

String scattering amplitudes and mutual information in confining backgrounds: The partonic behavior

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Using AdS/CFT methods in various holographic confining backgrounds, we present the connections between the behaviors of string scattering amplitudes and mutual information. We specifically lay down the analogies between the logarithmic branch cut behavior of the string scattering amplitude in $4d$, $\mathcal{A}_4$, at low and medium Mandelstam variable s , which is due to the dependence of the string tension on the holographic radial coordinate, and the branch cut behavior observed in mutual information and critical distance D_c at low cut-off variable u_{KK} . In both cases, as s or u_{KK} increases, the peaks of the branch cuts fade away in the form of $\text{Re}[\mathcal{A}_4] \propto s^{-1}$. Next, the modular flow and modular Hamiltonian as the intermediate quantities will be used to further shed light on this connection. We then explain how mutual information itself can act as a probe of chaos in various scenarios. For two examples of Compton scattering (of a photon entangled with another photon off a target) and also the decay of a highly excited string into two tachyons, the pattern of entanglement entropy and the change in the mutual information will be examined. Finally, the kink-kink and kink-antikink scatterings, as simple models of scattering in confining geometries, are employed to probe the fractal structures in the scatterings of topological defects.

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I. INTRODUCTION

In 1968, the SLAC laboratory observed that at high energies and fixed angles, the hadronic scattering amplitudes fall off based on a power law in the Mandelstam variable s (the center-of-mass energy), and not, as previously imagined, in an exponential form of s . That observation was incompatible with the expectations from string theory, therefore, further studies were needed from the theoretical perspective. On the other hand, using holography, interesting connections between string scattering amplitude and entanglement entropy have been found, which could help in solving this puzzle; for instance, see Ref. [1–4]. Additionally, the correspondences between strings and black holes, discussed for instance in [5], pointed out that a lot of information about quantum gravity and quantum information problems from the structures of string amplitudes could be obtained. In fact, as there are links between the chaotic behavior of the excited strings

and the chaoticity of black holes, one could expect that one of the best physical quantities to employ first would be the scattering amplitude, since the perturbative string theory has been formulated using it.

Building on these connections [1–4,6–14], we can now move to the richer mixed correlation measures, such as negativity and mutual information, and further establish the correlations between the behaviors of scattering amplitudes and the pattern of quantum entanglement in holography.

In [15], the scattering amplitude of highly excited strings has been studied within the DDF (Del Giudice, Di Vecchia, Fubini [16]) formalism, whereby the excited string is produced by photons that are scattered off of an initial tachyon repeatedly. We use this formalism to analyze the connections between the behaviors of string scatterings and the mixed quantum information measures.

In Fig. 1, three examples of scattering problems in the confining background of a mixed entanglement wedge constructed from two infinite strips with width L and distance D are shown. This figure is similar to Fig. 1 of [17], but here we specifically consider a confining background with an end wall positioned at u_{KK} constructed from the mixed correlations of two strips. The classical chaos can be studied using the model of the scattering of pinballs (left part of Fig. 1), where the angle of the outgoing particle is very sensitive to the impact parameter. This angle

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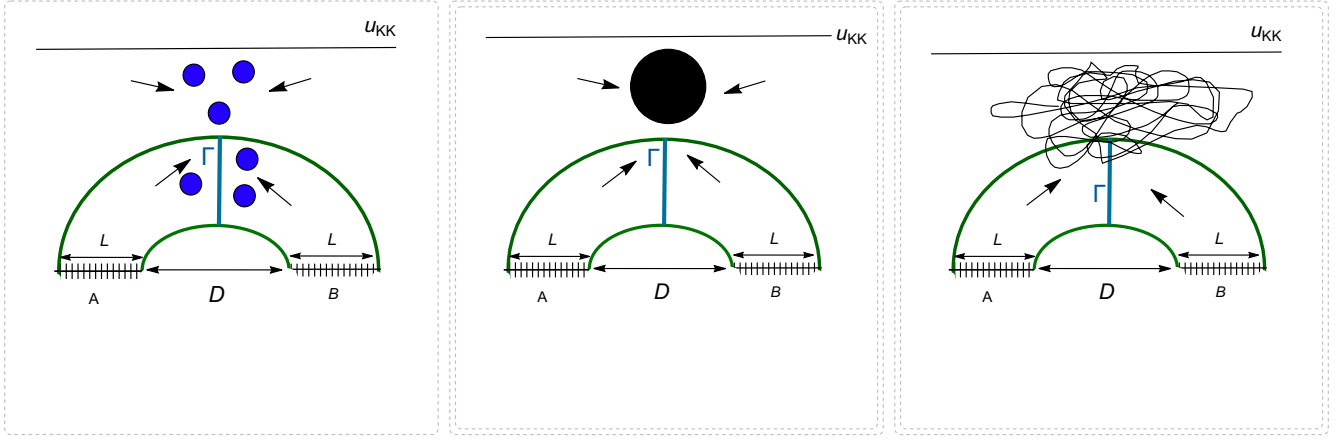


FIG. 1. In the left part, the scattering of “classical” pinballs, in the middle, the “quantum” and highly chaotic scattering process off of a black hole, and in the right part, the scattering off of highly excited strings in a confining background with an end wall positioned at u_{KK} are shown.

is related to several other parameters of the setup as well. However, this scattering process is not sensitive to the “quantum” mutual information of the two systems A and B .

In the middle part of Fig. 1, the scattering of particles off a black hole system in the background of a confining geometry is sketched. In this scenario, any perturbation in the parameters of the system could cause a large change in the state of the outgoing Hawking quanta, where the out-of-time-order correlator grows exponentially. Therefore, this setup can efficiently capture the highly chaotic nature of black holes. The level of chaos in this case is higher than the pinball example in the sense that it has both the “classical” and “quantum” chaoticity. In this setup, depending on the size of the black hole, the position of the wall u_{KK} , and the lengths of L and D , various phases could exist.

In the right part of Fig. 1, a sketch of a complicated and highly excited open string structure, positioned in the confining and mixed background, is shown. As will be shown later, in this case there is an exponentially large number of internal states and a strong dependence on the parameters of the system. Therefore, the scattering amplitude displays a high level of chaos and high sensitivity to various parameters of the system. Also, based on the relative sizes of the parameters in this setup and the strength of the correlations between the two systems, a particular saddle can dominate at any specific configuration. Thus, a very rich phase structure for this system can be envisioned.

Entanglement entropy has already been employed in various works to find connections between energy conditions and quantum information theory. For example, in [18], using the second-order shape deformation of the geometric entanglement entropy, a conjectural lower bound on the expectation value of the components of the energy-momentum tensor has been found, which has been dubbed the quantum null energy condition. In addition, the

properties of the modular Hamiltonian under shape deformations for any state ψ can help in this regard [18,19].

The difference between our setup and Fig. 1 of [17] is that here the background is constructed from the entanglement of two mixed correlated subsystems, and there is also a wall at the end of the geometry. The modular flow in holography defines the proper internal time flow for an observer in this system. Our observation here is related to the result of [20], where the Connes cocycle flow, kink transform, and the bulk Weyl tensor shock have all been connected with each other.

For this system, one can imagine the nonequilibrium modular flows and the “internal modular time” (instead of the real physical time) [21,22], which describe the pattern of entanglement. The modular flows [23] from these two sources of mixed subsystems hit the wall at the end of the geometry, and then slide or reflect back [24], constructing a chaotic structure with those observed peaks, where, by increasing the Mandelstam variable s , these peaks would decay.

The measure of mutual information in quantum field theories can detect the patterns of entanglement in a more precise way, as it is UV regulated. Therefore, mutual information is a superior measure to extract the universalities in the structures of the entanglement of the system. It could also describe the intrinsic behaviors of strings in a more exhaustive way. In addition, as pointed out in [25], compared to entanglement entropy, mutual information is less constrained by the symmetries and hence it can illustrate better the details, such as the full spectrum of the Conformal field theories (CFTs) directly related to the behaviors of the fundamental strings.

Specifically, the singularities in the scattering amplitudes, which have been observed in [26], would form logarithmic branch cuts (rather than poles). These erratic behaviors and branch cuts, which are more pronounced for small s and fade away at larger s , can be explained by the

creation of bound states in a confining background. The same erratic behavior and peak structures due to these bound states can be detected through mutual information as well, as we observed in our numerical studies in [27,28].

In [26], specifically, the partonic behavior of scattering processes at large and medium values of the Mandelstam variable s in the holographic confining backgrounds has been explored. This observation was based on the works of Polchinski and Strassler [29], whose main result was writing the $4d$ amplitude in terms of the $10d$ scattering amplitude as

$$\mathcal{A}_4(s, t, u) \propto \int dr \sqrt{-g} \times \mathcal{A}_{10}[\tilde{s}(r), \tilde{t}(r), \tilde{u}(r)], \quad (1.1)$$

where $\mathcal{A}_{10}$ is the $10d$ string amplitude, r is the holographic radial coordinate, and $\Psi(r)$ is the wave function of the scattered state. In the holographic QCD models with a nontrivial dilaton field such as Maldacena–Nuñez [30] or Witten–QCD [31,32], where the dilaton field depends on the radial coordinate, the above relation should instead be written as

$$\begin{aligned} \mathcal{A}_4(s, t, u) = & \int_{u_{KK}}^{\infty} du \sqrt{-g} e^{-2\phi} \left(\prod_{i=1}^4 \psi_i(u) \right) \\ & \times \mathcal{A}_{10}[\tilde{s}(r), \tilde{t}(r), \tilde{u}(r)]. \end{aligned} \quad (1.2)$$

Unlike theories such as $\mathcal{N} = 4$ SYM, the spectrum of QCD is discrete, which is the result of the IR end wall and the boundary condition that it imposes. This discrete spectrum can also affect the behavior of quantum information correlation measures. Moreover, the entanglement entropy and mutual information can also detect the asymptotic freedom where the gauge couplings and the binding energies become smaller at that point. On the other hand, as found in [26], the scattering amplitude itself could act as a measure that captures the passage from soft to hard scattering, which is the “bending” trajectory in the confining models. The “quantized” behavior of such binding energies could also be detected at lower energies.

In this setup, most of the closed strings reside near the end wall, where instead of vanishing, this accumulation causes the wave function to become peaked, which might seem counterintuitive. This feature shows its signatures in the entanglement patterns and mixed-correlations saddles as well. Based on such partonic behavior of hadron scattering [9,26,33], one would expect that the entanglement patterns could detect such behavior as well and the direct connections between these two quantities could be constructed more firmly, as we demonstrate here.

This paper is organized as follows. In Sec. II, we discuss the fractal structures arising from the phase transitions and chaos in confining geometries due to the presence of the wall. These structures have been captured by the mutual

information and critical distance D_c , as noted in our previous works [27,28,34,35]. In Sec. III, similar to the results of [26], we explore the scattering amplitude in $10d$ and $4d$, the structures of its zeros and poles and their statistics, which demonstrate the appearance of chaos. We also discuss how the structures of the zeros behave similarly for both mutual information and string scattering amplitude.

In Sec. IV, we use the modular flow as an instrumental concept to further establish the relations between the mutual information and string scattering amplitude. In Sec. V, we will show that the mutual information itself can detect the chaotic behavior. There, we also discuss quantum scattering on a leaky torus and using this geometrical structure, further explore the chaotic behaviors of strings and also the chaotic spread of mutual information in confining geometries. In Sec. VI, we analyze two examples: Compton scattering with a witness and also the decay of a highly excited string into two tachyons, where for each case we discuss the behavior of entanglement entropy or mutual information during these processes. Next, in Sec. VII, as another example of chaotic structure in confining geometries, we consider the kink-kink and kink-antikink scattering, which similar to the behavior of D_c and mutual information in confining geometries, they show fractal structures with periodic patterns. Finally, we give concluding remarks in Sec. VIII.

II. CRITICAL DISTANCE IN CONFINING BACKGROUNDS

In [36,37], it has been shown that, in order to calculate the entanglement entropy for a general confining background with the geometry of

$$\begin{aligned} ds^2 = & \alpha(u) [\beta(u) du^2 + dx^\mu dx_\mu] + g_{ij} d\theta^i d\theta^j, \\ u_{KK} < u < \infty, \quad & x^\mu (\mu = 0, 1, \dots, d), \\ & \theta^i (i = d+2, \dots, 9), \end{aligned} \quad (2.1)$$

in terms of the minimum of the holographic radial coordinate, the length of a strip and its corresponding Ryu–Takayanagi surface, u_0 , can be written as

$$\begin{aligned} L(u_0) = & 2 \int_{u_0}^{\infty} du \sqrt{\frac{\beta(u)}{\frac{H(u)}{H(u_0)} - 1}}, \\ S(u_0) = & \frac{V_0}{2G_N} \int_{u_0}^{\infty} du \sqrt{\frac{\beta(u)H(u)}{1 - \frac{H(u_0)}{H(u)}}}, \end{aligned} \quad (2.2)$$

where here $H(u) = e^{-4\phi} V_{\text{int}}^2 \alpha(u)^d$ and $V_{\text{int}} = \int d\vec{\theta} \sqrt{\det[g_{ij}]}$. The corresponding setup is shown in Fig. 2.

For two strips of width L and separation distance D , the mutual information is given by

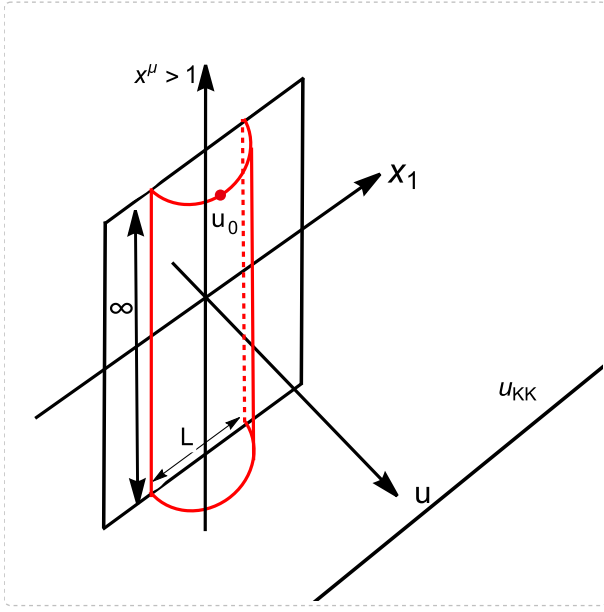


FIG. 2. The setup of one strip and its corresponding Ryu-Takayanagi surface in a confining background.

$$I(D, L) = 2S(L) - S(D) - S(2L + D), \quad (2.3)$$

and the critical distance D_c is defined by the condition $I(D_c, L) = 0$. The parameter D_c (as can be shown in Fig. 1) has been used in our previous works [27,28,35,38,39] to probe the quantum correlation properties of different geometries.

