

Off-shell form factor: Factorization is violated

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We study the Sudakov form factor on the Coulomb branch of $\mathcal{N} = 4$ SYM which endows only external states with masses, and implies that the former is off-shell in the traditional sense. Our consideration is performed at three-loop order in the near mass-shell limit. We use a combination of tools to perform required calculations centered around the method of regions as the main go-to formalism for the asymptotic expansion of emerging parametric Feynman integrals. Explicit separation of quantum loops in terms of hard, collinear, and ultrasoft modes allows us to explore the factorization properties of this infrared-sensitive quantity. While the hard region is cleanly separated from the rest, the ultrasoft-collinear modes remain intertwined. We exhibit effects of factorization violation explicitly in the momentum space making use of the infrared power counting.

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I. INTRODUCTION

The Sudakov form factor [1] is the simplest infrared-sensitive matrix element, which describes the production of a colored particle pair by a gauge-invariant operator source from the vacuum. It thus represents the simplest laboratory to explore gauge dynamics. To use the full predictive power of perturbative quantum field theory, one must establish a clear separation of physics at different space-time scales involved. Intuitively, this should occur due to the immunity of the short-distance, aka hard, scattering to long wavelength, aka infrared (IR), (nearly) real emissions. The latter can affect the former only incoherently. IR components *per se* can be classified according to the scaling pattern of their momenta with respect to a soft scale intrinsic to processes with gauge bosons. If the latter are predominantly emitted in the direction of their parent, they define the collinear regime, while if all components of gauge momenta are small, they correspond to the (ultra)soft domain. Classically, one anticipates that (ultra)soft modes feel only the overall color charge of the fast propagating jet of collinear emissions and, similarly to the hard factorization alluded to above, influences it only incoherently. These general arguments require proof. Proofs are observable specific and are not universal. They are not warranted.

Typically, factorization theorems rely on Landau equations [2,3] and establish pinch surfaces of amplitudes/cross sections where propagators go on-shell and blow up. These allow one to devise their decomposition in terms of various IR components. One of the central elements in these considerations is the ability to use eikonal, or Grammer-Yennie, approximation. Then, a precise form for various incoherent building blocks can be established in terms of matrix elements of certain nonlocal operators. Over the years, a variety of techniques has been used for these purposes: (i) original QCD-field formulation [4–11], (ii) soft-collinear effective theory that introduces individual modes for each IR regime [12–14], and (iii) effective field theory [15,16].

There are processes where the above factorization theorems are known to be broken. One of the most well-studied cases corresponds to the situation that receives leading power-counting contributions from the so-called Glauber gluons [17–21]. The latter are distinguished from other IR modes by the fact that they possess transverse (with respect to some collinear directions) momentum which is much larger than their energy/longitudinal components. These induce interactions between initial and final states and are known to invalidate strict factorization of amplitudes [19,22]. They can vanish, however, at a cross-section level if observables are sufficiently inclusive [19,22–24], or ruin them for noninclusive ones altogether [25–27]. The latter effects manifest themselves through the appearance of superleading logarithms at high perturbative orders. Apparently, a different violation of strict factorization takes place in exceptional kinematics when some of the initial and final state momenta become collinear [21,28,29].

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However, its physical origin is related to the emergence of superleading logarithms [29].

In this paper, we address hard-ultrasoft-collinear factorization of the off-shell Sudakov form factor. As compared to the aforementioned works by other authors, we will rely on the method of regions [30] (see also Refs. [31–33] for extensive and concise reviews, respectively) as a tool for the classification and extraction of IR physics from perturbative graphs order by order in perturbation theory. This framework has the advantage of providing a precise definition for leading momentum regions in IR power counting. Having established their integral form, one can then devise matrix elements of operators that reproduce them in the perturbative expansion. This task was accomplished at two loops in Ref. [34]. Presently, we push it to the next perturbative order, i.e., three loops.

In this work, the off-shell Sudakov form factor is studied in a “toy” gauge theory, the maximally supersymmetric Yang-Mills theory, aka $\mathcal{N} = 4$ sYM, on the Coulomb branch. The reason for this is twofold. First, this model provides a simple way to construct a gauge-invariant off-shell generalization of on-shell amplitudes. Second, the off-shell Sudakov form factor in the near mass-shell limit possesses an exact all-orders form, with the off-shellness serving as an IR regulator, so it allows one to study (ultra) soft-collinear physics unobstructed by regularization issues.

Our subsequent presentation is organized as follows. In the next section, we define the off-shell Sudakov form factor and introduce IR power counting for integration momentum in one-loop graphs. Next, we recall our results at two-loop order before turning to the three-loop consideration within the method of regions. In Sec. VIA, we factor out the hard region from the ultrasoft-collinear modes and discuss complications in separating the latter two. In Sec. VIB, we pinpoint the origin of the obstruction. Finally, we conclude.

II. SUDAKOV FORM FACTOR AND IR POWER COUNTING

The Sudakov form factor $\mathcal{F}_2$ in $\mathcal{N} = 4$ sYM is defined as a transition matrix element

$$(2\pi)^4 \delta^{(4)}(q - P_1 - P_2) \mathcal{F}_2 = \int d^4x e^{-iq \cdot x} \langle P_1, P_2 | \mathcal{O}(x) | 0 \rangle \quad (2.1)$$

of the half-BPS operator built from the sextet of scalar fields ϕ_{AB} of the model

$$\mathcal{O}(x) = \frac{1}{2N_c} \text{tr}_{\text{adj}} \phi_{12}^2(x). \quad (2.2)$$

This is the lowest component of the stress-tensor multiplet and, thus, it does not renormalize. On the Coulomb branch

of the theory, which can be obtained by a generalized dimensional reduction from the six-dimensional $\mathcal{N} = (1, 1)$ sYM [35–37], one can furnish only external legs $P_{1,2}$ with masses [37,38], while keeping all other lines massless. We will interpret the former as virtualities. Moreover, we will pass to the Euclidean kinematics and choose all invariants to be spacelike, i.e.,

$$q^2 = -Q^2, \quad P_i^2 = -m^2, \quad Q^2 \gg m^2 \rightarrow 0. \quad (2.3)$$

Here, we also imposed a strong ordering on m and Q since we will be only interested in the near mass-shell limit. Keeping m small but nonvanishing will serve as a regulator of IR divergences. This setup differs from the on-shell case [39], where one needs to introduce an additional regulator to tame IR singularities. This is traditionally done by continuing the dimension of space-time away from four [40,41] or by means of a Higgs mechanism [42]. In the following consideration, we will set $Q^2 = 1$ without loss of generality since up to an overall scale $\mathcal{F}_2$ depends on the dimensionless variable $t = m^2/Q^2$, such that with this choice the off-shell Sudakov form factor is a function of the virtuality t only. We will use t as the IR scale throughout our consideration.

In this paper, we will study (2.1) within perturbation theory. More specifically, we will calculate it to the highest currently achievable order, i.e., three loops. The expansion

$$\mathcal{F}_2 = \sum_{\ell} g^{2\ell} \mathcal{F}_2^{(\ell)} \quad (2.4)$$

runs in the 't Hooft coupling

$$g^2 \equiv e^{-\epsilon \gamma_E} \frac{g_{\text{YM}}^2 N_c}{(4\pi)^{D/2}}. \quad (2.5)$$

For the reasons that will become clear very shortly, we define it in $D = 4 - 2\epsilon$ dimensions.

The tree-level form factor is normalized to one, $\mathcal{F}_2^{(0)} = 1$. The next three orders $\ell = 1, 2, 3$ were calculated in Refs. [43,44] employing the method of canonical differential equations [45] for basis momentum integrals at each perturbative order that were constructed [41] with the unitarity-cut sewing procedure of Refs. [46,47]. What was found as a result of that consideration is that the off-shell Sudakov form factor exponentiates

$$\log \mathcal{F}_2 = -\frac{1}{2} \Gamma_{\text{oct}}(g) \log^2 t - D(g) \quad (2.6)$$

for the IR double logarithmic part as well as its finite part $D(g)$ and the first three terms in the 't Hooft expansion coincide with the octagon anomalous dimensions Γ_{oct} and the overall normalization of the so-called octagon form factor [48]

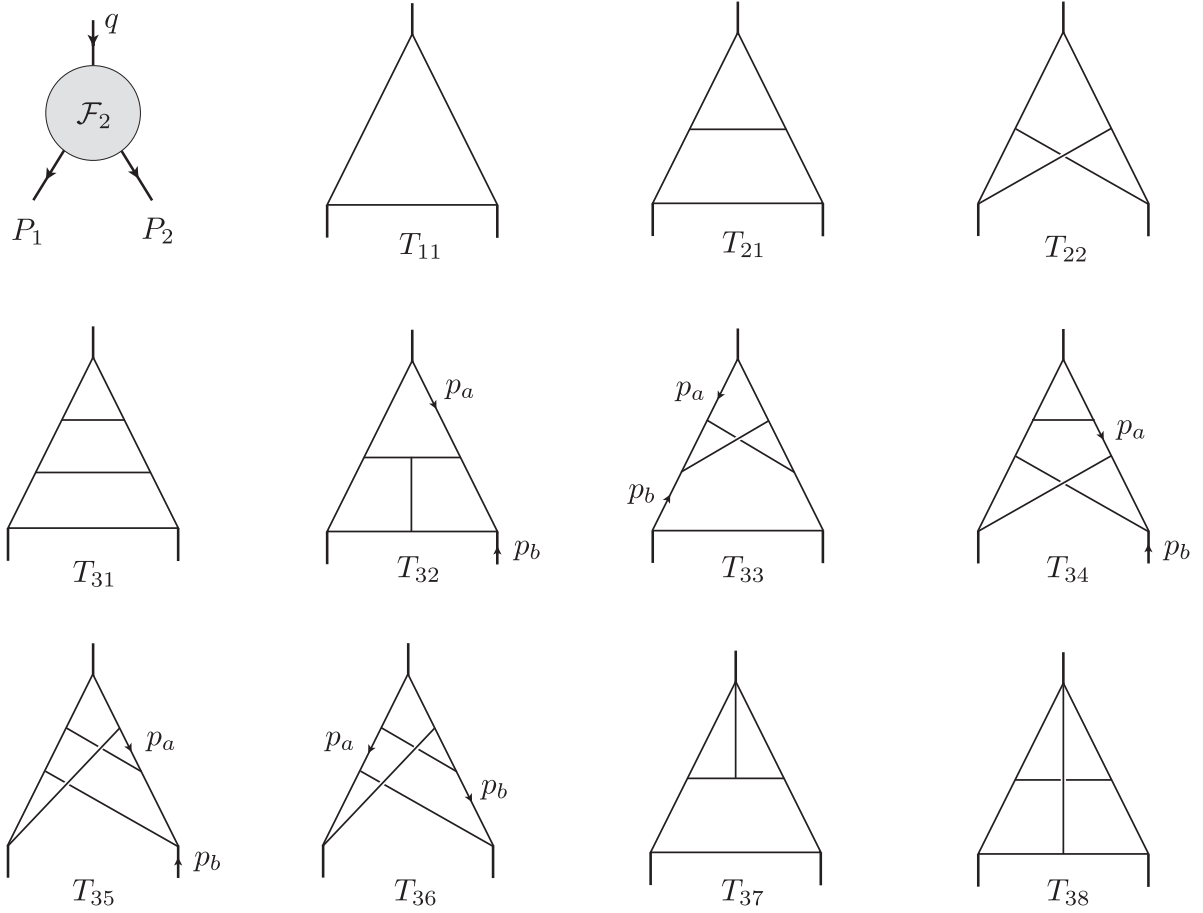


FIG. 1. Generic representation for the Sudakov form factor (top leftmost figure) and contributing momentum integrals to three-loop order. Arrows on perturbative graphs imply the presence of an irreducible numerator $(p_a + p_b)^2$.

$$\Gamma_{\text{oct}}(g) = \frac{2}{\pi^2} \log \cosh(2\pi g), \quad D(g) = \frac{1}{4} \log \frac{\sinh(4\pi g)}{4\pi g}. \quad (2.7)$$

This was yet another manifestation of an observation that a four-point correlator of heavy half-BPS operators can be interpreted as a scattering amplitude of massive W-bosons [38]. With this IR-sensitive observable possessing such a simple all-orders form, it is only natural to test its factorization properties.

