

Dynamics of black hole in dark matter halo: Quasinormal modes

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In this paper, we explore the scalar, electromagnetic, and gravitational perturbations of a Schwarzschild black hole embedded within a Dehnen-(1, 4, γ) dark matter halo. We focus on three distinct values of γ , each yielding a unique central behavior for the dark matter distribution: $\gamma = 1$, where the central density vanishes; $\gamma = 0$, where it remains finite; and $\gamma = 1$, where it diverges. Employing higher-order Wentzel-Kramers-Brillouin and asymptotic iteration methods, we calculate the characteristic oscillation frequencies, namely, the quasinormal modes and present their temporal evolution using a time-domain approach. Our findings indicate that dark matter amplifies the gravitational pull of the black hole. While it reduces the actual oscillation frequencies of the perturbations, its impact on the damping rates of the fundamental modes is negligible. However, in the nonfundamental modes, both the real and imaginary parts of the quasinormal frequencies exhibit a pronounced decay.

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I. INTRODUCTION

Although our theoretical studies treat black holes as isolated entities, in reality, they continuously interact with surrounding baryonic matter. Additionally, observations reveal the presence of nonbaryonic matter which is not made up of quarks or baryons, such as dark matter, which constitutes approximately 27% of the Universe's total mass-energy content and accounts for about 85% of all matter [1]. Dark matter is one of the most intriguing and mysterious components of the Universe and plays a crucial role in the structure and evolution of the cosmos [2]. Since it does not emit, absorb, or reflect light, it is invisible. Consequently, it is observable solely through its gravitational interactions, such as the galaxy rotation curve, where stellar velocities within galaxies appear inconsistent with the predictions based solely on visible matter [3]. Another indication is gravitational lensing, where the lensing effects observed in galaxy clusters diverge from those predicted by

general relativity [4]. Furthermore, the large-scale structure of the Universe, as indicated by the distribution of galaxies and galaxy clusters, strongly suggests the presence of dark matter [5]. In the absence of dark matter, the large-scale structure observed in the Universe today would not have developed in its current form. Thus, dark matter serves as the gravitational infrastructure essential for the formation of galaxies and galaxy clusters. Dark matter is often referred to as cold dark matter, because it moves slowly compared to the speed of light [6]. Despite the existence of a bunch of observational data indicating the existence of dark matter, the exact nature of dark matter remains unknown. Given the limited number of known physical properties, numerous theoretical models of dark matter have been developed. One such model is the weakly interacting massive particle model, which postulates dark matter as a hypothetical particle interacting through gravitational and the weak nuclear forces [7]. There are also other models that postulate dark matter as hypothetical particles, such as axions [8], sterile neutrinos [9,10].

One more important aspect that interests the entire scientific community is how dark matter behaves under strong gravitational field regime, especially in the Galactic Center where it is occupied by the supermassive black hole. In most models, dark matter is considered to form halos around black holes and has a strong influence on their dynamics and growth [11,12]. Studying black hole-dark matter systems would help to investigate the distribution and density of dark matter [13]. By using observational data

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of black holes, one can constrain theoretical models of dark matter, such as the presence or absence of dark matter spikes, which can test predictions of cold dark matter models [14]. Recent breakthrough events such as the detection of gravitational waves by LIGO and Virgo scientific collaboration [15,16] and the first ever image of the supermassive black holes at the centers of the Milky Way [17] and M87 [18] galaxies captured by the Event Horizon Telescope project may open new windows to explore the nature of dark matter. The dynamics of shadows and particles around black holes immersed in a dark matter medium have been explored across various dark matter distributions and black hole solutions [19–21]. Conversely, the coalescence of black holes has the potential to disturb the surrounding dark matter, thereby potentially imprinting signatures onto gravitational wave signals. Analyzing these phenomena can yield indirect evidence of dark matter [22]. As a result, the study of gravitational waves emitted by black holes emerges as a critical tool for detecting possible dark matter imprints. The ringdown, which represents the final phase in the coalescence of black holes, is analyzed within the framework of perturbation theory. Therefore, quasinormal modes which are the characteristic frequencies of perturbations associated with diverse black hole types and dark matter distributions have been under great interest [23–27].

In this paper, the gravitational, scalar, and electromagnetic perturbations of a Schwarzschild black hole embedded within a Dehnen-(1, 4, γ) dark matter halo are examined, and the effects of the dark matter halo core radius and the nature of the dark matter distribution on the quasinormal frequencies of these perturbations are investigated. The paper is organized as follows. In Sec. II, the derivations of the solution for a Schwarzschild black hole embedded within a dark matter halo are presented. In Sec. III, the equations of motion for gravitational, electromagnetic, and scalar perturbations, along with their respective effective potentials, are derived in separate subsections. Subsequently, in Sec. IV, the formalisms of the WKB (Wentzel-Kramers-Brillouin) approximation, asymptotic iteration, and physics-informed neural networks methods for calculating the quasinormal frequencies of these perturbations are outlined. In Sec. V, the numerical values of the quasinormal modes, calculated using the methods outlined in the preceding section, are presented, and their additional characteristics are discussed. Finally, in Sec. VI, the primary findings obtained throughout the paper are summarized. Throughout the paper, we adopt the metric signature $(-, +, +, +)$ and set the speed of light and the Newtonian gravitational constant equal to unity, $c = G = 1$.

II. BACKGROUND

In this paper, we employ the Dehnen-(α, β, γ) type dark matter distribution, which is expressed as [28,29]

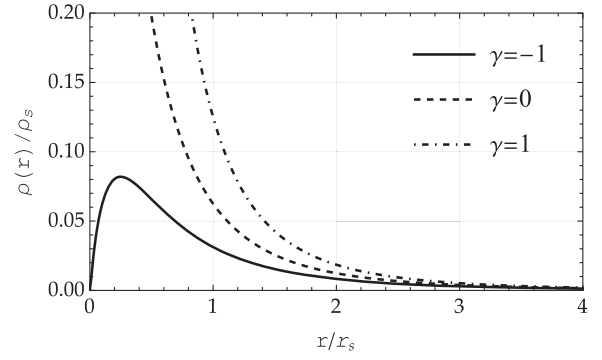


FIG. 1. The radial profiles of the dark matter density for selected three distinct values of γ . Credit to Ref. [30] for the picture.

$$\rho(r) = \rho_s \left(\frac{r_s}{r} \right)^\gamma \left[1 + \left(\frac{r}{r_s} \right)^\alpha \right]^{(\gamma-\beta)/\alpha}, \quad (1)$$

where ρ_s and r_s represent the central halo density and the core radius of the dark matter halo, respectively. In this paper, we specifically focus on the model (1, 4, γ), and the value of the γ is specified later. The dark matter density profile diminishes as $1/r^4$ at large distances, irrespective of the γ value. It should be noted that, according to [28], γ must fall within the interval $\gamma \in [0, 3)$ to ensure the density diverges as $1/r^4$ at the spacetime's center. In this study, however, we deliberately step outside this prescribed range to investigate the impact of central density behavior on the surrounding dynamics. For this purpose, we adopt three specific γ values: $\gamma = -1$, where the central density tends to zero; $\gamma = 0$, where it stays finite; and $\gamma = 1$ (similar to a Hernquist profile), where it becomes infinite, as illustrated in Fig. 1. These three aforementioned dark matter density distributions are given below,

$$\rho(r) = \rho_s \frac{r}{r_s} \left(1 + \frac{r}{r_s} \right)^{-5}, \quad \text{for } \gamma = -1, \quad (2)$$

$$\rho(r) = \rho_s \left(1 + \frac{r}{r_s} \right)^{-4}, \quad \text{for } \gamma = 0, \quad (3)$$

$$\rho(r) = \rho_s \frac{r_s}{r} \left(1 + \frac{r}{r_s} \right)^{-3}, \quad \text{for } \gamma = 1. \quad (4)$$

Given that the derivations of the relativistic solutions for the Schwarzschild black hole embedded within a dark matter halo, characterized by density functions (2)–(4), have been presented in our previous work [30], they are not repeated here with great detail. The formalism underlying these solutions can be found in [31,32]. Instead, a concise overview of key aspects is provided for the case of $\gamma = 0$ corresponding to the density profile given by (3).