In [27], we found that, in the background of confining geometries, the critical distance between two strips as a

measure of mixed correlations shows singularities at smaller values of u_{KK} and statistically smooths out in larger u_{KK} . This behavior is especially pronounced in the Witten-QCD geometry; see Fig. 19 of [27], which is shown again here in Fig. 3. Similar jumps can also be seen in the background of Klebanov-Tseytlin geometry [40,41], as presented here in Fig. 4.

The potential in the Witten-QCD geometry can be written as $V(r) \sim r^{4/3}$, which follows a warped metric structure. Due to confinement at large distances, it grows softer than a linear potential. The geometry is cigarlike, and the time direction is periodic, avoiding the conical singularity.

Similar to the results of [26], it can be seen that, in the Witten-QCD background, the peaks of D_c , which correspond to the peaks of mutual information and mixed correlation measures, are larger in the medium (or rather smaller) regions of u_{KK} , which correspond to the smaller values of Mandelstam variable s . For larger values of s (and larger t , with fixed angle), which correspond to larger u_{KK} , the peaks fade away. This is the result of asymptotic freedom, where the gauge couplings and the binding energies of the bound states are small. However, from Figs. 3 and 4, it can be seen that there is no peak at larger u_{KK} , which is due to the fact that the binding energy, length, and the mass of the strings become smaller in those setups. Note also that, similar to [13] and the case of tripartite information $I^{[n=3]}$, the measures can become negative with abrupt, singular drops.

Note that this is similar to the result of [42] which describes the kink-antikink scattering in a quantum vacuum. In their model, they consider two fields, $\varphi(t, x)$ with

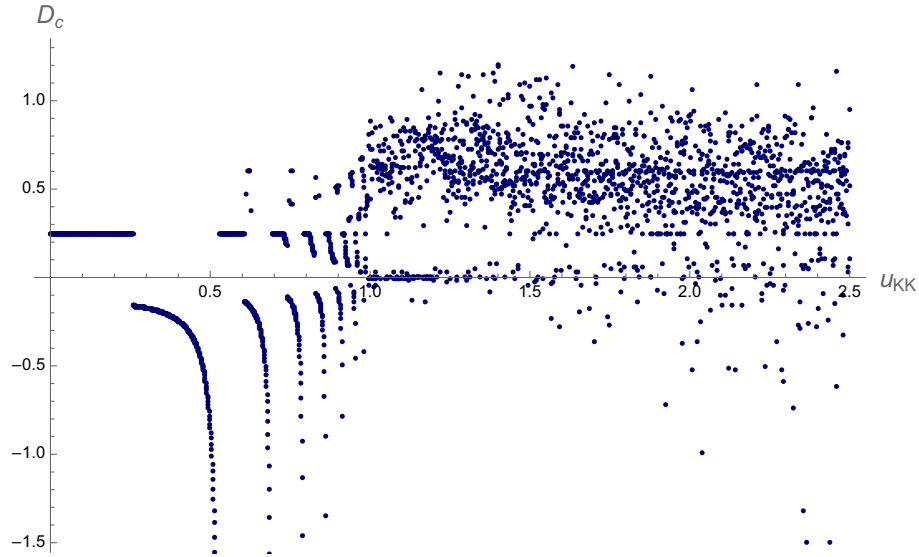


FIG. 3. The behavior of the critical distance, D_c , derived from the mutual information between two mixed strips, as a function of the IR wall position, u_{KK} , in the Witten-QCD background. This figure can be compared with Fig. 6 of [26] based on the behavior of the real parts of the amplitude. The behavior of the branch cuts and the decay at large s are very similar to the pattern of Mutual Information (MI) and D_c in this model.

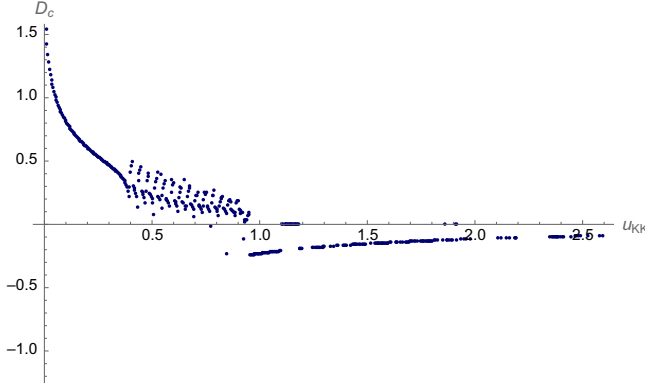


FIG. 4. Left: The relationship between the critical distance D_c between two strips and u_{KK} , in the background of Klebanov–Tseytlin geometry.

mass m and a scalar field $\psi(t, x)$ with mass μ and coupling constant λ . The exchange of energy between the classical field configuration and quantum bath is analogous to the process of strings hitting the end wall in confining models and therefore similar patterns in the observables are expected. As the coupling constant λ increases, from Figs. 4 and 5 of [42], which are the plots of E_ϕ versus t , one can see that the cascading decay becomes an oscillatory decay and the bound states become long-lived, weakly radiating objects. This is similar to Fig. 3, where the bound states and breatherlike phase decay to radiating phase. Moreover, the phase shifts in kink-antikink scattering are related to the memory of past interactions being restored in the quantum mutual information. Both quantities also exhibit nonlinear behavior. Furthermore, just as there exists a critical distance and a threshold for mutual information, there would also be a certain energy threshold where above which kinks and antikinks can interact significantly. We will explain the kink-antikink model further in Sec. VII.

The behavior of the critical distance D_c versus u_{KK} , and the potential in the background of the Sakai–Sugimoto [43], is shown in Fig. 5. As this figure has been created from the “real” part of the entanglement entropy, $\text{Re}(S(u_0))$ versus the real part of the strip size $\text{Re}(L(u_0))$, one would expect that, similarly in the “real” part of the amplitude, at bigger values of s or u , a decaying power law behavior should be observed. This is indeed what one can see from the behavior in Fig. 5. The relation for the amplitude follows $\text{Re}[\mathcal{A}_4] \sim s^{-1}$ and $\text{Im}[\mathcal{A}_4] \sim s^{2-\Delta/2}$, where Δ is the sum of the scaling dimensions of the operators that contribute as $\prod_{i=1} \psi_i(r) \sim r^{-\Delta}$. (In QCD models, Δ can be replaced by the twist operator).

Also note that among the models we consider here, the Sakai–Sugimoto is the one with the most similarities to the model studied in [44] for kink-antikink scattering. As found there, two separate regions exhibit the highest amount of radiation, which are also the most chaotic regions. In the

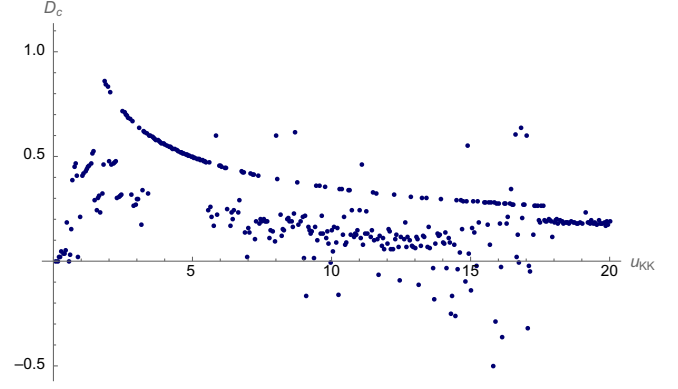


FIG. 5. Top: The relationship between the critical distance D_c and u_{KK} , in the background of the Sakai–Sugimoto geometry. Again, one can see that the peaks become less pronounced as u_{KK} increases and fade away at large values of s . The decay behavior at large u_{KK} follows a power law.

mutual information or D_c , we similarly notice two distinct peaks: one around $u_{KK} \simeq 2$ and one around $u_{KK} \simeq 17$, which correspond to $v \simeq 0.39$ and $v \simeq 0.075$ in the kink-antikink scattering case. Note that larger values of u_{KK} correspond to lower velocities.

As shown in Fig. 6, in the background of AdS-soliton, the transition from regular to chaotic regimes can be clearly observed in the behavior of D_c . This plot should be compared with Fig. 7 of [45], where the transition from regular to random occurs at large u_{KK} values. Note that larger values of u_{KK} correspond to smaller values of the discrete levels or “eigenvalues.” The peaks become less pronounced as u_{KK} increases and fade away at large values of s . This is similar to the case in [45], where, as energy and s increase, the singularities fade away. However, the general behaviors of mutual information or D_c versus u_{KK} are not exactly the same across different geometries

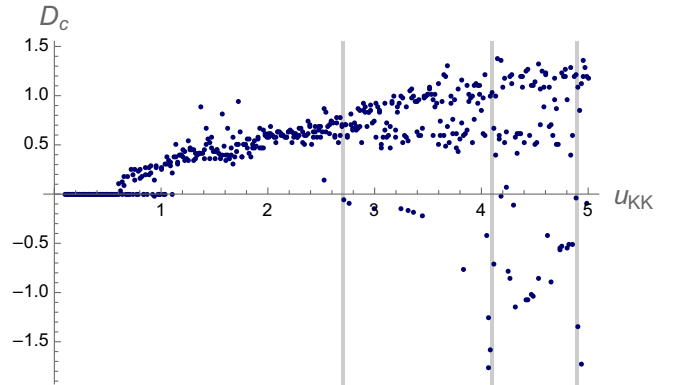


FIG. 6. The relationship between the critical distance D_c between the two strips versus u_{KK} , in the background of AdS-soliton geometry, and its potential. One can compare these diagrams with the plots of Crofton form presented in [27,28] as they highlight the specific characteristics of each geometry.

and depend on other background factors such as flux, genus, etc.

For the case of hard wall confining geometries such as Witten–QCD, the decay at large u_{KK} follows a decreasing power law, similar to its behavior in terms of s . As explained in [45], the cause of this behavior in the large energy domain is asymptotic freedom, where the gauge coupling becomes weaker and the binding energy of the bound states vanishes. This is analogous to the case where, as u_{KK} increases and asymptotic freedom is approached, the binding energy decreases, which—based on the first law of entanglement entropy—leads to smaller peaks in the entanglement entropy, in mutual information, and also in D_c .

For the case of AdS-soliton, however, as one can analyze the geometry using the potential and also its Crofton form as in [27,28], the well at low holographic coordinates u is positive. For bigger values of u , the Crofton form, while being negative, approaches zero, which is different from the case of hard-wall models such as Witten–QCD and Sakai–Sugimoto models. Therefore, using the Crofton form, one can track the behavior of critical distance D_c . For the case of AdS-soliton, the bound states could be created at larger u_{KK} values, while for smaller u_{KK} , the critical distance D_c would be zero, as the Crofton form ω is infinite there.

For interpreting this result, one can visualize the spread of wave functions between the two entangled strips in spacetime, where they interfere with each other, collide with the wall at the end of the confining geometry, ripple back, and create the chaotic structures and the peaks that fade away at larger values of s . These waves are sourced from the two entangled mixed subregions, creating the “modular chaos” that forms the shape of the peaks in the plots of mutual information.

Also, it has been demonstrated, for instance in [23], that modular chaos structures, similar to those of entanglement and mutual information, can reconstruct the holographic bulk geometry. Furthermore, in [26], the connections between “chaos” and the peak structures of the string amplitude have been discussed. Consequently, one would expect that the peak structures of string amplitudes and those of entanglement are connected through “modular chaos,” as we will prove.

The disruption of entanglement entropy, mutual information, or D_c in this case could serve as a probe of chaos. However, from Fig. 6, it can be seen that the “edge of chaos” is not very clear. This can also be related to the result of [46], where they showed that the performance of a reservoir computer at the edge of chaos, where the largest Lyapunov exponent in a dynamical system shifts from negative to positive, might be optimum or poor, as the actual performance would depend on various other parameters as well. It was previously believed that the greater mutual information, or critical distance, between the fit

signal $h(t)$ and the training signal $g(t)$ in reservoir computers leads to an improved performance. Also, at the edge of chaos, the modular flow and the string scattering amplitudes could fluctuate.

III. STRING SCATTERING AMPLITUDE IN CONFINING BACKGROUNDS

Now, turning attention on the closed string amplitude side, the ten-dimensional amplitude, as presented in [26] can be expressed as:

$$\begin{aligned} \mathcal{A}_{10} &= 4^4 \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^2}{(n!)^2} \frac{(1+c_+^4+c_-^4)}{\alpha' s/4 - (n+1)} \\ &\quad \times \frac{\Gamma((n+1)c_+)\Gamma((n+1)c_-)}{\Gamma(1-(n+1)c_+)\Gamma(1-(n+1)c_-)} \\ &\equiv \sum_{n=0}^{\infty} \frac{\mathcal{R}(\theta)}{\alpha' s/4 - (n+1)}, \end{aligned} \quad (3.1)$$

where $c_{\pm} = \frac{1}{2}(1 \pm \cos \theta)$. Here, larger values of α' correspond to larger values of u_{KK} and higher energy levels. In terms of Mandelstam variable s and the scattering angle θ , the amplitude can also be written as:

$$\begin{aligned} \mathcal{A}_{10} &= \frac{1}{32} \alpha'^4 s^4 (\cos(2\theta) + 7)^2 \frac{\Gamma(-\frac{1}{4}(\alpha' s))}{\Gamma(\frac{\alpha' s}{4} + 1)} \\ &\quad \times \frac{\Gamma(-\frac{1}{8}\alpha' s(\cos(\theta) - 1))}{\Gamma(\frac{1}{8}(-\alpha' s + \alpha' s \cos(\theta) + 8))} \\ &\quad \times \frac{\Gamma(\frac{1}{8}\alpha' s(\cos(\theta) + 1))}{\Gamma(1 - \frac{1}{8}\alpha' s(\cos(\theta) + 1))}. \end{aligned} \quad (3.2)$$

In [26], the behavior of ten-dimensional string amplitude $\mathcal{A}_{10}$ as a function of the Mandelstam variable s in confining holographic backgrounds such as Witten–QCD has been shown. The specific confining backgrounds considered there included hard-wall, soft-wall, and Witten–QCD models, showing similar singularity patterns to entanglement entropy and mutual information, as shown in Fig. 7. Subsequently, using the relations (1.1) and (1.2), $\mathcal{A}_{10}$ can be integrated and the scattering amplitude in $4d$ can be derived. As for our main figure, the behavior of $\text{Re}[\mathcal{A}_4]$ for different values of θ is shown in Fig. 8, which should be compared with the behavior of D_c in various confining backgrounds.