At one loop, it receives a single contribution¹

$$\mathcal{F}_2^{(1)} = -2T_{11} \quad (2.8)$$

from a triangle Feynman integral, see Fig. 1, built from the products of the scalar propagators $D(k) = [k^2 + i0]^{-1}$

$$T_{11} = - \int_k D(k) D(k - P_1) D(k + P_2), \quad (2.9)$$

¹We follow the nomenclature of Ref. [34] and thus keep the same sign assignments.

over the dimensionally regularized measure

$$\int_k \equiv e^{\epsilon \gamma_E} \mu^{2\epsilon} \int \frac{d^D k}{i\pi^{D/2}}. \quad (2.10)$$

We will set $\mu = 1$ in what follows. To detect the pinch surfaces as $t \rightarrow 0$, we employ the Landau criterium that the inverse propagators have to vanish or be proportional to external momenta [2]. To discuss it in a transparent fashion, it is instructive to use the Sudakov decomposition for the loop momentum

$$k = (k_+, k_-, k_\perp) = k_+ p_1 + k_- p_2 + k_\perp, \quad (2.11)$$

in terms of two lightlike vectors p_1 and p_2 , with $p_1 \cdot p_2 = -\frac{1}{2}$, which encode predominant directions of the nearly on-shell external legs

$$P_1 = p_1 + t p_2, \quad P_2 = p_2 + t p_1, \quad (2.12)$$

respectively. Here, without loss of generality we set $P_{1,+} = P_{2,-} = 1$. In this manner, one uncovers the IR scaling for the loop momentum in powers of t . They are

$$P_1\text{-collinear: } k_{c1} \sim (t^0, t, \sqrt{t}), \quad (2.13)$$

$$P_2\text{-collinear: } k_{c2} \sim (t, t^0, \sqrt{t}), \quad (2.14)$$

$$\text{ultrasoft: } k_{us} \sim (t, t, t). \quad (2.15)$$

These are the leading IR singular regions. Folding in the scaling of the integration measure (2.10) along with the behavior of scalar propagators, one finds that T_{11} has the following IR scalings:

$$T_{11}^{c1} \sim O(t^{-\varepsilon}), \quad T_{11}^{c2} \sim O(t^{-\varepsilon}), \quad T_{11}^{us} \sim O(t^{-2\varepsilon}), \quad (2.16)$$

and asymptotes to

$$T_{11}^{c1} = - \int_k D(k) D(k - P_1) D_{\text{eik}}(k + P_2), \quad (2.17)$$

$$T_{11}^{c2} = - \int_k D(k) D_{\text{eik}}(k - P_1) D(k + P_2), \quad (2.18)$$

$$T_{11}^{us} = - \int_k D(k) D_{us}(k - P_1) D_{us}(k + P_2). \quad (2.19)$$

Here, we introduced the eikonal and ultrasoft approximations for scalar propagators

$$\begin{aligned} D_{\text{eik}}(k_i \pm P_j) &\equiv [\pm 2k_i \cdot p_j]^{-1}, \\ D_{us}(k_i \pm P_j) &\equiv [P_j^2 \pm 2k_i \cdot p_j]^{-1}. \end{aligned} \quad (2.20)$$

To match the original integral (2.9) to its IR decomposition, we have to account for a t -independent remainder. This corresponds to all components of the loop momentum being of order $O(t^0)$,

$$\text{hard: } k_h = (t^0, t^0, t^0). \quad (2.21)$$

This defines the hard region,

$$T_{11}^h = - \int_k D(k) D(k - P_1) D(k + P_2)|_{t=0} \quad (2.22)$$

with the external virtuality exactly set to zero. In other words, its integrand is identical to the massless case. If we now allow for the (components of the) loop momentum to cover the entire real axis, we get the representation advocated within the method of regions [30],

$$T_{11} = T_{11}^h + t^{-\varepsilon}(T_{11}^{c1} + T_{11}^{c2}) + t^{-2\varepsilon}T_{11}^{us}. \quad (2.23)$$

Here for presentation clarity, we explicitly pulled out the IR scalings (2.16) out of the integral definitions (2.17)–(2.19). Now, it becomes transparent why we insisted on keeping the dimension of space-time away from four. The above-approximated integrals T_{11}^{c1} , T_{11}^{c2} , and T_{11}^{us} are divergent in

the ultraviolet, while m takes on the role of an IR “cutoff.” T_{11}^h is infrared divergent, on the other hand, and is regularized dimensionally. Explicit expressions for these integrals are well known [14,34] and are quoted for the reader’s convenience in Appendix A 1. Summing up contributions of these regions, we observe the cancellation of the ε dependence and confirm Eq. (2.6) with just the leading g^2 terms kept in Eq. (2.7).

It is important to stress that the method of regions has the status of experimental mathematics. However, up to now, there are no known examples where it is found to fail. We refer the interested reader to Refs. [33,49] for discussions of possible ways to prove this strategy.

This one-loop analysis suggests a multiplicatively factorized form for the near mass-shell form factor in terms of incoherent momentum components: hard, collinear, and ultrasoft,

$$\begin{aligned} \mathcal{F}_2 &= (1 - 2g^2 T_{11}^h)(1 - 2g^2 t^{-\varepsilon} T_{11}^{c1})(1 - 2g^2 t^{-\varepsilon} T_{11}^{c2}) \\ &\quad \times (1 - 2g^2 t^{-2\varepsilon} T_{11}^{us}) + O(g^4), \end{aligned} \quad (2.24)$$

with each factor corresponding to its own momentum region and reflecting Taylor expansion of the integral T_{11} in the vicinity of the corresponding pinch surfaces. Does this persist at higher loops? This is what we are set to answer in the rest of this paper.

Analyses of higher loop integrands with proper implementation of IR power counting (2.13) is very challenging since it depends on the routing of loop momenta. So, an exhaustive enumeration of all leading regions is next to impossible. To circumvent this problem, the Feynman parameter integral representation is a far better choice [50], and moreover, it can be recast in an entirely geometric way by associating a *Newton polytope* to each integrand as a convex hull of its vertices [51]. We refrain from an in-depth review of this formalism since we recently did just that in Refs. [34,44], but we will shed light on it from the perspective of Ref. [52].

Since the two-loop integrals T_{21} and T_{22} were recently addressed in Ref. [34], with results quoted in Appendix A 2 to the required order $O(\varepsilon^2)$, we jump straight to three loops.

III. SOLVING INTEGRALS WITH MELLIN-BARNES METHODS

To reveal regions for a given Feynman integral, we apply a systematic algorithm [50,51,53] implemented in the computer code ASY [51,53] that unambiguously determines lower facets of a Newton polytope connected with two Symanzik polynomials of the Feynman parametric representation. In fact, the expansion by regions can be applied with `asy` to any parametric integral over R_+^N with

²For other domains, one should first map it to R_+^N and then proceed with ASY. Examples of its application to integrals, which are not Feynman integrals, can be found in [54,55].

the integrand given by the product of polynomials raised to powers depending linearly on the regularization parameter ε .

In the case of Feynman integrals, it proves more convenient to apply the code ASY as a part of the FIESTA5 distribution package [56,57] by executing the command `SDExpandAsy` because within it all basic properties of Feynman integrals, including the folklore Cheng-Wu theorem³ [58], are already taken into account. At this point, information about propagators, indices, and the desired orders of the ε expansion and the expansion in the parameter t is chosen.

In our previous paper [44], we evaluated the asymptotic behavior of the three-loop integrals T_{3n} , shown in Fig. 1 for $n = 1, 2, \dots, 8$, at leading order in the limit $t \rightarrow 0$ as a terminating series in powers of $\log t$, with the largest power being $\log^6 t$. We relied mostly on the method of canonical differential equations [45,59,60] with partial checks from the method of regions. Presently, our goal is to evaluate *individual* contributions from all the regions in the expansion of the integrals T_{3n} . We can slightly simplify the task at hand by lumping up two integrals together, i.e., T_{35} and T_{36} , since they have identical propagator structures, possess the same coefficients in the Sudakov form factor, and differ only in the presence of irreducible numerators. Thus, we evaluate the asymptotics of the combined integral $T_{3,56} = T_{35} + T_{36}$ from the same family with the following propagators and irreducible numerators:

$$\left\{ -k_1^2, -k_2^2, -k_3^2, -(k_3 - P_1)^2, -(k_2 + P_2)^2, -(k_1 + k_2 + k_3 - P_1)^2, -(k_1 + k_2 + k_3 + P_2)^2, \right. \\ \left. -(k_2 + k_3 - P_1)^2, -(k_1 + k_2 + P_2)^2, -(k_3 - P_1 - P_2)^2, -(k_1 + k_2)^2, -(k_1 + k_3)^2 \right\}. \quad (3.1)$$

Here, k_i ($i = 1, 2, 3$) are the loop momenta, and the minus signs reflect our choice of the Euclidean kinematics. The $T_{3,56}$ is determined by the sum of its two Feynman integrals $F(1, 1, \dots, 1, -1, 0, 0) + F(1, 1, \dots, 1, 0, -1, 0)$.

To reveal all regions contributing at leading order in the limit $t \rightarrow 0$, we run `SDExpandAsy` for the integrals T_{3n} , $i = 1, 2, 3, 4, 56, 7, 8$. For example, in the $n = 56$ case, we obtain 41 contributions corresponding to the regions with region vectors,

$$\mathbf{r} = \left\{ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 1, 1, 1, 1, 2, 1, 1, 0\}, \right. \\ \{2, 0, 1, 1, 1, 2, 2, 1, 2, 0\}, \{1, 0, 1, 0, 1, 1, 2, 0, 1, 0\}, \{0, 1, 1, 0, 0, 1, 1, 1, 0, 0\}, \\ \{1, 0, 1, 1, 1, 1, 2, 1, 2, 0\}, \{1, 0, 1, 1, 0, 1, 1, 1, 1, 0\}, \{1, 0, 1, 0, 0, 1, 1, 0, 1, 0\}, \\ \{0, 0, 1, 1, 0, 1, 1, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0, 1, 0, 1, 0\}, \{0, 0, 0, 1, 1, 1, 1, 1, 1, 0\}, \\ \{1, 0, 0, 0, 0, 1, 1, 1, 1, 0\}, \{0, 0, 1, 0, 1, 0, 2, 0, 1, 0\}, \{2, 0, 1, 0, 1, 2, 1, 0, 1, 0\}, \\ \{1, 1, 1, 0, 1, 1, 1, 1, 0, 0\}, \{2, 0, 1, 0, 1, 2, 2, 0, 2, 0\}, \{1, 0, 0, 0, 1, 1, 1, 0, 1, 0\}, \\ \{2, 1, 1, 0, 1, 2, 2, 1, 2, 0\}, \{1, 0, 1, 0, 1, 1, 1, 0, 0, 0\}, \{1, 0, 0, 1, 1, 2, 0, 1, 0, 0\}, \\ \{1, 1, 1, 0, 0, 2, 1, 1, 1, 0\}, \{1, 0, 1, 1, 1, 2, 1, 1, 0, 0\}, \{1, 0, 0, 1, 1, 2, 1, 1, 1, 0\}, \\ \{1, 1, 0, 0, 1, 1, 1, 1, 1, 0\}, \{1, 1, 1, 1, 0, 2, 1, 2, 1, 0\}, \{0, 0, 0, 0, 0, 1, 1, 1, 0, 0\}, \\ \{1, 0, 1, 1, 1, 2, 2, 1, 1, 0\}, \{1, 1, 0, 1, 1, 2, 1, 2, 1, 0\}, \{1, 1, 0, 1, 1, 2, 0, 2, 0, 0\}, \\ \{0, 1, 1, 1, 0, 2, 1, 2, 0, 0\}, \{1, 1, 1, 1, 1, 2, 1, 2, 0, 0\}, \{2, 0, 1, 1, 1, 2, 1, 1, 1, 0\}, \\ \{1, 0, 0, 0, 0, 1, 0, 1, 0, 0\}, \{0, 1, 0, 0, 0, 1, 0, 1, 0, 0\}, \{1, 0, 0, 0, 1, 1, 0, 0, 0, 0\}, \\ \{1, 1, 1, 0, 1, 1, 2, 1, 1, 0\}, \{0, 0, 0, 1, 0, 1, 0, 1, 0, 0\}, \{1, 1, 0, 0, 1, 1, 0, 1, 0, 0\}, \\ \{0, 1, 0, 1, 0, 2, 0, 2, 0, 0\}, \{2, 1, 1, 0, 1, 2, 1, 1, 1, 0\} \left. \right\}. \quad (3.2)$$

In fact, their number remains intact at any order of the small t expansion for this integral, i.e., they manifest themselves starting from the leading order, t^0 , and running to $t^{-6\varepsilon}$. The corresponding scalings associated with the above region vectors are

$$\left\{ t^0, t^{-3\varepsilon}, t^{-5\varepsilon}, t^{-3\varepsilon}, t^{-5\varepsilon}, t^{-6\varepsilon}, t^{-4\varepsilon}, t^{-\varepsilon}, t^{-2\varepsilon}, t^{-2\varepsilon}, t^{-3\varepsilon}, t^{-6\varepsilon}, t^{-2\varepsilon}, t^{-6\varepsilon}, t^{-5\varepsilon}, t^{-5\varepsilon}, \right. \\ t^{-4\varepsilon}, t^{-2\varepsilon}, t^{-3\varepsilon}, t^{-6\varepsilon}, t^{-6\varepsilon}, t^{-5\varepsilon}, t^{-5\varepsilon}, t^{-5\varepsilon}, t^{-\varepsilon}, t^{-4\varepsilon}, t^{-3\varepsilon}, t^{-4\varepsilon}, t^{-4\varepsilon}, t^{-5\varepsilon}, t^{-5\varepsilon}, \\ \left. t^{-4\varepsilon}, t^{-4\varepsilon}, t^{-3\varepsilon}, t^{-3\varepsilon}, t^{-3\varepsilon}, t^{-4\varepsilon}, t^{-3\varepsilon}, t^{-2\varepsilon}, t^{-6\varepsilon}, t^{-4\varepsilon} \right\}. \quad (3.3)$$

³It manifests the GL(1) redundancy of the Feynman parameter integration measure.

