

Radiated angular momentum from spinning black hole scattering trajectories

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Using the worldline quantum field theory approach, we derive solutions to the equations of motion for spinning massive bodies up to quadratic order in spins. At leading post-Minkowskian (PM) order, these trajectories are obtained in the time domain, and at subleading order, they are obtained in the frequency domain. Our approach incorporates diagrammatic techniques and modern Feynman integration technologies and includes a family of loop integrals different to those seen in asymptotic PM calculations. Our results provide a new mechanism for computing the radiated angular momentum involved in gravitational scattering, which we reproduce at 2PM order up to linear spins. We have established a framework for computing higher-order effects to further extend the high-precision frontier in analytical gravitational wave physics and push predictions for the radiated angular momentum to higher perturbative orders.

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I. INTRODUCTION

The gravitational two-body problem has held a foundational position in physics since the days of Newton. In its general relativistic version, it not only represents a fascinating intellectual challenge—given its conceptual simplicity paired with its technical complexity—but has become increasingly important due to the advent of gravitational wave astronomy [1]. Today, gravitational wave detectors routinely observe gravitational waves emitted from binary black hole and neutron star systems [2], and future third-generation observatories will profoundly advance this in the coming decade [3–6], calling for high-precision theoretical predictions from Einstein gravity to enable data analysis and interpretation.

Within perturbative frameworks to address the scattering variant of the relativistic two-body problem, an expansion in Newton’s constant G —the post-Minkowskian (PM) expansion—is most natural [7–11]. Here, dramatic progress in the past years was made by translating this classical physics problem to a scattering problem in perturbative quantum

field theory (QFT), thereby unleashing state-of-the-art techniques developed for collider physics [12–49]. In doing so, we today have control up to the fifth order in PM perturbation theory (5PM) for the scattering observables of the scattering angle and radiated 4-momentum [50,51] (see Refs. [52,53] for related work in electrodynamics and supergravity), as well as up to the next-to-leading order for the far-field scattering waveform [21,54–67]. Importantly, spin degrees of freedom may also be included in these effective field theory descriptions [22,55,68–77] and have been similarly advanced to the 5PM order (in a physical PM counting) [78–81].

These QFT-inspired approaches are designed to compute asymptotic quantities: the change in momentum or spins of bodies involved in the scattering process, scattering angles, and the far-field waveform that encodes radiated linear and angular momentum carried by the gravitational wave pulse. Of these, the *radiated angular momentum* is one of the more intricate to obtain [82]: current methods include extracting it from the far-field waveform [21,39,55], linear response relations [39,79,81,83,84], or the eikonal [85–87]. More recently, the use of Dirac brackets on an exponential representation of the S -matrix has also been explored [88]. So far, the explicit form of the full time-dependent *trajectories* has also not been obtained with QFT methods. Current state of the art is 2PM order, derived by integrating the equations of motion directly [8,9,89–94].

Here, we employ the worldline quantum field theory (WQFT) approach [14,22,23,76,95] to directly generate solutions to the equations of motion of the gravitational

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two-body problem through diagrammatic techniques, thereby also including spinning degrees of freedom for the first time. At leading 1PM order, we provide explicit results up to quadratic order in spins in the time domain, and at subleading 2PM order, we provide linear-in-spin results in Fourier space. Using these new results, we are able to reproduce the radiated angular momentum at order G^2 . Under the intrinsic choice of Bondi-Metzner-Sachs frame [96], the radiated angular momentum starts at G^2 , while the radiated momentum is delayed to G^3 [39,97].

Our results are achieved by incorporating integration by parts, differential equations, and the method of regions techniques from the modern technical toolbox of collider physics. Interestingly, we encounter a family of one-loop Feynman integrals that is a subset of Feynman integrals contributing to the gravitational wave form at $\mathcal{O}(G^3)$ [57,60–62]. Our work builds on classical works for the leading-order dynamics [8,9,89–92] and in particular recent 2PM works [93,94] that established and solved the equations of motion in the nonspinning case.

II. CLASSICAL BLACK HOLE SCATTERING TRAJECTORIES

Our starting point is an effective description of spinning black holes as point particles, described by a one-dimensional worldline action modelling the spins via bosonic oscillators [76]. This approach is superior to the previous $\mathcal{N} = 2$ supersymmetric formalism [22], as it is not restricted to quadratic spin order and is more practicable for not needing to deal with anticommuting objects. Up to terms quadratic in spin, the i th black hole action is ($i = 1, 2$)

$$\frac{S^{(i)}}{m_i} = - \int d\tau_i \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + i \bar{\alpha}_{i,\mu} \frac{D\alpha_i^\mu}{d\tau_i} + \frac{1}{2} R_{\mu\nu\rho\sigma} \bar{\alpha}_i^\mu \alpha_i^\nu \bar{\alpha}_i^\rho \alpha_i^\sigma \right]. \quad (1)$$

The worldline trajectory $x_i^\mu(\tau_i)$ is augmented by a complex vector $\alpha_i^\mu(\tau_i)$, which is a bosonic oscillator mode. Here, $\frac{D\alpha_i^\mu}{d\tau_i} = \dot{\alpha}_i^\mu + \Gamma^\mu_{\nu\rho} \dot{x}_i^\nu \alpha_i^\rho$ is the covariant worldline derivative employing the Levi-Civita connection. These oscillators encode the spin tensor of the compact object [76],

$$S_i^{\mu\nu}(\tau_i) = -2m_i i \bar{\alpha}_i^{[\mu}(\tau_i) \alpha_i^{\nu]}(\tau_i), \quad (2)$$

antisymmetrizing with unit weight. The time-dependent spin vector is correspondingly defined by

$$S_i^\mu(\tau_i) = \frac{1}{2} \epsilon^\mu_{\nu\rho\sigma} S_i^{\nu\rho}(\tau_i) \dot{x}_i^\sigma(\tau_i). \quad (3)$$

We consider scattering only of Kerr black holes and not spinning neutron stars, which would necessitate an additional quadratic-in-spin term [76]. In Eq. (1), we have

adopted a generalized proper time gauge in which

$$g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu = 1 - \frac{1}{4m_i^2} R_{\mu\nu\rho\sigma} S_i^{\mu\nu} S_i^{\rho\sigma}. \quad (4)$$

Inclusion of the quadratic-in-spin term implies that this is *not* a proper-time gauge.

The complete two-body system is described by an action comprising the two black holes, coupled to bulk gravity and including a de Donder gauge-fixing term for the graviton,

$$S = -\frac{2}{\kappa^2} \int d^D x \sqrt{-g} R[g] + S_{\text{g.f.}} + \sum_{i=1}^2 S^{(i)}, \quad (5)$$

where $\kappa = \sqrt{32\pi G}$. Anticipating the need for dimensional regularisation, we work in $D = 4 - 2\epsilon$ dimensions. As we are interested in scattering solutions to the classical equations of motion, we perform the background-field expansions:

$$x_i^\mu(\tau_i) = b_i^\mu + v_i^\mu \tau_i + z_i^\mu(\tau_i), \quad (6a)$$

$$\alpha_i^\mu(\tau_i) = \alpha_{-\infty,i}^\mu + \alpha_i^{\prime\mu}(\tau_i), \quad (6b)$$

$$S_i^\mu(\tau_i) = m_i(\alpha_i^\mu + \alpha_i^{\prime\mu}(\tau_i)), \quad (6c)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \quad (6d)$$

Moreover $z_i^\mu(\tau_i)$, $\alpha_i^{\prime\mu}(\tau_i)$, $\alpha_i^{\prime\mu}(\tau_i)$, and $\kappa h_{\mu\nu}(x)$ are expanded in G according to

$$z_i^\mu(\tau_i) = G z_i^{(1)\mu}(\tau_i) + G^2 z_i^{(2)\mu}(\tau_i) + \mathcal{O}(G^3), \quad (7a)$$

$$\alpha_i^{\prime\mu}(\tau_i) = G \alpha_i^{(1)\mu}(\tau_i) + G^2 \alpha_i^{(2)\mu}(\tau_i) + \mathcal{O}(G^3), \quad (7b)$$

$$a_i^{\prime\mu}(\tau_i) = G a_i^{(1)\mu}(\tau_i) + G^2 a_i^{(2)\mu}(\tau_i) + \mathcal{O}(G^3), \quad (7c)$$

$$\kappa h_{\mu\nu}(x) = G h_{\mu\nu}^{(1)}(x) + G^2 h_{\mu\nu}^{(2)}(x) + \mathcal{O}(G^3). \quad (7d)$$

In terms of the background variables, all defined at past infinity, we have the relative impact parameter $b^\mu = b_2^\mu - b_1^\mu$, incoming momenta $p_i^\mu = m_i v_i^\mu$, Pauli-Lubanski spin vectors a_i^μ , and the initial spin tensor $S_{-\infty,i}^{\mu\nu} = -2m_i i \bar{\alpha}_{-\infty,i}^{[\mu} \alpha_{-\infty,i}^{\nu]}$. The incoming velocities satisfy $v_1^2 = v_2^2 = 1$, and $\gamma = v_1 \cdot v_2 = 1/\sqrt{1-v^2}$ is the dimensionless boost factor. There is also the unit-normalized angular momentum vector

$$\hat{l}^\mu = -\epsilon^\mu_{\nu\rho\sigma} \frac{b^\nu v_1^\rho v_2^\sigma}{|b| \sqrt{\gamma^2 - 1}} \quad (8)$$

and the dual velocities,

$$w_1^\mu := \frac{\gamma v_2^\mu - v_1^\mu}{\gamma^2 - 1}, \quad w_2^\mu := \frac{\gamma v_1^\mu - v_2^\mu}{\gamma^2 - 1}. \quad (9)$$

These definitions ensure that $v_i \cdot w_j = \delta_{ij}$.

Using retarded worldline propagators fixes the background parameters such that the black holes are incoming from past infinity. This will lead to a logarithmic divergence in the trajectory, which we subtract off later on. To efficiently generate perturbative solutions to the classical equations of motion, we use the WQFT approach [14,23], exploiting the fact that the tree-level one-point functions of $\langle z_i^\mu \rangle$ and $\langle h_{\mu\nu} \rangle$ solve the classical equations of motions. We will subsequently drop the expectation value notation for convenience. In WQFT, perturbative PM classical solutions are represented as sums over tree-level Feynman diagrams. The graviton propagator (in de Donder gauge) is denoted by a wiggly line:

$$\mu \xrightarrow{k} \rho = i \frac{\eta_{\mu(\rho} \eta_{\sigma)\nu} - \frac{1}{D-2} \eta_{\mu\nu} \eta_{\rho\sigma}}{(k^0 + i0^+)^2 - \mathbf{k}^2}. \quad (10)$$

For the two worldline fields in energy space, $z_i^\mu(\omega)$ and $\alpha_i^\mu(\omega)$, we collect these in a composite vector [76],

$$\mathcal{Z}_I^\mu(\omega) = \{z^\mu(\omega), \alpha'^\mu(\omega), \tilde{\alpha}'^\mu(\omega)\}, \quad (11)$$

with flavor index I . The retarded propagator of this flavoured worldline field then reads

$$\begin{array}{c} \mu \\ \vdots \\ I \end{array} \xrightarrow{\omega} \begin{array}{c} \nu \\ \vdots \\ J \end{array} = -i \frac{\eta^{\mu\nu}}{m} \begin{pmatrix} \frac{1}{(\omega+i0)^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\omega+i0} \\ 0 & \frac{1}{\omega+i0} & 0 \end{pmatrix}_{IJ}. \quad (12)$$

As dictated by causality and the in-in formalism [23,98,99], we exclusively use retarded worldline propagators. In this diagram, the arrow indicates causality flow and the energy ω travels in the same direction. The Feynman vertices emerging from Eq. (5) have been collected in Refs. [14,51] for the nonspinning case and in Ref. [76] for the spinning case.