The reason for such peculiar behaviors in generating these singularities in the form of the branch cuts at low energies is the dependence of the string tension on the holographic radial coordinate. It can be noted that in $10d$ or $11d$ geometries, the amplitudes drop faster compared to lower- d geometries, which is similar to the flattening of singularities in the mixed-correlation measures in higher dimensions.

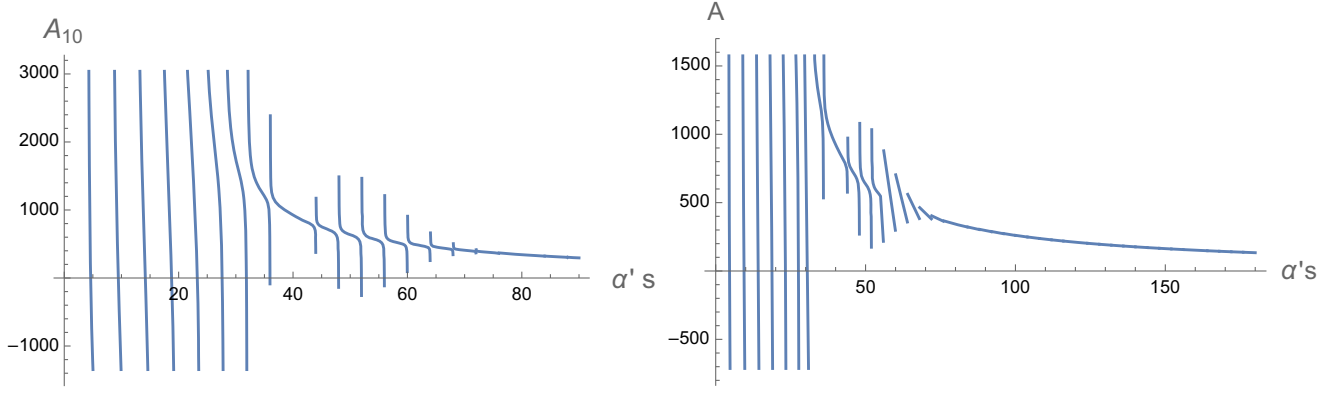


FIG. 7. The relationship between $\mathcal{A}_{10}$ versus $\alpha's$ for $\cos\theta = 0.8$. The weakening pattern of the branch cut singularities at large Mandelstam variable s is similar to the behavior of D_c versus u_{KK} in confining backgrounds.

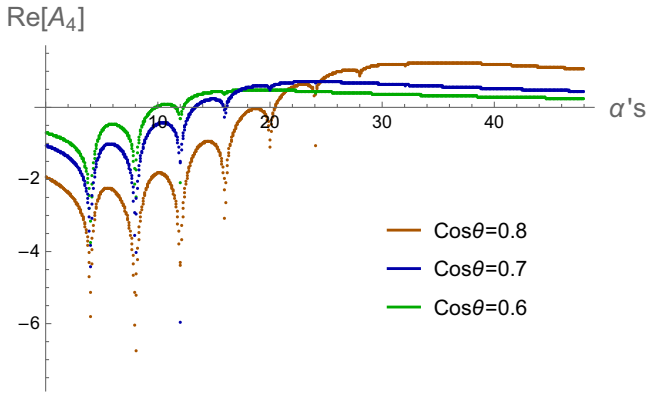


FIG. 8. The relationship between the real part of $[\mathcal{A}_4]$ versus $\alpha's$ for different values of θ , in the hard-wall confining model, created based on the results of [26]. The main behaviors are similar to the case of Witten–QCD. The logarithmic branch cut singularities at lower s , and the damping patterns at larger s , i.e., $[\mathcal{A}_4] \sim s^{-1}$, are similar to the behaviors of the critical distance D_c (coming from MI), observed in various confining backgrounds.

The connections between s and θ have been further explored in Fig. 9, as in the interval of $0 < \theta < 3\pi$, nearly four branch cuts, and in the interval of $0 < \theta < 6\pi$, nearly seven branch cuts can be found. Owing to the fact that the geometry is $10d$, we are allowed to consider such values for the angle θ , as we are considering scattering from highly excited strings in $10d$. Since we can consider bigger spins in the structures of chaotic and excited strings, and specifically as we consider higher dimensions, it would be necessary for θ to take larger than 2π . In general, with the upper bound of $n\pi$, $n + 1$ branch cuts would be present. Also, it can be seen that for bigger s , the branch cuts become dispersed, which is also the case for the mixed-correlation quantum information measures such as mutual information and D_c .

One could see that in the simple case of fixed t , as shown in Fig. 10, $\Delta S_E \propto (\log s)^2$. For the mixed-correlation measures and in the presence of a hard wall which creates the “four” main saddles [35,47], the behaviors become more interesting, as shown in Figs. 3 and 4. Since the parameter D_c is related to the sum of several entanglement

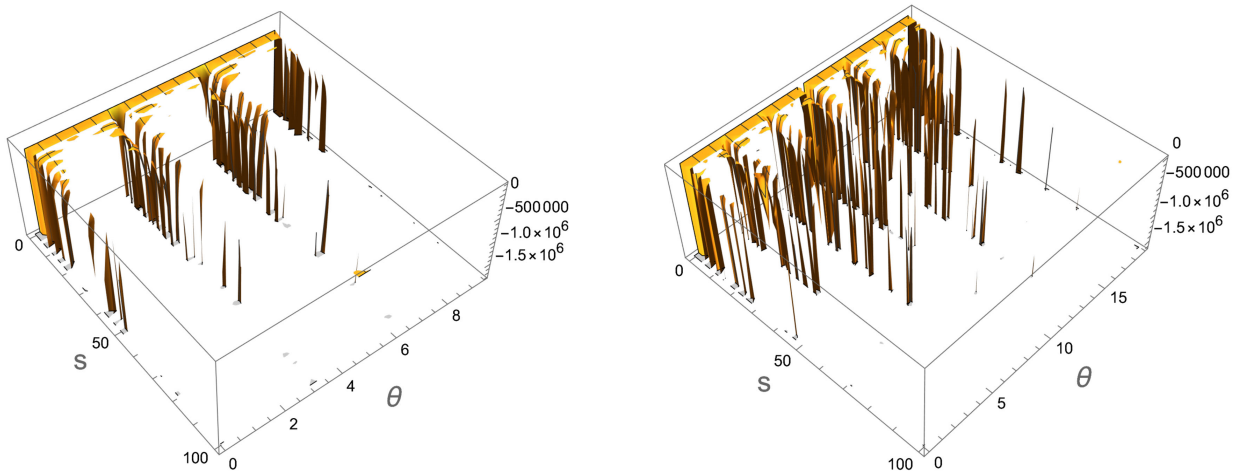
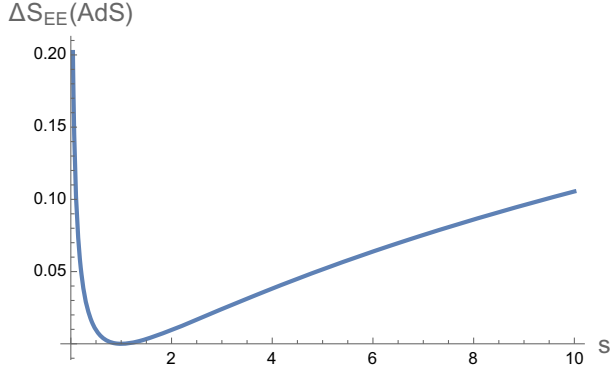
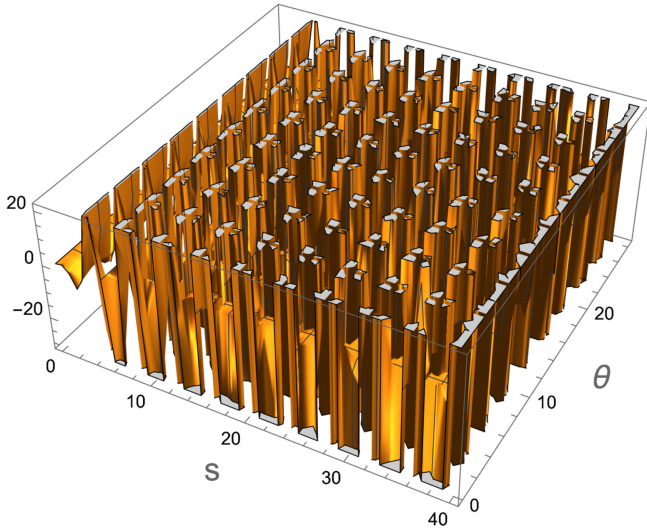
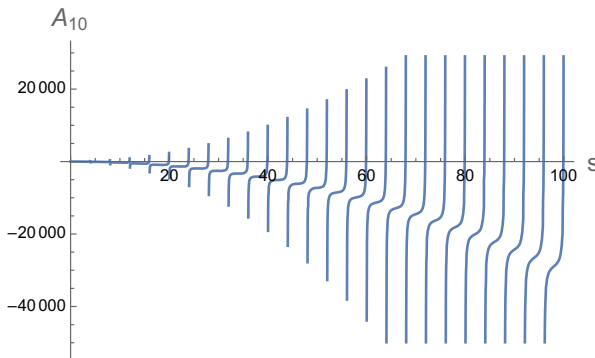


FIG. 9. The 3D plot of the $10D$ scattering amplitude, $\mathcal{A}_{10}$, versus s and θ . We set $\alpha' = 1$ here.

FIG. 10. The relationship between ΔS_{EE} and s in pure AdS.FIG. 11. The relationship between $\mathcal{A}_{10}$ versus s and θ for $\alpha' = 1$ in the fixed-angle approximated limit.

entropies, Fig. 10 can be compared to the confining and mixed setup of Figs. 3 and 4.

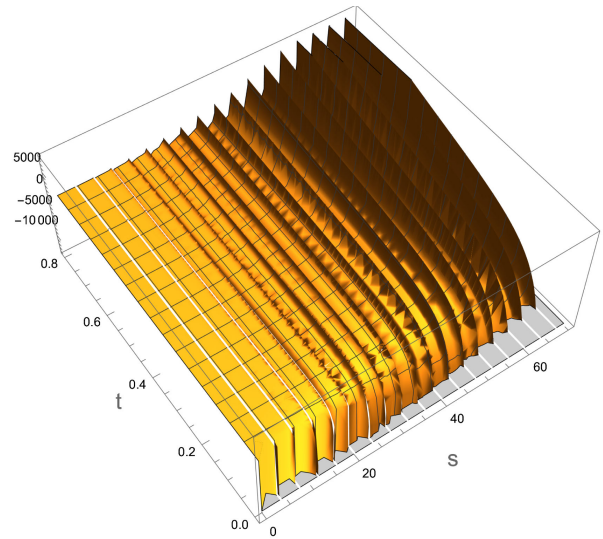
Note that the scattering process will always be dominated by the interior regions that are closer to the IR wall, and therefore the position of the wall is the main factor in determining the saddles. In fact, the mixed/random phase is dominant closer to that wall, which is one of the reasons that the phase transitions can be probed effectively using MI and D_c versus u_{KK} .

Now let us consider two interesting limits. The fixed angle limit approximation of $s \rightarrow \infty$ while s/t is being fixed has been studied in [26], where the behavior of the real part of the A_{10} is shown in Fig. 11. Note that this limit is not physical as there are imaginary contributions. Therefore, the behavior in this limit is not similar to the behavior of quantum mixed-correlation measures, unlike the case we saw for the full physical case of A_{10} without this approximation.

On the other hand, the Regge limit, where $s \rightarrow \infty$ while t is fixed, would give the figures of Fig. 12. In this case the full phase diagram cannot be exposed, while for finding the connections, having the full spectrum is needed.

Other than comparing the structures of singularities and branch cuts in the plot of D_c (MI) and string scattering amplitude as we have done previously, one can find more direct and analytical connections between various quantum information measures and scattering amplitudes.

In an older work, [1], it has been shown that any interaction between particles would change the entanglement entropy (EE), and this change of EE has been directly linked to the scattering amplitude. This is also the case for all other mixed correlation measures such as mutual information and negativity. So the scattering amplitude should also be able to track the “four saddles” of the mixed correlations studied in [35,47].

FIG. 12. The relationship between $\mathcal{A}_{10}$ versus s and θ for $\alpha' = 1$ in the Regge limit.