In this manner, we find Feynman parametric integrals that enter as their accompanying coefficients. It is our goal to calculate them next as a Laurent series in the ε parameter. As usual, the use of the Cheng-Wu theorem is instrumental for the simplification of required calculations.

We postpone the discussion of the t^0 contribution until later and focus on the rest. We are left with the scalings $t^{-j\varepsilon}$ where $j = 1, 2, \dots, 6$. Our first approach to evaluate these parametric integrals in the ε expansion is the Mellin-Barnes (MB) method (see, e.g., Ref. [32]), which is based on the use of the simple formula

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{B^z}{A^{\lambda+z}} \Gamma(\lambda+z) \Gamma(-z). \quad (3.4)$$

It allows one to partition a complicated polynomial in Feynman parameters in terms of its two “simpler” components A and B . In this equation, the contour $\mathcal{C}$ runs along the imaginary axis $-i\infty$ to $+i\infty$ of the complex z plane such that all poles of $\Gamma(\dots + z)$ are to its left while the ones of $\Gamma(\dots - z)$ are to its right. Repeated use of the above formula allows one to perform all parametric integrals in terms of Euler gamma functions, yielding a multifold MB integral. Of course, one attempts to arrive at as simple a final representation as possible with the fewest number of complex integrations. In most cases, we made this step with an extensive use of the code MBCREATE.M [61]. However, sometimes it helped to derive MB representations by hand, especially in circumstances when one of the indices in a given integral was -1 since it inevitably led to very cumbersome numerators in contributing regions and visual lumping of terms was far more advantageous than the code could offer.

Having determined MB representations for all parametric integrals, the next step was to resolve singularities in ε and construct corresponding Laurent series. The goal here is to represent a given integral as a linear combination of MB integrals whose ε expansions can be performed under the integral sign. There are two public codes MB.M and MBRESOLVE.M [62,63] where this algorithm is implemented automatically. Both of them are based on integration strategies developed in [64,65]. We relied on MBRESOLVE.M [63].

The next step is to evaluate MB integrals emerging as the coefficients in the Laurent expansion in ε . Here the command DoAllBarnes from Kosower’s⁴ BARNESROUTINES.M automatically applies the first and the second Barnes lemmas (and their corollaries) whenever possible and thereby performs some integrations in terms of the Euler gamma functions.

If, after all these possibilities were exhausted, some MB integrals are still left, one can turn to high-accuracy

numerical evaluations and subsequent application of the PSLQ algorithm [66] to obtain analytic results in terms of a given basis of transcendental numbers. Here, we relied on the knowledge of uniform transcendental weights of all coefficients in the ε expansion: the coefficients of ε^{-6} are just rational numbers, the terms with ε^{-5} are absent (since we use the standard normalization of Feynman integral by multiplying the result by $e^{\gamma_E \varepsilon}$ per loop), the terms with ε^{-4} , ε^{-3} , ε^{-2} are proportional to ζ_2 , ζ_3 , ζ_4 , respectively, while ε^{-1} and ε^0 are accompanied by a linear combination of $\zeta_3 \zeta_2$ and ζ_5 , and, respectively, ζ_3^2 and ζ_6 with rational coefficients. For one-fold MB integrals, one can achieve an arbitrarily high precision, so that an analytic result is guaranteed. Our experience shows that, for two-fold MB integrals, the precision of 20 decimal places is sufficient to arrive at an analytic result as well. Moreover, we have found that in this fashion, even three-fold integrals can be successfully tackled as well, since sufficiently high precision can indeed be reached. However, above that, this strategy is hopeless. So we had to turn to other means, as we describe next. Those cases were analyzed using the method introduced in [52] as well as the method of canonical differential equations.

IV. SOLVING INTEGRALS WITH TROPICAL GEOMETRY

Let us provide a brief introduction to the formalism of Ref. [52], which was instrumental in finding the ε expansion of contributions of some relevant regions.⁵ The region integrals are examples of the so-called *Euler integrals* that admit the following generic form:

$$I(\mathbf{c}; \mathbf{s}) = \int_{\mathbb{R}_{\geq 0}^n} \frac{d\alpha}{\alpha} \prod_{i=1}^m P_j(\alpha, s)^{c_j}, \quad \frac{d\alpha}{\alpha} \equiv \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_n}{\alpha_n}, \quad (4.1)$$

where $P_j(\alpha, s)$ are polynomials in the integration variables with coefficients collectively denoted by $\mathbf{s}$,

$$P_j(\alpha, s) = \sum_{\mathbf{m} \in \mathbb{Z}^n} s_{\mathbf{m}} \alpha^{\mathbf{m}}, \quad \alpha^{\mathbf{m}} := \alpha_1^{m_1} \dots \alpha_n^{m_n}. \quad (4.2)$$

In the present case, all exponents c_j depend linearly on the common parameter ε , i.e., the dimensional regulator, $c_j = c_{j,1}\varepsilon + c_{j,0}$.

Since all of the expansion coefficients $\mathbf{s}$ are non-negative in the current application to the Sudakov integrand, singularities arise solely from the boundaries of the integration regions. There are none in the bulk. The Laurent expansion of $I(\varepsilon, \mathbf{s})$ in ε can then be obtained

⁴All required MB tools can be downloaded from <https://gitlab.com/feynmanintegrals/MB>.

⁵This section was made possible largely due to a collaboration with Giulio Salvatori. His method, introduced in Ref. [52], was indispensable for the successful calculation of contributions of some of the most complicated regions at three-loop order.

with the so-called *subtraction formula* introduced in Ref. [52]. This allows us to express $I(\epsilon, \mathbf{s})$ as a linear combination of *locally finite* integrals, i.e., they can be expanded in ϵ directly *under* the integral sign and then integrated term by term.⁶

In order to apply the aforementioned formula, conditions of the corresponding theorem 6 from Ref. [52] must be met. These can be formulated as geometrical properties satisfied by the Newton polytope of Symanzik polynomials determining the integrand,

$$\mathcal{P} := \text{Newt} \prod_{j=1}^m P_j. \quad (4.3)$$

The singularity structure of Feynman integrals is related to the facet, rather than the aforementioned vertex, description of the associated Newton polytope [67,68]. It becomes indispensable in the current consideration, and we refer to [52,67] for a precise definition of the corresponding cutout conditions. Let us consider a collection of vectors $\rho \in \mathbb{Z}^n$ that are outward-pointing normals to the facets, i.e., codimension one facets, of $\mathcal{P}$. These define the boundaries of domains of linearity. We say that two rays ρ and ρ' are *compatible* if the corresponding facets of $\mathcal{P}$ intersect in a lower-dimensional facet of the polytope. For each vector ρ , we consider a rescaling of the integration variables $\alpha_i \rightarrow \alpha_i \lambda^{-\rho_i}$, $i = 1, \dots, n$, under which the integrand of (4.1) transforms as

$$\prod_{j=1}^m P_j(\alpha_i \lambda^{-\rho_i}) = \lambda^{-\text{Trop}(\rho)} \times \mathcal{J}(\lambda), \quad (4.4)$$

where $\mathcal{J}(\lambda)$ is a holomorphic function of λ . The exponent $\text{Trop}(\rho)$ controls the divergence of the integral in the vicinity of a corner of the integration domain that was exposed by the rescaling. If $\text{Trop}(\rho) = a + b\epsilon$ and $a > 0$, the integrand has a powerlike divergence, if $a = 0$ it possesses a logarithmic divergence, and if $a < 0$ the integrand is finite at that corner. From now on, we will solely focus on the divergent rays and their corresponding facets.

The conditions of theorem 6 require that

- (1) $a = 0$ for all divergent rays (log divergences only).
- (2) No divergent ray can be expressed as a linear combination of the divergent rays that are compatible with it, i.e., all cones that they span are simplicial.

Assuming these are satisfied, theorem 6 of [52] states that the following subtraction formula holds:

⁶Local finiteness is a more stringent condition than just finiteness, which is merely the statement of the absence of poles in ϵ in the integrated result.

$$I(\mathbf{c}; \mathbf{s}) = \sum_{F \in \partial \mathcal{P}} \text{Vol}(F) I_F^{\text{ren}}. \quad (4.5)$$

Here, the sum runs over divergent facets of $\mathcal{P}$ of all possible dimensions. If a facet F is the intersection of several facets ρ it contributes with a “volume” [67] (see also [68])

$$\text{vol}(F) = \prod_{\rho \in F} \frac{1}{\text{Trop}(\rho)}, \quad (4.6)$$

which is responsible for an overall pole in ϵ , due to the fact that $\text{Trop}(\rho) = \mathcal{O}(\epsilon)$ on a divergent ray. The volume factor multiplies an integral I_F^{ren} , which is of the same dimensionality as the facet F . The integrand of I_F^{ren} is constructed starting from the integrand of $I(\mathbf{c}, \mathbf{s})$ and “restricting” it on the facet F , i.e., keeping in each polynomial P_j only the monomials that live on the facet F . The resulting integrand can then be made finite by subtracting from it contributions coming from facets of the polytope that are compatible with F . This procedure ensures that the integrals I_F^{ren} are locally finite. Therefore, we can evaluate all contributions appearing in the right-hand side of the subtraction formula by simply expanding all *integrands* in ϵ and evaluating emerging integrals term-by-term.

If one of the two conditions of the theorem is not satisfied, [52] provides a lifeline. For divergences stronger than logarithmic, one can apply the Nilsson-Passare analytical continuation⁷ [70,71] on the rays that break the first criterion, resulting in an expansion of the original integrand into a linear combination of integrands that do satisfy the criterium. This approach also helps if the second condition is not fulfilled. Alternatively, if one encounters a non-simplicial cone, one needs to triangulate it into simplicial ones. However, the fact that we generally deal with polytopes that are not permutahedra, one needs to perform this nonunique decomposition on a case-by-case basis. The author of [52] warns us that this is, in general, an expensive procedure, and indeed, we confirm this in our calculation, as we will explain shortly.

Having reviewed the formalism of [52], we now explain how we have applied it to the problem at hand. We have constructed Newton polytopes for each region integral using the code POLYMAKE of Ref. [72]. For the integrals satisfying the requirements alluded to above, we then applied the subtraction formula (4.5) [52].

As an example, for one of the regions contributing to the graph T_{32} , namely with the region vector $\mathbf{r} = \{0, 0, 0, 0, 1, 1, 1, 0, 1, 0\}$, we find the divergent rays shown in the left panel of Fig. 2. These are the first integer

⁷The Nilsson-Passare procedure is based on a repeated integration-by-parts along facets’ normals and is similar in spirit to the original integration-by-parts of ’t Hooft and Veltman, which allows one to construct a meromorphic continuation of momentum integrals in space-time dimension [69].