It was assumed that the pure dark matter has a spherically symmetric spacetime given by

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (5)$$

By equalizing the tangential velocity of the test particle given by

$$v_{\text{ig}}^2(r) = r \frac{d \ln \sqrt{F(r)}}{dr}, \quad (6)$$

to the Keplerian velocity $v_{\text{ig}}^2 = M_{\text{DM}}/r$, as indicated in $M_{\text{DM}}(r) = 4\pi \int_0^r r'^2 \rho(r') dr'$, one can find the explicit form of the metric function $F(r)$ as

$$F(r) = \exp \left(-\frac{4\pi\rho_s r_s^3 (r_s + 2r)}{3(r_s + r)^2} \right). \quad (7)$$

If the core of the dark matter halo is small, the metric function (7) can be written as

$$F(r) \approx 1 - \frac{4\pi\rho_s r_s^3 (r_s + 2r)}{3(r_s + r)^2}. \quad (8)$$

Assuming that the spacetime (5) is a solution of the Einstein field equations, $G_\mu^\nu = \kappa(T_\mu^\nu)_{\text{DM}}$, we can find the energy-momentum tensor of the dark matter in terms of the Einstein tensor as

$$\begin{aligned} \kappa(T_t^t)_{\text{DM}} &= \kappa(T_r^r)_{\text{DM}} = \frac{F'(r)}{r} + \frac{F(r)}{r^2} - \frac{1}{r^2}, \\ \kappa(T_\theta^\theta)_{\text{DM}} &= \kappa(T_\phi^\phi)_{\text{DM}} = \frac{F''(r)}{2} + \frac{F'(r)}{r}. \end{aligned} \quad (9)$$

Thus, we have derived the pure dark matter spacetime characterized by the energy-momentum tensor (9). The next step is to construct a spherically symmetric black hole of mass M within general relativity, embedded in the previously established dark matter distribution. Let us assume the spacetime of this system is in the following form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (10)$$

The metric functions of (10) are associated with the ones of (5) through the relations

$$f(r) = F(r) + F_1(r), \quad g(r) = F(r) + F_2(r). \quad (11)$$

By solving the Einstein field equations for the geometry (10) with the energy-momentum tensor (9), we obtain the following expressions for the unknown metric functions $F_1(r)$ and $F_2(r)$:

$$\begin{aligned} F_1(r) &= F_2(r) = -\frac{2M}{r}, \\ f(r) &= g(r) = F(r) + F_1(r). \end{aligned} \quad (12)$$

Thus, for the dark matter density profile $\gamma = 0$, the metric function takes the following form:

$$f(r) = \exp \left(-\frac{4\pi\rho_s r_s^3 (r_s + 2r)}{3(r_s + r)^2} \right) - \frac{2M}{r}. \quad (13)$$

By repeating similar calculations for the values $\gamma = 1$ and $\gamma = 1$, we obtain the following solutions, respectively:

$$f(r) = \exp \left(-\frac{2\pi\rho_s r_s^3 (3r^2 + 3rr_s + r_s^2)}{3(r_s + r)^3} \right) - \frac{2M}{r}, \quad (14)$$

$$f(r) = \exp \left(-\frac{4\pi\rho_s r_s^3}{r_s + r} \right) - \frac{2M}{r}. \quad (15)$$

The main characteristic of the black hole is its event horizon radius that represents a boundary beyond which even light cannot escape the gravitational pull of the black hole. The event horizon radius is determined by the divergence of the spacetime line element, specifically through the radial metric tensor, which is in our case $f(r) = 0$. In the context of a black hole with the dark matter models (2)–(4), the event horizon radius of the black hole cannot be found analytically. Therefore, we illustrate the dependence of the exact event horizon radii on the dark matter halo core parameters for each of these three type of black holes in Fig. 2.

III. PERTURBATIONS

In this section, we examine the black hole's response within a dark matter halo to gravitational perturbations. Furthermore, we analyze the behavior of scalar and electromagnetic configurations in the field of the black hole when the perturbations were induced. Below we consider these three distinct perturbations separately.

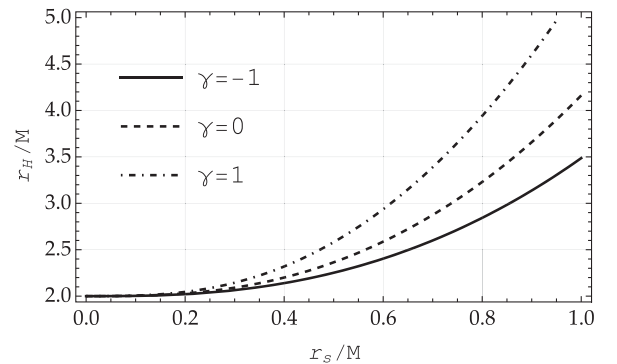


FIG. 2. The radial profiles of the dark matter density for $\rho_s = 0.5$ and selected three distinct values of γ .

A. Axial gravitational perturbations

In this subsection, we study axial (odd-parity) gravitational perturbation of the black holes immersed in the dark matter halo with the density profiles given by (2)–(4) with line element (10). We denote the background metric tensor $\bar{g}_{\mu\nu}$ and the linear perturbations $h_{\mu\nu}$ which is, in general, a function of space and time, and consequently, the final metric tensor can be written as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ ($|\bar{g}_{\mu\nu}| \gg |h_{\mu\nu}|$). Due to the spherical symmetry of the spacetime metric, the gravitational perturbations can be decomposed into tensor spherical harmonics and separated into odd (axial) and even (polar) modes in terms of the parity as $h_{\mu\nu} = h_{\mu\nu}^{\text{ax}} + h_{\mu\nu}^{\text{pol}}$. The odd parity perturbations, $h_{\mu\nu}^{\text{odd}}$, in the Regge-Wheeler gauge [33] are written in terms of two perturbation functions $h_0(r)$ and $h_1(r)$ as the following:

$$h_{\mu\nu}^{\text{ax}} = \begin{pmatrix} 0 & 0 & 0 & h_0(t, r) \\ * & 0 & 0 & h_1(t, r) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \sin \theta \partial_\theta P_\ell(\cos \theta), \quad (16)$$

where $*$ denotes the symmetric components, and $P_\ell(\cos \theta)$ is the Legendre polynomials with order ℓ and $m = 0$ for the spherical symmetry. As we have shown in Sec. II, the line element of the black hole in dark matter halo (10) is a solution of the Einstein equations with the energy-momentum tensor (9) as $G_{\mu\nu} = \kappa(T_{\mu\nu})_{\text{DM}}$.

The gravitational perturbations of the system are determined by solving the Einstein equations,

$$\delta G_{\mu\nu} = \delta(\kappa T_{\mu\nu})_{\text{DM}}, \quad (17)$$

with respect to the perturbations functions. In certain literature, such as [34–37], the perturbation of dark matter is often neglected by assuming that axial gravitational perturbations are inherently decoupled from any scalar field. However, in our calculations, we explicitly account for these perturbations. To demonstrate the contribution of dark matter perturbations, we introduce a constant coefficient ξ into the energy-momentum tensor of the dark matter as

$$\delta(\kappa T_{\mu\nu})_{\text{DM}} = \begin{pmatrix} 0 & 0 & 0 & \xi h_0(t, r)(T_\phi^\phi)_{\text{DM}} \\ * & 0 & 0 & \xi h_1(t, r)(T_\phi^\phi)_{\text{DM}} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \times \sin \theta \partial_\theta P_\ell(\cos \theta). \quad (18)$$