Working in momentum (energy) space, the full trajectory of the i th black hole is given by

$$\mathcal{Z}_i^\mu(\tau_i) = \int_\omega e^{-i\omega\tau_i} \langle \mathcal{Z}_i^\mu(\omega) \rangle, \quad (13)$$

where we introduced the shorthand notation $\int_\omega := \int \frac{d\omega}{2\pi}$. The one-point function $\langle \mathcal{Z}_i^\mu(\omega) \rangle$ is given by a sum over diagrams with a single outgoing line, carrying energy ω :

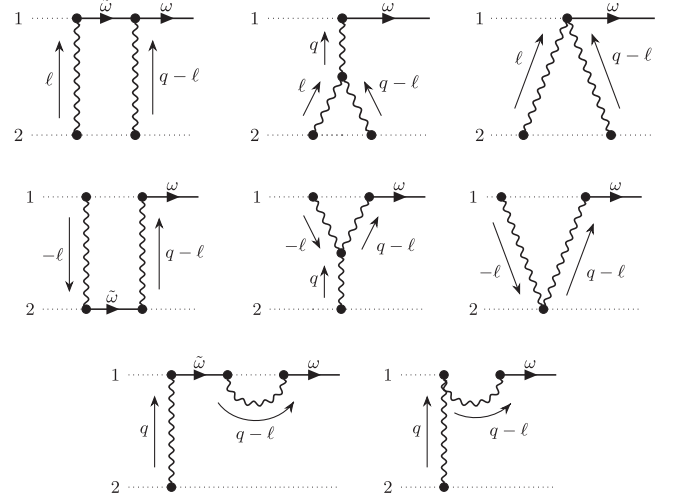


FIG. 1. Diagrams contributing to the subleading scattering trajectory $\mathcal{Z}_1^{(2)\mu}(\tau)$, including spin. The three diagrams in the first row are proportional to m_2^2 , and contribute in the probe limit $m_1 \ll m_2$; the five diagrams in the second and third rows are proportional to $m_1 m_2$.

$$\langle \mathcal{Z}_1^\mu(\omega) \rangle = q \uparrow \text{ (diagram) } \xrightarrow{\omega} \mathcal{Z}_1^\mu(\omega). \quad (14)$$

In schematic form, the full trajectory of the first black hole (BH) is given by (we write $\int_q := \int \frac{d^D q}{(2\pi)^D}$)

$$\begin{aligned} \mathcal{Z}_1^\mu(\tau_1) &= \int_q \int_\omega e^{i(q \cdot b - \omega \tau_1)} \delta(q \cdot v_1 - \omega) \delta(q \cdot v_2) \mathcal{Z}_1^\mu(\omega) \\ &= \int_q e^{iq \cdot \tilde{b}(\tau_1)} \delta(q \cdot v_2) \mathcal{Z}_1^\mu(q), \end{aligned} \quad (15)$$

where q^μ is the total momentum exchanged between the two worldlines and we have introduced the shifted impact parameter:

$$\tilde{b}^\mu(\tau_1) := b^\mu - v_1^\mu \tau_1. \quad (16)$$

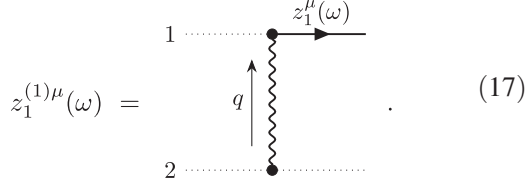
This extends the relative impact parameter $b^\mu = b_2^\mu - b_1^\mu$ along the undeflected trajectory of the first black hole.

The diagrams involved in the trajectory contain those used to compute the momentum impulse Δp_1^μ plus so-called “mushroom” diagrams, compare Figure 1. There are two important differences, though. First, we include the worldline propagators (12) on the outgoing line, unlike for the impulse where this line is cut. Second, we leave the energy on the outgoing line finite ($\omega \neq 0$) unlike in the impulse where the outgoing energy is set to zero. The Fourier transform on ω in Eq. (15) yields $\omega = q \cdot v_1$, and

so there are now two mass-dimension scales present: $|q|$, which is the overall momentum transfer, and $q \cdot v_1$, which is the energy on the outgoing line.

III. LEADING-ORDER TRAJECTORY

Let us begin with the nonspinning case. Without loss of generality, in the following we consider the trajectory of BH1. At leading-PM order, a single diagram contributes:



$$z_1^{(1)\mu}(\omega) = \text{diagram} \quad (17)$$

Therefore, using the WQFT Feynman rules,

$$z_1^{(1)\mu}(\tau_1)_{S^0} = -i \frac{m_2 \kappa^2}{8} \int_q \frac{\tilde{\delta}(q \cdot v_2)}{q^2 (q \cdot v_1 + i0)^2} e^{iq \cdot \tilde{b}(\tau_1)} \times [2(q \cdot v_1)(2\gamma v_2^\mu - v_1^\mu) - (2\gamma^2 - 1)q^\mu]. \quad (18)$$

The Fourier transform on ω (13) has been performed here, yielding $\omega = q \cdot v_1$ and the shifted impact parameter $\tilde{b}^\mu(\tau_1)$ (16). Integrating out the external energy scale in this way is familiar from work on the gravitational waveform [55] and yields the same kind of Fourier transforms on the total exchanged momentum q^μ . The $i0$ prescription on the graviton is in this case immaterial.

To evaluate the leading trajectory, we need two Fourier transform integrals both belonging to the family

$$I_{\alpha,\beta}^{\mu_1 \dots \mu_n} = \int_q \frac{\tilde{\delta}(q \cdot v_2)}{(q^2)^\alpha (q \cdot v_1 + i0)^\beta} e^{iq \cdot \tilde{b}(\tau_1)} q^{\mu_1} \dots q^{\mu_n}. \quad (19)$$

These are derived in Appendix B:

$$I_{1,1} = \frac{i}{4\pi\sqrt{\gamma^2 - 1}} \log(S(\tau)) \Big|_{-\infty}^{\tau_1}, \quad (20a)$$

$$I_{1,2}^\mu = -\frac{i}{4\pi\sqrt{\gamma^2 - 1}} \times \left[\frac{b^\mu}{|b|^2} \left(\frac{D(\tau)}{\sqrt{\gamma^2 - 1}} + \tau \right) - w_1^\mu \log(S(\tau)) \right] \Big|_{-\infty}^{\tau_1}. \quad (20b)$$

We have adopted the notation of Ref [94]:

$$D(\tau) = \sqrt{|b|^2 + (\gamma^2 - 1)\tau^2}, \quad (21a)$$

$$S(\tau) = \frac{\sqrt{\gamma^2 - 1}\tau + D(\tau)}{|b|}. \quad (21b)$$

$|\tilde{b}|_2$, referred to as $|\tilde{b}|_2$ in Ref. [55], measures the length of the shifted impact parameter in the rest frame of the second black hole. As $S(\tau \rightarrow -\infty) \rightarrow 0$, we encounter a $\log(0)$ divergence in our result for $\langle z_1^\mu(\tau_1) \rangle_{S^0}$. As mentioned above, this is due to our choice of fixing the zeroth-order parameters at $\tau_i = -\infty$. This implies that the black holes have already travelled an infinite distance once they reach a finite τ_i , yielding a divergent constant. See, e.g., Refs. [61,94] for discussions on this intrinsic logarithmic divergence and its relation to the long-range effects of gravity.

To obtain a finite result from the trajectory, we subtract the same result evaluated at a finite reference time:

$$\begin{aligned} z_1^{(1)\mu}(\tau_1)_{S^0, \text{ph}} &= z_1^{(1)\mu}(\tau_1)_{S^0} - z_1^{(1)\mu}(0)_{S^0} \\ &= m_2 \frac{1}{(\gamma^2 - 1)^{3/2}} (\gamma(2\gamma^2 - 3)v_2^\mu + v_1^\mu) \log(S(\tau_1)) \\ &\quad + m_2 (2\gamma^2 - 1) \frac{S(\tau_1) - 1}{(\gamma^2 - 1)} \frac{b^\mu}{|b|}. \end{aligned} \quad (22)$$

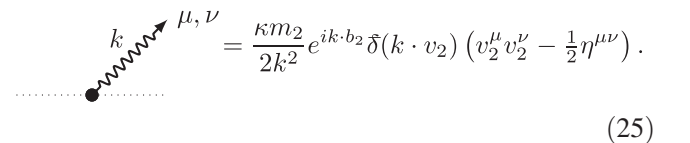
In fact, the trajectory is dependent on the frame choices and thus the conventions and boundary conditions one works with. Our trajectory is consistent with previous works in the WQFT formalism [14,22,23,76,95]. While the authors of Ref. [93] fix their boundary conditions differently, the above result agrees with theirs when acting on it with the projector $P_\nu^\mu = \delta_\nu^\mu - v_1^\mu v_{1,\nu}$. As further confirmation of our result, we reproduce the 1PM momentum impulse:

$$\begin{aligned} \Delta p_1^{(1)\mu} &= m_1 (\dot{z}_1^\mu(\tau_1 \rightarrow \infty) - \dot{z}_1^\mu(\tau_1 \rightarrow -\infty)) \\ &= m_1 m_2 \frac{(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{b^\mu}{|b|^2} + \mathcal{O}(S). \end{aligned} \quad (23)$$

Finally, at this leading-PM order, plugging the background field expansions (6) into the on-shell condition (4) for the nonspinning case, $g_{\mu\nu} \dot{x}_1^\mu \dot{x}_1^\nu = 1$, gives

$$0 = G(2v_1 \cdot \dot{z}_1^{(1)}(\tau_1) + h_{\nu\rho}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\nu v_1^\rho). \quad (24)$$

Unlike the corresponding on-shell condition on the momentum impulse Δp_1^μ , which at this perturbative order is simply $v_1 \cdot \Delta p_1 = 0$, here we also require the static contribution to the gravitational waveform produced by the second black hole:



$$\text{diagram} = \frac{\kappa m_2}{2k^2} e^{ik \cdot b_2} \tilde{\delta}(k \cdot v_2) (v_2^\mu v_2^\nu - \frac{1}{2} \eta^{\mu\nu}). \quad (25)$$

Putting together the pieces,

$$2v_1 \cdot \dot{z}_1^{(1)}(\tau_1) = -h_{\mu\nu}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\mu v_1^\nu = \frac{2m_2(2\gamma^2 - 1)}{D(\tau_1)}, \quad (26)$$

thus verifying the consistency requirement (24).

Moving on to the spinning case, we compute the trajectory up to quadratic order in spins. For compactness, we present here the results only up to linear order in spins—

for the full quadratic order, we refer the reader to our [100] repository submission. With a single diagram contributing, the main difference to the nonspinning case is that we require higher-rank integrals of the family $I_{\alpha,\beta}^{\mu_1 \dots \mu_n}$ (19), up to $n = 3$. As described in Appendix B, these integrals are obtained as b -derivatives of the scalar integrals with the same powers of the propagators. The 1PM linear-in-spin trajectory is then given by

$$\begin{aligned} z_1^{(1)\mu}(\tau_1)_{S^1, \text{ph}} = & -\frac{m_2}{\hat{D}(\tau_1)|b|} \left[\gamma \bar{\tau}_1 w_1^\mu a_1 \cdot \hat{l} - \bar{\tau}_1 v_2^\mu (a_1 \cdot \hat{l} + 2a_2 \cdot \hat{l}) - \frac{\hat{b}^\mu}{v} [(\hat{D}(\tau_1)(2\bar{\tau}_1 - 1) + 2\bar{\tau}_1^2 + 1)a_1 \cdot \hat{l} \right. \\ & + 2(\hat{D}(\tau_1)(\bar{\tau}_1 - 1) + \bar{\tau}_1^2 + 1)a_2 \cdot \hat{l}] - \frac{\hat{l}^\mu}{(\gamma v)^3} [\bar{\tau}_1(2\gamma((\gamma v)^2 \hat{D}(\tau_1) + \bar{\tau}_1)a_2 \cdot \hat{b} + \gamma v a_2 \cdot v_1) \\ & \left. - \gamma v a_1 \cdot v_2) + \gamma(\gamma v)^2(\hat{D}(\tau_1)(2\bar{\tau}_1 - 1) + 2\bar{\tau}_1^2 + 1)a_1 \cdot \hat{b}] \right], \end{aligned} \quad (27)$$

where we have used

$$\bar{\tau}_1 := \frac{\tau_1}{|b|\gamma v}, \quad \hat{D}(\tau_1) := \frac{D(\tau_1)}{|b|}. \quad (28)$$

The spinning correction to the trajectory is finite since spin effects do not affect the gravitational long-range interactions which cause the logarithmic drift in the trajectory discussed above. As in the nonspinning case (23), we checked this result by confirming that it reproduces the momentum impulse in the asymptotic limit, an expression which may be found in Ref. [22]. We also checked the on-shell condition (4), which at leading-PM order becomes

$$\begin{aligned} 2v_1 \cdot \dot{z}_1^{(1)}(\tau_1) + h_{\mu\nu}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\mu v_1^\nu \\ = \frac{1}{2m_2^2} \partial_\nu \partial_\sigma h_{\mu\rho}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) S_{-\infty,1}^{\mu\nu} S_{-\infty,1}^{\rho\sigma}. \end{aligned} \quad (29)$$

Here, we have used $R_{\mu\nu\rho\sigma} = \frac{\kappa}{2}(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}) + \mathcal{O}(\kappa^2)$. While it can be ignored at linear order in spin, at quadratic order in spin, the Riemann

curvature tensor enters the calculation. We confirm that Eq. (29) holds at every point in time τ_1 .

To obtain the leading-order spin deflection $a_1^{(1)\mu}(\tau_1)$, we first derived an expression for $\alpha_1^{(1)\mu}(\tau_1)$. Like the deflection $z_1^{(1)\mu}(\omega)$ (17), the corresponding frequency-space result is given by a single diagram:

$$\alpha_1^{(1)\mu}(\omega) = \begin{array}{c} \text{Diagram: A vertical wavy line labeled } q \text{ connects two horizontal lines. The top line is labeled } 1 \text{ and the bottom line is labeled } 2. \text{ An arrow labeled } \alpha_1^\mu(\omega) \text{ points to the right from the top line.} \end{array} \quad (30)$$

We have checked that the charge $\alpha_1 \cdot \bar{\alpha}_1$ is conserved and that the spin-supplementary condition (SSC) $S_i^{\mu\nu} \dot{x}_{i,\nu} = 0$ holds at all times, which is implied by $\alpha_i \cdot \dot{x}_i = 0$. Using Eqs. (2) and (3), we are consequently able to obtain the leading-order change in the Pauli-Lubanski spin vector,

$$\begin{aligned} a_1^{(1)\mu}(\tau_1)_{S^1} = & \frac{m_2}{\hat{D}(\tau_1)|b|} \left[-2(a_1 \cdot \hat{l})\hat{l}^\mu - 2(a_1 \cdot b - \gamma \hat{\tau}_1 a_1 \cdot v_2)\hat{b}^\mu - (a_1 \cdot b((1 - 2\gamma^2)\gamma v \hat{D}(\tau_1) + (\gamma v)^2 \hat{\tau}_1) + 3\gamma a_1 \cdot v_2)w_1^\mu \right. \\ & \left. - \frac{(a_1 \cdot b((2\gamma^2 - 1)\hat{D}(\tau_1)v + \gamma \hat{\tau}_1 v^2) - a_1 \cdot v_2)v_2^\mu}{v^2} \right], \end{aligned} \quad (31)$$

where $\hat{\tau}_1 := \tau_1/|b|$.