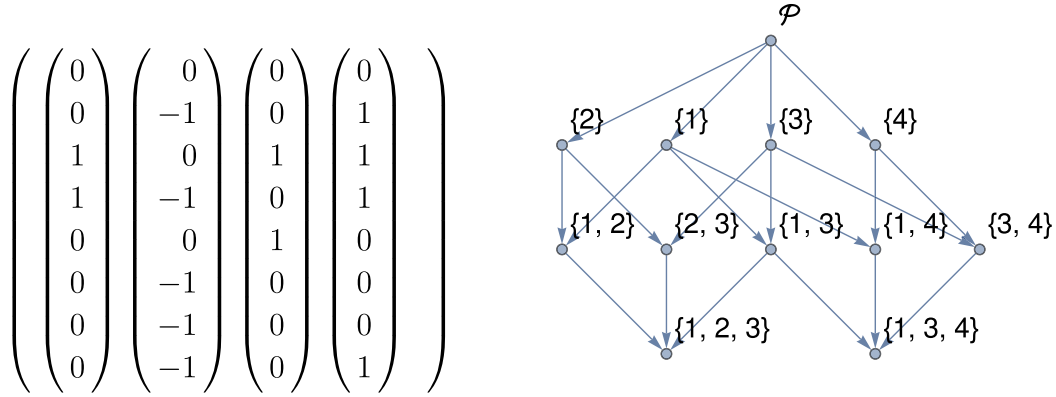


FIG. 2. Divergent rays of a region contributing to diagram T_{33} , listed as columns of a matrix (left panel). The lattice of divergent facets of the corresponding Newton polytope (right panel). Each node represents a face of the polytope where the facets with the listed normals meet.

vectors along the positive direction of the outward-pointing normal to a facet of the Newton polytope $\mathcal{P}$ defined in Eq. (4.3). We listed only facets that are responsible for a pole in ε , i.e., such that $\text{Trop}(\mathcal{I})[\rho] = \mathcal{O}(\varepsilon)$. The subset of $\partial\mathcal{P}$ inducing higher order poles in ε is shown in its combinatorial structure in the right panel of Fig. 2. These are dual to the cones formed by taking a positive combination of compatible rays. Each node in the partially ordered set corresponds to a face F of $\mathcal{P}$ of codimension equal to the number of compatible rays. In particular, at the lower level of the partially ordered set, we find the cones responsible for the leading pole in ε . They are computed by means of the formula (4.5) from [52] in terms of convergent integrals of the lowest dimension. Conversely, the top level of the diagram is a single integral of the highest dimension. Its integrand is the result of the renormalization map, described in Ref. [52], applied to the integrand (4.1). This is the only term in (4.5) that does not produce a pole in ε .

As a next step, we evaluated emerging locally finite integrals from the previous step by expanding their integrands in ε and integrating the series term-by-term using HYPERINT of [73]. The algorithm devised in [73] integrates one variable at a time, expressing the result in terms of Goncharov polylogarithms. This works only for *linearly reducible* integrals, those for which there is at least one order of the integration variables where at each step the partial integral is a polylogarithmic function with rational prefactors such that both the letters and the poles of the prefactor are linear in the next integration variable. An appealing feature of this code is that it provides a way to safely ignore spurious divergent contributions appearing in intermediate integration steps with the use of the shuffle regularization. This means that if we know that an integral $\int(\mathcal{I}_1 - \mathcal{I}_2)$ is convergent, even though the individual integrals $\int\mathcal{I}_i$ are divergent, HYPERINT is able to integrate each integrand separately while systematically dropping divergent terms. The latter are guaranteed to cancel when adding the two up together. The only restriction to this

procedure is that both integrals must be calculated by integrating variables in the *same* order. Accordingly, in order to evaluate an integral of a locally finite integrand, written as a combination of Euler integrands in the same integration domain, we listed all the admissible integration orders for each of the individual integrands and then kept only those that are common.

In principle, HYPERINT should be able to integrate locally finite integrands using any common order. However, different orders required very different CPU times for the integrals to be carried over, with some simply never terminating. As a practical solution to this issue, we merely attempted various orders simultaneously until we found one that terminated successfully.

With this strategy, we were able to evaluate all the contributions of regions for diagrams $T_{3,1}$, $T_{3,2}$, $T_{3,3}$, $T_{3,4}$, and $T_{3,7}$. Among these, the regions $T_{31}^{\text{h-h-us}}$, $T_{31}^{\text{c-c-us}}$, $T_{32}^{\text{h-h}}$, and $T_{32}^{\text{h-h-c}}$ proved to be untractable by the Mellin-Barnes approach described in the previous section.

On the other hand, not all of the region integrals met the above criteria. For example, for the diagram $T_{3,56}$ the region with vector $\mathbf{r} = \{1, 1, 1, 0, 1, 1, 2, 1, 1, 0\}$ yields an integral whose divergent rays are depicted in Fig. 3. The fourth divergent ray is compatible with all the others, and it belongs to their linear span, breaking the second criterion. Applying once a Nilsson-Passare analytical continuation [70,71] along the fourth vector, we reduce the original integral to a combination of integrals satisfying the required criteria. In this manner, we can therefore calculate them individually with the same formula as before. The caveat is that we had to reach a higher order in the ε expansion. Although we always find them to be linearly reducible, the sheer amount of integrands adds up quickly and becomes very time consuming to be a practical strategy.⁸ This issue

⁸To address this issue, we consider it highly desirable to develop a proper parallelization of core functionalities of HYPERINT.

$$\begin{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} & \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \end{pmatrix}$$

FIG. 3. The normals to the Newton polytope for a region of diagram $T_{3,56}$, listed as columns of a matrix.

was particularly severe for diagrams $T_{3,56}$ and $T_{3,8}$, for which multiple regions broke the requirements necessary for the subtraction formula. Luckily, the contributions of regions for $T_{3,8}$ proved tractable by the Mellin-Barnes approach, and the contributions of regions for $T_{3,56}$ were calculated with differential equations, so that every region was covered by at least one of the three methods.

V. SOLVING INTEGRALS WITH DIFFERENTIAL EQUATIONS

We now return to the remaining, hard regions with t^0 behavior. These are obtained from *ab initio* integrals by setting $t = 0$. Their integrands do not simplify sufficiently for $t = 0$ to warrant using their parametric representations directly for subsequent calculations. Instead, it is much more efficient to reduce the original massless on-shell momentum integrals to a set of master integrals by means of integration-by-parts relations. These master integrals were calculated before [74–77] and are available up to transcendentality weight eight from Ref. [78].

Finally, we needed to calculate contributions of individual regions for integrals that yielded multifold MB integrals that could not be tamed numerically with sufficient precision. All of them appeared in $T_{3,56}$. Here again, we have applied the method of the differential equations [45,59,60]. In fact, this method was applied earlier to the three-loop massless vertex integrals [79] at the so-called symmetrical point, i.e., $p_1^2 = p_2^2 = q^2 = -1$. By adopting this earlier analysis in [44], we evaluated our integrals T_{3n} at general values of Q^2 and m^2 , with $p_1^2 = p_2^2 = -m^2$ and then deduced results for their $t \rightarrow 0$ asymptotics by expanding the thus-obtained analytical results.

Presently, we need to evaluate, however, the contributions of *individual* regions for the leading order in the small- t expansion of $T_{3,56}$. As in the previous analysis, we start from differential equations at generic values of t for the 77 primary master integrals. We introduce a new variable, i.e., $t = -(z-1)^2/z$, that rationalizes and, therefore, avoids potential square roots in resulting differential equations.

Then turn to their canonical form [45] using the available public codes like, e.g., CANONICA [80],

$$\frac{\partial}{\partial t} \mathbf{f} = \varepsilon \mathbf{A} \cdot \mathbf{f}. \quad (5.1)$$

Here $\mathbf{f}$ is a vector of canonical basis elements and $\mathbf{A}$ is a corresponding matrix with elements being functions of z but independent of ε .

Since we only need the leading asymptotics of the 77 elements of the canonical basis in the limit $t \rightarrow 0$, these can be extracted from the matrix exponent $\exp(\mathbf{A}_0 \log t) \cdot \mathbf{b}$, where $\mathbf{A}_0$ is the $1/t$ coefficient of $\mathbf{A}$. The rest of the terms are irrelevant for our purposes. Matrix elements of this matrix exponent are linear combinations of the powers $t^{-j\varepsilon}$ with $j = 0, 1, \dots, 6$. Here $\mathbf{b}_i = \{b_i\}_i$, $i = 1, 2, \dots, 77$, are the integration constants. They are uniformly transcendental linear combinations of the form $b_{ij}\varepsilon^j$, where $j = -2, -1, \dots, 4$. Once these boundary conditions are fixed, we obtain the leading asymptotics of the elements of the canonical basis in the limit $t \rightarrow 0$. Integration-by-parts identities provide us with a relation of $T_{3,56}$ to the found elements of the canonical basis in the limit $t \rightarrow 0$. To fix b_i , it was sufficient to apply explicit analytic solutions for the master integrals at general t obtained in [79] and then to expand corresponding expressions in the leading order of the limit $t \rightarrow 0$.

In this manner, we deduced the Laurent expansion of contributions to all relevant regions up to $O(\varepsilon^0)$. They are listed in Appendix A 3. A more detailed presentation of three-loop results is given in the Supplemental Material [81].

VI. FACTORIZATION

Now we are in a position to assemble our findings in the three-loop form factor. Contributing graphs, enumerated according to the nomenclature of Fig. 1, at each consecutive ℓ -loop order, enter the Sudakov form factor with integer coefficients $\alpha_{\ell n}$. $\mathcal{F}_2^{(1)}$ was quoted in Eq. (2.8), so $\alpha_{11} = -2$. For the rest, we have

$$\mathcal{F}_2^{(2)} = \sum_{n=1}^2 \alpha_{2n} T_{2n}, \quad \mathcal{F}_2^{(3)} = \sum_{n=1}^8 \alpha_{3n} T_{3n}, \quad (6.1)$$

with

$$\alpha_{2n} = \{4, 1\}, \quad \alpha_{3n} = \{-8, -2, 4, 4, -4, -4, -4, 2\}. \quad (6.2)$$

Here, the two- and three-loop integrals are decomposed over contributing regions as

$$T_{2n} = T_{2n}^{\text{h-h}} + t^{-\varepsilon} T_{2n}^{\text{h-c}} + t^{-2\varepsilon} (T_{2n}^{\text{c-c}} + T_{2n}^{\text{h-us}}) + t^{-3\varepsilon} T_{2n}^{\text{c-us}} + t^{-4\varepsilon} T_{2n}^{\text{us-us}}, \quad (6.3)$$

$$T_{3n} = T_{3n}^{\text{h-h-h}} + t^{-\varepsilon} T_{3n}^{\text{h-h-c}} + t^{-2\varepsilon} (T_{3n}^{\text{h-c-c}} + T_{3n}^{\text{h-c-us}}) + t^{-3\varepsilon} (T_{3n}^{\text{c-c-c}} + T_{3n}^{\text{h-c-us}}) + t^{-4\varepsilon} (T_{3n}^{\text{c-c-us}} + T_{3n}^{\text{h-us-us}}) + t^{-5\varepsilon} T_{3n}^{\text{c-us-us}} + t^{-6\varepsilon} T_{3n}^{\text{us-us-us}}, \quad (6.4)$$

where we did not split collinear regions up further with respect to legs P_1 and P_2 . We can immediately verify the correctness of the total sum against Eq. (2.6), which was obtained solely on the basis of determining master integrals through the method of differential equations. Indeed, it confirms the exponentiation of the Sudakov form factor flawlessly.