If the coefficient $\xi = 0$, it corresponds to the scenario where the perturbation of the dark matter is neglected, while $\xi = 1$ indicates that the perturbation of the dark matter is taken into account. In terms of the perturbed metric tensor and energy-momentum tensor, the Einstein equations (17) provide three independent following differential equations:

$$(t, \phi): \partial_t^2 h_1 - \partial_r^2 h_0 + \frac{2}{r} \partial_t h_1 + \frac{r(rf'' + 2f' - \xi rF'' - 2\xi F') + 2f + L^2 - 2}{r^2 f} h_0 = 0, \quad (19)$$

$$(r, \phi): \partial_t^2 h_1 - \partial_r^2 h_0 + \frac{2f^2}{r} \partial_r h_1 + f \frac{r[r(f'' - \xi F'')] + 4f' - 2\xi F'] + L^2 - 2}{r^2} h_1 = 0, \quad (20)$$

$$(\theta, \phi): \partial_t h_0 = f \partial_r (f h_1), \quad (21)$$

where L^2 is the square of the angular momentum operator whose explicit form is given by

$$L^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right). \quad (22)$$

The eigenvalue of the square of the angular momentum operator is found through the Legendre equation as

$$L^2 P_\ell(\cos \theta) = \ell(\ell + 1) P_\ell(\cos \theta), \quad (23)$$

where the value of the multipole number is within the range of $\ell = 2, 3, 4, \dots$. By differentiating (21) with respect

to the radial coordinate and eliminating the h_0 term from (20), we arrive at the following second order differential equation:

$$-\frac{\partial_t^2 h_1}{f^2} + \partial_r^2 h_1 + \left(\frac{3f'}{f} - \frac{2}{r} \right) \partial_r h_1 + \left[\frac{f'^2}{f^2} - \frac{4rf' - \xi r(rF'' + 2F') + L^2 - 2}{r^2 f} \right] h_1 = 0. \quad (24)$$

By introducing the “tortoise” (Regge-Wheeler) coordinate, r_* , defined by $dr_* = dr/f(r)$ and a new function

$$\Psi_{\text{gr}}(t, r) = \frac{f(r)}{r} h_1(t, r), \quad (25)$$

Eq. (24) can be reformulated into the following wave equation form:

$$-\frac{\partial^2 \Psi_{\text{gr}}}{\partial t^2} + \frac{\partial^2 \Psi_{\text{gr}}}{\partial r_*^2} - V_{\text{gr}} \Psi_{\text{gr}} = 0, \quad (26)$$

where V is the effective potential that for the gravitational perturbation reads as

$$V_{\text{gr}} = f \left[\frac{(\ell+2)(\ell-1)}{r^2} + \frac{r^2 f'' + r f' + 2f}{r^2} - \xi \frac{r F'' + 2F'}{r} \right]. \quad (27)$$

It is important to note that the tortoise coordinate ranges from negative infinity to positive infinity in relation to the radial coordinate. Here, negative infinity corresponds to the radius of the event horizon, while positive infinity aligns with the positive infinity of the radial coordinate. Thus, it considers only the outside of the black hole.

B. Scalar perturbation

We here consider the perturbation of the massless scalar field in the field of the black hole in the dark matter halo (10). The dynamics of the scalar field, $\Phi(t, r, \theta, \phi)$, is governed by the Klein-Gordon equation which is given by $\square \Phi(t, r, \theta, \phi) = 0$. When the perturbation is induced to the scalar field as $\Phi(t, r, \theta, \phi) + \delta \Phi(t, r, \theta, \phi)$, the perturbation term $\delta \Phi(t, r, \theta, \phi)$ must also satisfy the Klein-Gordon equation whose covariant form is given by

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \delta \Phi(t, r, \theta, \phi) = 0. \quad (28)$$

In order to separate the angular components within the Klein-Gordon equation (28), the scalar field is expressed in terms of the spherical harmonics as

$$\delta \Phi(t, r, \theta, \phi) = \frac{\Psi_{\text{sc}}(t, r)}{r} Y_\ell(\theta, \phi), \quad (29)$$

and derive the subsequent equation for the function $\Psi_{\text{sc}}(t, r)$ as presented below,

$$-\frac{\partial_t^2 \Psi_{\text{sc}}}{f^2} + \partial_r^2 \Psi_{\text{sc}} + \frac{f'}{f} \partial_r \Psi_{\text{sc}} - \left[\frac{\ell(\ell+1)}{r^2 f} + \frac{f'}{r f} \right] \Psi_{\text{sc}} = 0. \quad (30)$$

By substituting the radial coordinate with the Regge-Wheeler coordinates, r_* , the Klein-Gordon equation (30) can be reformulated into the master wave equation form (26), with the following effective potential:

$$V_{\text{sc}} = f \left[\frac{\ell(\ell+1)}{r^2} + \frac{f'}{r} \right]. \quad (31)$$

Since the scalar field has a monopole configuration, the multipole starts with $\ell = 0, 1, 2, \dots$

C. Electromagnetic perturbations

In this subsection, we derive the equations of motion for linear perturbations of the electromagnetic field of the black holes in the dark matter halo (10). Our main goal is to obtain the master equation in the form of the master wave equation (26) with the effective potential. In order to study the dynamics of electromagnetic perturbation, we introduce the following linear perturbation into the 4-potential of the electromagnetic field, by assuming that these perturbations do not alter the spacetime geometry,

$$A_\mu = \bar{A}_\mu + \delta A_\mu, \quad (32)$$

where the perturbation, δA_μ , is a function of the spacetime coordinates. It is well-known that the electromagnetic perturbation can be decomposed into tensor spherical harmonics and be separated into odd (axial), δA_μ^- , and even (polar), δA_μ^+ , modes in terms of the parity as

$$\delta A_\mu = \delta A_\mu^{\text{odd}} + \delta A_\mu^{\text{even}}. \quad (33)$$

As it has already been demonstrated in [38–42] that if the background spacetime is not associated with the nonlinear electrodynamics, the equations of motion for both polar and axial electromagnetic perturbations are equivalent and yield identical effective potentials, in concordance with the findings of [43,44]. Therefore, we here consider only axial electromagnetic perturbation which is given by

$$\delta A_\mu^{\text{odd}} = \begin{pmatrix} 0 \\ 0 \\ \Psi_{\text{em}}(t, r) \partial_\phi Y_\ell(\theta, \phi) / \sin \theta \\ -\Psi_{\text{em}}(t, r) \sin \theta \partial_\theta Y_\ell(\theta, \phi) \end{pmatrix}. \quad (34)$$

The electromagnetic field in the curved spacetime is governed by the Maxwell equations,

$$\nabla_\nu F^{\mu\nu} = 0, \quad (35)$$

where $F_{\mu\nu}$ is the electrodynamic field tensor that can be written in terms of the electromagnetic potential as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By simplifying the Maxwell equations in linear orders of the perturbation function, we obtain the following differential equation:

$$-\frac{\partial_t^2 \Psi_{\text{em}}}{f^2} + \partial_r^2 \Psi_{\text{em}} + \frac{f'}{f} \partial_r \Psi_{\text{em}} - \frac{\ell(\ell+1)}{r^2 f} \Psi_{\text{em}} = 0. \quad (36)$$

By substituting the radial coordinate with the Regge-Wheeler coordinates, r_* , the Maxwell equation (36) can be reformulated into the master wave equation form (26), with the following effective potential:

$$V_{\text{em}} = f \frac{\ell(\ell+1)}{r^2}. \quad (37)$$

Since the electromagnetic field has a vector nature and it is the fundamentally dipole field, the multipole starts with $\ell = 1, 2, 3, \dots$

As demonstrated across these three distinct types of perturbations, each can be expressed in the form of the master wave equation (26) with varying scattering potentials. When the perturbations are treated as harmonically time dependent as

$$\Psi_k(t, r) = \psi_k(r) e^{-i\omega_k t}, \quad (38)$$

the master wave equation (26) can be rewritten as the Schrödinger-like equation with scattering potential as

$$\partial_{r_*}^2 \psi_k + (\omega_k^2 - V_k) \psi_k = 0, \quad (39)$$

where a subscript k refers the type of perturbations as $\{k = \text{gr, sc, em}\}$, and ω is referred to as the quasinormal frequency, as it is represented by complex numbers. The full solutions to the perturbation equations, which incorporate these frequencies, are known as quasinormal modes. Given the explicit forms of the scattering potentials for the perturbations, we can analyze their shapes and dependence on spacetime parameters. The asymptotic flatness of the spacetime ensures that the effective potentials tend to zero at infinity and vanish at the event horizon. Additionally, the height of the effective potentials increases with the multipole number. Figure 3 illustrates further characteristics of these potentials, revealing that they act as potential barriers whose height decreases as the dark matter core radius

increases. Furthermore, the effective potential for the scalar perturbation has the largest value, while that of the gravitational perturbation has the smallest height.