IV. SUBLEADING TRAJECTORY

The subleading trajectory $\mathcal{Z}_1^{(2)\mu}$ comprises two mass sectors which are proportional to m_2^2 and $m_1 m_2$. All contributing diagrams are displayed in Fig. 1, with terms carrying m_2^2 contributing in the probe limit $m_1 \ll m_2$. Just as at 1PM order, we

perform the trivial Fourier transform integral on the outgoing energy ω , which introduces the shifted impact parameter $\tilde{b}^\mu(\tau_1) = b^\mu - v_1^\mu \tau_1$ in the exponential of the q -Fourier transform (15). Unlike at 1PM order, though, and given the complexity involved, we will not perform this Fourier transform explicitly; instead, $\mathcal{Z}_1^{(2)\mu}(q)$ is the focus of this section. Our object of interest will ultimately be the total radiated angular momentum from the system J_{rad}^μ , which is negative the asymptotic change in total angular momentum of the system ΔJ^μ . In Sec. V, we will show how the latter is directly extracted from $\mathcal{Z}_i^\mu(q)$ —performing the Fourier integrals in the simpler asymptotic limit $\tau_i \rightarrow \pm\infty$.

A. Integration

The 2PM trajectory $\mathcal{Z}_1^{(2)\mu}$ comprises one-loop integrals of two different families, emerging from the two mass sectors m_2^2 and $m_1 m_2$. The integral families, which are always multiplied by an overall $\delta(q \cdot v_2)$, are

$$I_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}(|q|, q \cdot v_1, \gamma) = \int_{\ell} \frac{\delta(\ell \cdot v_2)}{(\ell^2)^{\nu_1} ((\ell - q)^2)^{\nu_2} (\ell \cdot v_1 + i0)^{\nu_3}}, \quad (32a)$$

corresponding to the m_2^2 terms, and

$$J_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}(|q|, q \cdot v_1, \gamma) = \int_{\ell} \frac{\delta(\ell \cdot v_1)}{(\ell^2)^{\nu_1} ((\ell - q)^2 + i0 \operatorname{sgn}(\ell^0 - q^0))^{\nu_2} (\ell \cdot v_2 + i0)^{\nu_3}}, \quad (32b)$$

corresponding to the $m_1 m_2$ terms. We stress that, due to the suppressed multiplication with $\delta(q \cdot v_2)$, these integrals are not related by symmetry and depend on *two* dimensionful parameters: $|q|$ and $q \cdot v_1$. This is unlike the single-scale integrals emerging in the calculation of the momentum impulse that cannot depend on $q \cdot v_1$ due to the presence of $\delta(q \cdot v_1)$. Here, we may still assume that $q \cdot v_2 = 0$, which leads to simplifications of the loop integrands. Interestingly, the integral families form a subset of the integral families arising in the one-loop gravitational waveform [57,60–62].

While a complete account of the loop integration is contained in Appendix C, we remark here on some important features. By introducing $\hat{q} \cdot v_1 = q \cdot v_1 / |q|$, the overall mass dimension is relegated to overall powers of $|q|$, and thus factored from the integrals. All nontrivial functional dependence is therefore contained in γ and $\hat{q} \cdot v_1$. In the probe-limit family (32a), the dependence on γ can be established by entering the frame $v_2^\mu = (1, \mathbf{0})$. In both families, dependence on $\hat{q} \cdot v_1$ is fixed by the method of differential equations (DEs), considering also integration-by-parts identities between integrals of the same family.

Boundary conditions on these integrals are provided by the limit $\hat{q} \cdot v_1 \rightarrow 0$. In particular, we keep the full dependence on γ in the boundary integrals. The $\omega \rightarrow 0$ limit comprises two integration regions (after a shift $\ell \rightarrow \ell + q$):

$$\ell^\mu \sim (\hat{q} \cdot v_1, 1), \quad \ell^\mu \sim (\hat{q} \cdot v_1, \hat{q} \cdot v_1). \quad (33)$$

The first of these produces boundary integrals that are familiar from one-loop computations of asymptotic quantities such as the momentum impulse Δp_1^μ . The other is a new region that requires separate evaluation, and we comment on it in Appendix C. Note that, despite the apparent similarity, these regions are entirely unrelated to the potential and radiative regions used when fixing γ -dependent integrals in the limit $v \rightarrow 0$ —both regions contribute to both mass sectors.

To further illustrate the structure of the integrals, we present the master integrals of each family. First, the probe-limit family comprises three master integrals [normalized by a factor of $(4\pi)^{-2\epsilon} e^{\gamma_E \epsilon}$],

$$I_{0,1,1}^{1\text{-loop}} = \frac{i|q|^{-2\epsilon}}{8\pi\gamma v} \left(\frac{1}{\epsilon} + i\pi + 2 \log[(\hat{q} \cdot v_1)_N^{-1}] \right) + \mathcal{O}(\epsilon), \quad (34a)$$

$$I_{1,1,0}^{1\text{-loop}} = \frac{1}{8} |q|^{-1-2\epsilon} + \mathcal{O}(\epsilon), \quad (34b)$$

$$I_{1,1,1}^{1\text{-loop}} = \frac{i|q|^{-2\epsilon}}{8\pi|q|^2\gamma v} \left(\frac{1}{\epsilon} - i\pi - 2 \log[(\hat{q} \cdot v_1)_N^{-1}] \right) + \mathcal{O}(\epsilon), \quad (34c)$$

where we defined $(\hat{q} \cdot v_1)_N := \hat{q} \cdot v_1 / (\gamma v)$. Meanwhile, the four masters of integral family $J_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}$ are

$$J_{0,1,0}^{1\text{-loop}} = i|q|^{1-2\epsilon} \frac{\hat{q} \cdot v_1}{4\pi} + \mathcal{O}(\epsilon), \quad (35a)$$

$$J_{0,1,1}^{1\text{-loop}} = -\frac{i|q|^{-2\epsilon}}{8\pi\gamma v} \left(\frac{1}{\epsilon} + i\pi + 2 \operatorname{arccosh}(\gamma) + 2 \log[(\hat{q} \cdot v_1)_N^{-1}] \right) + \mathcal{O}(\epsilon), \quad (35b)$$

$$J_{1,1,0}^{1\text{-loop}} = -\frac{i|q|^{-1-2\epsilon}}{8\pi\sqrt{(\hat{q} \cdot v_1)^2 + 1}} \times \left(i\pi + 2 \operatorname{arctanh} \left(\frac{\hat{q} \cdot v_1}{\sqrt{(\hat{q} \cdot v_1)^2 + 1}} \right) \right) + \mathcal{O}(\epsilon), \quad (35c)$$

$$J_{1,1,1}^{1\text{-loop}} = \frac{i|q|^{-2\epsilon}}{8\pi|q|^2\gamma v} \left(-\frac{1}{\epsilon} + i\pi + 2 \operatorname{arccosh}(\gamma) + 2 \log[(\hat{q} \cdot v_1)_N^{-1}] \right) + \mathcal{O}(\epsilon). \quad (35d)$$

Notice the appearance in these integrals of $\text{arccosh}(\gamma)$, which in a calculation of, e.g., the momentum impulse shows up only in two-loop integrals—for us here, it will ultimately contribute to the radiated angular momentum.

B. Frequency-domain trajectory

Evaluation of the diagrams in Fig. 1 yields the 2PM trajectory in q -space. This comprises results for $z_1^{(2)\mu}(q)$ and $\alpha_1^{(2)\mu}(q)$ —and consequently $a_1^{(2)\mu}(q)$ —up to linear order in spin. To illustrate our results here, we present the nonspinning part of $z_1^{(2)\mu}(q)$ only, with linear-in-spin provided in our [100] submission. Our result is split into two individually gauge-invariant mass sectors,

$$z_1^{(2)\mu}(q) = C_1^{(m_2^2)} w_1^\mu + C_2^{(m_2^2)} v_2^\mu + C_3^{(m_2^2)} \hat{q}^\mu + C_1^{(m_1 m_2)} w_1^\mu + C_2^{(m_1 m_2)} v_2^\mu + C_3^{(m_1 m_2)} \hat{q}^\mu, \quad (36)$$

with

$$C_1^{(m_2^2)} = -\frac{\pi m_2^2 |q|^{-2-2\epsilon}}{\gamma v (\hat{q} \cdot v_1)^2} \left[\frac{2((1-2\gamma^2)^2 + 2(\hat{q} \cdot v_1)^2)}{\epsilon} + (-8\gamma^2 - 3i\pi(\gamma v)^3 \hat{q} \cdot v_1 + 4(\hat{q} \cdot v_1)^2(2\gamma^2 - i\pi - 2\log[(\hat{q} \cdot v_1)_N^{-1}]) + 4) \right] + \mathcal{O}(\epsilon), \quad (37a)$$

$$C_2^{(m_2^2)} = -\frac{4\pi\gamma m_2^2 |q|^{-2-2\epsilon}}{(\gamma v)^3} \left[\frac{1}{\epsilon} + \frac{(\hat{q} \cdot v_1(2\gamma^2 - i\pi - 2\log[(\hat{q} \cdot v_1)_N^{-1}]) - 2) - 3i\pi(\gamma v)^3}{\hat{q} \cdot v_1} \right] + \mathcal{O}(\epsilon), \quad (37b)$$

$$C_3^{(m_2^2)} = -\frac{\pi m_2^2 |q|^{-2-2\epsilon}}{(\gamma v)^3 \hat{q} \cdot v_1} \left[\frac{2(1-2\gamma^2)}{\epsilon} + \frac{1}{2(\hat{q} \cdot v_1)} (3i\pi\gamma v(5\gamma^4 - 6\gamma^2 + 1) - 7i\pi(\gamma v)^3(\hat{q} \cdot v_1)^2 - 4\hat{q} \cdot v_1(4(\gamma v)^2\gamma^2 - (2\gamma^2 - 1)(i\pi + 2\log[(\hat{q} \cdot v_1)_N^{-1}]))) \right] + \mathcal{O}(\epsilon); \quad (37c)$$

$$C_1^{(m_1 m_2)} = \pi m_1 m_2 |q|^{-2-2\epsilon} \left[-\frac{2\gamma((1-2\gamma^2)^2 + (6-4\gamma^2)(\hat{q} \cdot v_1)^2)}{\gamma v \epsilon (\hat{q} \cdot v_1)^2} + \frac{1}{6} \left(-\frac{24\gamma(2\gamma^2 - 3)(2\text{arccosh}(\gamma) + i\pi + 2\log[(\hat{q} \cdot v_1)_N^{-1}])}{\gamma v} - \frac{24\gamma(-2\gamma^2 + 2\gamma^2(\hat{q} \cdot v_1)^2 + 1)}{\gamma v (\hat{q} \cdot v_1)^2} + \frac{2(61\gamma^4 - 80\gamma^2 + 24\gamma^2(2\gamma^2 - 3)(\hat{q} \cdot v_1)^6 + 2(70\gamma^4 - 105\gamma^2 - 1)(\hat{q} \cdot v_1)^4 + (168\gamma^4 - 233\gamma^2 - 7)(\hat{q} \cdot v_1)^2 - 5)}{((\hat{q} \cdot v_1)^2 + 1)^3} - \frac{3(i\pi + 2\text{arctanh}(\frac{\hat{q} \cdot v_1}{\sqrt{(\hat{q} \cdot v_1)^2 + 1}}))}{\hat{q} \cdot v_1 ((\hat{q} \cdot v_1)^2 + 1)^{7/2}} (3(5\gamma^4 - 8\gamma^2 + 3) + 8(2\gamma^4 - 3\gamma^2 + 1)(\hat{q} \cdot v_1)^6 + 8(7\gamma^4 - 10\gamma^2 + 3)(\hat{q} \cdot v_1)^4 + 25(2\gamma^4 - 3\gamma^2 + 1)(\hat{q} \cdot v_1)^2) \right) \right] + \mathcal{O}(\epsilon), \quad (37d)$$