Anticipating the hard-ultrasoft-collinear factorization, we write an ansatz for $\mathcal{F}_2$,

$$\mathcal{F}_2 = h(\varepsilon) J(\varepsilon, \sqrt{t}) S(\varepsilon, t), \quad (6.5)$$

where the hard h , jet J , and ultrasoft S functions receive contributions solely from hard, collinear, and ultrasoft modes, respectively. Explicitly,

$$h(\varepsilon) = 1 + g^2 \alpha_{11} T_{11}^{\text{h}} + g^4 \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{h-h}} + g^6 \sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-h-h}} + O(g^8), \quad (6.6)$$

$$J(\varepsilon, \sqrt{t}) = 1 + g^2 \alpha_{11} t^{-\varepsilon} T_{11}^{\text{c}} + g^4 t^{-2\varepsilon} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-c}} + g^6 t^{-3\varepsilon} \sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{c-c-c}} + O(g^8), \quad (6.7)$$

$$S(\varepsilon, t) = 1 + g^2 \alpha_{11} t^{-2\varepsilon} T_{11}^{\text{us}} + g^4 t^{-4\varepsilon} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{us-us}} + g^6 t^{-6\varepsilon} \sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{us-us-us}} + O(g^8), \quad (6.8)$$

where $T_{11}^{\text{c}} \equiv T_{11}^{\text{c1}} + T_{11}^{\text{c2}}$ from Eqs. (2.17) and (2.18). This form implies that all mixed contributions, such as for instance, $T^{\text{h-c}}$, should arise from products of individual regions from lower loop order. This statement can be confirmed or rebutted with our explicit three-loop results.

A. Hard factorization

Let us begin this analysis with the mixed contributions involving at least one hard region. At two loops, it is not hard to check that

$$\sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{h-c}} = 4 T_{11}^{\text{h}} T_{11}^{\text{c}}, \quad \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{h-us}} = 4 T_{11}^{\text{h}} T_{11}^{\text{us}}. \quad (6.9)$$

Analogously, we find at three loops

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-h-c}} = -2 T_{11}^{\text{c}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{h-h}}, \quad \sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-h-us}} = -2 T_{11}^{\text{us}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{h-h}}, \quad (6.10)$$

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-c-c}} = -2 T_{11}^{\text{h}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-c}}, \quad \sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-c-us}} = -2 T_{11}^{\text{h}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-us}}, \quad (6.11)$$

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-us-us}} = -2 T_{11}^{\text{h}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{us-us}}. \quad (6.12)$$

These relations prove the expected factorization of the hard region from the ones with ultrasoft-collinear modes up to three-loop order.

B. Ultrasoft-collinear factorization: A two-loop obstruction

We now turn to separating collinear from ultrasoft regions. A quick inspection shows, however, that it is broken starting from two loops. Namely,

$$\sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-us}} \neq 4 T_{11}^{\text{c}} T_{11}^{\text{us}}, \quad (6.13)$$

and this implies that the alleged multiplicative factorization (6.5) is violated.

Let us zoom in on this “problematic” region. Since the double ladder and cross ladder yield, up to an overall numerical factor, identical expressions for this region, i.e., $4 T_{21}^{\text{c-us}} = T_{22}^{\text{c-us}}$, we can focus on just one of these integrals. We consider T_{21} which reads

$$T_{21} = \int_{k_1, k_2} D(k_1)D(k_2)D(k_1 - P_1)D(k_1 + k_2 - P_1) \\ \times D(k_1 + P_2)D(k_1 + k_2 + P_2). \quad (6.14)$$

In the $k_2 - k_1$ ordering there are two identical contributions, i.e., c1-us and c2-us, defining the ultrasoft-collinear region. Again, it suffices to discuss just one, say, c1-us determined by the vector $\mathbf{r}_{\text{c1-us}} = \{0, 1, 1, 1, 1, 2\}$, which unambiguously specifies the IR scaling of the integrand's propagators. Namely, according to IR counting rules (2.13), $D(k_1)$ and $D(k_2)$ behave as

$$D(k_1) \sim t^{-2}, \quad D(k_2) \sim t^{-1}, \quad (6.15)$$

and do not change their form since both terms in the light-cone decomposition $k_i^2 = k_{i\perp}^2 - k_{i+}k_{i-}$ scale the same way. The propagator $D(k_1 + k_2 + P_2) \sim t^0$ admits the eikonal form

$$D(k_1 + k_2 + P_2) = D_{\text{eik}}(k_2 + P_2), \quad (6.16)$$

while $D(k_1 - P_1) \sim t^{-1}$ and $D(k_2 + P_2) \sim t^{-1}$ are ultrasoft in this region

$$D(k_1 - P_1) = D_{\text{us}}(k_1 - P_1), \\ D(k_1 + P_2) = D_{\text{us}}(k_1 + P_2). \quad (6.17)$$

Finally, $D(k_1 + k_2 - P_1) \sim t^{-1}$ possesses two additive contributions

$$D(k_1 + k_2 - P_1) = [(k_2 - p_1)^2 + 2k_1 \cdot (k_2 - p_1)]^{-1}. \quad (6.18)$$

To achieve the desired factorization of the ultrasoft from the collinear region, we would like to drop the second term, which entangles the two-loop momenta. However, this is not justified since both of them provide parametrically leading-order effects according to the IR counting rules.

To unravel the effect of this intertwining, we calculate the ultrasoft loop first. To this end, we use the Schwinger-time representation for the ultrasoft propagators (6.17) and the “offending” one (6.18), where we temporarily introduce $K_2 = k_2 - p_1$,

$$\mathcal{L}_1 = \int_{k_1} D(k_1)D_{\text{us}}(k_1 - P_1)D_{\text{us}}(k_1 + P_2)D(k_1 + K_2) \\ = -ie^{\epsilon\gamma_E} \int_0^\infty d\alpha e^{i\alpha K_2^2} \int_0^\infty d\beta_1 d\beta_2 e^{-it(\beta_1 + \beta_2)} \\ \times \frac{\Gamma(1 - \epsilon)}{[-2\beta_2 p_2 \cdot (\alpha K_2 - \beta_1 p_1)]^{1 - \epsilon}}, \quad (6.19)$$

where we extracted the leading IR asymptotics in the denominator after the Fourier transform in k_1 . Next,

integrating over β_2 is straightforward as well as α after the rescaling $\beta_1 \rightarrow \alpha\beta_1$. Finally, changing the integration variable β_1 to x via the map $\beta = \bar{x}/x$, we get

$$\mathcal{L}_1 = e^{\epsilon\gamma_E} \frac{\Gamma(\epsilon)\Gamma(1 - \epsilon)\Gamma(1 + \epsilon)}{t^{2\epsilon}[-K_2^2]} \\ \times \int_0^1 dx \bar{x}^{\epsilon-1} \left[1 + 2t \frac{x p_2 \cdot K_2}{\bar{x}[-K_2^2]} \right]^{\epsilon-1}. \quad (6.20)$$

We now see explicitly that the second term in the square bracket of the integrand is the obstruction for factorization. It cannot be ignored since, while its numerator is suppressed by t , it gets enhancement from the P_1 -collinear kinematics of the denominator. It is easy to calculate the resulting x integral yielding a closed-form geometric series. However, we will not do it since the integral form is more appropriate for the subsequent evaluation of the k_2 loop. It yields the following ϵ -exact expression:

$$T_{21}^{\text{c1-us}} = e^{2\epsilon\gamma_E} \frac{\Gamma^3(\epsilon)\Gamma(-\epsilon)\Gamma^2(1 - \epsilon)\Gamma(1 + 2\epsilon)}{\Gamma^2(1 + \epsilon)\Gamma(1 - 2\epsilon)}. \quad (6.21)$$

Its Laurent expansion coincides with the result obtained from the parametric integral to $\mathcal{O}(\epsilon^3)$. Comparing it with the product of the one-loop expressions, we conclude that

$$T_{21}^{\text{c1-us}} = \frac{1}{4} z_1(\epsilon) T_{11}^{\text{c}} T_{11}^{\text{us}} \quad \text{with} \quad z_1(\epsilon) = \frac{\Gamma(1 + 2\epsilon)}{\Gamma^2(1 + \epsilon)}. \quad (6.22)$$

This implies that by accounting for the mismatch factor z_1 in (6.13), it becomes

$$\sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-us}} = 4 z_1(\epsilon) T_{11}^{\text{c}} T_{11}^{\text{us}}. \quad (6.23)$$

such that the multiplicative separation of the collinear and ultrasoft modes (6.5) can be mended by twisting the mode functions, $(h, J, S) \rightarrow (h/\sqrt{z_1}, \sqrt{z_1}J, \sqrt{z_1}S)$ along with a finite renormalization of the 't Hooft coupling [34] to this perturbative order. However, this does not seem to be possible already at three loops.

C. Broken ultrasoft-collinear factorization

Going to three loops, first, we need to reinspect Eq. (6.12) since it contains a mixed two-loop region, which is not a part of the incoherent components (6.6), (6.7), and (6.8). But, we already know that it will be broken when $T_{2n}^{\text{c-us}}$ is naively recast in terms of the product $T_{11}^{\text{c}} T_{11}^{\text{us}}$ as was observed in Eq. (6.13). To save the day, it needs an adjustment factor. Indeed, it is the very same z_1^2 as in Eq. (6.23), so that Eq. (6.12) is superseded by

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{h-c-us}} = -8z_1(\varepsilon) T_{11}^{\text{h}} T_{11}^{\text{c}} T_{11}^{\text{us}}. \quad (6.24)$$

This is the harbinger of a problem. Now, the twisting implemented at two loops does not work since it produces a single factor of $\sqrt{z_1}$ rather than the required z_1 in the simple, multiplicative factorization. Though replacing the product with a convolution could reproduce it, we will not dwell on it since further considerations of the remaining three-loop mixed regions involving only collinear and ultrasoft modes introduce irreparable complications in our attempt to separate them.

An inspection of both c-c-us and c-us-us regions demonstrates that they cannot simply be represented in terms of lower-loop incoherent components; instead, they require additional adjustment factors. With a little ingenuity, they can be found. The equations

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{c-c-us}} = -2z_1^2(\varepsilon) z'_{21}(\varepsilon) T_{11}^{\text{us}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{c-c}}, \quad (6.25)$$

$$\sum_{n=1}^8 \alpha_{3n} T_{3n}^{\text{c-us-us}} = -2z_1^2(\varepsilon) z'_{22}(\varepsilon) T_{11}^{\text{c}} \sum_{n=1}^2 \alpha_{2n} T_{2n}^{\text{us-us}}, \quad (6.26)$$

hold to $O(\varepsilon^0)$ for

$$z'_{21}(\varepsilon) = 1 + \frac{1}{2}\zeta_4\varepsilon^4 + \left(\frac{7}{2}\zeta_5 - \frac{19}{4}\zeta_2\zeta_3\right)\varepsilon^5 + \left(\frac{2819}{192}\zeta_6 + \frac{65}{8}\zeta_3^2\right)\varepsilon^6 + O(\varepsilon^7), \quad (6.27)$$

$$z'_{22}(\varepsilon) = 1 + 4\zeta_4\varepsilon^4 - (2\zeta_5 + 8\zeta_2\zeta_3)\varepsilon^5 + \left(\frac{2617}{48}\zeta_6 + \frac{13}{2}\zeta_3^2\right)\varepsilon^6 + O(\varepsilon^7). \quad (6.28)$$

Notice that the factorization breaking is postponed to order ε^4 in the z_{2n} factors when z_1^2 , determined at earlier loop order, is included. Unfortunately, z_{21} and z_{22} are different, which prevents us from twisting the two-loop incoherent components of J and S to accommodate this violation.

The inability to separate the collinear and ultrasoft regions necessitates a modification in Eq. (6.5) with a factorization-restoring factor $\mathcal{R}$

$$J(\varepsilon, \sqrt{t})S(\varepsilon, t) \rightarrow J(\varepsilon, \sqrt{t})S(\varepsilon, t)\mathcal{R}(\varepsilon, t), \quad (6.29)$$

with

$$\begin{aligned} \mathcal{R}(\varepsilon, t) = & 1 + 4g^4(z_1 - 1)t^{-3\varepsilon}T_{11}^{\text{c}}T_{11}^{\text{us}} \\ & - 2g^6\left[(z_{21} - 1)t^{-4\varepsilon}T_{11}^{\text{us}}\sum_{n=1}^2\alpha_{2n}T_{2n}^{\text{c-c}} \right. \\ & \left. + (z_{22} - 1)t^{-5\varepsilon}T_{11}^{\text{c}}\sum_{n=1}^2\alpha_{2n}T_{2n}^{\text{us-us}}\right] + O(g^6), \end{aligned} \quad (6.30)$$

and the z_{2n} factors being

$$z_{21}(\varepsilon) = 1 + \frac{17}{4}\zeta_4\varepsilon^4 + \left(\frac{7}{2}\zeta_5 - \frac{49}{4}\zeta_2\zeta_3\right)\varepsilon^5 + \left(\frac{283}{6}\zeta_6 + \frac{137}{8}\zeta_3^2\right)\varepsilon^6 + O(\varepsilon^7), \quad (6.31)$$

$$z_{22}(\varepsilon) = 1 + 19\zeta_4\varepsilon^4 - (2\zeta_5 + 29\zeta_2\zeta_3)\varepsilon^5 + \left(\frac{6217}{48}\zeta_6 + \frac{49}{2}\zeta_3^2\right)\varepsilon^6 + O(\varepsilon^7). \quad (6.32)$$

Considering the fact that at each ℓ th order of the perturbation theory, the Laurent series possesses the leading singularity $O(\varepsilon^{-2\ell})$, the breaking of the ultrasoft-collinear factorization is pushed to the subleading order $O(\varepsilon^{-2})$. This is the order at which the violation first occurred at two loops, as was clarified in Sec. VIB.