IV. QUASINORMAL MODES

In this section, we focus on calculating the quasinormal frequencies for gravitational, scalar, and electromagnetic perturbations. Since the frequency ω is a complex number with both real and imaginary parts, its computation is highly complex and cannot be performed analytically. To address this, we employ well-established (semi)analytic approximation methods, such as the WKB method and the asymptotic iteration method, along with numerical calculations. Below, we provide a brief overview of these methods and their application to our calculations.

A. WKB method

To calculate the quasinormal frequencies in (26) using the effective potentials for gravitational (27), scalar (31), and electromagnetic (37) perturbations, appropriate boundary conditions must be imposed. Given the form of the effective potential, which asymptotically approaches zero at spatial infinity and vanishes at the event horizon of the black hole, we choose a waveform that satisfies these boundary conditions. Specifically, we require the waveform to exhibit purely ingoing behavior at the event horizon of the black hole and purely outgoing behavior at spatial infinity. This ensures that the solution is physically consistent and aligns with the expected behavior of perturbations in the spacetime as

$$\psi(r) = \begin{cases} e^{-i\omega r_*} & \text{at } r_* \rightarrow -\infty, \quad (r \rightarrow r_H), \\ e^{i\omega r_*} & \text{at } r_* \rightarrow \infty, \quad (r \rightarrow \infty). \end{cases} \quad (40)$$

The WKB method is a powerful semianalytical approach for approximating the complex frequencies of quasinormal modes. It is an inherently approximate technique, with

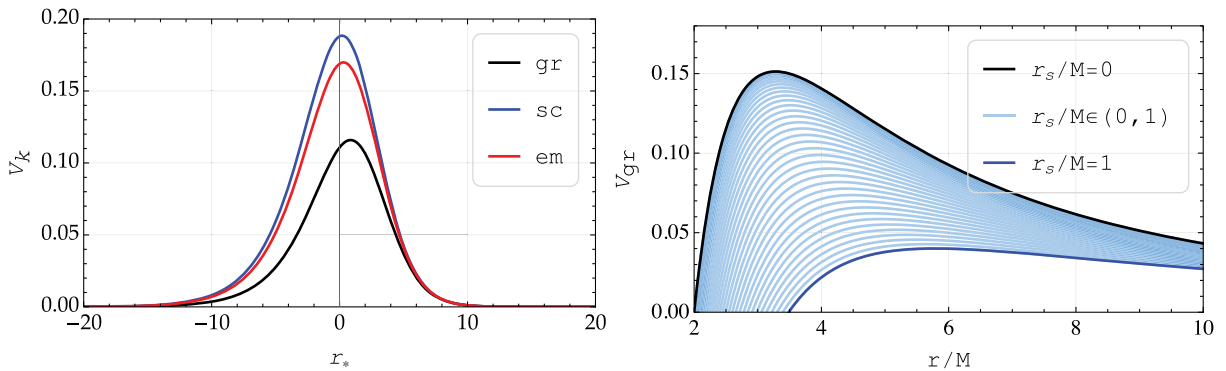


FIG. 3. Left panel: the effective potential for the three types of perturbations studied in this paper is shown as a function of the tortoise coordinate. Right panel: radial profile of the effective potential for gravitational perturbations of a black hole within a dark matter halo is displayed, with varying core radius values in the range $r_s/M \in [0, 1]$ with increments of 0.025. The parameters are set to $\ell = 2$, $\rho_s = 0.5$.

higher-order corrections improving its accuracy. The method was first applied to black hole quasinormal mode calculations by Schutz and Will [45]. It was later extended to third order by Iyer and Will [46,47], to sixth order by Konoplya [48], and more recently, to thirteenth order by using Padé approximations by Matyjasek and Opala [49] as

$$\omega^2 = V_0 + A_2(\mathcal{K}^2) + A_4(\mathcal{K}^2) + \dots - i\mathcal{K}\sqrt{-2V_2}[1 + A_3(\mathcal{K}^2) + A_5(\mathcal{K}^2) + \dots]. \quad (41)$$

where V_0 is the maximum value of the effective potential at r_0 , and $\mathcal{K}$ and $A_k(\mathcal{K}^2)$ are half-integer and correction terms with order k being polynomials in $\mathcal{K}^2$ with rational coefficients, respectively. As demonstrated in [50], the accuracy of the WKB method diminishes, or its applicability becomes limited, in the following scenarios: (i) during superradiance, where the rotation of a black hole causes an incident wave to reflect with increased amplitude, (ii) in unstable black holes, where the effective potential takes negative values outside the event horizon, (iii) in quasi-resonance modes, where the peak and asymptotic value of the effective potential are equal, leading to a very small damping rate of the perturbations, and (iv) in cases where the effective potential is asymptotically nonconstant. Nevertheless, this method has very good accuracy in cases where the potential barrier is deep and narrow. As we have shown in Fig. 3 and discussed in a previous section, we observe that as the values of the multipole ℓ increases, the effective potential becomes more suitable for the application of the WKB method. However, when the dark matter core radius r_s increases, it results in a reduction in the height of the effective potential, consequently diminishing the accuracy of the WKB method. Therefore, to determine the more accurate values of the quasinormal frequencies, we need to employ other methods alongside the WKB method to cross-verify and compare the results.

B. Asymptotic iteration method

We employ the improved asymptotic iteration method (AIM) [51–55]. We first write the wave equation (39) in

terms of the radial coordinate as

$$\psi_k'' + \frac{f'}{f}\psi_k' + \frac{\omega_k^2 - V_k}{f^2}\psi_k = 0, \quad (42)$$

where the prime denotes the derivative with respect to the radial coordinate ($\iota = \partial_r$). Further details of the AIM require the explicit forms of the wave equation. Therefore, we present here the AIM for the scalar perturbation ($k = \text{sc}$) of the black hole in the dark matter halo with metric function (15). For simplicity, let us write the metric function (15) in terms of its zeros as

$$f(r) = \frac{(r - r_1)(r - r_H)}{r(r + r_s)}, \quad (43)$$

where r_1 and r_H are the inner and event horizons of the black hole, respectively. Now, we introduce a new variable instead of the radial coordinate, defined by $x = 1/r$ and in terms of x , the wave equation (42) takes the following form:

$$\psi'' + \left(\frac{2}{x} + \frac{f'}{f}\right)\psi' + \left[\frac{\omega^2}{x^4 f^2} + \frac{x f' - \ell(\ell + 1)}{x^2 f}\right]\psi = 0, \quad (44)$$

where the prime denotes the derivative with respect to the new variable x . To avoid the divergence of the wave function at spatial infinity ($x \rightarrow 0$), we choose the wave function in the following form:

$$\psi = e^{i\omega r_s} \chi(x), \quad (45)$$

and the wave equation (44) reduces to the following form in terms of the new variable χ :

$$\chi'' + 2A\chi' + B\chi = 0, \quad (46)$$

where

$$A = \frac{x\{x[-r_H(2xr_s + 3) + r_1(xr_H(3xr_s + 4) - 2xr_s - 3) + r_s] + 2\} - 2i\omega(xr_s + 1)^2}{2x^2(r_1x - 1)(xr_H - 1)(xr_s + 1)},$$

$$B = -\frac{x[r_H + r_s + r_1 - r_1xr_H(xr_s + 2)] + \ell(\ell + 1)(xr_s + 1)^2}{x^2(r_1x - 1)(xr_H - 1)(xr_s + 1)}.$$