$$C_2^{(m_1 m_2)} = \pi m_1 m_2 |q|^{-2-2\epsilon} \left[\frac{2((\gamma v)^2(1-2\gamma^2)^2 + (4\gamma^4 - 6\gamma^2)(\hat{q} \cdot v_1)^2)}{(\gamma v)^3 \epsilon (\hat{q} \cdot v_1)^2} + \frac{1}{6} \left(-\frac{24(2\gamma^2 - 3)\gamma^2(2\text{arccosh}(\gamma) + i\pi + 2\log[(\hat{q} \cdot v_1)_N^{-1}])}{(\gamma v)^3} - \frac{24(2\gamma^2 + 6\gamma^2(\hat{q} \cdot v_1)^2 - 1)}{\gamma v (\hat{q} \cdot v_1)^2} + \frac{3(i\pi + 2\text{arctanh}(\frac{\hat{q} \cdot v_1}{\sqrt{(\hat{q} \cdot v_1)^2 + 1}}))\gamma(3(5\gamma^2 + 7) + 8(2\gamma^2 + 1)(\hat{q} \cdot v_1)^6 + 8(7\gamma^2 + 5)(\hat{q} \cdot v_1)^4 + (50\gamma^2 + 53)(\hat{q} \cdot v_1)^2)}{\hat{q} \cdot v_1 ((\hat{q} \cdot v_1)^2 + 1)^{7/2}} - \frac{2\gamma(61\gamma^4 - 70\gamma^2 + 24\gamma^2(2\gamma^2 - 1)(\hat{q} \cdot v_1)^6 + 2(70\gamma^4 - 55\gamma^2 + 21)(\hat{q} \cdot v_1)^4 + 3(56\gamma^4 - 57\gamma^2 + 25)(\hat{q} \cdot v_1)^2 + 33)}{(\gamma v)^2 ((\hat{q} \cdot v_1)^2 + 1)^3} \right) \right] + \mathcal{O}(\epsilon), \quad (37e)$$

$$\begin{aligned}
C_3^{(m_1 m_2)} = & \frac{\pi m_1 m_2 |q|^{-2-2\epsilon}}{\hat{q} \cdot v_1} \left[\frac{2\gamma(-4\gamma^4 + 8\gamma^2 - 3)}{(\gamma v)^3 \epsilon} \right. \\
& + \frac{1}{6(\hat{q} \cdot v_1)} \left(\frac{12(4\gamma^4 - 8\gamma^2 + 3)\gamma(2 \operatorname{arccosh}(\gamma) + i\pi + 2 \log[(\hat{q} \cdot v_1)_N^{-1}])\hat{q} \cdot v_1}{(\gamma v)^3} + \frac{96\gamma^3 \hat{q} \cdot v_1}{\gamma v} \right. \\
& \left. \left. - \frac{3 \left(i\pi + 2 \operatorname{arctanh} \left(\frac{\hat{q} \cdot v_1}{\sqrt{(\hat{q} \cdot v_1)^2 + 1}} \right) \right) (15\gamma^2 + 16\gamma^2 (\hat{q} \cdot v_1)^6 + (52\gamma^2 - 2)(\hat{q} \cdot v_1)^4 + (46\gamma^2 - 5)(\hat{q} \cdot v_1)^2 - 3)}{((\hat{q} \cdot v_1)^2 + 1)^{7/2}} \right. \right. \\
& \left. \left. + \frac{2\hat{q} \cdot v_1 (5\gamma^4 + 30\gamma^2 + 12(2\gamma^2 - 1)(\hat{q} \cdot v_1)^6 + (80\gamma^2 - 44)(\hat{q} \cdot v_1)^4 + (20\gamma^4 + 71\gamma^2 - 55)(\hat{q} \cdot v_1)^2 - 23)}{(\gamma v)^2 ((\hat{q} \cdot v_1)^2 + 1)^3} \right) \right] + \mathcal{O}(\epsilon). \quad (37f)
\end{aligned}$$

The $1/\epsilon$ poles which are present in $z_1^\mu(q)$ either cancel or yield finite terms after Fourier integration in all asymptotic observables that we have computed. Thus, they do not affect our argument that the trajectory is a suitable tool to efficiently derive asymptotic observables, in particular the radiated angular momentum. Analogously, we found the spin trajectory which is made use of in later checks. However, we do not display it but refer to our [100] repository submission.

C. Checks

We performed two main checks on the subleading trajectory (36). First, as at leading order, we confirmed that it reproduces the momentum impulse Δp_1^μ in the asymptotic limit $\tau_1 \rightarrow \pm\infty$, wherein the q -Fourier transform simplifies due to $q \cdot v_1 = 0$. We have

$$\begin{aligned}
\Delta p_1^\mu &= m_1 \int_{-\infty}^{\infty} d\tau_1 \frac{d^2 z_1^\mu(\tau_1)}{d\tau_1^2} \\
&= m_1 \int_q e^{iq \cdot b} \delta(q \cdot v_1) \delta(q \cdot v_2) (-iq \cdot v_1)^2 z_1^\mu(q), \quad (38)
\end{aligned}$$

having inserted Eq. (15) and performed the τ_1 -integration. Inserting the subleading trajectory (36), the Fourier integrals are well known, and we reach agreement with the literature—see Ref. [22] both for the Fourier transform and momentum impulse. In a similar manner, using the time-dependent spin trajectory, we have also reproduced the 2PM spin kick [22].

Our other main check is the on-shell condition $g_{\mu\nu}(x_1) \dot{x}_1^\mu \dot{x}_1^\nu = 1$. At 2PM order, expansion of the on-shell condition in terms of the background fields (7) yields

$$\begin{aligned}
0 = & 2v_1 \cdot \dot{z}_1^{(2)} + (\dot{z}_1^{(1)})^2 + 2h_{\mu\nu}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\mu \dot{z}_1^{(1)\nu} \\
& + h_{\mu\nu}^{(2)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\mu v_1^\nu + \partial_\rho h_{\mu\nu}^{(1)}(b_1^\mu + \tau_1 v_1^\mu) v_1^\mu v_1^\nu \dot{z}_1^{(1)\rho}. \quad (39)
\end{aligned}$$

Besides the trajectory corrections $z_1^{(n)}$, this check requires inclusion of the metric corrections $h_{\mu\nu}^{(n)}$. The first correction $h_{\mu\nu}^{(1)}$ was provided in Eq. (25); the subleading correction is represented by the six diagrams in Fig. 2. Like the trajectory, these diagrams are organized into mass sectors carrying either m_2^2 —representing a subleading contribution to the Kerr metric—or $m_1 m_2$, whose contributions represent the leading-order gravitational waveform from a scattering of two spinning black holes [55].

The check on the subleading on-shell condition is most easily performed in momentum space. Inserting the Fourier

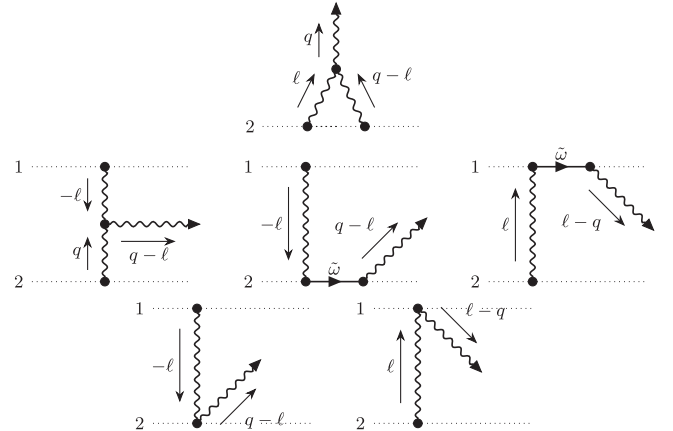


FIG. 2. Diagrams contributing to the subleading metric $h_{\mu\nu}^{(2)}$. The top diagram carries m_2^2 and represents a 2PM contribution to the Kerr metric; the five remaining diagrams carry $m_1 m_2$ and represent the leading-order scattering waveform of two spinning black holes first derived in Ref. [55].

transform on the trajectory (15), the condition (39) is rendered as

$$0 = \int_q e^{iq \cdot \tilde{b}} \delta(q \cdot v_2) \left\{ 2(-iq \cdot v_1) v_1 \cdot z_1^{(2)}(q) + h_{\mu\nu}^{(2)}(q) v_1^\mu v_1^\nu \right. \\ \left. + \int_\ell \delta(\ell \cdot v_2) \left[(-i\ell \cdot v_1)(-i(q-\ell) \cdot v_1) z_1^{(1)}(\ell) \cdot z_1^{(1)}(q-\ell) \right. \right. \\ \left. \left. + (2(-i(q-\ell) \cdot v_1) \delta_\rho^\nu + (-i\ell_\rho) v_1^\mu) v_1^\mu h_{\mu\nu}^{(1)}(\ell) z_1^{(1)\rho}(q-\ell) \right] \right\}. \quad (40)$$

Here, we introduced the notation

$$h_{\mu\nu}^{(n)}(b_1^\mu + \tau_1 v_1^\mu) = \int_k e^{ik \cdot \tilde{b}(\tau_1)} \delta(k \cdot v_2) h_{\mu\nu}^{(n)}(k). \quad (41)$$

Time derivatives on the trajectory are replaced by factors of $-i(q \cdot v_1)$, while derivatives ∂_μ on the waveform are replaced by $-iq^\mu$. We have also used

$$\int_{q_1} e^{iq_1 \cdot \tilde{b}} g(q_1) \int_{q_2} e^{iq_2 \cdot \tilde{b}} h(q_2) = \int_{q, \ell} e^{iq \cdot \tilde{b}} g(\ell) h(q-\ell), \quad (42)$$

which is derived by identifying $q^\mu = q_1^\mu + q_2^\mu$ and $\ell^\mu = q_1^\mu$. We need only demonstrate vanishing of the terms in curly brackets of Eq. (40) to verify the on-shell condition (39).

To verify Eq. (40), we must perform the integral on ℓ^μ present for certain terms. Quite conveniently, the integrals involved all belong to the one-loop probe family $I_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}$ (32a) already used when evaluating the 2PM trajectory. The only new subtlety is that these one-loop integrals on ℓ^μ can have worldline propagators of the kind $1/(\ell \cdot v_1 + i0)$ and $1/((q-\ell) \cdot v_1 + i0)$. In the latter case, we can simply relabel $\ell \rightarrow q-\ell$ to bring our integrals into the form of Eq. (32a); however, if both worldline propagators appear together, then we have an over complete set. We therefore apply the partial fraction identity,

$$\frac{1}{(\ell \cdot v_1 + i0)((q-\ell) \cdot v_1 + i0)} \\ = \frac{1}{(q \cdot v_1 + i0)} \left(\frac{1}{(q-\ell) \cdot v_1 + i0} + \frac{1}{\ell \cdot v_1 + i0} \right), \quad (43)$$

and $(q \cdot v_1 + i0)^{-1}$ can be pulled outside the integral. The resulting terms can then be relabeled as above. Given these

ingredients, we can confirm that the on-shell condition (39) holds as expected up to linear order in spin.

The involvement of the full gravitational scattering waveform makes this highly nontrivial. Similarly, we can verify local preservation of the SSC $\dot{x}_{i,\mu} S_i^{\mu\nu} = 0$, which follows from

$$\alpha_i \cdot \pi_i = 0, \quad (44)$$

with the covariant momentum $\pi_i^\mu = m_i \dot{x}_i^\mu + \mathcal{O}(R^2)$ [76]. The analog of Eq. (40) is

$$0 = \int_q e^{iq \cdot \tilde{b}} \delta(q \cdot v_2) \left\{ (-iq \cdot v_1) \alpha_{1,-\infty} \cdot z_1^{(2)}(q) + \alpha_1^{(2)}(q) \cdot v_1 \right. \\ \left. + \int_\ell \delta(\ell \cdot v_2) (-i(q-\ell) \cdot v_1) \alpha_1^{(1)}(\ell) z_1^{(1)}(q-\ell) \right\}. \quad (45)$$

This involves the Fourier transforms of α_1 , $\bar{\alpha}_1$, and z_1 . By similar calculations, we have moreover confirmed that

$$(\alpha_1)^2 = \text{const}, \quad (\bar{\alpha}_1)^2 = \text{const}, \quad \bar{\alpha}_1 \cdot \alpha_1 = \text{const}. \quad (46)$$

These quantities are related to the spin length by

$$-\frac{1}{2} S_1^{\mu\nu} S_{1,\mu\nu} = (\bar{\alpha}_1 \cdot \alpha_1)^2 - \bar{\alpha}_1^2 \alpha_1^2, \quad (47)$$

and so the spin length is itself also conserved.

V. RADIATED ANGULAR MOMENTUM

Using our new results for the full scattering trajectories, let us finally compute the overall change in angular momentum ΔJ^μ —and thus the total radiated angular momentum $J_{\text{rad}}^\mu = -\Delta J^\mu$. The angular momentum vector J^μ , defined with respect to the center-of-momentum frame, is

$$J^\mu = \frac{1}{2} \epsilon^\mu_{\nu\rho\sigma} J^{\nu\rho} \hat{P}^\sigma, \quad (48)$$

where $\hat{P}^\mu = \frac{1}{E} (m_1 v_1^\mu + m_2 v_2^\mu)$ is the unit vector pointing in the direction of the total momentum, $E = M\sqrt{1+2\nu(\gamma-1)}$ is the total incoming energy, and $M = m_1 + m_2$, $\nu = m_1 m_2 / M^2$. At the perturbative PM orders we are dealing with in this paper, and given that momentum loss due to gravitational wave emission begins only at 3PM order [39,97], we may safely assume that the total energy E and momentum P^μ (and therefore $\hat{P}^\mu$) are conserved. The total angular momentum tensor is given by a sum of orbital and spin components:

$$J^{\mu\nu} = L^{\mu\nu} + S_1^{\mu\nu} + S_2^{\mu\nu}, \quad (49)$$

$$L^{\mu\nu} = 2x_1^{[\mu} p_1^{\nu]} + 2x_2^{[\mu} p_2^{\nu]}. \quad (50)$$

The two parts of the orbital angular momentum, labeled, respectively, by the subscripts 1 and 2, are parametrized by separate time parameters τ_i . Inserting into Eq. (48), we learn that

$$J^\mu = L^\mu + \sum_{i=1}^2 \Lambda_i^\mu{}_\nu S_i^\nu, \quad (51a)$$

$$\Lambda_i^\mu{}_\nu = \dot{x}_i \cdot \hat{P} \delta_\nu^\mu - \dot{x}_i^\mu \hat{P}_\nu, \quad (51b)$$

where the angular momentum vector L^μ is given by

$$L^\mu = \frac{1}{2} \epsilon^\mu{}_{\nu\rho\sigma} L^{\nu\rho} \hat{P}^\sigma = \frac{1}{E} \epsilon^\mu{}_{\nu\rho\sigma} (x_1 - x_2)^\nu p_1^\rho p_2^\sigma. \quad (52)$$

Notice that $J^\mu \neq L^\mu + S_1^\mu + S_2^\mu$, because the spin vectors S_i^μ are defined with respect to their respective rest frames $\dot{x}_i^\mu$ (3). The boosts $\Lambda_i^\mu{}_\nu$ are therefore needed to bring them into the center-of-mass frame where L^μ is also defined.