Then, by employing perturbation theory, the disc scattering amplitude of two heavy quark-antiquark pairs, which are being accelerated in the opposite directions, has been found in [2]. By taking the entanglement entropy of the quark and antiquark, which is approximately $S_{EE} = \sqrt{\lambda}$, where λ is the 't Hooft coupling, the connections between these quantities can be demonstrated in a more rigorous and analytical way. In addition, the entanglement entropy can be written in terms of the Wilson loop, $\langle W \rangle$ as $S_E = (1 - c\lambda\partial_\lambda) \log \langle W \rangle$, [please check Eq. (1.1) [48] which is correct in general d -dimensions, and also [1,49]]. Thus, at large 't Hooft couplings λ , the change of EE at leading order in λ could be related to the gluon scattering amplitude as [1]

$$\Delta S_E \sim \frac{(1 - \frac{1}{2}c)\sqrt{\lambda}}{8\pi} \left(\log \frac{s}{t} \right)^2. \quad (3.3)$$

Also, in [2], the amplitude and then entanglement entropy between two accelerating quark and antiquarks have been found from the four-point function

$$g(x_f, \tilde{x}_f; x_i, \tilde{x}_i) = \int_0^\infty \frac{dT}{T} \int [dx^\mu] e^{iS[x,T]} \int_0^\infty \frac{d\tilde{T}}{\tilde{T}} \int [d\tilde{x}^\mu] e^{i\tilde{S}[\tilde{x},\tilde{T}]} W[x, \tilde{x}], \quad (3.4)$$

where

$$S = \int_0^1 d\sigma \left[\frac{(\dot{x}^\mu(\sigma))^2}{4T} - M^2 T + \frac{E}{2} (x^1 \dot{x}^0 - x^0 \dot{x}^1) \right], \\ \tilde{S} = \int_0^1 d\sigma \left[\frac{(\dot{\tilde{x}}^\mu(\sigma))^2}{4\tilde{T}} - M^2 \tilde{T} - \frac{E}{2} (\tilde{x}^1 \dot{\tilde{x}}^0 - \tilde{x}^0 \dot{\tilde{x}}^1) \right]. \quad (3.5)$$

Also, $W[x, \tilde{x}] = \langle 0 | \text{Tr} U[x] \tilde{U}[\tilde{x}] | 0 \rangle$, where

$$U[x] = \mathcal{T} e^{i \int_{-\infty}^{\infty} d\tau [A_\mu(x(\tau)) \dot{x}^\mu(\tau) + \sqrt{-\dot{x}(\tau)^2} \phi^1(x(\tau))]}, \\ \tilde{U}[\tilde{x}] = \left(\mathcal{T} e^{i \int_{-\infty}^{\infty} d\tau [A_\mu(\tilde{x}(\tau)) \dot{\tilde{x}}^\mu(\tau) - \sqrt{-\dot{\tilde{x}}(\tau)^2} \phi^1(\tilde{x}(\tau))]} \right)^\dagger. \quad (3.6)$$

So, one can see that increasing the mass M would decrease S and $\tilde{S}$, the four-point function, and the amplitude.

Similar analytical connections between the mixed correlation measures such as mutual information, negativity, entanglement, and complexity of purification on one hand, and the string scattering amplitude on the other hand, need to be found. For instance, the scattering fermions, the spin-momentum and helicity-momentum correlations, and also the variations of mutual information have been found in [50], which could be extended to other measures as well. Also, the connections between string scattering and

evolution of the Ryu–Takayanagi surface and reflected entropy have been studied in [51].

Another interesting observation is the link between the change of entanglement entropy and the Bremsstrahlung of radiative corrections, $\lambda \partial_\lambda \log \langle W \rangle$, in the case of accelerating quark-antiquark pairs, which is proportional to the Bremsstrahlung function [1,52]. So the “spikes” that we have noticed during the phase transitions in the plot of D_c could be related to K-lines in Bremsstrahlung radiation.

Note that our observation of the connections is more precise for closed strings (glueballs). In the case of open strings, which describe mesons or baryons, there are two kinds of entanglement entropy, which make finding the connections between mutual information and scattering amplitude more challenging. In that case, the entanglement entropy includes contributions from the correlations between the strings' end points on the gluons (or flavor branes) and the correlations between the gluons themselves. This makes the framework and building the connections that we are interested in somehow more complicated.

IV. MUTUAL INFORMATION, MODULAR HAMILTONIAN, AND BINDING ENERGY

In various papers, the entanglement entropy and mutual information have already been used to extract the universalities in the properties of renormalization group flows at various energies [53,54]. Therefore, it is expected that they could also detect the quantized behaviors at low energies which occur due to the creation of binding energy, as found using the scattering amplitude in [26] and as we inquired into in Sec. III. We could also use the modular Hamiltonian, modular flows, and shape dependence, similar to the setup of [25], to formulate such connections. We should model the scattering process and its effects on the modular flows as shape perturbations, similar to the setup of [25]. Additionally, we can show that modular flows could also detect the partonic behaviors of [26] at low energies.

The mutual information between two spherical regions with radii R_A and R_B , separated by a large distance L , can be written as [25,55–57]

$$I_{A,B} = \mathcal{N}_\Delta \frac{\sqrt{\pi} \Gamma(2\Delta + 1)}{4\Gamma(2\Delta + \frac{3}{2})} \left(\frac{R_A R_B}{L^2} \right)^{2\Delta} + \dots \quad (4.1)$$

In the above relation, Δ is the conformal dimension of an internal scalar primary operator that carries the mixed correlation between the spherical regions A and B . The string scattering process would affect this scalar field and the weights of the “carriers,” and therefore the total correlations.

I present the behavior of the mutual information I_{AB} versus Δ , for the case of $L > R_{A,B}$, in Fig. 13. One can see that, as Δ increases, the mutual information decreases. Also, for any specific value of Δ , a smaller distance L

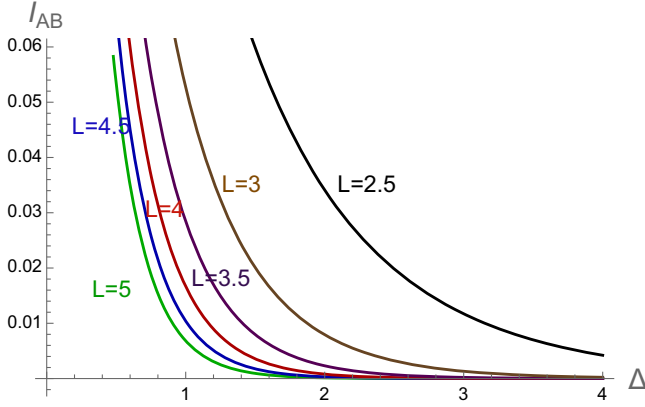


FIG. 13. The relationship between mutual information I_{AB} and the conformal dimension of an internal scalar primary operator Δ , for various distances L between the spherical regions. The same behavior is expected for the string scattering amplitude versus the mass of the field.

corresponds to larger mutual information I_{AB} between the two spheres.

The scaling dimension and the mass of the field ϕ are connected with each other as $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$. So, for the case of $L > R_{A,B}$, bigger Δ leads to bigger mass m which makes the strings scattering process between the strings more difficult, leading to smaller amplitudes. This is consistent with the relation (3.4) as well. Furthermore, this can be checked from Fig. 3 of [58], where the interactions of string coherent states in the DDF formalism have been studied and the open string profiles at different average masses and their time evolutions were shown. There, one could see that by increasing the mass, their profiles shrink, making the string scattering amplitude decrease. This can signal how scattering amplitude and mutual information are related.

Note that in other models of CFT, such as $O(N)$, when the scaling dimension increases, the system becomes more stable against fluctuations of the magnetic field or the electric field. The charge in these models increases the anomalous dimension Δ [59], and also breaks the correlations based on a decreasing power-law function, as shown in Fig. 13.

The scattering process of an accelerating string or quark moving toward a sphere or another static quark can be modeled as a perturbative deformation of shape and modular flow [25]. For any Hamiltonian H , the thermal density matrix can be written as $\rho_\beta = \frac{e^{-\beta H}}{\text{tr}(e^{-\beta H})}$, and conversely, for any invertible density matrix ρ , a thermal Hamiltonian can be constructed as $K_\rho \equiv -\log \rho$, where K_ρ is the modular Hamiltonian of the state ρ . Therefore, any density matrix can be written in exponential form as $\rho_V = c e^{-H}$. An interesting aspect of the modular theory of these operators or Tomita–Takesaki theory [60–62], is

that it can even be formulated for quantum systems where the density matrices cannot be defined. The modular Hamiltonian can be written as the sum of a local part and a nonlocal part as $H = H_{\text{loc}} + H_{\text{non-loc}}$. These components can, in fact, encode the complete entanglement data of the system.

For any Hilbert space $\mathcal{H}$ with a von Neumann algebra $\mathcal{A}$ and a state $|\Omega\rangle \in \mathcal{H}$, the antilinear operator S_Ω can be defined as [63]

$$S_\Omega(a|\Omega\rangle) = a^\dagger|\Omega\rangle, \quad a \in \mathcal{A}. \quad (4.2)$$

The modular operator $\Delta_\Omega \equiv S_\Omega^\dagger S_\Omega$ can be constructed using the operator S_Ω , and then, using Δ_Ω , the modular Hamiltonian can be redefined as $K_\Omega \equiv -\log \Delta_\Omega$. Next, the modular flow of operators can be set out via a map as

$$a \mapsto e^{iK_\Omega t} a e^{-iK_\Omega t} = \Delta_\Omega^{-it} a \Delta_\Omega^{it}. \quad (4.3)$$

This is in fact a unitary operator which implements an “internal time flow,” which was the basis for us to connect the mutual information and string scattering amplitude. For the case of a system with more than one interval, the modular evolution is nonlocal, which causes the “teleportation” between the disjoint intervals [64]. This can demonstrate that for connecting scattering amplitude and quantum information, one should in fact take the “mixed correlation measures” for multiregions rather than the entanglement entropy.

An interesting connection can be seen from the relation of the modular Hamiltonian H which can be written using the decomposition in terms of the eigenvectors as

$$H(x, y) = 2\pi \sum_k \int_{-\infty}^{\infty} ds s \Psi_s^k(y) \Psi_s^{k*}(x), \quad (4.4)$$

where $\Psi(s) \equiv \Psi(x^\mu(s))$, and s_1 and s_2 are length parameters along the spherical set V . Also $\Psi(x)$ s are the Dirac fields. So the sum of the scattered eigenvectors from the scattering process could reconstruct the modular Hamiltonian and the trajectories of modular flow.

Note that the local part of the Hamiltonian can be written as

$$H_{\text{loc}} = \pi i \left(2 \left(\frac{dz(x)}{dx} \right)^{-1} \partial_x + \frac{d}{dx} \left(\frac{dz(x)}{dx} \right)^{-1} \right) \delta(x - y), \quad (4.5)$$

and the nonlocal part can be written as

$$H_{\text{non-loc}} = -2\pi i \sum_{l, x_l(z(x)) \neq x} \frac{1}{(x - y)} \left(\frac{dz}{dy} \right)^{-1} \delta(y - x_l(z(x))), \quad (4.6)$$

which mixes the finite number of points, i.e., $x_i(z)$, from each interval, which are related to the peaks of D_c . So, the singularities and the zeros in the plots of scattering amplitude and also D_c are mostly related to the “nonlocal” correlations.

Due to the scattering process, the boundary of one of the subsystems changes as $\tilde{x}^\mu = x^\mu + \zeta^\mu$. The strings that are being created from the vacuum for an instant in time would move along the geodesics, which are determined by the linear response of the metric deformation. Then, the scattering process, similar to [25], would be proportional to the deformation and “modular flow,” $\nabla^\mu \xi^\nu$, leading to the metric deformation $\delta g^{\mu\nu} = 2\nabla^\mu \xi^\nu$, which then leads to the change of the reduced density matrix as

$$\delta \rho_A = U^\dagger \circ \rho_{\tilde{A}} \circ U - \rho_A \simeq \frac{1}{2} \int \delta g^{\mu\nu} \rho_A \tilde{T}_{\mu\nu}. \quad (4.7)$$

From the first law of entanglement entropy, one can deduce that, in general, the linear response of the entanglement entropy is proportional to the correlation of the contour integral of stress tensor with the modular Hamiltonian as [25] $\delta S_A \propto \oint_{z \sim \partial A} dz \langle T(z) H_A \rangle$. Then, due to this scattering process and the resulting shape deformation, the entanglement entropy of one subregion would change as

$$\delta S_A = \int dx \sqrt{g} \nabla^\mu \xi^\nu(x) \langle T_{\mu\nu}(x) H_A \rangle + \mathcal{O}(\zeta^2). \quad (4.8)$$

This equation should be compared with the equation

$$\mathcal{A}_4(s, t, u) = \int_{U_A}^\infty dU \sqrt{g} e^{-2\phi} \left(\prod_{i=1}^4 \psi_i(U) \right) \mathcal{A}_{10}(\tilde{s}, \tilde{t}, \tilde{u}). \quad (4.9)$$

From these two equations, one can see how the ten-dimensional scattering amplitude and the wave functions that are being scattered are related to the average of the correlation of the energy-momentum tensor and modular Hamiltonian. In other words, we claim that in an elastic scattering process we have the relation $\nabla^\mu \xi^\nu(x) \langle T_{\mu\nu}(x) H_A \rangle \propto e^{-2\phi} (\prod_{i=1}^4 \psi_i(U)) \mathcal{A}_{10}(\tilde{s}, \tilde{t}, \tilde{u})$. This is consistent with the result of [51], where they showed that the scattering distance can be identified as the entanglement wedge cross section. There, they also found connections between the string worldsheet and the Ryu–Takayanagi surface, as well as mutual information and the geometric BV (Batalin–Vilkovisky) formalism master equation [51]. So the string scattering amplitude can be written in terms of the integration of the wave functionals.

This connection can further be checked phenomenologically by applying a magnetic field. In [65], it has been proposed that the magnetic field can be adjusted such that

the amplitude vanishes for any kinematic setup. Similar studies in [34,66] have been performed for the case of entanglement entropy, where it has been shown that for any size of subregion, the magnetic field can be tuned to make the entanglement entropy vanish. This then further manifests this connection.

Equation (4.6) shows that the change in entanglement entropy only picks up the residue of the simple poles of $\langle T(z) H_A \rangle$ as

$$\langle T(z) H_A \rangle = \cdots + \frac{\text{Res}}{z - \partial A} + \cdots \quad (4.10)$$

For the case of mutual information, as has been found in [25], the relation can be extended and the linear response of mutual information, $\delta I_{A,B}$, can be encoded in the “susceptibility” of shape deformation as

$$\frac{\delta I_{A,B}}{\delta \xi(X^i)} = (i) \oint_{|z|=\tilde{z}} dz \langle (T_{zz}(z, \tilde{z}, X^i) + T_{z\bar{z}}(z, \tilde{z}, X^i)) \Delta \tilde{H}_{A,B} \rangle + \text{H.c.}, \quad (4.11)$$

which picks up the residue of simple poles of $\langle T_{zz} \Delta \tilde{H}_{A,B} \rangle$ and is independent of the radial coordinate. This is exactly similar to the case of string scattering amplitude, where the structure of the residues of its poles were discussed in [26]. So for the interval $A \equiv \{-\frac{l}{2} \leq x \leq \frac{l}{2}\}$, the modular flow which induces the change $\partial^\mu T_{\mu\nu}$ along H_A follows a specific trajectory and a contour integral which is related to the trajectory of the strings that are being created from the vacuum, i.e., $\partial S_A \propto \oint_{z \sim \partial A} \langle T(z) H_A \rangle$. This, as mentioned in [25], takes the residue of the simple pole in $\langle T(z) H_A \rangle$ where the poles that are emerging in the integral of the modular Hamiltonian are related to the poles of the string scattering amplitude $\mathcal{A}_4$.