From the appropriate boundary condition at the black hole horizon, r_H , we write the function χ in the following form:

$$\chi(x) = \left(x - \frac{1}{r_H}\right)^{\frac{-i\omega}{\kappa_H}} u(x), \quad (47)$$

where the surface gravity of the black hole horizon is given by

$$\kappa_H = \frac{r_H - r_1}{2r_H(r_H + r_s)}. \quad (48)$$

By substituting the wave function (47) and the surface gravity (48) into the wave equation (46), we obtain the following wave equation:

$$u'' = \lambda_0 u' + s_0 u, \quad (49)$$

with

$$\begin{aligned} \lambda_0 &= \frac{2i\omega r_H(xr_s + 1)[xr_s(2xr_H - 1) + 2x^2r_H^2 - 1] + r_1^2x^2[-xr_H(3xr_s + 4) + 2xr_s + 3]}{x^2(r_1 - r_H)(r_1x - 1)(xr_H - 1)(xr_s + 1)} \\ &\quad - \frac{xr_H[xr_H(2xr_s + 3) - xr_s - 2] + r_1[x^2r_H^2(3xr_s + 4) - xr_s - 2] - 2i\omega(xr_s + 1)(2x^3r_H^2r_s + 2x^3r_H^3 - xr_s - 1)}{x^2(r_1 - r_H)(r_1x - 1)(xr_H - 1)(xr_s + 1)}, \\ s_0 &= \frac{(r_1 - r_H)^2(xr_s + 1)^2\ell(\ell + 1) - r_H^2(r_H + r_s)[4x\omega^2r_H^2(xr_s + 1) + 2\omega r_H(2\omega(xr_s + 1)^2 + ix(xr_s + 2)) - x]}{x^2(r_1 - r_H)^2(r_1x - 1)(xr_H - 1)(xr_s + 1)} \\ &\quad + \frac{r_1r_H[4x^2\omega^2r_H^4(xr_s + 1) + r_H^2(-x^2(xr_s + 2) + 4\omega^2(xr_s + 1)^3 + 4ix\omega(xr_s + 1)^2)]}{x^2(r_1 - r_H)^2(r_1x - 1)(xr_H - 1)(xr_s + 1)} \\ &\quad + \frac{2x\omega r_H^3[2xr_s(2x\omega r_s + ix + 3\omega) + 3ix + 2\omega] + r_H[-x + 2\omega r_s(xr_s(2x\omega r_s + ix + 4\omega) + 2(\omega + ix))] - 2xr_s}{x^2(r_1 - r_H)^2(r_1x - 1)(xr_H - 1)(xr_s + 1)}. \end{aligned} \quad (50)$$

By performing n sequential differentiations of equation (49) concerning x , the subsequent equation is obtained [52],

$$u^{(n+2)} = \lambda_n u' + s_n u, \quad (51)$$

where the coefficients satisfy the following recurrence relations:

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1}, \\ s_n &= s'_{n-1} + s_0 \lambda_{n-1}. \end{aligned} \quad (52)$$

The coefficients λ_n and s_n are subsequently represented as Taylor series expansions about an arbitrary point x_0 , as follows:

$$\begin{aligned} \lambda_n &= \sum_{i=0}^{\infty} c_n^i (x - x_0)^i, \\ s_n &= \sum_{i=0}^{\infty} d_n^i (x - x_0)^i, \end{aligned} \quad (53)$$

where c_n^i and d_n^i represent the i th coefficients of λ_n and s_n , respectively, in terms of their Taylor series expansions. By substituting Eq. (53) into Eq. (52), the subsequent expressions can be derived,

$$c_n^i = (i + 1)c_{n-1}^{i+1} + d_{n-1}^i + \sum_{k=0}^i c_0^k c_{n-1}^{i-k},$$

$$d_n^i = (i + 1)d_{n-1}^{i+1} + \sum_{k=0}^i d_0^k c_{n-1}^{i-k}. \quad (54)$$

For sufficiently large values of n , the asymptotic behavior of the coefficients can be inferred as follows:

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}}. \quad (55)$$

Finally, we derive the subsequent recurrence relation, which is utilized for the calculation of quasinormal modes,

$$d_n^0 c_{n-1}^0 - c_n^0 d_{n-1}^0 = 0. \quad (56)$$

By determining the respective values of λ_0 and s_0 , and selecting an appropriate number for iterations, n , the quasinormal modes of the perturbations can be ascertained.

C. Physics-informed neural networks method

The quasinormal frequencies shown in Tables I–III reveal excellent agreement between the WKB method and AIM calculations. To further validate these results, we employ an additional approach: the physics-informed neural networks (PINNs) method. This technique, rooted in deep learning foundations [56,57], has recently emerged as a powerful tool for computing quasinormal modes [58,59]. Unlike traditional numerical methods such as WKB or

AIM, PINNs directly incorporate the underlying differential equation and boundary conditions into the training of a neural network, offering a mesh-free and flexible framework [57]. In [58], it was shown that the PINNs method successfully computes fundamental quasinormal modes for both four- and higher-dimensional Schwarzschild black holes, thanks to the similar structure of their differential equations. Thus, PINNs can achieve exceptional accuracy, with deviations of less than 0.01% compared to established methods like the WKB approximation and Leaver's continued fraction methods. This high level of agreement highlights PINNs as a promising approach deserving further exploration. The method provides several significant advantages, including the elimination of grid-based discretization through direct use of the tortoise coordinate, the ability to train multiple modes simultaneously, and the flexibility to extend to modified gravity potentials, among other benefits. However, while PINNs offer a versatile approach to solving differential equations, their application to quasinormal mode problems faces certain limitations. These include sensitivity to the initial guess of quasinormal frequencies, the influence of loss function weight choices (especially for boundary conditions), and difficulties in accurately implementing asymptotic boundary conditions. We provide a brief overview of the PINNs method and its underlying working principles. PINNs utilize deep neural networks to approximate the wavefunction $\psi(r)$ and treat the complex frequency ω as a trainable parameter. By minimizing a loss function that enforces both the governing equation and the boundary conditions, the method simultaneously solves for the eigenfunction and eigenvalue, confirming and complementing results from conventional techniques. The PINNs method leverages neural networks to solve differential equations by embedding the governing physics directly into the loss function. For calculating quasinormal modes, we solve the Schrödinger-like wave equation (39). The PINNs approach approximates $\psi_k(r)$ with a neural network $\psi_\xi(r)$, parameterized by weights ξ , and treats ω_k as a trainable parameter. The key feature of PINNs is the construction of a physics-informed loss function. For collocation points $\{r_i\}_{i=1}^N$ sampled in the domain $r \in [r_H, r_\infty]$, the residual of the Schrödinger-like equation,

$$\mathcal{R}(r_i) = \frac{d^2 \psi_\xi}{dr_*^2}(r_i) + (\omega^2 - V(r_i))\psi_\xi(r_i), \quad (57)$$

is minimized in a mean squared error sense. Additionally, boundary conditions are enforced via a penalty term $\mathcal{L}_{BC}$ that implements the physical requirements:

- (i) Purely ingoing waves at the event horizon $r = r_H$,
- (ii) Decaying solutions at spatial infinity $r \rightarrow \infty$.

The total loss function is

$$\mathcal{L}(\xi, \omega) = \frac{1}{N} \sum_{i=1}^N |\mathcal{R}(r_i)|^2 + \mathcal{L}_{BC}. \quad (58)$$

A typical form for the boundary loss is

$$\mathcal{L}_{BC} = |\psi_\xi(r_H) - e^{-i\omega r_*(r_H)}|^2 + |\psi_\xi(r_\infty)|^2, \quad (59)$$

where r_∞ is chosen sufficiently large to approximate infinity numerically. The network parameters ξ and the complex frequency ω are optimized iteratively to minimize $\mathcal{L}$. The Adam optimizer is first used for rapid convergence, followed by L-BFGS (Limited-memory Broyden–Fletcher–Goldfarb–Shanno) for more precise minimization. Convergence is monitored through the decrease in loss and stability of the computed frequency. The general implementation of the PINNs method follows the computational procedure given by Algorithm 1. We must add the following notes that must be taken into account before implementation:

- (i) The neural network architecture and hyperparameters (e.g., number of layers, neurons, learning rate) are tuned based on the problem's complexity and desired accuracy.
- (ii) The choice of collocation points and boundary condition enforcement significantly affects convergence. We use $N \approx 1000$ – 5000 points for robust results.
- (iii) PINNs are particularly advantageous for quasinormal modes due to their ability to handle complex geometries and nonlinear potentials without requiring explicit discretization of the differential equation.