VII. CONCLUSIONS

In this work, we considered factorization properties of the off-shell Sudakov form factor at higher orders of perturbative expansion. We observed that the hard factorization is valid, i.e., short-distance physics is indeed incoherent from the IR physics governing its near mass-shell behavior. However, we found a violation of the IR factorization for the collinear and ultrasoft modes. These remain entangled. It starts at two loops in subleading orders of the ε expansion. It is unsettling that an IR-sensitive observable as simple as the Sudakov form factor defies traditional soft-collinear factorization. This appears to suggest that all on-shell scattering amplitudes on this Coulomb branch will follow suit. In this paper, we did not address the question of the operator definition for the joint ultrasoft-collinear functions. This could be of value, and our findings will be reported elsewhere.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: ε EXPANSION FOR REGIONS

In this appendix, we present Laurent expansions for regions at one, two, and three loops. Whenever there is an exact result in ε , we quote it instead.

1. One loop

At one-loop, contributions of all regions can be evaluated as exact functions of ε :

$$T_{11}^h = e^{\varepsilon\gamma_E} \frac{\Gamma^2(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}, \quad (\text{A1})$$

$$T_{11}^c = T_{11}^2 = e^{\varepsilon\gamma_E} \frac{\Gamma^2(-\varepsilon)\Gamma(\varepsilon)}{2\Gamma(-2\varepsilon)}, \quad (\text{A2})$$

$$T_{11}^{us} = e^{\varepsilon\gamma_E} \Gamma(1-\varepsilon)\Gamma(\varepsilon)^2. \quad (\text{A3})$$

2. Two loops

At two loops, the results are required to order $O(\varepsilon^2)$ compared to $O(\varepsilon^0)$ of Ref. [34]. They are

$$T_{21}^{h-h} = \frac{1}{4\varepsilon^4} + \frac{5\pi^2}{24\varepsilon^2} + \frac{29\zeta_3}{6\varepsilon} + \frac{3\pi^4}{32} + \varepsilon \left(\frac{329\zeta_5}{10} - \frac{107\pi^2\zeta_3}{36} \right) + \varepsilon^2 \left(\frac{1723\pi^6}{60480} - \frac{833\zeta_3^2}{18} \right), \quad (\text{A4})$$

$$T_{21}^{h-c} = -\frac{1}{\varepsilon^4} - \frac{\pi^2}{3\varepsilon^2} - \frac{22\zeta_3}{3\varepsilon} - \frac{7\pi^4}{30} + \varepsilon \left(\frac{50\pi^2\zeta_3}{9} - \frac{478\zeta_5}{5} \right) + \varepsilon^2 \left(\frac{622\zeta_3^2}{9} - \frac{113\pi^6}{756} \right), \quad (\text{A5})$$

$$T_{21}^{h-us} = \frac{e^{2\varepsilon\gamma_E} \Gamma(1-\varepsilon)\Gamma(-\varepsilon)^2\Gamma(\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}, \quad (\text{A6})$$

$$T_{21}^{c-c} = \frac{1}{2\varepsilon^4} + \frac{\pi^2}{12\varepsilon^2} + \frac{5\zeta_3}{3\varepsilon} + \frac{163\pi^4}{720} + \varepsilon \left(\frac{329\zeta_5}{5} - \frac{13\pi^2\zeta_3}{18} \right) + \varepsilon^2 \left(\frac{6451\pi^6}{30240} - \frac{164\zeta_3^2}{9} \right), \quad (\text{A7})$$

$$T_{21}^{c-us} = -\frac{1}{\varepsilon^4} - \frac{\pi^2}{3\varepsilon^2} + \frac{14\zeta_3}{3\varepsilon} - \frac{\pi^4}{15} + \varepsilon \left(\frac{14\pi^2\zeta_3}{9} + \frac{62\zeta_5}{5} \right) + \varepsilon^2 \left(-\frac{98\zeta_3^2}{9} - \frac{2\pi^6}{189} \right), \quad (\text{A8})$$

$$T_{21}^{us-us} = e^{2\varepsilon\gamma_E} \Gamma(-\varepsilon)^2\Gamma(2\varepsilon)^2, \quad (\text{A9})$$

and

$$T_{22}^{h-h} = \frac{1}{\varepsilon^4} - \frac{\pi^2}{\varepsilon^2} - \frac{83\zeta_3}{3\varepsilon} - \frac{59\pi^4}{120} + \varepsilon \left(\frac{79\pi^2\zeta_3}{6} - \frac{587\zeta_5}{5} \right) + \varepsilon^2 \left(\frac{2567\zeta_3^2}{9} + \frac{59\pi^6}{1512} \right), \quad (\text{A10})$$

$$T_{22}^{h-c} = -\frac{4}{\varepsilon^4} + \frac{200\zeta_3}{3\varepsilon} + \frac{7\pi^4}{5} + \frac{8\pi^2}{3\varepsilon^2} + \varepsilon \left(\frac{2408\zeta_5}{5} - \frac{256\pi^2\zeta_3}{9} \right) + \varepsilon^2 \left(\frac{257\pi^6}{378} - \frac{3272\zeta_3^2}{9} \right), \quad (\text{A11})$$

$$T_{22}^{h-us} = 0, \quad (\text{A12})$$

$$T_{22}^{c-c} = \frac{6}{\varepsilon^4} - \frac{54\zeta_3}{\varepsilon} - \frac{209\pi^4}{180} - \frac{4\pi^2}{3\varepsilon^2} + \varepsilon \left(\frac{89\pi^2\zeta_3}{9} - \frac{1722\zeta_5}{5} \right) + \varepsilon^2 \left(\frac{752\zeta_3^2}{3} - \frac{185\pi^6}{252} \right), \quad (\text{A13})$$

$$T_{22}^{c-us} = -\frac{4}{\varepsilon^4} + \frac{56\zeta_3}{3\varepsilon} - \frac{4\pi^4}{15} - \frac{4\pi^2}{3\varepsilon^2} + \varepsilon \left(\frac{56\pi^2\zeta_3}{9} + \frac{248\zeta_5}{5} \right) + \varepsilon^2 \left(-\frac{392\zeta_3^2}{9} - \frac{8\pi^6}{189} \right), \quad (\text{A14})$$

$$T_{22}^{us-us} = 2e^{2\varepsilon\gamma_E} \Gamma(-\varepsilon)\Gamma(2\varepsilon)^2[\Gamma(-\varepsilon) + 2\Gamma(-2\varepsilon)\Gamma(1+\varepsilon)]. \quad (\text{A15})$$

3. Three loops

Finally, at three loops, the expansion was obtained to $O(\epsilon^0)$:

$$T_{31}^{\text{h-h-h}} = \frac{1}{36\epsilon^6} + \frac{11\pi^2}{144\epsilon^4} + \frac{61\zeta_3}{36\epsilon^3} + \frac{497\pi^4}{5760\epsilon^2} + \frac{1}{\epsilon} \left(\frac{47\pi^2\zeta_3}{144} + \frac{1469\zeta_5}{60} \right) + \frac{49\zeta_3^2}{72} + \frac{36331\pi^6}{622080}, \quad (\text{A16})$$

$$T_{31}^{\text{h-h-c}} = -\frac{1}{6\epsilon^6} - \frac{7\pi^2}{24\epsilon^4} - \frac{13\zeta_3}{2\epsilon^3} - \frac{127\pi^4}{576\epsilon^2} + \frac{1}{\epsilon} \left(\frac{47\pi^2\zeta_3}{24} - \frac{679\zeta_5}{10} \right) + \frac{149\zeta_3^2}{4} - \frac{103561\pi^6}{725760}, \quad (\text{A17})$$

$$T_{31}^{\text{h-c-c}} = \frac{1}{6\epsilon^6} + \frac{13\pi^2}{72\epsilon^4} + \frac{23\zeta_3}{6\epsilon^3} + \frac{443\pi^4}{2880\epsilon^2} + \frac{1}{\epsilon} \left(\frac{519\zeta_5}{10} - \frac{205\pi^2\zeta_3}{72} \right) - \frac{539\zeta_3^2}{12} + \frac{86657\pi^6}{725760}, \quad (\text{A18})$$

$$T_{31}^{\text{h-h-us}} = \frac{1}{4\epsilon^6} + \frac{13\pi^2}{48\epsilon^4} + \frac{19\zeta_3}{4\epsilon^3} + \frac{299\pi^4}{1920\epsilon^2} + \frac{1}{\epsilon} \left(\frac{657\zeta_5}{20} - \frac{89\pi^2\zeta_3}{48} \right) - \frac{383\zeta_3^2}{8} + \frac{661\pi^6}{10752}, \quad (\text{A19})$$

$$T_{31}^{\text{c-c-c}} = -\frac{1}{18\epsilon^6} - \frac{\pi^2}{24\epsilon^4} - \frac{13\zeta_3}{18\epsilon^3} - \frac{659\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{19\pi^2\zeta_3}{24} - \frac{143\zeta_5}{10} \right) + \frac{839\zeta_3^2}{36} - \frac{93301\pi^6}{2177280}, \quad (\text{A20})$$

$$T_{31}^{\text{h-c-us}} = -\frac{1}{2\epsilon^6} - \frac{3\pi^2}{8\epsilon^4} - \frac{5\zeta_3}{2\epsilon^3} - \frac{81\pi^4}{320\epsilon^2} + \frac{1}{\epsilon} \left(\frac{17\pi^2\zeta_3}{8} - \frac{447\zeta_5}{10} \right) + \frac{167\zeta_3^2}{4} - \frac{40469\pi^6}{241920}, \quad (\text{A21})$$

$$T_{31}^{\text{c-c-us}} = \frac{1}{6\epsilon^6} + \frac{\pi^2}{8\epsilon^4} - \frac{\zeta_3}{2\epsilon^3} + \frac{29\pi^4}{192\epsilon^2} + \frac{1}{\epsilon} \left(\frac{149\zeta_5}{10} - \frac{17\pi^2\zeta_3}{24} \right) - \frac{25\zeta_3^2}{4} + \frac{17443\pi^6}{103680}, \quad (\text{A22})$$

$$T_{31}^{\text{h-us-us}} = \frac{\Gamma(1-\epsilon)^2\Gamma(-\epsilon)^2\Gamma(\epsilon)^2\Gamma(2\epsilon)^2}{\Gamma(1-2\epsilon)\Gamma(\epsilon+1)}, \quad (\text{A23})$$

$$T_{31}^{\text{c-us-us}} = \frac{2\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)}, \quad (\text{A24})$$

$$T_{31}^{\text{us-us-us}} = \frac{\Gamma(1-\epsilon)^3\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)^2}{\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)^2}, \quad (\text{A25})$$

for T_{31}

$$T_{32}^{\text{h-h-h}} = \frac{1}{36\epsilon^6} + \frac{7\pi^2}{144\epsilon^4} + \frac{55\zeta_3}{36\epsilon^3} + \frac{5329\pi^4}{51840\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1171\pi^2\zeta_3}{432} + \frac{1199\zeta_5}{60} \right) + \frac{353\zeta_3^2}{8} + \frac{2606843\pi^6}{13063680}, \quad (\text{A26})$$

$$T_{32}^{\text{h-h-c}} = -\frac{1}{6\epsilon^6} - \frac{13\pi^2}{72\epsilon^4} - \frac{35\zeta_3}{6\epsilon^3} - \frac{2561\pi^4}{8640\epsilon^2} - \frac{1}{\epsilon} \left(\frac{247\pi^2\zeta_3}{72} + \frac{739\zeta_5}{10} \right) - \frac{497\zeta_3^2}{12} - \frac{306353\pi^6}{725760}, \quad (\text{A27})$$

