Z^{\text{MS,pert}}(\mu)}$ and $\frac{Z^{\text{RI,latt}}(\xi, \mu)}{Z^{\text{MS,pert}}(2 \text{ GeV})}$ are finite and can be obtained by the perturbative calculations [38].

Figure 15 presents a comparison between the extrapolated lattice results $\tilde{Z}^{\text{RI,latt}}(\xi, \mu)$ (colored data points) and their perturbative counterparts $\tilde{Z}^{\text{RI,pert}}(\xi, \mu)$ (colored lines) for both scalar (upper panel) and tensor (lower panel) quark bilinear operators. The observed agreement primarily demonstrates the effectiveness of the $a^2 \mu^2$ polynomial in describing the discrepancy between lattice computations and perturbative calculations. A more rigorous comparison would require continuum extrapolation using lattice data at multiple spacing values.

V. SUMMARY

In this work, we first establish the empirical dependence of RI/MOM renormalization constants for quark bilinear operators on Landau gauge-fixing precision. We demonstrate that the deviation scales universally as $\sim \theta^{1/2}$ across operators (scalar, tensor, energy momentum tensor), RI/MOM scales, and fermion discretizations.

Thus, imprecise Landau gauge fixing differs from ξ gauge fixing with $\xi \propto \theta$, owing to the distinct parameter dependencies of physical quantities ($\sim \theta^{1/2}$ versus ξ), which originate from the respective distributions of deviations from the Landau gauge. It further allows us to rule out an interpretation of imprecise gauge fixing within the soft covariant gauge framework [48,49] based on the continuum quantization proposal [50,51]. Perturbatively, the soft covariant gauge approaches the Landau gauge in the $M^2 \rightarrow \infty$ limit with a $1/M^2$ correction [52]. If our imprecise gauge fixing were equivalent to a soft covariant gauge with some M^2 , we would expect deviations scaling as $1/M^2$. In contrast, our observed scaling is $\sim \theta^{1/2} \propto 1/M$, which is qualitatively different. Moreover, as noted in the literature [49], reaching the $1/M^2 \rightarrow 0$ limit in the soft covariant gauge corresponds to a weighted average of different minimum of θ and is then numerically challenging. Thus, it is unlikely to be achieved by imprecise gauge fixing for a given minimum of θ .

Building on this analysis, we introduce a precision-extrapolation procedure that eliminates gauge-fixing residuals in the ξ gauge. Applying this method, our lattice calculations of $Z_{S,T}^{\text{RI}}$ for $\xi \leq 1$ achieve 0.3% consistency with three-loop perturbative results.

Nevertheless, the empirical form for the θ dependence of the ξ -dependent quantities remain an assumption. Unlike the Landau gauge, the ξ gauge-fixing problem is ill posed, so the precision extrapolation here should be viewed as a practical tool rather than a rigorously justified limit.

Given that precision sensitivity grows significantly at small lattice spacings a or low RI/MOM scales μ (e.g., $\mu \lesssim 1 \text{ GeV}$), this extrapolation method is especially valuable for accurate infrared parton studies. It also paves the way for systematic investigations of ξ -dependent quark and gluon propagators and their nonperturbative infrared interactions under controlled gauge-fixing uncertainties. As such, this work provides a foundation for improving a range of phenomenological models of nonperturbative QCD.

Furthermore, improved gauge-fixing conditions [47] could significantly suppress the ξ -dependent $\mathcal{O}(a^2 \mu^2)$ discretization errors observed in Fig. 12, making this an important direction for future study. At the same time, the precision-extrapolation method becomes unreliable for $\xi \gtrsim 1.2$, as the minimal achievable residual $\theta_{\min} \sim 10^{2.9\xi-7.3}$ drastically reduces the number of viable data points. Extending these calculations to larger ξ values will therefore require the development of more sophisticated gauge-fixing algorithms.

The calculations were performed using the Chroma software suite [53] with QUDA [46,54,55] and GWU codes [56,57] through then HIP programming model [58].

ACKNOWLEDGMENTS

We thank the MILC Collaboration for providing their HISQ gauge configuration and Ying Chen for valuable comments and suggestions. The numerical calculations were carried out on the ORISE Supercomputer, HPC Cluster of ITP-CAS and Advanced Computing East China Subcenter. This work is supported in part by National Key R&D Program of China No. 2024YFE0109800, NSFC Grants No. 12293060, No. 12293062, No. 12435002, No. 12447101, No. 12447102, and No. 12175030, the Science and Education Integration Young Faculty Project

of University of Chinese Academy of Sciences, and the Strategic Priority Research Program of Chinese Academy of Sciences, Grant No. YSBR-101.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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