The asymptotic change ΔJ^μ is given by

$$\Delta J^\mu = J^\mu(\tau_1, \tau_2 \rightarrow +\infty) - J^\mu(\tau_1, \tau_2 \rightarrow -\infty), \quad (53)$$

and using Eq. (51), we have

$$\begin{aligned} L_{S^0}^{(1)\mu}(\tau_1, \tau_2) &= \frac{m_1 m_2}{\hat{D}(\tau_1)^3 |b| E} \left[\frac{m_2 \hat{b}^\mu (\gamma^2 v a_1 \cdot \hat{b} - a_1 \cdot v_2 (\hat{D}(\tau_1)^3 + \gamma^3 \hat{\tau}_1^3 v^3)) - 2\gamma^4 \hat{\tau}_1^2 v^3 a_2 \cdot \hat{b} + 2\gamma a_2 \cdot v_1 (\hat{D}(\tau_1)^3 + \gamma^3 \hat{\tau}_1^3 v^3)}{\gamma v} \right. \\ &+ \frac{(\gamma m_1 + m_2) w_1^\mu (\gamma^2 v a_1 \cdot \hat{b} (2\hat{D}(\tau_1)^3 + 2\gamma^3 \hat{\tau}_1^3 v^3 + 3\gamma \hat{\tau}_1 v) - a_1 \cdot v_2 + 2\gamma(\gamma v a_2 \cdot \hat{b} (\hat{D}(\tau_1)^3 + \gamma^3 \hat{\tau}_1^3 v^3 + 2\gamma \hat{\tau}_1 v) + a_2 \cdot v_1))}{2} \\ &- \frac{m_1 v_2^\mu (\gamma^2 v a_1 \cdot \hat{b} (2\hat{D}(\tau_1)^3 + 2\gamma^3 \hat{\tau}_1^3 v^3 + 3\gamma \hat{\tau}_1 v) - a_1 \cdot v_2 + 2\gamma(\gamma v a_2 \cdot \hat{b} (\hat{D}(\tau_1)^3 + \gamma^3 \hat{\tau}_1^3 v^3 + 2\gamma \hat{\tau}_1 v) + a_2 \cdot v_1))}{2} \\ &\left. + \frac{\hat{\epsilon}^\mu (a_1 \cdot \hat{\epsilon} (m_1 - \gamma m_2) - 2\gamma a_2 \cdot \hat{\epsilon} (\gamma m_1 v^2 + 2m_2 (\gamma^2 \hat{\tau}_1^2 v^2 + 1)))}{2} \right] + (1 \leftrightarrow 2), \quad (56) \end{aligned}$$

where $\hat{D}(\tau)$ was introduced in Eq. (28). Meanwhile, the spin contribution is given by

$$\begin{aligned} \Delta J^\mu &= \Delta L^\mu + \sum_i \frac{1}{m_i} (\Delta p_i \cdot \hat{P} S_i^\mu - S_i \cdot \hat{P} \Delta p_i^\mu) \\ &+ \sum_i \frac{1}{m_i} ((p_i + \Delta p_i) \cdot \hat{P} \Delta S_i^\mu - \Delta S_i \cdot \hat{P} (p_i^\mu + \Delta p_i^\mu)). \end{aligned} \quad (54)$$

This involves the change in the orbital angular momentum vector ΔL^μ . As with the trajectory, we compute ΔJ^μ at leading (1PM) and subleading (2PM) order. At leading order, this merely confirms that no angular momentum is radiated, thus providing another check on the trajectory.

A. Leading order

At leading-PM order, we can write down the angular momentum as a function of the proper times of both massive bodies $\tau_{1,2}$. In the nonspinning case, inserting the time-dependent trajectory into Eq. (52), we have

$$\begin{aligned} L_{S^0}^{(1)\mu}(\tau_1, \tau_2) &= \frac{m_1 m_2 |b| \hat{l}^\mu [\gamma(3 - 2\gamma^2) m_1 + (3 - 4\gamma^2) m_2]}{2\gamma v D(\tau_1) E} \\ &+ (1 \leftrightarrow 2). \end{aligned} \quad (55)$$

It is clear that the difference $\Delta L^\mu = L^\mu(+\infty, +\infty) - L^\mu(-\infty, -\infty)$ vanishes, as $D(\tau_i) = D(-\tau_i)$. Thus, as $\Delta J^\mu = \Delta L^\mu$, there is no dissipation, and angular momentum is conserved. Working up to quadratic order in spin, we require both the orbital and spin contributions to the total angular momentum J^μ . The orbital component is presented here up to linear order in spin,

$$\begin{aligned}
& \sum_i \Lambda_i^\mu S_i^{(1)\nu} \\
&= \frac{m_1 m_2}{|b|E} \left[\hat{b}^\mu \left[\frac{2(m_1 + \gamma m_2)(a_1 \cdot v_2(2\gamma^2 \hat{D}(\tau_1) \hat{\tau}_1^2 v^2 + \hat{D}(\tau_1) + 2\gamma^3 \hat{\tau}_1^3 v^3 + 2\gamma \hat{\tau}_1 v) - v a_1 \cdot \hat{b}(\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + \gamma^2 \hat{\tau}_1^2 v^2 + 1))}{\hat{D}(\tau_1)^2 v S(\tau_1)} \right. \right. \\
&\quad \left. \left. - \frac{a_1 \cdot \hat{b}((2\gamma^2 - 1)m_1 + 2\gamma m_2)(\hat{D}(\tau_1) + \gamma \hat{\tau}_1 v)}{\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + \gamma^2 \hat{\tau}_1^2 v^2 + 1} - \frac{(2\gamma^2 - 1)m_2 a_1 \cdot v_2(\hat{D}(\tau_1) + \gamma \hat{\tau}_1 v)}{\gamma v \hat{D}(\tau_1)} \right] \right. \\
&\quad \left. - \frac{(\gamma m_1 + m_2) w_1^\mu (2\gamma^2 v a_1 \cdot \hat{b}(4\gamma^2 \hat{D}(\tau_1) \hat{\tau}_1^2 v^2 + \hat{D}(\tau_1) + 4\gamma^3 \hat{\tau}_1^3 v^3 + 3\gamma \hat{\tau}_1 v) + (4\gamma^2 - 1)a_1 \cdot v_2(2\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + 2\gamma^2 \hat{\tau}_1^2 v^2 + 1))}{S(\tau_1)(\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + \gamma^2 \hat{\tau}_1^2 v^2 + 1)} \right. \\
&\quad \left. + \frac{m_1 v_2^\mu (2\gamma^2 v a_1 \cdot \hat{b}(4\gamma^2 \hat{D}(\tau_1) \hat{\tau}_1^2 v^2 + \hat{D}(\tau_1) + 4\gamma^3 \hat{\tau}_1^3 v^3 + 3\gamma \hat{\tau}_1 v) + (4\gamma^2 - 1)a_1 \cdot v_2(2\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + 2\gamma^2 \hat{\tau}_1^2 v^2 + 1))}{S(\tau_1)(\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + \gamma^2 \hat{\tau}_1^2 v^2 + 1)} \right. \\
&\quad \left. + \frac{a_1 \cdot \hat{\ell} \hat{\ell}^\mu}{(\gamma v \hat{\tau}_1 S(\tau_1) + |b|)} \left[- \frac{2(m_1 + \gamma m_2)(\gamma \hat{D}(\tau_1) \hat{\tau}_1 v + \gamma^2 \hat{\tau}_1^2 v^2 + 1)^2}{\hat{D}(\tau_1)^2 S(\tau_1)} - ((2\gamma^2 - 1)m_1 + 2\gamma m_2) S(\tau_1) \right] \right] + (1 \leftrightarrow 2). \quad (57)
\end{aligned}$$

Full results up to quadratic order in spin are provided in the [100] submission. Summing up these two contributions to obtain $J^\mu(\tau_1, \tau_2)$ (51), we have confirmed that $\Delta J^\mu = 0$ at this perturbative order, and thus there is no radiated angular momentum.

B. Subleading order

Finally, in this section, we calculate ΔJ^μ up to $\mathcal{O}(G^2)$ precision, i.e., a one-loop result up to linear order in spin. As our result for the subleading trajectory (36) is given only in momentum space, we take a slightly different approach here than used at leading-order 1PM. The change in the orbital angular momentum vector L^μ is given by

$$\Delta L^\mu = \frac{1}{2} \epsilon^{\mu}_{\nu\rho\sigma} \Delta L^{\nu\rho} \hat{P}^\sigma. \quad (58)$$

Meanwhile, the 2PM expression for the angular momentum tensor $L^{\mu\nu}$ decomposes as follows:

$$\begin{aligned}
L^{(2)\mu\nu}(\tau_1, \tau_2) &= 2 \sum_{i=1}^2 \left(x_i^{(2)[\mu}(\tau_i) p_i^{(0)\nu]} + x_i^{(1)[\mu}(\tau_i) p_i^{(1)\nu]}(\tau_i) \right. \\
&\quad \left. + x_i^{(0)[\mu}(\tau_i) p_i^{(2)\nu]}(\tau_i) \right). \quad (59)
\end{aligned}$$

We evaluate these three contributions separately, ultimately obtaining results for the radiated angular momentum in agreement with the literature [78,94].

To evaluate the first contribution, it suffices to plug in our momentum space result and take the asymptotic limit. The procedure is analogous to Eq. (38):

$$\begin{aligned}
& \Delta(x_1^{(2)[\mu} p_1^{(0)\nu]}) \\
&= m_1 \int_{-\infty}^{\infty} d\tau_1 \frac{d}{d\tau_1} \left(\int_q e^{iq \cdot \tilde{b}(\tau_1)} \delta(q \cdot v_2) z_1^{(2)[\mu}(q) v_1^{\nu]} \right) \\
&= m_1 \int_q e^{iq \cdot b} \delta(q \cdot v_1) \delta(q \cdot v_2) (-iq \cdot v_1) z_1^{(2)[\mu}(q) v_1^{\nu]}. \quad (60)
\end{aligned}$$

For concreteness, we have specialized to the trajectory of BH1. Tackling the second term in Eq. (59) involves computing products of Fourier transforms, so we again make use of convolutions (42):

$$\begin{aligned}
& \Delta(x_1^{(1)[\mu} p_1^{(1)\nu]}) \\
&= \Delta \left(\int_{q, \ell} e^{iq \cdot \tilde{b}(\tau_1)} \delta(q \cdot v_2) \delta(\ell \cdot v_2) z_1^{(1)[\mu}(\ell) p_1^{(1)\nu]}(q - \ell) \right) \\
&= \int_q e^{iq \cdot b} \delta(q \cdot v_1) \delta(q \cdot v_2) (-iq \cdot v_1) \\
&\quad \times \int_{\ell} \delta(\ell \cdot v_2) z_1^{(1)[\mu}(\ell) p_1^{(1)\nu]}(q - \ell). \quad (61)
\end{aligned}$$

The resulting one-loop integral family, again, is the same encountered earlier. While it is also feasible to obtain this contribution in position space, we perform the calculation in momentum space in order to maintain the link to higher perturbative orders. As for the third term in Eq. (59), using the zeroth-order trajectory $x_1^{(0)\mu} = b_1^\mu + \tau_1 v_1^\mu$, we have

$$\Delta(x_1^{(0)[\mu} p_1^{(2)\nu]}) = b_1^{[\mu} \Delta p_1^{(2)\nu]} + v_1^{[\mu} \Delta(\tau_1 p_1^{(2)\nu]}). \quad (62)$$

The second term is evaluated using an integration-by-parts identity. We decompose it as

$$\begin{aligned}
v_1^{[\mu} \Delta(\tau_1 p_1^{(2)\nu]}) &= v_1^{[\mu} \int_{-\infty}^{\infty} d\tau \frac{d}{d\tau} \left(\tau \int_q e^{iq \cdot \tilde{b}(\tau)} \delta(q \cdot v_2) p^{(2)\nu]}(q) \right) \\
&= v_1^{[\mu} \int_q e^{iq \cdot b} (\delta(q \cdot v_1) + (q \cdot v_1) \delta'(q \cdot v_1)) \delta(q \cdot v_2) p^{(2)\nu]}(q) \\
&= -v_1^{[\mu} \int_q e^{iq \cdot b} \delta(q \cdot v_1) \delta(q \cdot v_2) (q \cdot v_1) \frac{d}{d(q \cdot v_1)} [p(q)^{(2)\nu}]. \quad (63)
\end{aligned}$$

The remaining contribution is a Fourier integral over $v_1 \cdot \partial_q p(q)^{(2)\nu}$ by virtue of the chain rule.