Similarly, the scattering amplitude, $\mathcal{A}_4$, can be written as the integral of the metric, and the shape deformation due to the scattering process, and also in terms of the correlation functions between the stress tensor and modular Hamiltonian, i.e., $\langle T_{\mu\nu} \Delta H \rangle$. In terms of the modular flow, it can be written as $\langle T_{\mu\nu} \mathcal{O}(-is_1) \mathcal{O}(-is_2) \rangle$. Therefore, the logarithmic branch cuts observed in the string scattering process and its trajectories could be described by the modular trajectories and the zeros in the correlations between stress tensor and modular Hamiltonian. As mentioned in [25], in the shape deformation, the main contributing modes are the ones that change the position of the cut-off tube, instead of deforming it. So at low energies, these modes are the one that are related to the binding energy.

Actually, in [67], the scattering amplitude in terms of the modular flow has been written as

$$\begin{aligned}
& \mathcal{A}(\{\phi_l^i, \phi_r^j\} \rightarrow \{\chi^k\}) \\
&= \text{Tr}(\rho^{1-is} \chi^1 \dots \chi^n \rho^{is} \phi_l^1(t_{l1}) \dots \phi_l^i(t_{li}) \phi_r^1(t_{r1}) \dots \phi_r^j(t_{rj})),
\end{aligned} \quad (4.12)$$

where $\{\chi^k\}$ is the set of bulk operators which act on a thermofield double state, ρ is the probe and $\phi_l^1(t_{l1}) \dots \phi_l^i(t_{li}) \phi_r^1(t_{r1}) \dots \phi_r^j(t_{rj})$ is the boundary state which generates the particles. So through this link between the scattering amplitude and modular flow, the direct relations between mutual information and scattering amplitude could be derived. Therefore, the information that scattering amplitude could catch, such as detecting the binding energies at low s , could be captured by modular flow, entanglement entropy, and mutual information as well. However, as explained before, the correlation of string amplitude with mutual information is more direct and physical compared with entanglement entropy.

For example, the mutual information between two spheres can be written as a logarithmic function which has branch cuts [25], and this gives signals for the reasons behind the logarithmic branch cut behavior observed in string scattering amplitudes in [26].

In [25], actually, the linear response of the mutual information between the two spherical regions due to the scattering process or any other perturbative effects like shape deformation has been written as

$$\begin{aligned}
\delta I_{A,B} &= N_d \left(\frac{\partial I_{A,B}}{\partial \ln \rho} \right) \\
&\times \int d\Omega_{d-2} \left(\frac{L^2 - R_A^2}{L^2 + R_A^2 - 2LR_A \cos \alpha} \right)^{d-1} \zeta(\Omega). \quad (4.13)
\end{aligned}$$

This results in a form of susceptibility that resembles the integral used to calculate the scattering process amplitude. So one could see that

$$\begin{aligned}
& \int_{u_{KK}}^{\infty} du \prod_{i=1}^4 \psi_i(u) \\
& \propto \delta S \propto \delta^{-(d-2)} \int_{S_{d-2}} d\Omega_{d-2} \xi(\Omega) \\
& \propto \delta^{-(d-2)} \int_{S_{d-2}} d\Omega_{d-2} \sum_{\ell, m} \xi^{(\ell, m)} \mathcal{J}(\Omega)^{-1} \Phi_{A,B}^{(\ell, m)}(\Omega). \quad (4.14)
\end{aligned}$$

So the modes of the scattered strings can be written in terms of the decomposition modes of the deformation vector. This is consistent with the fact that the entanglement has a nonlocal structure.

Also, for the case of two spheres, we could write the approximate relation

$$\begin{aligned}
& N_d \left(\frac{\partial I_{A,B}}{\partial \ln \rho} \right) \left(\frac{L^2 - R_A^2}{L^2 + R_A^2 - 2LR_A \cos \alpha} \right)^{d-1} \\
& \simeq \sqrt{-g} e^{-2\phi} A_{10}[\tilde{s}(r), \tilde{t}(r), \tilde{u}(r)], \quad (4.15)
\end{aligned}$$

connecting the change of mutual information and $10d$ scattering amplitude.

Another point related to the interconnections between the modular Hamiltonian and string scattering is the additive behavior in s_k , as the response of mutual information can be written as [25]

$$\delta I_{AB} = \int ds_k \delta I_{AB}(s_k). \quad (4.16)$$

We can claim that this additive behavior arises from the connections between mutual information, modular flow, and the string scattering process.

A. Time-dependent mutual information

Another interesting direction to examine the connections between the spread of quantum information in spacetimes and string scattering amplitudes is the recent development in defining the spacetime mutual information (STMI) and characterizing its behavior, as in [68–71]. In [68], the spacetime mutual information $J(A:B)$ has been defined by optimizing the relative entropy over the choice of gates V_A and V_B as

$$\begin{aligned}
J_N(A:B) &= \frac{1}{N} \sup_{V_A, V_B} S(\sigma_{W,1}^{(N)} | \sigma_{W,0}^{(N)}), J(A:B) \\
&= \lim_{N \rightarrow \infty} J_N(A:B). \quad (4.17)
\end{aligned}$$

These gates would actually act as the ancilla-system coupling, where W plays the role of an idler that carries the information encoded in A to the future, where these gates and W are arbitrary. This notion is based on the hypothesis testing interpretation of relative entropy [72], as relative entropy provides the fastest rate at which one can identify a wrong hypothesis before taking the measurement.

One could first consider a black box, which in our study here is the confining geometry. In the box, there are either N copies of quantum state ρ or N copies of quantum state σ . Then one can do measurements on this N -copied system and determine whether it is $\rho^{\otimes}$ or $\sigma^{\otimes}$. If one makes a hypothesis that the state is $\sigma^{\otimes}$ and carries out the measurement, then the probability of a measurement outcome could be found by assuming the state is $\sigma^{\otimes N}$. If the initial hypothesis is correct, then the probability will reach 1 in the large N , but if it is incorrect, and the state be ρ , the probability $P_N(\rho|\sigma)$ would be smaller. This probability has a lower bound as $P_N(\rho|\sigma) \geq e^{-NS(\rho|\sigma)}$, where $S(\rho|\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$. For two spacelike separated

regions A , B , the mutual information is a relative entropy $I(A:B) = S(\rho_{AB}|\rho_A \otimes \rho_B)$.

For the case where σ is not full rank, there would be states which never appear in σ and in this case the relative entropy would diverge. One could imagine a qubit system with states $|0\rangle$ and $|1\rangle$. Then, if $\sigma = |0\rangle\langle 0|$, the probability of seeing $|1\rangle$ is zero, so as $P_N = 0$ then $S(\rho|\sigma) \rightarrow \infty$. This case corresponds to the branch cuts behavior of mutual information as we have found in the previous section, which is due to the creation of bound states.

The expression for $J_N(A:B)$ can also be written as [68]

$$J_N(A:B) = \sup_V \frac{1}{N} [I^{(N)}(B:W) + S(\rho_{B^N}|\rho_{B,0}^{\otimes N})]. \quad (4.18)$$

From the above relation, the condition for the divergence of relative entropy is that $\rho_{B,0}$ should not be full rank and ρ_{B^N} has a nonzero probability to be in the null space of $\rho_{B,0}^{\otimes N}$. As explained in [68], if in a system there is a conserved charge and the original state $\rho_{B,0}$ is supported in a charge range $[q_1, q_2]$, then if one can tune V to change the charge of B to go beyond this range, then $J(A:B)$ would diverge. This exactly corresponds to the creation of bound states in the confining geometries, as there are fluxes and conserved charges which can change the charge of the subsystem B . This corresponds to a certain set of ancilla which has a finite probability to show up and can cause this effect. Also, such divergences have a logarithmic nature which is similar to the low-energy logarithmic singularities and logarithmic branch points of [26,45], which replace the poles in the original flat space amplitude.

Note that the full amplitude has the relation [26]

$$\mathcal{A}_4(s, t, u) = \frac{r_0^4}{R^4} \frac{1}{4 - \Delta} \sum_{n=0}^{\infty} \frac{\mathcal{R}_n(\theta)}{n+1} {}_2F_1 \left(1; \frac{\Delta}{2} - 2; \frac{\Delta}{2} - 1; \frac{\alpha'_0 s}{4(n+1)} \right), \quad (4.19)$$

where for even $\Delta = 2(k+1) > 2$, and for any integer k , the hypergeometric function can be written as a logarithmic function as

$$\begin{aligned} {}_2F_1(1; k; k+1; z) \\ = -\frac{k \log(1-z)}{z^k} - \sum_{l=1}^{k-1} \frac{k}{(k-l)z^l}. \end{aligned} \quad (4.20)$$

On the other hand, for factorized initial state, as found in [68], the relative entropy could be written as

$$\begin{aligned} S(\rho_{BW}|\rho_{BW,0}) \\ = -S(\tilde{\mathcal{N}}(\rho_W)) + S(\rho_W) - \text{Tr} \rho_W \mathcal{N}^\dagger \log \mathcal{N}(\rho_{\text{in}}), \end{aligned} \quad (4.21)$$

where ρ_W could also be written as

$$\rho_W = C \exp [\tilde{\mathcal{N}}^\dagger \log \tilde{\mathcal{N}}(\rho_W) - \mathcal{N}^\dagger \log \mathcal{N}(\rho_{\text{in}})], \quad (4.22)$$

and $\mathcal{N}$ here are quantum channels. So the radial direction in confining geometries where the integral is taken could be related to the quantum channels to explain the logarithmic divergences. The time evolution then can be defined by a quantum channel $\mathcal{N}$, and also the holographic direction can be mapped to the time direction, where its integral could be considered as a time interval.

For unitary channels $\mathcal{N}: A \rightarrow B$, the quantities are

$$\rho_W = \frac{\rho_{\text{in}}^{-1}}{\text{Tr} \rho_{\text{in}}^{-1}}, \quad J_1 = \log \text{Tr} \rho_{\text{in}}^{-1}. \quad (4.23)$$

For the case of single-qubit system, the divergent part of relative entropy can be found from [68],

$$S(\rho_{BW}|\rho_{BW,0}) \approx -\text{Tr} \mathcal{N}_{\text{dph}}(\rho_W) \log \mathcal{N}_{\text{dph}}(\rho_{\text{in}}), \quad (4.24)$$

where $\mathcal{N}_{\text{dph}}$ is a dephasing channel. Then, the STMI has been found as

$$J_1(A:B) \approx -2 \log \epsilon. \quad (4.25)$$

The STMI diverges as the initial state approaches an eigenstate of the evolution and is independent of the expansion coefficients $0 \leq p \leq 1$. This result has been generalized to many-body localized systems as well. This approach to the eigenstate of the evolution is dual to the bound state creation.

For instance, one could take A as a single qubit at the initial time $t = 0$ and B the same qubit at time t , with initial pure states on A of the form $\rho_{\text{in}} = |\chi\rangle\langle\chi|$, where

$$|\chi\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle. \quad (4.26)$$

As the STMI quantifies the information preserved by the system, during bound state creation, one would expect it to diverge, since the system can get closer to the initial state (4.26). This state is an eigenstate of the conserved operator σ^3 , and getting closer to it would make the STMI larger.

In addition, in [70], a family of entanglement measures for time-separated subsystems based on von Neumann entanglement entropy or Tsallis entropy have been proposed. For a one-dimensional free fermion chain with the Hamiltonian

$$H = \sum_s \psi(s) \bar{\psi}(s+1) + \text{H.c.}, \quad (4.27)$$

their measure, for Euclidean complex coordinates $z = s + i\tau$ and for two intervals $[z_1, z_2] \cup [z_3, z_4]$, leads to

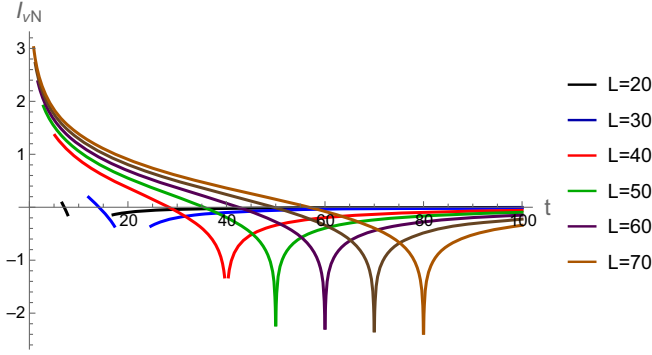


FIG. 14. The relation between time-dependent mutual information in (4.30) for free fermions and various subsystem sizes L , separated by a time interval t .

$$\text{Tr} \rho^n = \left(\frac{\epsilon^4}{x \bar{x} (z_2 - z_1)(\bar{z}_2 - \bar{z}_1)(z_4 - z_3)(\bar{z}_4 - \bar{z}_3)} \right)^{-\frac{n-1}{12n}}, \quad (4.28)$$

where

$$x = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_3 - z_1)(z_4 - z_2)}, \quad (4.29)$$

and the numerical results match the CFT prediction, which is

$$I_{\nu N} = \frac{1}{3} \text{Re} \log \left(1 - \frac{L^2}{t^2} \right). \quad (4.30)$$

For different L , the relation between $I_{\nu N}$ and time t is displayed in Fig. 14.

It can be seen that increasing L would increase the time that the branch cut happens and it can magnify its peak, but the two branches become sharper themselves. Note that the Mandelstam variable s for a 2-particle $\rightarrow$ 2-particle process has the relation $s = (p_1 + p_2)^2 = (p'_1 + p'_2)^2$ where $p_1^2 = m_1^2, p_2^2 = m_2^2, p'_1^2 = m_1'^2, p'_2^2 = m_2'^2$. For bigger time interval t between the two subsystems, the transferred energy and information, and therefore s , would be smaller. This concept is shown in Fig. 15.