The final columns of Tables I–III present quasinormal frequencies computed using the PINNs method, with initial guesses selected from the WKB method. The quasinormal mode frequencies obtained via PINNs show excellent agreement with those from WKB and AIM methods, typically matching up at least to three significant digits.

D. Ringdown waveform

In this subsection, we investigate the temporal evolution of the scalar, electromagnetic, and gravitational perturbations of the Schwarzschild black hole in the dark matter halo by adopting the characteristic integration method proposed by Gundlach *et al.* [60]. Moreover, we mostly follow the algorithm and notations presented in [39,61]. The method employs light-cone variables, encompassing the retarded time coordinate $du \equiv dt - dr_*$ as well as the advanced time coordinate $dv \equiv dt + dr_*$. Initial conditions are specified on the two null surfaces $u = u_0$ and $v = v_0$. As a result, the wave equation (26) can be rewritten in terms of the light-cone coordinates as follows:

$$4 \frac{\partial^2 \Psi}{\partial u \partial v} + V \Psi = 0. \quad (60)$$

Solving this equation requires the use of numerical methods. We discretize the (u, v) space into a finite grid with a uniform spacing of Δ between adjacent grid points. The numerical scheme used to solve this equation is as follows:

Algorithm 1. PINNs for quasinormal modes calculation.

Input:

Black hole parameters: mass M , dark matter core radius r_s , dark matter core density ρ_s .

Effective potential $V(r)$, defined for the specific spacetime and perturbation.

Tortoise coordinate $r_*(r)$, mapping the radial domain $r \in [r_H, \infty)$, where r_H is the event horizon.

Number of collocation points N for residual evaluation.

Output:

Complex quasinormal frequency $\omega = \omega_R + i\omega_I$.

Steps:

1. **Define the Wave Equation:** Use the Schrödinger-like equation to describe the quasinormal mode dynamics.
 2. **Construct Neural Network:** Define a feedforward neural network $\psi_\xi(r)$ with parameters ξ (weights and biases) to approximate $\psi(r)$. Typically, a deep network with 4–8 layers and 50–100 neurons per layer is used, with activation functions like tanh or sin.
 3. **Initialize Trainable Parameters:** Initialize the complex frequency ω as a trainable parameter (real and imaginary parts) and the neural network parameters ξ .
 4. **Select Collocation Points:** Choose N collocation points $\{r_i\}_{i=1}^N$ in the domain $r \in [r_H, \infty)$, typically using a uniform or Latin hypercube sampling to ensure good coverage.
 5. **Compute Residual:** At each collocation point r_i , calculate the residual of the wave equation (57), where the derivative $\frac{d^2\psi_\xi}{dr_*^2}$ is computed using automatic differentiation.
 6. **Define Loss Function:** Construct the loss function as (58), where $\mathcal{L}_{BC}$ enforces boundary conditions (e.g., $\psi_\xi(r_H) \sim e^{-i\omega r_*}$ for ingoing waves at the horizon and $\psi_\xi(r \rightarrow \infty) \sim e^{i\omega r_*}$ for outgoing waves). The boundary loss $\mathcal{L}_{BC}$ is typically a weighted sum of squared deviations from these conditions at boundary points.
 7. **Train the Network:**

Use an optimizer, such as Adam (with a learning rate of ~ 0.001) for initial training due to its efficiency in handling nonconvex loss landscapes, followed by L-BFGS for fine-tuning to achieve higher precision.

Iteratively update ξ and ω to minimize $\mathcal{L}(\xi, \omega)$ until convergence (e.g., when the loss falls below a threshold or gradient updates become negligible).
 8. **Output:** Return the optimized complex frequency ω , representing the quasinormal mode.
-

$$\Psi_N = \Psi_W + \Psi_E - \Psi_S - \frac{\Delta^2 V_S}{8} (\Psi_W + \Psi_E) + O(\Delta^4), \quad (61)$$

where the indices N , W , E , and S refer to grid points $N \equiv (u + \Delta, v + \Delta)$, $W \equiv (u + \Delta, v)$, $E \equiv (u, v + \Delta)$, and $S \equiv (u, v)$. In our calculations, we assume that the initial perturbation is a Gaussian function centered around the point r_{*c} and it takes the form

$$\begin{aligned} \Psi(t=0, r_*) &= A \exp(-(r_* - r_{*c})^2 / \sigma^2) \\ &= A \exp(-(v - v_c)^2 / \sigma^2). \end{aligned} \quad (62)$$

Since the time coordinate is expressed in terms of the light-cone coordinates as follows:

$$t = \frac{u + v}{2}, \quad (63)$$

the light-cone coordinates initially (at $t = 0$) have the relationship $u = -v$, and therefore,

$$r_* = \frac{v - u}{2} = v. \quad (64)$$

At the center of the body, $r_* = 0$, we initially put the boundary condition $\Psi(u_0, v_0) = 0$ which is considered along the line $u = v$ since for

$$r_* = \frac{v - u}{2} = 0 \Rightarrow u = v. \quad (65)$$

Our tortoise coordinate r_* is determined from the formula,

$$r_* = \int_0^r \frac{dr'}{f(r')}, \quad (66)$$

which implies that $r_* \geq 0$ for $r \geq 0$. We are therefore interested only in the region where $v \geq u$. In the integration loops the coordinates, u and v are determined by formulas

$$u = i_u \Delta, \quad \text{where } i_u = \{1, 2, \dots, N\}, \quad (67)$$

and

$$v = i_v \Delta, \quad \text{where } i_v = \{i_u + 1, i_u + 2, \dots, N\}. \quad (68)$$

For more details of the calculations through the time-domain analysis, read Refs. [62,63] and references therein.

V. NUMERICAL RESULTS

In this section, we demonstrate the quasinormal frequencies derived through the application of high-order WKB, AIM, and PINNs methods, and we present the temporal evolution of the perturbations. Tables I–III present the fundamental quasinormal frequencies for scalar, electromagnetic, and gravitational perturbations of the Schwarzschild black hole within a Dehnen-(1, 4, γ)-type dark matter halo. These frequencies are computed using the WKB, AIM, and PINNs methods for various dark matter core radii and multipole numbers, corresponding to $\gamma = -1$, $\gamma = 0$, and $\gamma = 1$, respectively. The quasinormal frequencies listed in the table indicate that the presence of dark matter surrounding the Schwarzschild black hole causes a reduction in both the perturbation frequencies and their corresponding damping rates. An increase in the dark matter core radius leads to a decay in both the real and imaginary parts of the quasinormal frequencies, similar to the case in the Hernquist-type matter around the static black hole [24,27]. The extent of these

TABLE I. The fundamental quasinormal frequencies for the scalar (first row), electromagnetic (second row), and gravitational (third row) perturbations of the black hole in the dark matter halo with Dehnen-type dark matter density profile $\gamma = -1$ calculated via the WKB (sixth order), AIM (after 40 iterations), and PINNs methods, where we set $\rho_s = 0.5$.