Adding all contributions to the angular momentum tensor (59) together, and combining to obtain the change in angular momentum vector ΔL^μ (58), we find

$$\begin{aligned}
\Delta J_{S^0}^{(2)\mu} &= -\frac{m_1 m_2 4(2\gamma^2 - 1)}{3|b|(\gamma v)^3} [(8 - 5\gamma^2)\gamma v + 3\gamma(2\gamma^2 - 3)\text{arccosh}(\gamma)] \hat{l}^\mu \\
&+ \frac{8i\pi m_1 m_2 \gamma(3 - 8\gamma^2 + 4\gamma^4)}{|b|(\gamma v)^3} \left[\int_q \delta(q \cdot v_1) \delta(q \cdot v_2) (b \cdot q) \left(e^{iq \cdot b} \left(\frac{1}{2\epsilon} - \frac{i\pi}{2} - \log[(\hat{q} \cdot v_1)_N^{-1}] \right) + (1 \leftrightarrow 2) \right) \right] \hat{l}^\mu, \quad (64)
\end{aligned}$$

where in this case $\Delta J^\mu = \Delta L^\mu$ due to the absence of spin. The entire second line cancels by symmetry: as the result can only depend on $|b|$ and $\gamma = v_1 \cdot v_2$, one may freely exchange particle labels, $v_1^\mu \leftrightarrow v_2^\mu$ and $b_1^\mu \leftrightarrow b_2^\mu$, implying $b^\mu \rightarrow -b^\mu$. The final result, given in the top line of Eq. (64), gains contributions only from the $m_1 m_2$ mass sector of the 2PM trajectory. This aligns with our expectations, as it is this mass sector which contains diagrams where gravitons can go on shell.

The only substantial difference in the spinning case, besides the higher tensor rank of the integrals involved (loop integrals and Fourier transforms), is that we must also include contributions from the change in spin vectors (51). Up to linear order in spin,

$$\Delta \left(\sum_{i=1}^2 \Lambda_i^\mu{}_\nu S_i^\nu \right) = \Delta \left(\sum_{i=1}^2 \dot{x}_{i,S^0} \cdot \hat{P} S_{i,S^1}^\mu - \dot{x}_i^\mu \hat{P} \cdot S_{i,S^1} \right). \quad (65)$$

Decomposing this into the contributing PM orders of the individual terms, we find that at 2PM

$$\begin{aligned}
&\Delta \left(\sum_{i=1}^2 \Lambda_i^\mu{}_\nu S_i^\nu \right) \\
&= \sum_{i=1}^2 (\hat{P} \cdot \Delta p_i^{(2)} a_i^\mu + (\hat{P} \cdot v_i) \Delta S_i^{(2)\mu} + (\hat{P} \cdot \Delta p_i) \Delta a_i^{(1)\mu} \\
&\quad - \hat{P} \cdot \Delta S_i^{(2)} v_i^\mu - (\hat{P} \cdot a_i) \Delta p_i^{(2)\mu} - (\hat{P} \cdot \Delta a_i^{(1)}) \Delta p_i^{(1)\mu}). \quad (66)
\end{aligned}$$

Taking this together with the change in the orbital angular momentum vector, we learn that overall

$$\begin{aligned}
\Delta J_{S^1}^{(2)\mu} &= -\frac{8\gamma m_1 m_2}{3(\gamma v)^2 |b|^2} [(8 - 5\gamma^2)\gamma v \\
&+ \gamma(6\gamma^2 - 9)\text{arccosh}(\gamma)] (\hat{l}^\mu (a_{12} \cdot \hat{l}) - b^\mu (a_{12} \cdot b)). \quad (67)
\end{aligned}$$

This result has components in the b^μ and $\hat{l}^\mu$ directions and agrees with the literature [55,81]. Divergent integrals of the kind in Eq. (64) could be shown to cancel by using the symmetry of these terms under the exchange of the black hole labels 1 and 2.

Finally, we specialize to aligned spins using

$$a_1^\mu = s_1 \hat{l}^\mu, \quad a_2^\mu = s_2 \hat{l}^\mu, \quad (68)$$

aligning the spin vectors with the orbital angular momentum of the system, so $a_i \cdot v_j = 0$ and $a_i \cdot b = 0$. The answer then simplifies to

$$\begin{aligned}
\Delta J_{S^1}^{(2)\mu} &= \frac{8\gamma m_1 m_2 (s_1 + s_2)}{3(\gamma v)^2 |b|^2} [(8 - 5\gamma^2)\gamma v \\
&+ \gamma(6\gamma^2 - 9)\text{arccosh}(\gamma)] \hat{l}^\mu. \quad (69)
\end{aligned}$$

As in the nonspinning case, $\Delta J^\mu = \Delta L^\mu$, and the result points in the direction of $\hat{l}^\mu$. There is no contribution due to the spin kick ΔS_i^μ as this observable vanishes.

VI. CONCLUSIONS

In this work, we have computed the trajectories of massive bodies involved in a classical gravitational two-body scattering in the presence of spin, using methods of perturbative quantum field theory. We established the 1PM

trajectories with spin effects in both momentum and coordinate space as well as the 2PM corrections in momentum space including linear-in-spin contributions. The underlying Feynman integrals contain an additional scale compared to asymptotic observables: alongside the momentum transfer q^μ (dual to the impact parameter b^μ), we encountered the quantity $q \cdot v_i$ (dual to the time parameter τ_i) for the trajectory of black hole i . At 2PM order, the Fourier transform to coordinate space remains challenging, mirroring similar difficulties in the next-to-leading-order waveform [57,59–63,65,67]. Some of the ideas presented in Ref. [64], namely, performing integration-by-parts reductions in the presence of exponential Fourier factors, might aid future computations.

Our work demonstrates that calculating scattering trajectories provides a novel pathway toward determining a crucial asymptotic quantity in the two-body scattering problem: the radiated angular momentum. Given the more cumbersome nature of the gravitational waveform, it is advantageous to have a direct computational pathway to J_{rad}^μ that bypasses the waveform at higher perturbative orders. Together with the radiated energy-momentum P_{rad}^μ , J_{rad}^μ will play a crucial role in the development of future gravitational waveform models (including effective one body) for the two-body problem informed by PM results [101–106].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [100].

APPENDIX A: LEADING-ORDER TRAJECTORY AT QUADRATIC ORDER IN SPIN

Here, we present the leading-PM trajectory at quadratic order in spin. We write the result in terms of its coefficients with respect to an orthogonal basis spanned by w_1^μ , v_2^μ , $\hat{b}^\mu$, and $\hat{l}^\mu$. The coefficients are sorted by dot products of the spin vectors of both black holes,

$$z_1^{(1)\mu}(\tau_1)_{S^2} = F_1 w_1^\mu + F_2 v_2^\mu + F_3 \hat{b}^\mu + F_4 \hat{l}^\mu. \quad (\text{A1})$$

The coefficients are

$$\begin{aligned} F_1 = & \frac{m_2}{\hat{D}(\tau_1)^3 |b|^2} \left[\frac{1}{2} a_1^2 \hat{\tau}_1 [3\gamma^2 + (2\gamma^4 - 3\gamma^2 + 1)\hat{\tau}_1^2 - 2] + a_1 \cdot \hat{b} \left[\gamma^2 \hat{\tau}_1 (2(\gamma v)^2 \hat{\tau}_1^2 + 3) a_2 \cdot \hat{b} + \frac{a_2 \cdot v_1}{v^2} \right] \right. \\ & + a_1 \cdot v_2 \left[-\frac{a_1 \cdot \hat{b} + a_2 \cdot \hat{b}}{\gamma v^2} - \gamma \hat{\tau}_1 (\gamma^2 \hat{\tau}_1^2 + 1) a_2 \cdot v_1 \right] + \gamma^2 \hat{\tau}_1 a_1 \cdot a_2 [(\gamma v)^2 \hat{\tau}_1^2 + 1] \\ & + \frac{(a_1 \cdot \hat{b})^2 (2\gamma^4 - 3\gamma^2 + 1) \hat{\tau}_1 (2\gamma^2 \hat{\tau}_1^2 v^2 + 3)}{2\gamma^2 v^2} + \frac{(a_1 \cdot v_2)^2}{2} \hat{\tau}_1 [(2\gamma^2 - 1)\hat{\tau}_1^2 + 3] + \frac{1}{2} a_2^2 \hat{\tau}_1 [-2\gamma^2 + (\gamma v)^2 \hat{\tau}_1^2 + 3] \\ & + \frac{(a_2 \cdot \hat{b})^2}{2} \hat{\tau}_1 [2(\gamma v)^2 \hat{\tau}_1^2 + 3] + \frac{(2\gamma^2 - 1) a_2 \cdot \hat{b} a_2 \cdot v_1}{\gamma^2 v^2} + \frac{1}{2} \hat{\tau}_1 (\hat{\tau}_1^2 - 4) (a_2 \cdot v_1)^2 - \frac{a_1 \cdot \hat{b} a_2 \cdot v_1 (\gamma^2 \hat{\tau}_1^2 v^2 + 1) \hat{D}(\tau_1)}{v^2} \\ & \left. + a_1 \cdot v_2 \left[\frac{(a_1 \cdot \hat{b} + a_2 \cdot \hat{b}) (\gamma^2 \hat{\tau}_1^2 v^2 + 1) \hat{D}(\tau_1)}{\gamma v^2} \right] - \frac{(2\gamma^2 - 1) a_2 \cdot \hat{b} a_2 \cdot v_1 (\gamma^2 \hat{\tau}_1^2 v^2 + 1) \hat{D}(\tau_1)}{\gamma^2 v^2} \right], \quad (\text{A2}) \end{aligned}$$

$$\begin{aligned}
F_2 = & \frac{m_2}{\hat{D}(\tau_1)^3 |b|^2} \left[a_1 \cdot \hat{b} \left[-\gamma \hat{\tau}_1 (2(\gamma v)^2 \hat{\tau}_1^2 + 3) a_2 \cdot \hat{b} - \frac{a_2 \cdot v_1}{\gamma v^2} \right] + a_1 \cdot v_2 \left[\frac{a_2 \cdot \hat{b}}{\gamma^2 v^2} + \hat{\tau}_1 (\gamma^2 \hat{\tau}_1^2 + 1) a_2 \cdot v_1 \right] \right. \\
& - a_1 \cdot a_2 \gamma \hat{\tau}_1 [(\gamma v)^2 \hat{\tau}_1^2 + 1] - a_2^2 \gamma \hat{\tau}_1 [(\gamma v)^2 \hat{\tau}_1^2 + 1] - \gamma \hat{\tau}_1 [2(\gamma v)^2 \hat{\tau}_1^2 + 3] (a_2 \cdot \hat{b})^2 - \frac{2a_2 \cdot \hat{b} a_2 \cdot v_1}{\gamma v^2} - \gamma \hat{\tau}_1^3 (a_2 \cdot v_1)^2 \\
& \left. + \frac{(\gamma a_1 \cdot \hat{b} a_2 \cdot v_1 - a_1 \cdot v_2 a_2 \cdot \hat{b}) [\gamma^2 \hat{\tau}_1^2 v^2 + 1] \hat{D}(\tau_1)}{\gamma^2 v^2} + \frac{a_2 \cdot \hat{b} a_2 \cdot v_1 [2\gamma^2 \hat{\tau}_1^2 v^2 + 2] \hat{D}(\tau_1)}{\gamma v^2} \right], \tag{A3}
\end{aligned}$$

with $\hat{D}(\tau)$ introduced in the main text (28). For a better overview, we split F_3 further: $F_3 = \sum_{i=1}^5 F_{3i}$,

$$\begin{aligned}
F_{31} = & \frac{m_2}{(\gamma v)^4 |b|^2} \left[a_1^2 \left[\frac{3\gamma^2 - 2(2\gamma^4 - 3\gamma^2 + 1)\gamma^4 \hat{\tau}_1^4 v^4 - 3(\gamma v)^4 (2\gamma^2 - 1)\hat{\tau}_1^2 + \gamma^4 (v^4 - 2) - 1}{2\hat{D}(\tau_1)^3} \right. \right. \\
& \left. \left. - \frac{3\gamma^2 + 2(2\gamma^4 - 3\gamma^2 + 1)\gamma \hat{\tau}_1 v + \gamma^4 (v^4 - 2) - 1}{2} \right] \right. \\
& + a_1 \cdot a_2 \left[-\frac{(2\gamma^4 - 3\gamma^2 + 1)(2\gamma^4 \hat{\tau}_1^4 v^4 + 3(\gamma v)^2 \hat{\tau}_1^2 + 1) + (2\gamma^4 - 3\gamma^2 + 1)(2\gamma \hat{\tau}_1 v - 1)\hat{D}(\tau_1)^3}{\hat{D}(\tau_1)^3} \right] \\
& + a_2^2 \left[\frac{-3\gamma^2 - 2(2\gamma^4 - 3\gamma^2 + 1)\gamma \hat{\tau}_1 v + 2\gamma^4 (v^4 + 1) + 1}{2} \right. \\
& \left. \left. - \frac{-3\gamma^2 + 2(2\gamma^4 - 3\gamma^2 + 1)\gamma^4 \hat{\tau}_1^4 v^4 + 3(\gamma v)^4 (2\gamma^2 - 1)\hat{\tau}_1^2 + 2\gamma^4 (v^4 + 1) + 1}{2\hat{D}(\tau_1)^3} \right] \right], \tag{A4}
\end{aligned}$$