Furthermore, in [69], using the pseudodensity matrix formalism, the timelike quantum mutual information has been defined where again the “branch cut behaviors” could be noticed in their phase space, specifically in their 3d plots of mutual information between a single qubit at multiple times versus the depolarizing parameter η , and the parameter p , $0 < p < 1$, which is a defining parameter of the initial state.

So, this analysis and the notion of timelike mutual information can shed more light on the connections between the quantum information behavior, its spreading, and the string scattering amplitude.

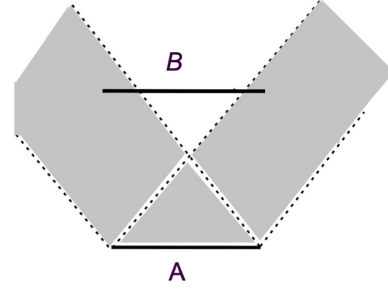


FIG. 15. The causal diamond between region A and B , where the operators on A can only have nonzero anticommutators with the operators that are in the shaded region. So, as time passes and region B moves outside of the shaded region, smaller parts of B can be affected by A , and s would decrease.

V. CHAOS, MUTUAL INFORMATION, AND STRING SCATTERING AMPLITUDE

In studies of the chaotic behavior of highly excited states (HES), decaying into two or three tachyons [45,73–76], it has been shown that the peaks of the angular dependence of the amplitude exhibit characteristics comparable to the β -ensemble in random matrix theory, thereby confirming their chaotic nature. This chaos is due to the random superposition of various spherical harmonics in the partial wave expansion of the amplitude [45]. One would think that the signature of this chaos would be present in mixed quantum measures such as mutual information as well. In fact, the pattern of successive peaks in the mutual information in the background of confining geometries, which can be seen from Figs. 3–5, also points to this chaotic behavior in the same phases where the damping of the peaks follows the same β -distribution.

The observed chaotic behavior in the scattering amplitude of highly excited strings, linked to the string/black hole correspondence, offers valuable insights into the inherent chaos characteristic of black holes. In [45], the authors demonstrate that multiplying the dressing factor of highly excited strings by the Veneziano amplitude results in a quantity that can interplay between them and create the transition from chaotic to regular spacing, as illustrated in their Fig. 7 of [45]. This can be written as

$$\mathcal{A}_{\text{HES}}(s, t) = \mathcal{A}_{\text{Ven}}(s, t) \mathcal{D}_{\text{HES}}(s, \theta), \quad (5.1)$$

where, at the chaotic phase, for the solution of $F_{\mathcal{D}}(\theta) \equiv \frac{d \log \mathcal{D}_{\text{HES}}}{d\theta} = 0$, the ratios of consecutive spacing follow the β -ensemble distribution. Also, at the edge of chaos, the peaks of the dressing factor $\mathcal{D}_{\text{HES}}$ form a complicated product of polynomials where the peaks are spaced erratically. An example, in the Regge limit, is shown in Fig. 16.

One would expect that various quantum information measures would show the same structure and exhibit a

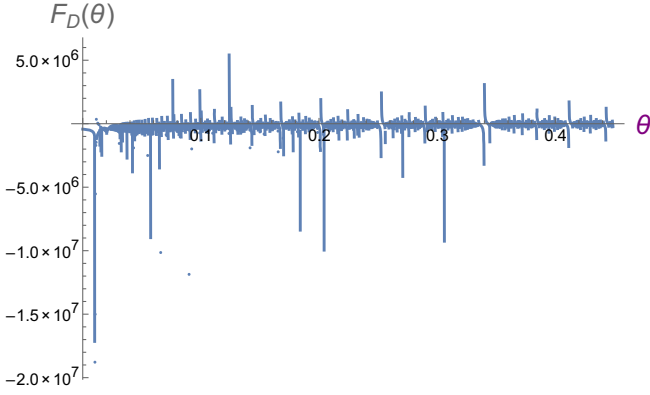


FIG. 16. The erratic behavior of the dressing factor $F_D(\theta)$ versus scattering angle θ , for $N = 100$, $J = 18$, [45].

transition between regular and chaotic behavior. For instance, in recent works such as [77], the Krylov complexity has been used to track the transition between integrable and chaotic regimes, which can also capture the eigenvalue statistics in the mixed-phase systems. Therefore, it is expected that Krylov complexity can be connected to high-energy string scattering amplitudes as well.

In our work, by varying the critical distance between two mixed states in a confining background, we observed a detectable transition from regular to chaotic behavior in the response of mutual information, as illustrated, for instance, in Fig. 6. This suggests a correlation between the scattering amplitude and mutual information, with both exhibiting similar behavior at the transition point. To solidify this link, one could compare the patterns in the peaks of mutual information with results from random matrix theory, potentially through an angular functional between the two mixed points, to further strengthen the connection.

In [45], it was also demonstrated that for a 3×3 matrix with eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$, the probability density function for the ratio $r = (\lambda_3 - \lambda_2)/(\lambda_2 - \lambda_1)$ in the β -ensemble is given by

$$f_\beta(r) = \frac{3^{\frac{3+\beta}{2}} \Gamma(1 + \frac{\beta}{2})^2}{2\pi \Gamma(1 + \beta)} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+\frac{3}{2}\beta}}, \quad (5.2)$$

which resembles the probability distribution function of the normalized mutual information in multiple-input multiple-output systems and the energy levels of a Coulomb fluid [78]. The plot of this distribution is shown in Fig. 17. It can be seen that by increasing β , the maximum of the distribution becomes larger and the peaks move farther apart. Since the case of $\beta \rightarrow \infty$ corresponds to a rigid structure or crystallization, one might infer that larger β corresponds to smaller mutual information between two mixed subsystems. This is indeed consistent with the result of [79], where it is shown that large β leads to conditional probabilities close to 1/2 and thus small mutual information, while small β leads to conditional probabilities close to 0 or 1, resulting in larger mutual information. This also aligns with our results here regarding the connection between string scattering amplitude and mutual information.

The source of the chaotic behaviors and partonic structures observed in the scattering amplitude and mutual information is the large fluctuations in the spin of intrinsically highly excited states, i.e., the complex superposition of many states with different spin s . Such superpositions would affect the mutual information and the spreading of quantum information in the hard wall confining model as well. As mentioned in [74], this huge degeneracy of string excitations, which have the same level N but generic spins, could create interesting chaotic structures. The precise impact of these spin fluctuations on quantum information measures, particularly mutual information and other mixed correlation metrics, requires further study.

As discussed in [74], the successive peaks of the chaotic scattering amplitude are expected to follow a log-normal distribution. This log-normal behavior is also observable in the pattern of peaks in the mutual information as a function of the critical distance. Notably, the same metric can be

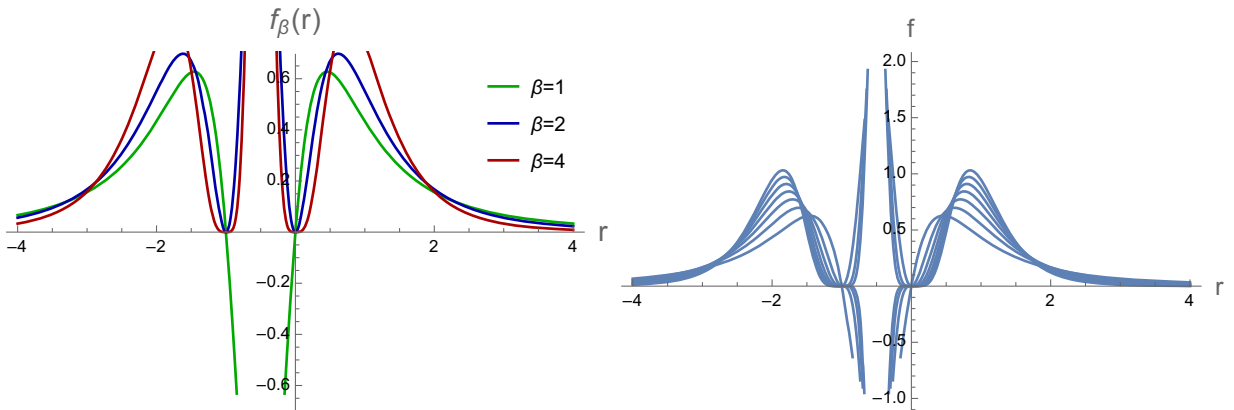


FIG. 17. The probability density functions for the ratio of eigenvalues in the β -ensemble, which are used as fitting models.

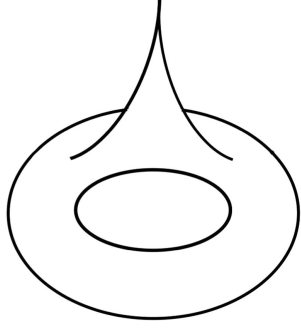


FIG. 18. The topology of a leaky torus embedded in Euclidean three-dimensional space [80] is shown. This geometry is smooth and orientable but not compact, with constant negative curvature and an infinite cusp. Also, it contains an open end where the phase shift of a particle that enters through it shows all the aspects of chaotic behavior, with great difficulty in its effective numerical computation. The chaos in this geometry could be compared with the chaos structures in confining backgrounds.

applied to both the decay process of a highly excited string state into two tachyons and to quantum scattering on a “leaky torus.” The topology of the leaky torus is illustrated in Fig. 18.

The geometry of the leaky torus H can be written as [80]

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad (5.3)$$

where the boundaries of H are

- (i) $x = -1, \quad 0 < y < \infty;$
- (ii) $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}, \quad 0 < x < 1;$
- (iii) $x = 1, \quad 0 < y < \infty;$
- (iv) $\left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$

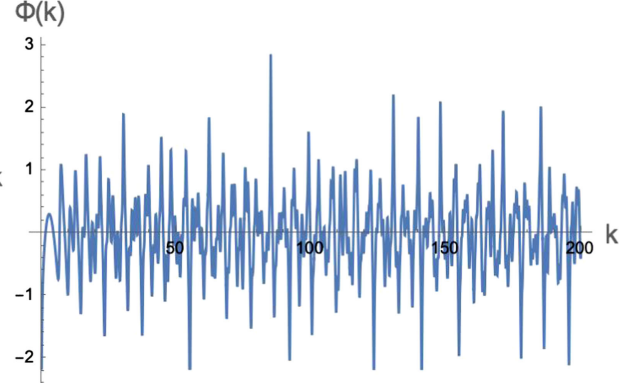
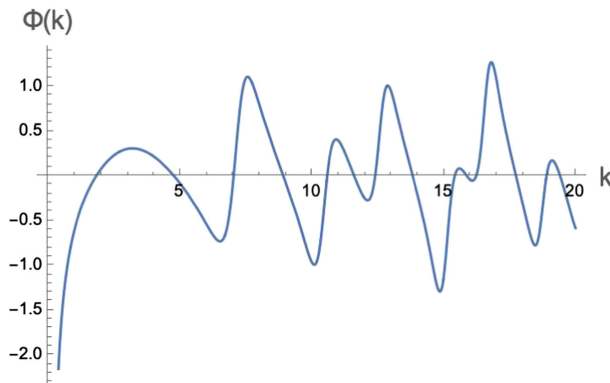


FIG. 19. The chaotic behavior of the phase shift as a function of the momentum of the incoming wave on a leaky torus is shown here, which can serve as a model of the spread of information in a confining geometry.

So, the pattern of the scatterings of the wave function on the leaky torus and the erratic behavior of its phase shift Φ as a function of momentum k , i.e., $\Phi(k) = \frac{\text{Im}[\zeta(1+2ik)]}{\text{Re}[\zeta(1+2ik)]}$ which is shown in Fig. 19 could be related to the spreading of information in a confining geometry with a hard wall. The spacings between successive maxima for both the function $\Phi(k)$ and $D_c(u_{KK})$ would align with the zeros on the critical line $\text{Re}[s] = \frac{1}{2}$. The free wave scatters from $y = \infty$, and the phase shift between the incoming and outgoing waves is measured at $y = y_0 > 0$.

Then, the problems of quantum state spreading on a leaky torus, the decay of highly excited strings (HES) into two tachyons, and the mutual information spreading in the presence of a hard wall in a confining system all become related and comparable to each other, each showing chaotic behavior, as the peaks of their parameters all correspond to zeros of the Riemann zeta function [74,80].

VI. MUTUAL INFORMATION AND COMPTON SCATTERING

To better understand the connections between the structures of entanglement entropy or mutual information of mixed systems on one side, and string scattering amplitudes on the other side, it would be interesting to consider their changes during various scattering processes—elastic (such as Rayleigh [81] and Mie [82]) scattering, or inelastic (such as Brillouin [83], Raman, and Compton) scattering.

There are several results from other works on changes in entanglement entropy during various scattering processes that align with our findings here. For instance, as shown in [82], the change in entanglement entropy versus different scaling lengths of the setup—such as the radius of a spherical scatterer in Mie scattering—is not monotonic but oscillatory and overall increasing. This behavior is consistent with the patterns of D_c and mutual information we observed in Sec. II.

One interesting form of inelastic scattering related to string scattering is Compton scattering, where the change in

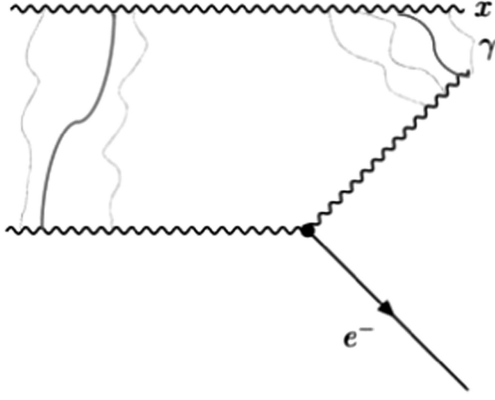


FIG. 20. The scattering process where a photon γ scatters off an electron e^- was studied in [84]. In the setup above, x is a witness particle (photon) that does not participate in the process but remains entangled with the second photon. One can imagine the two wiggly bosons as the end points of an open string, and then calculate the change in mutual information during this process, as given by relation (6.1).

entanglement entropy and mutual information during the process can be compared with the results obtained in previous sections. In fact, [84] discusses the entanglement entropy of the Compton scattering process in the presence of a witness. In their setup, one can imagine the two entangled photons as the endpoints of a string, with one participating in the Compton scattering and the other acting as a witness. The process is illustrated in Fig. 20.