$\gamma = -1$				
ℓ	r_s	WKB	AIM	PINNs
2	0	0.48364–0.09677 i	0.48364–0.09676 i	0.48364–0.09677 i
		0.45759–0.09501 i	0.45760–0.09500 i	0.45759–0.09501 i
		0.37362–0.08889 i	0.37367–0.08896 i	0.37362–0.08889 i
	0.2	0.47835–0.09562 i	0.47835–0.09562 i	0.47835–0.09562 i
		0.45261–0.09389 i	0.45261–0.09389 i	0.45261–0.09389 i
		0.36955–0.08785 i	0.36960–0.08792 i	0.36955–0.08785 i
	0.5	0.42039–0.08245 i	0.42039–0.08245 i	0.42039–0.08245 i
		0.39824–0.08103 i	0.39824–0.08103 i	0.39824–0.08103 i
		0.32513–0.07584 i	0.32515–0.07590 i	0.32513–0.07584 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i
3	0	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i
	0.2	0.66797–0.09536 i	0.66797–0.09536 i	0.66797–0.09536 i
		0.64972–0.09449 i	0.64972–0.09449 i	0.64971–0.09449 i
		0.59289–0.09161 i	0.59289–0.09161 i	0.59288–0.09161 i
	0.5	0.58702–0.08223 i	0.58702–0.08223 i	0.58702–0.08223 i
		0.57132–0.08152 i	0.57132–0.08152 i	0.57126–0.08152 i
		0.52133–0.07907 i	0.52133–0.07907 i	0.52133–0.07907 i
	4	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80917–0.09416 i
4	0	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80917–0.09416 i
	0.2	0.85792–0.09526 i	0.85792–0.09526 i	0.85791–0.09526 i
		0.84377–0.09473 i	0.84376–0.09473 i	0.84376–0.09473 i
		0.80033–0.09306 i	0.80033–0.09306 i	0.80019–0.09306 i
	0.5	0.75394–0.08215 i	0.75394–0.08215 i	0.75394–0.08215 i
		0.74176–0.08172 i	0.74176–0.08172 i	0.74176–0.08172 i
		0.70357–0.08030 i	0.70357–0.08030 i	0.70356–0.08030 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i

TABLE II. The same as Table I, but for $\gamma = 0$.

$\gamma = 0$				
ℓ	r_s	WKB	AIM	PINNs
2	0	0.48364–0.09677 i	0.48364–0.09676 i	0.48364–0.09677 i
		0.45759–0.09501 i	0.45760–0.09500 i	0.45759–0.09501 i
		0.37362–0.08889 i	0.37367–0.08896 i	0.37362–0.08889 i
	0.2	0.47639–0.09523 i	0.47639–0.09522 i	0.47639–0.09523 i
		0.45075–0.09351 i	0.45075–0.09350 i	0.45075–0.09351 i
		0.36803–0.08748 i	0.36808–0.08755 i	0.36803–0.08748 i
	0.5	0.39771–0.07797 i	0.39771–0.07796 i	0.39771–0.07797 i
		0.37675–0.07662 i	0.37675–0.07662 i	0.37675–0.07662 i
		0.30759–0.07172 i	0.30762–0.07177 i	0.30759–0.07172 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i

(Table continued)

TABLE II. (Continued)

$\gamma = 0$				
ℓ	r_s	WKB	AIM	PINNs
3	0	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i
	0.2	0.66523–0.09497 i	0.66523–0.09497 i	0.66522–0.09497 i
		0.64706–0.09410 i	0.64706–0.09410 i	0.64705–0.09410 i
		0.59046–0.09123 i	0.59046–0.09123 i	0.59046–0.09123 i
	0.5	0.55535–0.07776 i	0.55535–0.07776 i	0.55536–0.07776 i
		0.54049–0.07709 i	0.54049–0.07709 i	0.54049–0.07709 i
		0.49320–0.07477 i	0.49320–0.07477 i	0.49320–0.07477 i
	4	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80918–0.09416 i
4	0	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80917–0.09416 i
	0.2	0.85440–0.09486 i	0.85440–0.09486 i	0.85440–0.09486 i
		0.84031–0.09434 i	0.84031–0.09434 i	0.84030–0.09434 i
		0.79705–0.09267 i	0.79705–0.09267 i	0.79704–0.09267 i
	0.5	0.71326–0.07768 i	0.71326–0.07768 i	0.71326–0.07768 i
		0.70173–0.07727 i	0.70173–0.07727 i	0.70173–0.07727 i
		0.66560–0.07593 i	0.66560–0.07593 i	0.66560–0.07593 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i

TABLE III. The same as Tables I and II, but for $\gamma = 1$.

$\gamma = 1$				
ℓ	r_s	WKB	AIM	PINNs
2	0	0.48364–0.09677 i	0.48364–0.09676 i	0.48364–0.09676 i
		0.45759–0.09501 i	0.45760–0.09500 i	0.45760–0.09500 i
		0.37362–0.08889 i	0.37367–0.08896 i	0.37362–0.08889 i
	0.2	0.47250–0.09445 i	0.47250–0.09444 i	0.47250–0.09444 i
		0.44707–0.09274 i	0.44707–0.09273 i	0.44707–0.09273 i
		0.36503–0.08677 i	0.36508–0.08684 i	0.36503–0.08677 i
	0.5	0.35846–0.07038 i	0.35846–0.07037 i	0.35846–0.07037 i
		0.33954–0.06916 i	0.33954–0.06915 i	0.33954–0.06915 i
		0.27722–0.06472 i	0.27723–0.06477 i	0.27722–0.06472 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i
3	0	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i
	0.2	0.65980–0.09419 i	0.65980–0.09419 i	0.65979–0.09419 i
		0.64178–0.09333 i	0.64178–0.09333 i	0.64177–0.09333 i
		0.58564–0.09049 i	0.58564–0.09049 i	0.58563–0.09049 i
	0.5	0.50054–0.07019 i	0.50054–0.07019 i	0.50054–0.07019 i
		0.48712–0.06958 i	0.48712–0.06958 i	0.48712–0.06958 i
		0.44450–0.06748 i	0.44450–0.06748 i	0.44450–0.06748 i
	4	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80917–0.09416 i
4	0	0.86742–0.09639 i	0.86742–0.09639 i	0.86742–0.09639 i
		0.85310–0.09586 i	0.85310–0.09586 i	0.85309–0.09586 i
		0.80918–0.09416 i	0.80918–0.09416 i	0.80917–0.09416 i
	0.2	0.84742–0.09408 i	0.84742–0.09408 i	0.84741–0.09408 i
		0.83345–0.09357 i	0.83345–0.09357 i	0.83344–0.09357 i
		0.79054–0.09191 i	0.79054–0.09191 i	0.79054–0.09191 i
	0.5	0.64286–0.07012 i	0.64286–0.07012 i	0.64286–0.07012 i
		0.63246–0.06975 i	0.63246–0.06975 i	0.63246–0.06975 i
		0.59989–0.06853 i	0.59989–0.06853 i	0.59989–0.06853 i
	3	0.67537–0.09650 i	0.67537–0.09650 i	0.67537–0.09650 i
		0.65690–0.09562 i	0.65690–0.09562 i	0.65690–0.09562 i
		0.59944–0.09270 i	0.59944–0.09270 i	0.59944–0.09270 i

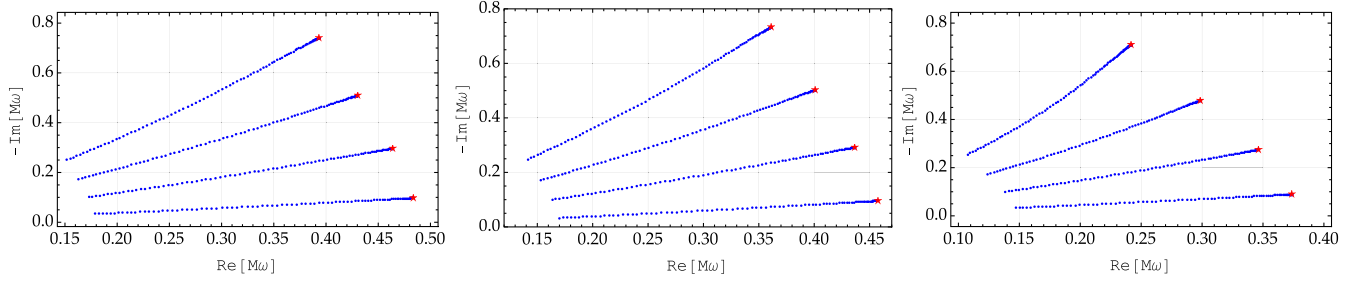


FIG. 4. Fundamental (lowest curve) and the first three nonfundamental (top three curves) quasinormal frequencies for scalar (left panel), electromagnetic (middle panel), and gravitational (right panel) perturbations of a Schwarzschild black hole embedded in a dark matter halo with Dehnen-(1, 4, 0) type profile for the dark matter core radius $r_s \in [0, M]$ with a step size of $0.01M$. The parameters are set as $\rho_s = 0.5$ and $\ell = 2$. The values marked with asterisks correspond to the quasinormal frequencies of the isolated Schwarzschild black hole.