$$T_{32}^{\text{h-c-c}} = \frac{5}{12\epsilon^6} + \frac{5\pi^2}{16\epsilon^4} + \frac{31\zeta_3}{4\epsilon^3} + \frac{6541\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{417\zeta_5}{4} - \frac{107\pi^2\zeta_3}{144} \right) - \frac{979\zeta_3^2}{24} + \frac{107749\pi^6}{290304}, \quad (\text{A28})$$

$$T_{32}^{\text{h-h-us}} = 0, \quad (\text{A29})$$

$$T_{32}^{\text{c-c-c}} = -\frac{1}{18\epsilon^6} - \frac{\pi^2}{24\epsilon^4} - \frac{13\zeta_3}{18\epsilon^3} - \frac{1657\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon} \left(\frac{59\pi^2\zeta_3}{216} - \frac{829\zeta_5}{30} \right) - \frac{25\zeta_3^2}{4} - \frac{42857\pi^6}{933120}, \quad (\text{A30})$$

$$T_{32}^{\text{h-c-us}} = -\frac{1}{2\epsilon^6} - \frac{3\pi^2}{8\epsilon^4} - \frac{5\zeta_3}{2\epsilon^3} - \frac{81\pi^4}{320\epsilon^2} + \frac{1}{\epsilon} \left(\frac{17\pi^2\zeta_3}{8} - \frac{447\zeta_5}{10} \right) + \frac{167\zeta_3^2}{4} - \frac{40469\pi^6}{241920}, \quad (\text{A31})$$

$$T_{32}^{\text{c-c-us}} = \frac{1}{6\epsilon^6} + \frac{13\pi^2}{72\epsilon^4} - \frac{\zeta_3}{6\epsilon^3} + \frac{1073\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{239\zeta_5}{10} - \frac{125\pi^2\zeta_3}{72} \right) + \frac{65\zeta_3^2}{12} + \frac{70433\pi^6}{725760}, \quad (\text{A32})$$

$$T_{32}^{\text{h-us-us}} = \frac{1}{4\epsilon^6} + \frac{13\pi^2}{48\epsilon^4} - \frac{5\zeta_3}{4\epsilon^3} + \frac{331\pi^4}{1920\epsilon^2} - \frac{1}{\epsilon} \left(\frac{65\pi^2\zeta_3}{48} + \frac{63\zeta_5}{20} \right) + \frac{25\zeta_3^2}{8} + \frac{1997\pi^6}{23040}, \quad (\text{A33})$$

$$T_{32}^{\text{c-us-us}} = -\frac{1}{6\epsilon^6} - \frac{7\pi^2}{24\epsilon^4} + \frac{3\zeta_3}{2\epsilon^3} - \frac{2513\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{205\pi^2\zeta_3}{72} + \frac{81\zeta_5}{10} \right) - \frac{73\zeta_3^2}{12} - \frac{4931\pi^6}{20736}, \quad (\text{A34})$$

$$T_{32}^{\text{us-us-us}} = \frac{1}{36\epsilon^6} + \frac{11\pi^2}{144\epsilon^4} - \frac{11\zeta_3}{36\epsilon^3} + \frac{6553\pi^4}{51840\epsilon^2} - \frac{1}{\epsilon} \left(\frac{299\pi^2\zeta_3}{432} + \frac{137\zeta_5}{20} \right) + \frac{\zeta_3^2}{8} + \frac{2086247\pi^6}{13063680}, \quad (\text{A35})$$

for T_{32}

$$T_{33}^{\text{h-h-h}} = -\frac{1}{36\epsilon^6} + \frac{17\pi^2}{144\epsilon^4} + \frac{149\zeta_3}{36\epsilon^3} + \frac{3701\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{967\zeta_5}{20} - \frac{229\pi^2\zeta_3}{144} \right) - \frac{11113\zeta_3^2}{72} + \frac{367151\pi^6}{4354560}, \quad (\text{A36})$$

$$T_{33}^{\text{h-h-c}} = \frac{1}{6\epsilon^6} - \frac{35\pi^2}{72\epsilon^4} - \frac{97\zeta_3}{6\epsilon^3} - \frac{5743\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{595\pi^2\zeta_3}{72} - \frac{2121\zeta_5}{10} \right) + \frac{3905\zeta_3^2}{12} - \frac{43447\pi^6}{80640}, \quad (\text{A37})$$

$$T_{33}^{\text{h-c-c}} = -\frac{1}{6\epsilon^6} + \frac{23\pi^2}{72\epsilon^4} + \frac{79\zeta_3}{6\epsilon^3} + \frac{1661\pi^4}{2880\epsilon^2} + \frac{1}{\epsilon} \left(\frac{2551\zeta_5}{10} - \frac{521\pi^2\zeta_3}{72} \right) - \frac{2089\zeta_3^2}{12} + \frac{88013\pi^6}{103680}, \quad (\text{A38})$$

$$T_{33}^{\text{h-h-us}} = -\frac{1}{4\epsilon^6} + \frac{19\pi^2}{48\epsilon^4} + \frac{45\zeta_3}{4\epsilon^3} + \frac{199\pi^4}{640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{903\zeta_5}{20} - \frac{167\pi^2\zeta_3}{48} \right) - \frac{1057\zeta_3^2}{8} + \frac{12911\pi^6}{483840}, \quad (\text{A39})$$

$$T_{33}^{\text{c-c-c}} = \frac{1}{18\epsilon^6} - \frac{5\pi^2}{72\epsilon^4} - \frac{83\zeta_3}{18\epsilon^3} - \frac{1709\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{127\pi^2\zeta_3}{72} - \frac{2491\zeta_5}{30} \right) + \frac{3001\zeta_3^2}{36} - \frac{637379\pi^6}{2177280}, \quad (\text{A40})$$

$$T_{33}^{\text{h-c-us}} = \frac{1}{2\epsilon^6} - \frac{7\pi^2}{24\epsilon^4} - \frac{27\zeta_3}{2\epsilon^3} - \frac{397\pi^4}{960\epsilon^2} + \frac{1}{\epsilon} \left(\frac{29\pi^2\zeta_3}{24} - \frac{993\zeta_5}{10} \right) + \frac{409\zeta_3^2}{4} - \frac{71867\pi^6}{241920}, \quad (\text{A41})$$

$$T_{33}^{\text{c-c-us}} = -\frac{5}{12\epsilon^6} - \frac{17\pi^2}{144\epsilon^4} + \frac{89\zeta_3}{12\epsilon^3} + \frac{319\pi^4}{3456\epsilon^2} + \frac{1}{\epsilon} \left(\frac{373\pi^2\zeta_3}{144} + \frac{213\zeta_5}{4} \right) - \frac{1549\zeta_3^2}{24} + \frac{32621\pi^6}{207360}, \quad (\text{A42})$$

$$T_{33}^{\text{h-us-us}} = 0, \quad (\text{A43})$$

$$T_{33}^{\text{c-us-us}} = -\frac{2\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)}, \quad (\text{A44})$$

$$T_{33}^{\text{us-us-us}} = -\frac{\Gamma(1-\epsilon)^3\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)^2}{\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)^2}, \quad (\text{A45})$$

for T_{33}

$$T_{34}^{\text{h-h-h}} = -\frac{1}{36\epsilon^6} - \frac{11\pi^2}{144\epsilon^4} - \frac{109\zeta_3}{36\epsilon^3} - \frac{59\pi^4}{1920\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1153\pi^2\zeta_3}{144} - \frac{5989\zeta_5}{60} \right) - \frac{1009\zeta_3^2}{72} - \frac{1155757\pi^6}{4354560}, \quad (\text{A46})$$

$$T_{34}^{\text{h-h-c}} = \frac{1}{6\epsilon^6} + \frac{7\pi^2}{24\epsilon^4} + \frac{21\zeta_3}{2\epsilon^3} + \frac{731\pi^4}{2880\epsilon^2} + \frac{1}{\epsilon} \left(\frac{2799\zeta_5}{10} - \frac{455\pi^2\zeta_3}{24} \right) - \frac{501\zeta_3^2}{4} + \frac{469033\pi^6}{725760}, \quad (\text{A47})$$

$$T_{34}^{\text{h-c-c}} = -\frac{5}{12\epsilon^6} - \frac{65\pi^2}{144\epsilon^4} - \frac{151\zeta_3}{12\epsilon^3} - \frac{3127\pi^4}{5760\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1973\pi^2\zeta_3}{144} - \frac{1211\zeta_5}{4} \right) + \frac{4915\zeta_3^2}{24} - \frac{34727\pi^6}{41472}, \quad (\text{A48})$$

$$T_{34}^{\text{h-h-us}} = 0, \quad (\text{A49})$$

$$T_{34}^{\text{c-c-c}} = \frac{1}{18\epsilon^6} + \frac{\pi^2}{24\epsilon^4} + \frac{37\zeta_3}{18\epsilon^3} + \frac{1907\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{2869\zeta_5}{30} - \frac{9\pi^2\zeta_3}{8} \right) - \frac{1799\zeta_3^2}{36} + \frac{135811\pi^6}{311040}, \quad (\text{A50})$$

$$T_{34}^{\text{h-c-us}} = \frac{1}{2\epsilon^6} + \frac{3\pi^2}{8\epsilon^4} + \frac{5\zeta_3}{2\epsilon^3} + \frac{81\pi^4}{320\epsilon^2} + \frac{1}{\epsilon} \left(\frac{447\zeta_5}{10} - \frac{17\pi^2\zeta_3}{8} \right) - \frac{167\zeta_3^2}{4} + \frac{40469\pi^6}{241920}, \quad (\text{A51})$$

$$T_{34}^{\text{c-c-us}} = -\frac{1}{6\epsilon^6} - \frac{\pi^2}{8\epsilon^4} + \frac{\zeta_3}{2\epsilon^3} - \frac{29\pi^4}{192\epsilon^2} + \frac{1}{\epsilon} \left(\frac{17\pi^2\zeta_3}{24} - \frac{149\zeta_5}{10} \right) + \frac{25\zeta_3^2}{4} - \frac{17443\pi^6}{103680}, \quad (\text{A52})$$

$$T_{34}^{\text{h-us-us}} = -\frac{1}{4\epsilon^6} - \frac{3\pi^2}{16\epsilon^4} + \frac{7\zeta_3}{4\epsilon^3} - \frac{163\pi^4}{1920\epsilon^2} + \frac{1}{\epsilon} \left(\frac{21\pi^2\zeta_3}{16} + \frac{93\zeta_5}{20} \right) - \frac{49\zeta_3^2}{8} - \frac{15269\pi^6}{483840}, \quad (\text{A53})$$

$$T_{34}^{\text{c-us-us}} = -\frac{2\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)},$$

$$T_{34}^{\text{us-us-us}} = -\frac{\Gamma(1-\epsilon)^3\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)^2}{\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)^2}, \quad (\text{A54})$$

for T_{34}

$$T_{3,56}^{\text{h-h-h}} = \frac{11}{36\epsilon^6} - \frac{43\pi^2}{144\epsilon^4} - \frac{307\zeta_3}{36\epsilon^3} - \frac{523\pi^4}{3456\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1483\pi^2\zeta_3}{144} - \frac{3257\zeta_5}{20} \right) - \frac{6193\zeta_3^2}{72} - \frac{1576517\pi^6}{4354560}, \quad (\text{A55})$$

$$T_{3,56}^{\text{h-h-c}} = -\frac{11}{6\epsilon^6} + \frac{113\pi^2}{72\epsilon^4} + \frac{235\zeta_3}{6\epsilon^3} + \frac{929\pi^4}{1728\epsilon^2} + \frac{1}{\epsilon} \left(\frac{3991\zeta_5}{10} - \frac{2545\pi^2\zeta_3}{72} \right) - \frac{2315\zeta_3^2}{12} + \frac{328597\pi^6}{725760}, \quad (\text{A56})$$

$$T_{3,56}^{\text{h-c-c}} = \frac{55}{12\epsilon^6} - \frac{127\pi^2}{48\epsilon^4} - \frac{301\zeta_3}{4\epsilon^3} - \frac{17593\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{6245\pi^2\zeta_3}{144} - \frac{1747\zeta_5}{4} \right) + \frac{15211\zeta_3^2}{24} - \frac{2311\pi^6}{10752}, \quad (\text{A57})$$