One can imagine that the two wiggly bosons are actually the two end points of an open string moving on the world sheet. Then, using this picture and repeating the procedures that have been done for the case of entanglement entropy as in [84], in terms of the Stokes parameters, one can calculate the change in the mutual information during this process. In low-energy Compton scattering, the final mutual information of the electron and the witness particle's polarization is nonzero. This is due to the scattering process where in spite of no direct interaction between the two particles, they become correlated, indicating a correlation between the string scattering and mutual information.

First, the initial pure state can be written as $|i\rangle = |e^-, r\rangle(\cos \eta |\gamma, \uparrow\rangle + e^{i\beta} \sin \eta |\gamma, \downarrow\rangle |x, \uparrow\rangle)$, where η is the entanglement parameter for the two photons γ , [84,85], and x can be considered as the two end points of a string. Therefore, the parameter η is proportional to the string tension T . The mass of the particles, on the other hand, is related to the energy of the string, which can be compared with the position of the hard wall in the confining geometry.

In terms of η , the mass of the electron e , and the initial energy of the photon, one can find the initial entanglement entropy of γ or x , the change in the entropy of the electron e , and the change in the entanglement entropy between x and e , and therefore the total change in mutual information during this process.

For example, the change in the entanglement entropy between the incoming electron e and the photon at one end x has been calculated in [84] as

$$\Delta S_{EE,ex} = -\frac{1}{2} \left(1 + 0.67 \frac{\sigma_T \omega^2}{V/T m^2} \right) \log \left(1 + 0.67 \frac{\sigma_T \omega^2}{V/T m^2} \right) - \frac{1}{2} \left(1 - 0.67 \frac{\sigma_T \omega^2}{V/T m^2} \right) \log \left(1 - 0.67 \frac{\sigma_T \omega^2}{V/T m^2} \right), \quad (6.1)$$

where $\sigma_T = \frac{8\pi r_e^2}{3}$ is the Thomson scattering cross section, ω is the initial energy of the photon, $V = (2\pi)^3 \delta^3(0)$ is the unnormalized volume, m is the rest mass of the electron, and T is a parameter in the S-matrix as in $S = 1 + iT$. One can also express the above relation, as well as the changes in entanglement entropy and mutual information, in terms of the Stokes parameter $\langle \sigma_z \rangle = -\frac{\sigma_T}{V/T} \frac{3\omega}{2m}$. In this process, increasing the ratio between the initial energies of γ and the electron, i.e., (ω/m) , leads to a greater jump in the mutual information during the scattering. The rest mass m here is directly related to the position of the end wall in the confining geometry.

For various energies ω and mass m , the behavior of $I(e, x)$ versus η is shown in Fig. 21. It can be seen that increasing m and decreasing the energy ω suppress the mutual information, as one would expect. Making an analogy between hard-wall and soft-wall QCD, one could claim that, if all other parameters be the same, the soft wall leads to higher mutual information compared to the hard wall.

One can also see an intrinsic periodic behavior in terms of the entanglement parameter here. One can approximately simulate the process of strings hitting the end wall of the confining geometry with the scattering process of one photon hitting an electron and the other photon acting as a witness, which is actually the other end point of the string. Then, the change in the mutual information during this process can be found. Based on the connections between mutual information and string scattering amplitude, the periodiclike patterns we observed in certain phases, as in Figs. 3 and 4, could be expected.

Recently, it has been shown that the fundamental string has a double-slit shape [86], and the images of the strings via the Veneziano amplitude have been constructed. The information creating these images could also reveal the entanglement structures of such strings. As the scattering amplitude shows that strings have a double-slit shape, a direct connection between the Veneziano amplitude and the mutual information of the two end points of strings can be envisioned.

Also, in [86], it has been shown that in the double-slit form of strings, the slits are at the end points, which could be modeled by the system in Fig. 1. This model can also help reveal the highly complicated structure of a long string. In confining backgrounds with a hard wall or soft wall, the chaotic behavior of strings leads to richer

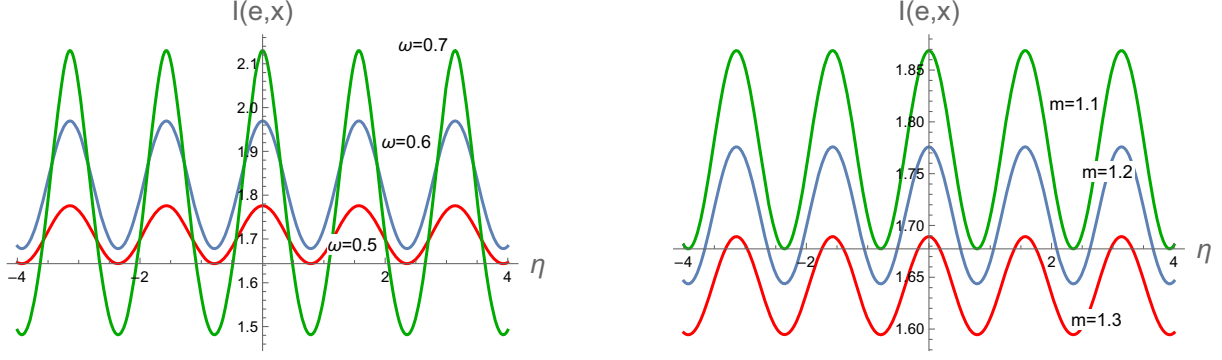


FIG. 21. The relation between the mutual information $I(e, x)$ between the two particles γ and x , versus the entanglement parameter η , is shown for different values of the mass and energy of the photon. The log functions are base 2, so the mutual information is expressed in bits.

structures. Thus, the similarities between the fractal structure of the scattering amplitude and the quantum information measures could reveal further such complex structures.

A. Two tachyons and one excited string amplitude

As another example to further explore the connections between string scattering amplitude and the spread of quantum information during the scattering process, one could consider the decay of a highly excited string into tachyons. Note that the probability of transitioning from initial states to the final state of the decay products is given by $P(f|i) \propto |\mathcal{A}(i \rightarrow f)|^2$, and with these probabilities, one can construct the density matrix ρ that describes the state of the system after scattering as $\rho = \sum_f P(f|i) |f\rangle \langle f|$. The entropy can then be written as $S = -\sum_f P(f|i) \log P(f|i)$ by which the mutual information between decay products of scattering processes such as tachyons can be computed. In a decay process of a highly excited string, the scattering amplitude determines the correlations between emitted particles, where these correlations are encoded in the off-diagonal terms of the reduced matrix, directly affecting the mutual information. In a chaotic system, the patterns of $P(f|i)$ are irregular, and the mutual information can capture these irregularities, which propagate into the correlations between the subsystems.

The excited string can be written as $\prod_{m=1}^{\infty} (\lambda \cdot A_{-m})^{n_m} |0\rangle$, [17], where $N = \sum_{m=1}^{\infty} m n_m$, and the m th mode of the string is excited n_m times, i.e., by n_m photons, each with polarization λ and momentum $-mq$. Here, m is an integer and q is a null vector. Then, the amplitude of the decay of this excited string into two tachyons can be written as [17],

$$\mathcal{A} \propto \prod_{m=1}^{\infty} (p_3 \cdot \zeta P_m(p_3 \cdot q))^{n_m},$$

$$p_3 \cdot \zeta = -\sqrt{\frac{N}{2}} \sin \alpha, \quad p_3 \cdot q = -\cos^2 \frac{\alpha}{2},$$

$$P_m(a) = \frac{(1 + ma)_{m-1}}{(m-1)!}, \quad (6.2)$$

where α is the relative angle between the tachyons' direction and the photons that create the string, and λ is the polarization of the DDF [16] photons, which are transverse to the momentum, i.e., $\lambda \cdot q = 0$. In addition, $(a)_m \equiv \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\dots(a+m-1)$ is the Pochhammer symbol, unifying the rising and falling factorials. The setup is shown in Fig. 22, where the correlations are depicted in gray lines.

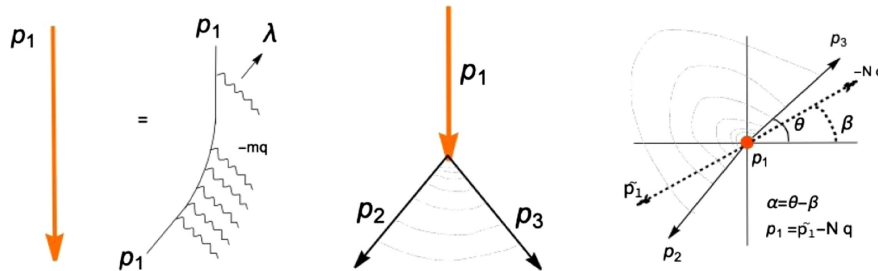


FIG. 22. On the left, the DDF mechanism of exciting a heavy string with n_m photons with momenta $-mq$ off an initial tachyon with momentum $\widetilde{p}_1$ is shown. In the middle, the momentum structure is depicted, where the highly excited string with momentum p_1 decays into tachyons with momenta p_2 and p_3 . As time progresses, the correlations among the decay products become suppressed. On the right, the angles between different momentum vectors are shown, as the amplitude depends on $\alpha = \theta - \beta$. The pattern of entanglement is also illustrated [17].

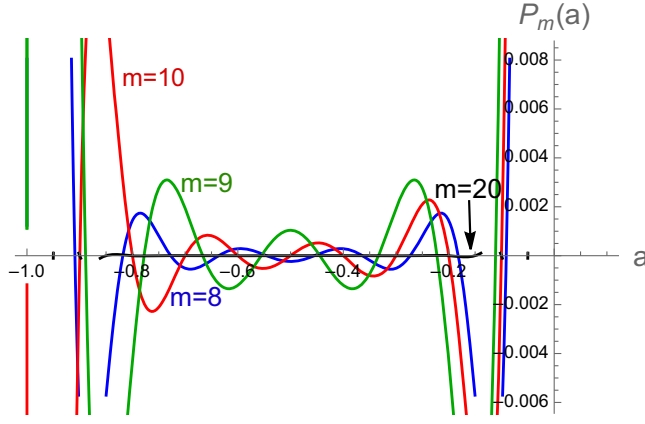


FIG. 23. The function $P_m(a)$, defined via the Pochhammer symbol $(a)_m$, for various values of m .

At early stages, due to the production of tachyons, the mutual information increases. Over time, as the particles separate and lose coherence, mutual information decreases due to decoherence effects and causal separation.

As the string modes n_m increase, the scattering amplitude and mutual information are also expected to increase. This is because, in a spacetime constructed from more entangled subsystems, the excited strings interact more easily with photons carrying momenta $-mq$, or with the null vacuum (vector) q . Therefore, one would expect a power-law relationship between the mutual information I and the string modes n_m as well.

Another interesting insight one can get from this formulation is on the source of the oscillatory behaviors in the mutual information. Based on the relation with scattering amplitude, this is rooted in the presence of the Pochhammer function of the Pochhammer symbol $(a)_m$, whose behavior is shown in Fig. 23. This function shows oscillatory behavior within the range of $-1 < a < 0$. As the parameter m increases, the oscillations become more pronounced, even though the peaks of the amplitude suppress. Additionally, for larger values of m , the function has more zeros and the amplitude is influenced by $P_m(a)^{n_m}$. For large m , this amplitude can also be estimated by a product of several sine functions, resulting in highly erratic behavior. As shown in [17], and by simplifying the Pochhammer function $\mathcal{A} \propto \prod_m (\sin(\pi m p_3 \cdot q))^{n_m}$.

In the limit of large m , which corresponds to larger or more excited strings, a greater number of modes n_m are stimulated, which in the confining case corresponds to bigger values of u_{KK} . Therefore, one would expect that at larger values of u_{KK} , the behavior of mutual information or D_c becomes more chaotic, as observed in Fig. 3. One could also expect the following approximate relation for the mutual information: $I \propto \prod_{q_b=1}^{q_l} P_{q_b}^{n_{q_b}}$, where q_b is the number of quantum bits correlated in the mixed system and q_l denotes the maximum number of qubits that can be incorporated into the subsystems. Therefore, mutual

information is proportional to the Pochhammer symbol raised to the power of the number of qubits in the system.

By any change in the angle of the outgoing tachyons θ , or $\alpha = \theta - \beta$, or the ingoing excited string β , the amplitude would show erratic behavior. In the confining case, changing u_{KK} would also change significantly the angle of the hitting strings to the wall, or the angle of two subsystems relative to each other and how they “see” each other [35,38]. Also, both the string amplitude and the mutual information would be sensitive to the occupation levels of the propagating strings.

Another connection between entanglement entropy or mutual information and scattering amplitude can be derived from the results of [87], where the authors first showed the relation

$$\Delta\mathcal{E}[\Omega^{\text{prod}}] = 4 \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right) \text{Im} \mathcal{A}_{\alpha\beta}^{\alpha\beta}(k_1 k_2 \rightarrow k_1 k_2), \quad (6.3)$$

where $\Delta\mathcal{E}$ is the change of entanglement of two scalars generated by scattering and can be related to mutual information. Also, $|\Omega^{\text{prod}}\rangle$ is the product of unentangled initial state, and $\mathcal{E}[|\Omega\rangle]$ is the linearized entropy, defined as $\mathcal{E}[|\Omega\rangle] = 1 - \text{Tr}_A[\tilde{\rho}^2]$. Additionally, $V \equiv (2\pi)^3 \delta_V^3(0)$ and $T \equiv 2\pi \delta_T(0)$ represent divergences in the spacetime volume. In confining geometries, this factor is directly related to the position of the wall u_{KK} . The amplitude here is elastic in the internal quantum numbers. Hence, from the above relation, it is again expected to see similar patterns of zeros and poles in both string scattering and mutual information, as noted in the first section.

Then, using the above relation, it has been demonstrated that the positivity of the string scattering amplitude is directly equivalent to the positivity of entropy, expressed as

$$\text{Im} \mathcal{A} > 0 \Leftrightarrow \mathcal{E}[|\Omega^{\text{prod}}\rangle] \geq 0, \quad (6.4)$$

Therefore, one could imagine that further information from quantum information theory, such as various inequalities—specifically in different backgrounds like confining geometries—could provide additional constraints and insights on scattering amplitudes.