$$\begin{aligned}
F_{32} = & \frac{m_2 a_1 \cdot v_2}{v^2 |b|^2} \left[\frac{\hat{\tau}_1 ((\gamma v)^2 \hat{\tau}_1^2 + 2) a_1 \cdot \hat{b}}{\gamma \hat{D}(\tau_1)^3} + \frac{\hat{\tau}_1 ((\gamma v)^2 \hat{\tau}_1^2 + 2) a_2 \cdot \hat{b}}{\gamma \hat{D}(\tau_1)^3} \right. \\
& + a_2 \cdot v_1 \left[\frac{2(\gamma v)^4 (2\gamma^2 - 1)\hat{\tau}_1^4 + (6\gamma^4 - 9\gamma^2 + 3)\hat{\tau}_1^2 + 2\gamma^2}{\gamma^3 v^2 \hat{D}(\tau_1)^3} + \frac{2((\gamma v)^2 \hat{\tau}_1^2 + 1)}{\gamma \hat{D}(\tau_1)^2} + \frac{2(2\gamma^2 - 1)(\gamma \hat{\tau}_1 v - 1)}{\gamma^3 v^2} \right] \\
& + a_1 \cdot v_2 \left[\frac{-5\gamma^2 + 2(1 - 2\gamma^2)\gamma^4 \hat{\tau}_1^4 v^4 + (-6\gamma^4 + 9\gamma^2 - 3)\hat{\tau}_1^2 - 2\gamma^4 (v^4 - 1) + 1}{2\gamma^4 \hat{D}(\tau_1)^3 v^2} - \frac{2((\gamma v)^2 \hat{\tau}_1^2 + 1)}{\hat{D}(\tau_1)^2} \right. \\
& \left. \left. + \frac{\gamma^2 + (2\gamma - 4\gamma^3)\hat{\tau}_1 v + 2\gamma^4 (v^4 + 1) - 1}{2\gamma^4 v^2} \right] \right], \tag{A5}
\end{aligned}$$

$$\begin{aligned}
F_{33} = & \frac{m_2 a_1 \cdot \hat{b}}{(\gamma v)^4 |b|^2} \left[a_2 \cdot \hat{b} \left[-3\gamma^2 - 4(2\gamma^4 - 3\gamma^2 + 1)\gamma \hat{\tau}_1 v + \gamma^4 (v^4 + 2) + 1 \right. \right. \\
& \left. \left. - \frac{-3\gamma^2 + 4(2\gamma^4 - 3\gamma^2 + 1)\gamma^4 \hat{\tau}_1^4 v^4 + 6(\gamma v)^4 (2\gamma^2 - 1)\hat{\tau}_1^2 + \gamma^4 (v^4 + 2) + 1}{\hat{D}(\tau_1)^3} \right] \right. \\
& + \frac{\hat{\tau}_1 a_2 \cdot v_1 [6\gamma^2 - (\gamma^2 - 1)^2 (2\gamma^2 - 1)\hat{\tau}_1^2 + \gamma^4 (v^4 - 4) - 2]}{\hat{D}(\tau_1)^3} \\
& \left. - a_1 \cdot \hat{b} \left[\frac{(2\gamma^4 - 3\gamma^2 + 1)(4\gamma^4 \hat{\tau}_1^4 v^4 + 6(\gamma v)^2 \hat{\tau}_1^2 + 1)}{2\hat{D}(\tau_1)^3} + \frac{(2\gamma^4 - 3\gamma^2 + 1)(4\gamma \hat{\tau}_1 v - 1)}{2} \right] \right], \tag{A6}
\end{aligned}$$

$$\begin{aligned}
F_{34} = & \frac{m_2 a_2 \cdot v_1}{(\gamma v)^4 |b|^2} \left[\frac{\hat{\tau}_1 a_2 \cdot \hat{b} [6\gamma^2 - (\gamma v)^4 (2\gamma^2 - 1)\hat{\tau}_1^2 + 2\gamma^4 (v^4 - 2) - 2]}{\hat{D}(\tau_1)^3} \right. \\
& \left. - a_2 \cdot v_1 \left[\frac{(2\gamma^2 - 1)(2\gamma^4 \hat{\tau}_1^4 v^4 + 3(\gamma v)^2 \hat{\tau}_1^2 + 2) + 2(2\gamma^2 - 1)(\gamma \hat{\tau}_1 v - 1)\hat{D}(\tau_1)^3}{2\hat{D}(\tau_1)^3} \right] \right], \tag{A7}
\end{aligned}$$

$$F_{35} = \frac{m_2(a_2 \cdot \hat{b})^2}{(\gamma v)^4 |b|^2} \left[\frac{-3\gamma^2 - 4(2\gamma^4 - 3\gamma^2 + 1)\gamma \hat{\tau}_1 v + \gamma^4(4v^4 + 2) + 1}{2} - \frac{-3\gamma^2 + 4(2\gamma^4 - 3\gamma^2 + 1)\gamma^4 \hat{\tau}_1^4 v^4 + 6(\gamma v)^4(2\gamma^2 - 1)\hat{\tau}_1^2 + \gamma^4(4v^4 + 2) + 1}{2\hat{D}(\tau_1)^3} \right]. \quad (\text{A8})$$

$$F_4 = \frac{m_2}{\hat{D}(\hat{\tau}_1)^3 |b|^2} \left[\frac{(2\gamma^2 - 1)a_1 \cdot \hat{b}(\gamma^2 \hat{\tau}_1^2 v^2 + 1)(2\gamma^2 \hat{\tau}_1^2 v^2 + 1)(a_1 \cdot \hat{\ell} + a_2 \cdot \hat{\ell})}{\gamma^2 v^2} + \frac{a_2 \cdot \hat{b}[a_1 \cdot \hat{\ell}(\gamma^2 + 2(2\gamma^2 - 1)\gamma^4 \hat{\tau}_1^4 v^4 + (6\gamma^4 - 9\gamma^2 + 3)\hat{\tau}_1^2) + a_2 \cdot \hat{\ell}(2(2\gamma^2 - 1)\gamma^4 \hat{\tau}_1^4 v^4 + (6\gamma^4 - 9\gamma^2 + 3)\hat{\tau}_1^2 + 1)]}{\gamma^2 v^2} + \frac{\hat{\tau}_1 a_2 \cdot v_1[(3\gamma^2 + (2\gamma^4 - 3\gamma^2 + 1)\hat{\tau}_1^2 - 2)a_1 \cdot \hat{\ell} + (4\gamma^2 + (2\gamma^4 - 3\gamma^2 + 1)\hat{\tau}_1^2 - 3)a_2 \cdot \hat{\ell}]}{\gamma^2 v^2} - \frac{\hat{\tau}_1 a_1 \cdot v_2(\gamma^2 \hat{\tau}_1^2 v^2 + 1)(a_1 \cdot \hat{\ell} + a_2 \cdot \hat{\ell})}{\gamma v^2} + \frac{(1 - 2\gamma^2)a_1 \cdot \hat{b}(1 - 2\gamma \hat{\tau}_1 v)(\gamma^2 \hat{\tau}_1^2 v^2 + 1)(a_1 \cdot \hat{\ell} + a_2 \cdot \hat{\ell})\hat{D}(\tau_1)}{\gamma^2 v^2} + \frac{a_2 \cdot \hat{b}(\gamma^2 \hat{\tau}_1^2 v^2 + 1)[\gamma a_1 \cdot \hat{\ell}(2(2\gamma^2 - 1)\hat{\tau}_1 v - \gamma) + a_2 \cdot \hat{\ell}(4\gamma^3 \hat{\tau}_1 v - 2\gamma \hat{\tau}_1 v - 1)]\hat{D}(\tau_1)}{\gamma^2 v^2} \right]. \quad (\text{A9})$$

In the results displayed above, we have introduced the dimensionless variable $\hat{\tau}_1 := \tau_1/|b|$.

APPENDIX B: 1PM FOURIER INTEGRALS

Here, we discuss the Fourier integrals (19) which were used in the 1PM results in Sec. III. The family is

$$I_{\alpha,\beta}^{\mu_1 \dots \mu_n} = \int_q \frac{\delta(q \cdot v_2)}{(q^2)^\alpha (q \cdot v_1 + i0)^\beta} e^{iq \cdot \tilde{b}} q^{\mu_1} \dots q^{\mu_n}. \quad (\text{B1})$$

We start by analyzing the subfamily $I_{\alpha,0}$ that does not include worldline propagators. The delta function is resolved by going into the rest frame of BH2, and the resulting integral is solved by Schwinger parametrization. The covariant solution is given below, where $P_2^{\mu\nu}$ projects onto the rest frame of BH2:

$$I_{\alpha,0} = \frac{(-1)^\alpha \Gamma(\frac{D-1}{2} - \alpha)}{4^\alpha \sqrt{\pi}^{D-1} \Gamma(\alpha)} (-\tilde{b} \cdot P_2 \cdot \tilde{b})^{\alpha - \frac{D-1}{2}}, \quad P_2^{\mu\nu} := \eta^{\mu\nu} - v_2^\mu v_2^\nu. \quad (\text{B2})$$

We generalize to $\beta \neq 0$, i.e., introduce retarded worldline propagators, using

$$\int_\omega e^{-i\omega\tau} \frac{f(\omega)}{(\omega + i0)^\beta} = (-1)^{\beta/2} \int_{-\infty}^\tau d\tau_1 \dots \int_{-\infty}^{\tau_{\beta-2}} d\tau_{\beta-1} \times \int_{-\infty}^{\tau_{\beta-1}} d\tau_\beta \int_\omega e^{i\omega\tau_\beta} f(\omega). \quad (\text{B3})$$

Powers of the worldline propagator are added by successive integration over the time parameter τ . Integrals of higher tensor rank $I_{\alpha,\beta}^{\mu_1 \dots \mu_n}$ are then obtained by differentiating with respect to the *unconstrained* b^μ :

$$I_{\alpha,\beta}^{\mu_1 \dots \mu_n} = (-i)^n \frac{\partial}{\partial b_{(\mu_n}} \dots \frac{\partial}{\partial b_{\mu_1)}} I_{\alpha,\beta}. \quad (\text{B4})$$

One does not impose $b \cdot v_i = 0$ before taking derivatives.

APPENDIX C: ONE-LOOP INTEGRALS

To compute 2PM trajectories, we introduced the following pair of one-loop integral families, in both cases with $q \cdot v_2 = 0$:

$$I_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}} = \int_\ell \frac{\delta(\ell \cdot v_2)}{(\ell^2)^{\nu_1} ((\ell - q)^2)^{\nu_2} (\ell \cdot v_1 + i0)^{\nu_3}}, \quad (\text{C1a})$$

$$J_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}} = \int_\ell \frac{\delta(\ell \cdot v_1)}{(\ell^2)^{\nu_1} ((\ell - q)^2 + i0 \text{sgn}(\ell^0 - q^0))^{\nu_2} (\ell \cdot v_2 + i0)^{\nu_3}}. \quad (\text{C1b})$$

Both integral families only depend on $\{|q|, \gamma, q \cdot v_1\}$. Introducing $\hat{q} \cdot v_1 := q \cdot v_1/|q|$ instead of $q \cdot v_1$ leaves $|q|$ as the only dimensionful quantity, and so its dependence in the integrals is simply determined by counting overall mass dimensions. This leaves two dimensionless parameters $X = \{\gamma, \hat{q} \cdot v_1\}$. As stated in the main text, the two integral families are spanned by the following masters:

$$\vec{I} = (I_{0,1,1}^{1\text{-loop}}, I_{1,1,0}^{1\text{-loop}}, I_{1,1,1}^{1\text{-loop}}), \quad \vec{J} = (J_{0,1,0}^{1\text{-loop}}, J_{0,1,1}^{1\text{-loop}}, J_{1,1,0}^{1\text{-loop}}, J_{1,1,1}^{1\text{-loop}}). \quad (\text{C2})$$

Final expressions for the I -integrals (34) and J -integrals (35) were given in the main text. In this Appendix, we

outline our procedure for deriving these expressions, using integration-by-parts reduction [107,108] to master integrals, differential equations involving $\hat{q} \cdot v_1$ and the method of regions for boundary integrals in the limit $\hat{q} \cdot v_1 \rightarrow 0$. This corresponds with the asymptotic limit $\tau_i \rightarrow \pm\infty$.