$$T_{3,56}^{\text{h-h-us}} = 0, \quad (\text{A58})$$

$$T_{3,56}^{\text{c-c-c}} = -\frac{37}{9\epsilon^6} + \frac{31\pi^2}{36\epsilon^4} + \frac{479\zeta_3}{9\epsilon^3} + \frac{2177\pi^4}{4320\epsilon^2} + \frac{1}{\epsilon} \left(\frac{859\zeta_5}{5} - \frac{449\pi^2\zeta_3}{36} \right) - \frac{8239\zeta_3^2}{18} + \frac{2801\pi^6}{120960}, \quad (\text{A59})$$

$$T_{3,56}^{\text{h-c-us}} = -\frac{2}{\epsilon^6} + \frac{26\zeta_3}{\epsilon^3} + \frac{39\pi^4}{80\epsilon^2} + \frac{1}{\epsilon} \left(\pi^2\zeta_3 + \frac{726\zeta_5}{5} \right) - 157\zeta_3^2 + \frac{71\pi^6}{189}, \quad (\text{A60})$$

$$T_{3,56}^{\text{c-c-us}} = \frac{13}{3\epsilon^6} + \frac{65\pi^2}{36\epsilon^4} - \frac{146\zeta_3}{3\epsilon^3} + \frac{293\pi^4}{864\epsilon^2} - \frac{1}{\epsilon} \left(\frac{205\pi^2\zeta_3}{9} + \frac{768\zeta_5}{5} \right) + \frac{983\zeta_3^2}{3} - \frac{977\pi^6}{51840}, \quad (\text{A61})$$

$$T_{3,56}^{\text{h-us-us}} = \frac{1}{4\epsilon^6} + \frac{13\pi^2}{48\epsilon^4} - \frac{5\zeta_3}{4\epsilon^3} + \frac{331\pi^4}{1920\epsilon^2} - \frac{1}{\epsilon} \left(\frac{65\pi^2\zeta_3}{48} + \frac{63\zeta_5}{20} \right) + \frac{25\zeta_3^2}{8} + \frac{1997\pi^6}{23040}, \quad (\text{A62})$$

$$T_{3,56}^{\text{c-us-us}} = -\frac{11}{6\epsilon^6} - \frac{53\pi^2}{24\epsilon^4} + \frac{37\zeta_3}{2\epsilon^3} - \frac{14731\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1787\pi^2\zeta_3}{72} + \frac{511\zeta_5}{10} \right) - \frac{1271\zeta_3^2}{12} - \frac{24185\pi^6}{20736}, \quad (\text{A63})$$

$$T_{3,56}^{\text{us-us-us}} = \frac{11}{36\epsilon^6} + \frac{31\pi^2}{48\epsilon^4} - \frac{115\zeta_3}{36\epsilon^3} + \frac{1601\pi^4}{1920\epsilon^2} - \frac{1}{\epsilon} \left(\frac{1085\pi^2\zeta_3}{144} + \frac{217\zeta_5}{20} \right) + \frac{1079\zeta_3^2}{72} + \frac{3868787\pi^6}{4354560}, \quad (\text{A64})$$

for the sum $T_{3,56} = T_{35} + T_{36}$

$$T_{37}^{\text{h-h-h}} = \frac{1}{36\epsilon^6} + \frac{13\pi^2}{432\epsilon^4} - \frac{7\zeta_3}{12\epsilon^3} + \frac{11\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{739\zeta_5}{60} - \frac{155\pi^2\zeta_3}{144} \right) + \frac{275\zeta_3^2}{24} + \frac{2267\pi^6}{1451520}, \quad (\text{A65})$$

$$T_{37}^{\text{h-h-c}} = -\frac{1}{6\epsilon^6} - \frac{5\pi^2}{72\epsilon^4} + \frac{17\zeta_3}{6\epsilon^3} + \frac{1007\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{181\pi^2\zeta_3}{72} + \frac{201\zeta_5}{10} \right) - \frac{241\zeta_3^2}{12} + \frac{34957\pi^6}{241920}, \quad (\text{A66})$$

$$T_{37}^{\text{h-c-c}} = \frac{1}{6\epsilon^6} + \frac{\pi^2}{72\epsilon^4} - \frac{19\zeta_3}{6\epsilon^3} - \frac{589\pi^4}{2880\epsilon^2} - \frac{1}{\epsilon} \left(\frac{115\pi^2\zeta_3}{72} + \frac{731\zeta_5}{10} \right) + \frac{73\zeta_3^2}{12} - \frac{16253\pi^6}{48384}, \quad (\text{A67})$$

$$T_{37}^{\text{h-h-us}} = \frac{1}{4\epsilon^6} + \frac{\pi^2}{48\epsilon^4} - \frac{11\zeta_3}{4\epsilon^3} - \frac{77\pi^4}{1920\epsilon^2} - \frac{1}{\epsilon} \left(\frac{11\pi^2\zeta_3}{48} + \frac{273\zeta_5}{20} \right) + \frac{121\zeta_3^2}{8} - \frac{5233\pi^6}{161280}, \quad (\text{A68})$$

$$T_{37}^{\text{c-c-c}} = -\frac{1}{18\epsilon^6} - \frac{\pi^2}{216\epsilon^4} + \frac{3\zeta_3}{2\epsilon^3} + \frac{589\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{41\pi^2\zeta_3}{72} + \frac{417\zeta_5}{10} \right) - \frac{67\zeta_3^2}{12} + \frac{460883\pi^6}{2177280}, \quad (\text{A69})$$

$$T_{37}^{\text{h-c-us}} = \frac{2\Gamma(1-\epsilon)^2\Gamma(-2\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon)^2\Gamma(2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1-2\epsilon)}, \quad (\text{A70})$$

$$T_{37}^{\text{c-c-us}} = \frac{2\Gamma(-2\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)\Gamma(3\epsilon)\Gamma(1-\epsilon)^3}{\Gamma(1-3\epsilon)\Gamma(1-2\epsilon)\Gamma(\epsilon+1)\Gamma(2\epsilon+1)} + \frac{\Gamma(-\epsilon)^2\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(1-\epsilon)^3}{\Gamma(1-2\epsilon)^2\Gamma(\epsilon+1)^2}, \quad (\text{A71})$$

$$T_{37}^{\text{h-us-us}} = 0, \quad (\text{A72})$$

$$T_{37}^{\text{c-us-us}} = \frac{2\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\epsilon+1)^2\Gamma(2\epsilon+1)}, \quad (\text{A73})$$

$$T_{37}^{\text{us-us-us}} = \frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)^2\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(3\epsilon)^2}{\Gamma(\epsilon+1)\Gamma(2\epsilon+1)^2}, \quad (\text{A74})$$

for T_{37} , and finally

$$T_{38}^{\text{h-h-h}} = \frac{1}{4\epsilon^6} - \frac{115\pi^2}{432\epsilon^4} - \frac{305\zeta_3}{36\epsilon^3} - \frac{3149\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1273\pi^2\zeta_3}{144} - \frac{6569\zeta_5}{60} \right) + \frac{9817\zeta_3^2}{72} - \frac{516863\pi^6}{4354560}, \quad (\text{A75})$$

$$T_{38}^{\text{h-h-c}} = -\frac{3}{2\epsilon^6} + \frac{11\pi^2}{8\epsilon^4} + \frac{75\zeta_3}{2\epsilon^3} + \frac{197\pi^4}{320\epsilon^2} + \frac{1}{\epsilon} \left(\frac{3609\zeta_5}{10} - \frac{275\pi^2\zeta_3}{8} \right) - \frac{1875\zeta_3^2}{4} + \frac{7837\pi^6}{26880}, \quad (\text{A76})$$

$$T_{38}^{\text{h-c-c}} = \frac{15}{4\epsilon^6} - \frac{331\pi^2}{144\epsilon^4} - \frac{827\zeta_3}{12\epsilon^3} - \frac{18763\pi^4}{17280\epsilon^2} + \frac{1}{\epsilon} \left(\frac{2095\pi^2\zeta_3}{48} - \frac{1927\zeta_5}{4} \right) + \frac{5503\zeta_3^2}{8} - \frac{62311\pi^6}{161280}, \quad (\text{A77})$$

$$T_{38}^{\text{h-h-us}} = 0, \quad (\text{A78})$$

$$T_{38}^{\text{c-c-c}} = -\frac{7}{2\epsilon^6} + \frac{193\pi^2}{216\epsilon^4} + \frac{965\zeta_3}{18\epsilon^3} + \frac{6131\pi^4}{8640\epsilon^2} + \frac{1}{\epsilon} \left(\frac{8413\zeta_5}{30} - \frac{1033\pi^2\zeta_3}{72} \right) - \frac{17335\zeta_3^2}{36} + \frac{208591\pi^6}{725760}, \quad (\text{A79})$$

$$T_{38}^{\text{h-c-us}} = \frac{2\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(-\epsilon)^3\Gamma(\epsilon)^2\Gamma(2\epsilon)}{\Gamma(1-2\epsilon)\Gamma(-3\epsilon)}, \quad (\text{A80})$$

$$T_{38}^{\text{c-c-us}} = \frac{\Gamma(1-\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(-\epsilon)^4}{\Gamma(-2\epsilon)^2\Gamma(\epsilon+1)^2} + \frac{2\Gamma(1-\epsilon)^2\Gamma(\epsilon)^2\Gamma(2\epsilon)^2\Gamma(-\epsilon)^3}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon+1)^2} \\ + \frac{2\Gamma(1-\epsilon)^2\Gamma(-2\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon)\Gamma(3\epsilon)\Gamma(-\epsilon)^2}{\Gamma(1-2\epsilon)\Gamma(-3\epsilon)\Gamma(\epsilon+1)\Gamma(2\epsilon+1)} + \frac{2\Gamma(1-\epsilon)^2\Gamma(-2\epsilon)\Gamma(\epsilon)^3\Gamma(3\epsilon)\Gamma(-\epsilon)^2}{\Gamma(1-2\epsilon)\Gamma(-3\epsilon)\Gamma(\epsilon+1)^2}, \quad (\text{A81})$$

$$T_{38}^{\text{h-us-us}} = \Gamma(1-\varepsilon)\Gamma(-\varepsilon)^2\Gamma(\varepsilon)^2\Gamma(2\varepsilon)^2, \quad (\text{A82})$$

$$T_{38}^{\text{c-us-us}} = \frac{2\Gamma(-\varepsilon)\Gamma(\varepsilon)^2\Gamma(2\varepsilon)^2\Gamma(3\varepsilon)\Gamma(1-\varepsilon)^3}{\Gamma(1-2\varepsilon)\Gamma(\varepsilon+1)^2\Gamma(2\varepsilon+1)} + \frac{2\Gamma(-\varepsilon)\Gamma(\varepsilon)^3\Gamma(2\varepsilon)\Gamma(3\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(\varepsilon+1)^2} \\ + \frac{2\Gamma(-\varepsilon)^2\Gamma(\varepsilon)^3\Gamma(2\varepsilon)\Gamma(3\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(-2\varepsilon)\Gamma(\varepsilon+1)^3} + \frac{2\Gamma(1-2\varepsilon)\Gamma(-\varepsilon)^2\Gamma(\varepsilon)^2\Gamma(2\varepsilon)^2\Gamma(3\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(-2\varepsilon)\Gamma(\varepsilon+1)\Gamma(2\varepsilon+1)}, \quad (\text{A83})$$

$$T_{38}^{\text{us-us-us}} = \frac{\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)^2\Gamma(3\varepsilon)^2\Gamma(\varepsilon)^4}{\Gamma(\varepsilon+1)^3} + \frac{2\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)^2\Gamma(2\varepsilon)\Gamma(3\varepsilon)^2\Gamma(\varepsilon)^3}{\Gamma(\varepsilon+1)^2\Gamma(2\varepsilon+1)} \\ + \frac{\Gamma(1-3\varepsilon)\Gamma(1-\varepsilon)^2\Gamma(2\varepsilon)^2\Gamma(3\varepsilon)^2\Gamma(\varepsilon)^2}{\Gamma(\varepsilon+1)^2\Gamma(2\varepsilon+1)}, \quad (\text{A84})$$

for T_{38} .

Mathematica-ready expressions for individual regions of the above three-loop results are given in the Supplemental Material [81].

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