VII. KINK SCATTERING, CHAOS, AND FRACTAL STRUCTURE

As another illustration of the connections between patterns of entanglement and mutual information, chaos, and scattering, we consider the case of kink-antikink and kink-kink scattering processes for solitons and compactons, which can model the exchange of information between subsystems and the delocalization of localized information. Depending on the system parameters, there could be a phase with creation of new excitations such as breathers, which is analogous to a generation of entanglement, or a phase with separated kinks and antikinks corresponding to

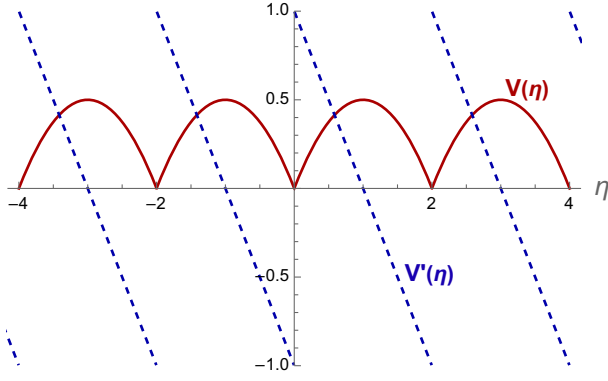


FIG. 24. The potential of a topological kink solution or a solitonic Skyrme model.

the decay phase. There is also a phase shift in the kink-antikink scattering which mirrors how mutual information can retain memory of past interactions in a quantum system, while both have nonlinear dynamics with complex evolution in a nonadditive way.

Also, in both mutual information and kink-antikink scattering, there is a threshold effect as the kinks and antikinks can only interact above a certain energy corresponding to a certain critical velocity. Around this critical velocity the behavior becomes fractal [44]. This critical velocity corresponds to the specific u_{KK} where the plots of D_c in a confining background with an end wall show the fractal behavior as well. This can be seen in our Fig. 4, where the fractal structures emerge around $u_{KK} \simeq 0.39$.

As an example of scattering in a topological kink solution, one first could consider a Heaviside-structured potential, similar to the one considered in [44] as

$$V(\eta) = \sum_{n=-\infty}^{\infty} \left(|\eta - 4n| - \frac{1}{2}(\eta - 4n)^2 \right) H_n(\eta),$$

$$H_n(\eta) := \theta(\eta - 4n + 2)\theta(\eta - 4n - 2), \quad (7.1)$$

where its behavior and its derivative are shown in Fig. 24. The confining solitonic Skyrme model can be considered as an example of having such potential. Here, η is a real scalar field in $1+1$ dimension with the action

$$S = \int dt dx \left[\frac{1}{2}(\partial_t \eta)^2 - \frac{1}{2}(\partial_x \eta)^2 - V(\eta) \right], \quad (7.2)$$

and with the BPS equation of $\frac{d\eta}{dx} = \pm \sqrt{2V(\eta)}$.

Note that this V-shaped potential is suitable for our analogy with QCD backgrounds, since this potential is also confining and can localize particles in certain regions. If the height of this potential increases, due to its specific shape, the subsystems become more spatially separated, and therefore the mutual information decreases. Also, this V-type potential induces more

decoherence and thermal fluctuations, which further reduce the mutual information.

Note that the most radiative case is shown to be the most chaotic. In chaotic dynamics, the correlations spread and get scrambled across the system, increasing the mutual information. So, as the energy levels $\delta_n = E_{n+1} - E_n$ or their ratios of successive spacing, and also the successive peaks of the scattering amplitude $\mathcal{A}(\alpha)$ as a function of the continuous kinematical variable α in the scattering problem can act as diagnostics of chaos, by analogy, the successive peaks of mutual information should play the same role as well. However, certain behaviors of the system depend on the shape of the potential—for instance, the specific model of a holographic QCD setup.

Since certain phenomenological and holographic QCD models, such as V_{QCD} , (see Ref. [88,89]), have shapes similar to these V-shape potentials in certain regions, and as shown in [44], using the collective coordinate method, one can similarly model mutual information in those systems using an internal mode as

$$I_s = I_k \left(b(x - a) + \frac{\pi}{2} \right) + c\chi(b(x - a)), \quad (7.3)$$

where, similar to the kink model, the internal mode can be written as

$$\chi(x) = \begin{cases} \sin(2x) + \frac{1}{2}\sin(4x) & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (7.4)$$

Here, a is the position of the center of momentum of the kink, b is a new coordinate to model the Lorentz contraction, and c is the amplitude of the internal mode, which can act as an additional coordinate. So, one expects that adding this new internal coordinate, such as Derrick mode, can help to model the spread of mutual information in various confining models with potentials that have similarities with the V-shaped model here. It has been shown that in models with nonanalytical potentials, the radiation is dominated by compact oscillons. Therefore, we could claim that in our setup, the radiation and the spread of mutual information in the confining backgrounds, which have nontrivial potentials, would be dominated by the oscillons as well. In fact, taking into account the causality and light-cone structures for the spread of information from each qubit in each of the subsystems, a structure similar to Fig. 25 could be imagined, which is similar to the shockwaves of [44].

In [44], it has been observed from their Figs. 4 and 5 that the regions with most emission of radiation match the regions of chaotic behavior. These are two separate regions: one at smaller velocity v with a smaller peak, and the other larger one at higher v with a bigger peak. These two plots match the behavior of mutual information or D_c , specifically in the case of the Sakai–Sugimoto model.

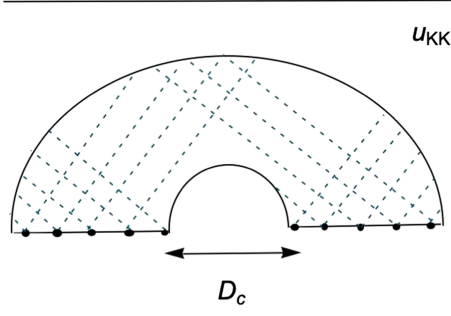


FIG. 25. Modeling the spread of mutual information between two mixed subsystems by kink-antikink scattering, by analogy with causality cones. This could point to the use of internal modes to describe the chaotic regions in our plots at specific parameters.

Generally, in the interactions between the compact topological defects—which can approximately model the spread of information in confining geometries with an end wall—two main classes of scattering can be identified. These are: the kink-antikink pairs that oscillate and radiate as shown in Fig. 26, and kink-antikink pairs that themselves emerge from collisions where a shockwave is formed, which then decays into a “cascade of compact oscillons” [44]. These are, in fact, the zeros that the authors of [45,74] observed, demonstrating chaos in the scattering process.

So, if the position of the wall, temperature, energy, or other parameters of the QCD system are perturbed, one could expect that oscillons would be emitted in a similarly chaotic pattern. This would also resemble the scattering of kink-antikink pairs or the scattering of compact oscillons in the signum-Gordon model. In our setup (Fig. 1), the shockwave borders also act as energy or information reservoirs.

The periodic relations between the kink-antikink parameters λ_1 and λ_2 and their velocities can be captured from the relation

$$u(x, t) = 4 \arctan \left[\left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \tan \left(\frac{v_1 - v_2}{4} \right) \right], \quad (7.5)$$

where from the tangent function, the periodic structure could be detected. Periodic structures in the zeros of the string scattering amplitude appear when the highly excited string states involved are nongeneric, as shown in Fig. 27, such as the states in the first Regge trajectory.

An interesting point is that higher scattering velocities result in smaller shockwaves and lower radiation intensities, while slower velocities produce the opposite effect. The oscillating behavior is only observable for velocities below a critical threshold, $v < v_c$. This explains the noise in the numerical studies of mutual information when the graviton is considered to be massive and the scattering velocities are suppressed. Increased noise also appears in the plots of entanglement entropy, complexity, and complexity of purification [39]. The critical velocity, v_c , depends on various parameters of the model, such as the graviton mass m , particle charge q , background dimensionality d , and the position of the wall u_{KK} . This critical velocity is the initial velocity of kinks where a single-bounce scattering changes to multibounce or annihilation.

Additionally, as mentioned in [44], close to the critical velocity v_c , the escape and no-escape cases alternate in a fractal manner, corresponding to the fractal structures around D_c , where the mutual information between the subsystems drops to zero. The “collective coordinate model” could be applied to describe all these collision

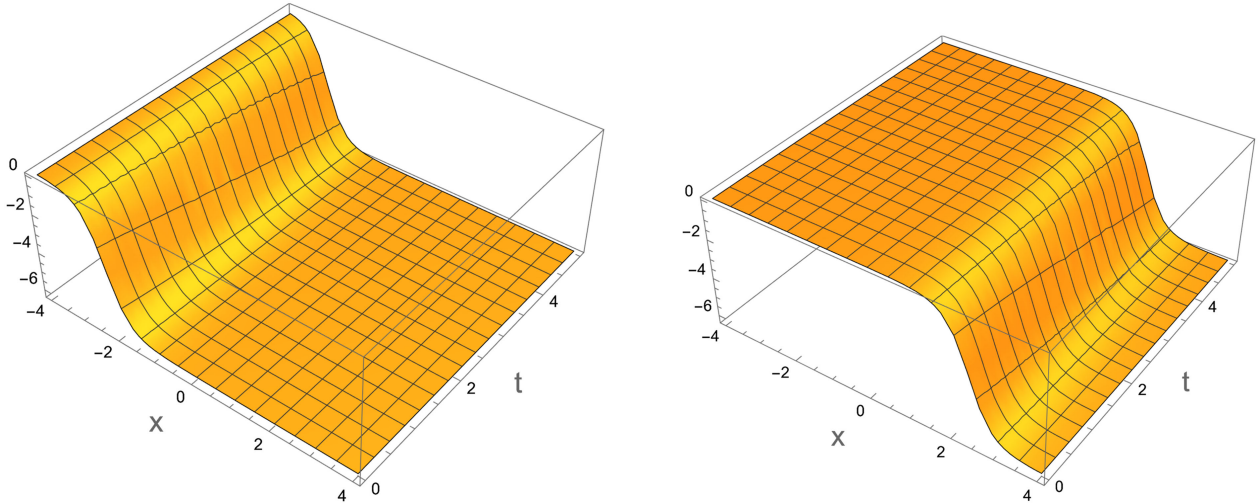


FIG. 26. The kink solution, given by the equation $u(x, t) = -4 \arctan(\exp(\lambda x + \frac{t}{\lambda} + \mu))$ with $\lambda > 0$, is shown on the left, while the anti-kink solution, with $\lambda < 0$ is shown on the right. The interaction and scattering of kink-antikink pairs can be compared to the spread of mutual information in confining models, where in both cases, critical phase transitions occur, and fractal structures emerge around them. The appearance of periodic behaviors within these fractallike structures has been observed in both cases.

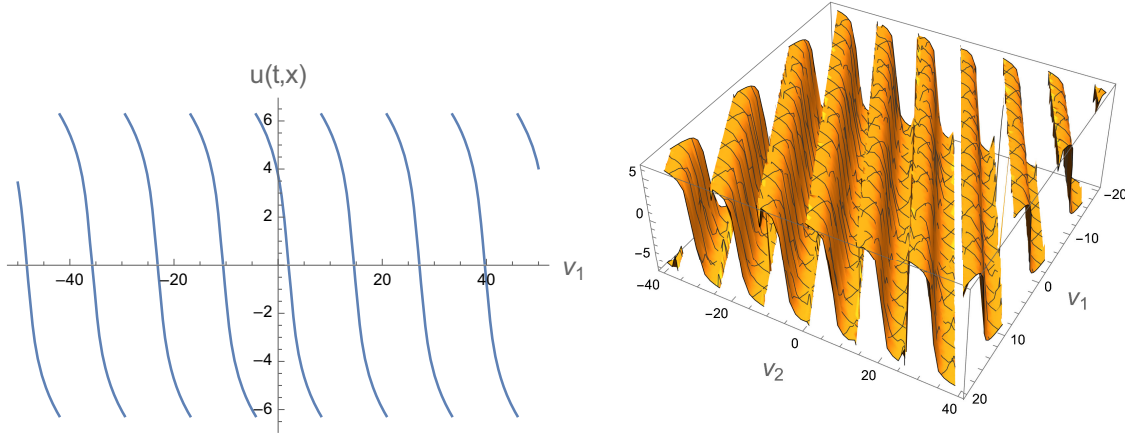


FIG. 27. The periodic structure in the behavior of kink and anti-kink velocities is shown, which can help model the periodic structures in the behavior of mutual information or D_c , as well as in the string scattering amplitude.

processes and also the structures of quantum information and entanglement.

VIII. CONCLUSION

In this work, we first showed that in holographic confining backgrounds, and at low Mandelstam variable s , there are connections between the logarithmic branch cut singularities of the closed string scattering amplitude and the behavior of the peaks of mutual information or critical distance D_c . These peaks result from the dependence of the string tension on the holographic radial coordinate and are a sign of the creation of bound states at low energies. The statistics of these peaks demonstrate chaos in both cases. One can imagine that the two mixed subregions spread quantum information across the spacetime in “modular time.” As the modular wavelets interfere with each other and reflect off the wall, they create the chaotic structures. The power-law decay of both quantities matches at higher energies. They also match based on the radial coordinate, as the real part of both follows the s^{-1} curve. These observations further reinforce the connections between the patterns of entanglement entropy and the string scattering amplitude.

To explain specific features of our results and elucidate the behavior of entanglement patterns, we implemented the modular flow and modular Hamiltonian as the foundation for introducing the “modular time.” Then, we drew connections between the scattering amplitude, binding energy, and the spread of mutual information in these setups. Specifically, we reviewed the recent developments in defining spacetime generalizations of mutual information, where again the same logarithmic branch cut behaviors—both in analytical and numerical studies—have been observed, and we connected these new concepts with the branch cut behaviors found in the string scattering

amplitude. These results could be helpful for using the spread of quantum information to analyze the behavior of high-energy strings.

Additionally, we discussed how mutual information in a setup with an end wall, similar to tripartite information, can detect chaos. The leaky torus model has also been used to simulate the spread of quantum information and the emergence of chaos in different confining backgrounds, which have an end wall in their IR that affects the chaotic structures.

In addition, the exchange of entanglement entropy and mutual information during Compton scattering of two photons, as well as the decay of a highly excited string into two tachyons, were explored, and the source of the periodic structures were explained. Finally, the scattering of kink-antikinks, the emergence of chaos in this scenario, and the links with the behaviors noted in previous sections were discussed.

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DATA AVAILABILITY

No data are associated with this manuscript.

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