1. Differential equations

Although the present integral families involve two parameters, it suffices to consider differential equations only in $\hat{q} \cdot v_1$ as all γ -dependence is fixed by the boundary

integrals. In the first family $I_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}$, we find it a natural choice to consider the differential equation with respect to $(q \cdot v_1)_N := \hat{q} \cdot v_1 / (\gamma v)$, as this allows us to pull all γ -dependence out of the integrals. This dependence is then given by an overall factor of $1/(\gamma v)^{\nu_3}$ —in short, the dependence on γ is made trivial by working in the frame $v_2^\mu = (1, \mathbf{0})$. The two differential equations are then given by (where we set $|q| = 1$ since its dependence is trivially determined by the mass scale of the integral):

$$\frac{d\vec{I}}{d(q \cdot v_1)_N} = A_I \vec{I}, \quad A_I = 2\epsilon \begin{pmatrix} -\frac{1}{(q \cdot v_1)_N} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{(q \cdot v_1)_N((q \cdot v_1)_N^2 - 1)} & -\frac{2}{((q \cdot v_1)_N^2 - 1)\gamma v} & \frac{(q \cdot v_1)_N}{(q \cdot v_1)_N^2 - 1} \end{pmatrix}, \quad (\text{C3a})$$

$$\frac{d\vec{J}}{d\hat{q} \cdot v_1} = A_J \vec{J}, \quad A_J = \begin{pmatrix} \frac{1-2\epsilon}{\hat{q} \cdot v_1} & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon}{\hat{q} \cdot v_1} & 0 & 0 \\ \frac{2\epsilon-1}{\hat{q} \cdot v_1(1+\hat{q} \cdot v_1^2)} & 0 & \frac{\hat{q} \cdot v_1(2\epsilon-1)}{1+\hat{q} \cdot v_1^2} & 0 \\ 0 & -\frac{2\epsilon(\gamma^2-1)}{\hat{q} \cdot v_1(1+\hat{q} \cdot v_1^2-\gamma^2)} & \frac{4\gamma\epsilon}{1+\hat{q} \cdot v_1^2-\gamma^2} & \frac{2\hat{q} \cdot v_1\epsilon}{1+\hat{q} \cdot v_1^2-\gamma^2} \end{pmatrix}. \quad (\text{C3b})$$

While the first DE is already in ϵ -form, we require a transformation in order to bring the differential equation represented by A_J into ϵ -form. We therefore act with

$$T_J = \begin{pmatrix} -\frac{2\epsilon-1}{2\epsilon\hat{q} \cdot v_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{(\hat{q} \cdot v_1)^2 + 1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{C4})$$

on the basis vector $\vec{J}$. Expressions for the master integrals are then built up order by order in ϵ , and our next task is to fix the corresponding constants of integration.

2. Evaluation of boundary integrals

To fix boundary conditions on our master integrals, we consider the limit $q \cdot v_1 \rightarrow 0$, wherein the integrals depend nontrivially only on γ . We claim that there are two regions

involved, characterized by the following scalings of the loop momentum ℓ :

$$\ell^\mu \sim (q \cdot v_1, 1), \quad \ell^\mu \sim (q \cdot v_1 - i0, q \cdot v_1 - i0). \quad (\text{C5})$$

We refer to the latter as the *soft* region. We will justify this claim in the next subsection; here, we focus on evaluating the corresponding boundary integrals. For the purposes of this discussion, we focus on the second (and more non-trivial) integral family $J_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}$. Boundary integrals for the family $I_{\nu_1, \nu_2, \nu_3}^{1\text{-loop}}$ are obtained by a similar procedure.

To illustrate our method, consider the specific master integral $J_{1,1,1}^{1\text{-loop}}$. Working in the inertial frame $v_1^\mu = (1, \mathbf{0})$, $v_2^\mu = \gamma(1, \mathbf{v})$, the integral is massaged into a (for our purposes) more convenient form,

$$\begin{aligned}
J_{1,1,1}^{1\text{-loop}} &= \int_{\ell} \frac{\delta(\ell \cdot v_1)}{\ell^2((\ell - q)^2 + \text{sgn}(\ell^0 - q^0)i0)(\ell \cdot v_2 + i0)} \\
&= \int_1 \frac{1}{(-\mathbf{l}^2)((q \cdot v_1)^2 - (\mathbf{l} - \mathbf{q})^2 - \text{sgn}(q \cdot v_1)i0)(-\gamma(\mathbf{l} \cdot \mathbf{v}_2) + i0)} \\
&= \frac{1}{\gamma} \int_1 \frac{1}{(\mathbf{l} + \mathbf{q})^2(-(\mathbf{q} \cdot v_1 - i0)^2 + \mathbf{l}^2)(-(\mathbf{q} \cdot v_1 - i0) - \mathbf{l} \cdot \mathbf{v})}, \tag{C6}
\end{aligned}$$

having used $q \cdot v_2 = 0$, which implies that $\mathbf{q} \cdot \mathbf{v} = q \cdot v_1$. Assuming the two scalings (C5) and expanding to leading order in $q \cdot v_1$, one arrives at (discarding the factor of γ in the denominator)

$$J_{1,1,1}^{1\text{-loop}} \xrightarrow{q \cdot v_1 \rightarrow 0} - \int_1 \frac{1}{(\mathbf{l} + \mathbf{q})^2 \mathbf{l}^2 (\mathbf{l} \cdot \mathbf{v})}, \quad \ell^\mu \sim (q \cdot v_1, 1) \tag{C7a}$$

$$\begin{aligned}
J_{1,1,1}^{1\text{-loop}} \xrightarrow{q \cdot v_1 \rightarrow 0} & - (q \cdot v_1 - i0)^{D-4} \frac{1}{\mathbf{q}^2} \int_1 \frac{1}{(\mathbf{l}^2 - 1)(1 + \mathbf{l} \cdot \mathbf{v} - i0)}, \\
\ell^\mu & \sim (q \cdot v_1, q \cdot v_1). \tag{C7b}
\end{aligned}$$

To arrive at the integrand in the latter soft region, we rescaled $\mathbf{l} \rightarrow q \cdot v_1 \mathbf{l}$. Hence, $\mathbf{l}$ is to be considered dimensionless in the lowermost of the two integrands. Computing these integrals yields the boundary conditions.

The first region $\ell^\mu \sim (q \cdot v_1, 1)$ is more straightforward to compute, as it corresponds simply with the one-loop integral family used in calculations of asymptotic quantities such as the impulse—see, e.g., Ref. [22]. The soft (S) region $\ell^\mu \sim (q \cdot v_1, q \cdot v_1)$ is more involved. We refer to the soft region of an integral by adding S as a subscript of the respective integral. In this region, it suffices to obtain $J_{0,1,0}^{1\text{-loop}}$ and $J_{S,0,1,1}^{1\text{-loop}}$. The other two masters (C2) are given by

$$J_{S,1,1,1}^{1\text{-loop}} = -\frac{1}{\mathbf{q}^2} J_{S,0,1,1}^{1\text{-loop}}, \quad J_{S,1,1,0}^{1\text{-loop}} = -\frac{1}{\mathbf{q}^2} J_{S,0,1,0}^{1\text{-loop}}. \tag{C8}$$

The master integral $J_{0,1,0}^{1\text{-loop}}$ is a tadpole integral, which is well known and can easily be solved,

$$J_{0,1,0}^{1\text{-loop}} = \left(\frac{-1}{4\pi}\right)^{\frac{D-1}{2}} \Gamma\left(1 - \frac{D-1}{2}\right) (-q \cdot v_1 + i0)^{D-3}. \tag{C9}$$

So, we are left only with $J_{S,0,1,1}^{1\text{-loop}}$ for which we need an expression.

To derive the boundary integral $J_{S,0,1,1}^{1\text{-loop}}$, we Schwinger parametrize the worldline propagator. The result is a one-dimensional parameter integral of a spherically symmetric Fourier transform:

$$J_{S,0,1,1}^{1\text{-loop}} = (q \cdot v_1 - i0)^{D-4} \int_0^\infty ds e^{-is} \int_1 \frac{e^{-is\mathbf{l} \cdot \mathbf{v}}}{\mathbf{l}^2 - 1}. \tag{C10}$$

The D -dimensional spherically symmetric Fourier integral is related to a Hankel transform of order $D/2 - 1$,

$$s^{\frac{D}{2}-1} \mathcal{F}[f(r)](s) = (2\pi)^{-\frac{D}{2}} \mathcal{H}_{\frac{D}{2}-1}[r^{\frac{D}{2}-1} f(r)](s), \tag{C11}$$

where $\mathcal{F}[f(r)](s)$ is the D -dimensional Fourier transform of $f(r)$. $\mathcal{H}_\nu[F](s)$ is the Hankel transform of the function F of order ν as a function of the variable s ,

$$\mathcal{H}_\nu[F](s) := \int_0^\infty F(r) J_{\frac{D}{2}-1}(sr) r dr, \tag{C12}$$

with J_ν the order- ν Bessel function of the first kind. The resulting one-parameter integral is then straightforwardly solved, and the ϵ -expanded result is given by

$$J_{S,0,1,1}^{1\text{-loop}} = \frac{i|q|^{-2\epsilon}}{8\pi\gamma v} \left(-\frac{1}{\epsilon} - i\pi - 2\text{arccosh}(\gamma) - 2\log[(\hat{q} \cdot v_1)_N^{-1}] \right) + \mathcal{O}(\epsilon). \tag{C13}$$

The full $q \cdot v_1 \rightarrow 0$ limit of the master integrals is given by the sum of both regions.

3. Identification of boundary regions

The two contributing regions (C5) to the integrals in Feynman parameter space in the $q \cdot v_1 \rightarrow 0$ limit were determined with the help of the *Mathematica* package ASYM [109,110]. In the Feynman parameter representation, $J_{1,1,1}^{1\text{-loop}}$ is given by

$$J_{1,1,1}^{1-\text{loop}} = (4\pi)^{-\frac{D-1}{2}} \Gamma\left(3 - \frac{D-1}{2}\right) \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \delta(1 - x_1 - x_2 - x_3) \mathcal{U}^{4-D} \mathcal{F}^{\frac{D-1}{2}-3}, \quad (\text{C14})$$

where

$$\mathcal{U} = -(x_1 + x_2), \quad \mathcal{F} = \frac{(2\mathbf{q}x_1 - \mathbf{v}x_3)^2}{4} - (x_1 + x_2)(\mathbf{q}^2x_1 - (q \cdot v_1)^2x_2 - (q \cdot v_1)x_3). \quad (\text{C15})$$

Without loss of generality, in this discussion, we drop the $i0$ in the scaling of the regions. Using `ASYM` [109,110], we identify the two regions (0,0,0) and (2,0,1), where the numbers characterize the scalings in $q \cdot v_1$ of each parameter x_i :

$$(0, 0, 0): (x_1, x_2, x_3) \sim (1, 1, 1), \quad (\text{C16a})$$

$$(2, 0, 1): (x_1, x_2, x_3) \sim ((q \cdot v_1)^2, 1, q \cdot v_1). \quad (\text{C16b})$$

In these two regions, the $\mathcal{U}$ and $\mathcal{F}$ polynomials become

$$(0, 0, 0): \mathcal{U} \rightarrow -(x_1 + x_2), \quad \mathcal{F} \rightarrow -\mathbf{q}^2x_1x_2 + \frac{\mathbf{v}^2x_3^2}{4}, \quad (\text{C17a})$$

$$(2, 0, 1): \mathcal{U} \rightarrow -x_2, \quad \mathcal{F} \rightarrow (q \cdot v_1)^2 \left(-\mathbf{q}^2x_1x_2 + x_2^2 + x_2x_3 + \frac{\mathbf{v}^2x_3^2}{4} \right). \quad (\text{C17b})$$

In the first case, the delta function is unchanged, and in the second, $\delta(1 - x_1 - x_2 - x_3) \rightarrow \delta(1 - x_2)$. The $q \cdot v_1 \rightarrow 0$ limit of the complete integral is given by the sum of both regions:

$$\begin{aligned} \lim_{q \cdot v_1 \rightarrow 0} J_{1,1,1}^{1-\text{loop}} &= (4\pi)^{-\frac{(D-1)}{2}} \Gamma\left(3 - \frac{(D-1)}{2}\right) \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \\ &\times \left[(q \cdot v_1)^{D-4} \delta(1 - x_2) (-x_2)^{4-D} \left(-\mathbf{q}^2x_1x_2 + x_2^2 + x_2x_3 + \frac{\mathbf{v}^2x_3^2}{4} \right)^{(D-1)/2-3} \right. \\ &\left. + \delta(1 - x_1 - x_2 - x_3) (-x_1 - x_2)^{4-D} \left(-\mathbf{q}^2x_1x_2 + \frac{\mathbf{v}^2x_3^2}{4} \right)^{(D-1)/2-3} \right]. \end{aligned} \quad (\text{C18})$$

This allows us to directly identify the momentum space regions (C5) with the parameter space regions (C16). Starting from the integrands (C7a) and (C7b), applying Feynman parametrization yields

$$J_{1,1,1}^{1-\text{loop}} \xrightarrow{q \cdot v_1 \rightarrow 0} \Gamma(3) \int \frac{d^{D-1}\mathbf{l}}{(2\pi)^{D-1}} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{\left[\mathbf{l}^2(x_1 + x_2) + \mathbf{q}^2x_1 - \frac{(-2\mathbf{q}x_1 + \mathbf{v}x_3)^2}{4(x_1 + x_2)} \right]^3}, \quad (\text{C19a})$$

$$J_{1,1,1}^{1-\text{loop}} \xrightarrow{q \cdot v_1 \rightarrow 0} (q \cdot v_1)^{D-4} \Gamma(3) \int \frac{d^{D-1}\mathbf{l}}{(2\pi)^{D-1}} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{\left[\mathbf{l}^2x_2 + \mathbf{q}^2x_1 - (x_2 + x_3) - \frac{(\mathbf{v}x_3)^2}{4x_2} \right]^3}, \quad (\text{C19b})$$

in the same two regions, respectively. Evaluating the remaining $\mathbf{l}$ integrals yields the two components in Eq. (C18), thus validating our claim. Similar considerations apply for the other master integrals.

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