changes also depends on the specific dark matter model, particularly the value of the parameter γ . For the models considered in this paper, namely, the cases where $\gamma = -1, 0$, and 1 , the quasinormal modes exhibit smaller frequency values for the $\gamma = 1$ case and larger values for the $\gamma = -1$ case. We also note that, although the relative heights of the effective potentials for the perturbations, as shown in Fig. 3, suggest that the scalar perturbation should exhibit the largest quasinormal modes, while the gravitational perturbation should exhibit the smallest among the three types considered, these expectations are quantitatively confirmed in Tables I–III. We must note that there can be raised a question regarding the relation between the tortoise coordinate r_* and the radial coordinate r , and its implications for numerical accuracy in the computation of quasinormal modes. In all methods presented in the paper, the tortoise coordinate is computed numerically with high accuracy. In the WKB method, it is obtained via precise numerical integration and interpolation of $dr/f(r)$, introducing negligible error due to the smoothness of $f(r)$. AIM does not use the tortoise coordinate, eliminating this concern. In the PINNs method,

the network is trained using values of $r_*(r)$ obtained from numerical integration.

In the above discussion, we have focused exclusively on the fundamental quasinormal modes of the perturbations. The nonfundamental quasinormal modes, however, remain of significant interest. Nonetheless, their importance is sometimes overlooked due to their more rapid damping rates. In Fig. 4, we illustrate how the frequency and damping rate of oscillations for scalar, electromagnetic, and gravitational perturbations of the black hole in the dark matter halo vary with the dark matter halo core radius, encompassing both the fundamental mode and a few overtones. Since the quasinormal modes of black hole perturbations presented in this paper span a wide range when calculated for the overtones $n = 0, 1, 2, 3$ and exhibit qualitatively similar yet quantitatively comparable values across all black holes considered, we display the quasinormal modes in Fig. 4 specifically for the black hole within a dark matter halo with $\gamma = 0$. Nevertheless, for all the spacetimes considered in this paper, we provide the numerical values of the nonfundamental quasinormal

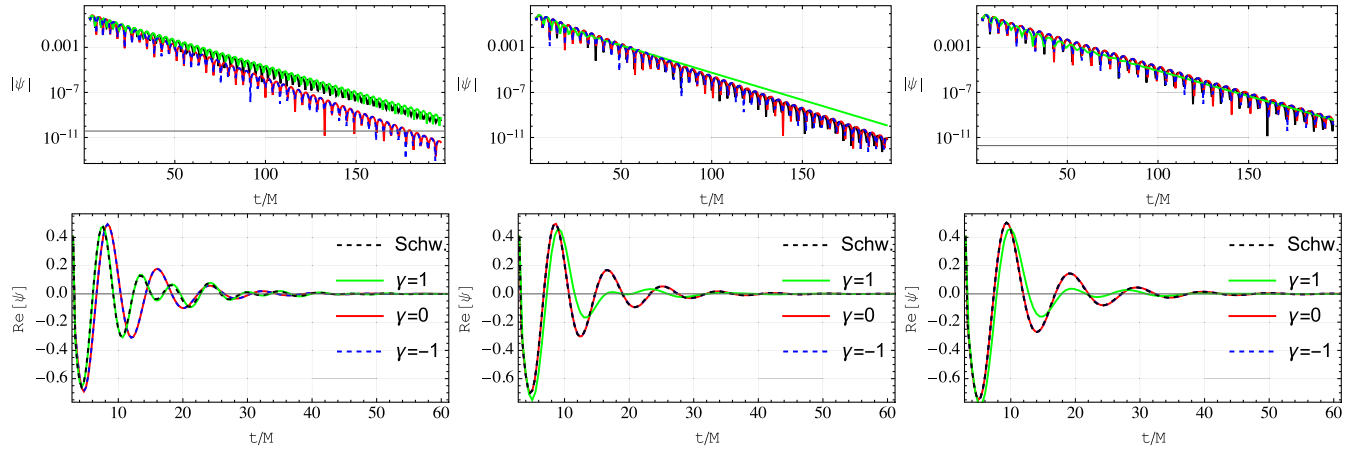


FIG. 5. Time-domain profiles of the scalar (left panel), electromagnetic (middle panel), and gravitational (right panel) perturbations of the Schwarzschild black hole are shown for two scenarios: in a vacuum (black) and in a dark matter halo with Dehnen-(1, 4, γ) type profiles for $\gamma = -1$ (blue), $\gamma = 0$ (red), and $\gamma = 1$ (green), where we set $\ell = 2$, $\rho_s = 0.2$, and $r_s = 0.2$.

frequencies in Tables IV–VI in Appendix A, corresponding to selected values of the dark matter core radius. From Fig. 4 and Tables IV–VI, it is evident that an increase in the overtone number slightly reduces the frequencies of the perturbations while significantly enhancing their damping rates, resulting in shorter-lived modes. Conversely, as dark matter reduces the damping rate of perturbations, the damping rates of neighbor overtones converge more closely for higher overtones as the radius of the dark matter halo core increases. It should be noted that Tables IV–VI indicate that the accuracy of the AIM decreases relative to the WKB method as the overtone number increases, consistent with the observation made in [52]. Similar discrepancies are observed between the WKB and PINNs methods at higher overtone numbers.

Finally, in Fig. 5 we present the time evolutions of the scalar, electromagnetic, and gravitational perturbations for all black holes considered in this paper for the fixed values of dark matter halo core parameters and multipole of perturbations. Given that the amplitude of the perturbations decays rapidly over time, as depicted in the bottom panels of the figure, a logarithmically scaled amplitude would provide greater insight into the duration of the perturbations before they reach equilibrium. As demonstrated in Tables I–III, small values of the dark matter halo core radius and density have a small impact on the frequency and damping rates of the perturbations. Consequently, in Fig. 5, the differences among the dark matter models considered in this paper are not particularly pronounced.

VI. CONCLUSION

In this paper, we study the scalar, electromagnetic, and gravitational perturbations of a static, spherically symmetric black hole embedded within a Dehnen-(1, 4, γ) type dark matter halo, characterized by a dark matter density that decays as $1/r^4$ at large distances and behaves as $1/r^\gamma$ at the center of the spacetime. Consequently, the dark matter density exhibits distinct behavior at the center of the spacetime depending on the value of γ . Here, we have chosen three specific values of γ : for $\gamma = -1$, the dark matter density vanishes; for $\gamma = 0$, it remains finite; and for $\gamma = 1$, it diverges, to explore the impact of the various dark matter densities on the perturbations of the black hole.

In this study, we have investigated the quasinormal modes of scalar, electromagnetic, and gravitational perturbations of a Schwarzschild black hole embedded within a Dehnen-(1, 4, γ)-type dark matter halo, employing high-order WKB and AIM methods. Our results reveal that the presence of dark matter reduces both the frequencies and damping rates of these perturbations, with the extent of this reduction influenced by the dark matter core radius and the parameter γ . Specifically, for $\gamma = -1, 0$, and 1 , the quasinormal frequencies exhibit larger values when $\gamma = -1$

and smaller values when $\gamma = 1$ as well as for the scalar perturbations display the largest modes and gravitational perturbations the smallest.

Extending our analysis to overtones, we find that higher overtones slightly decrease perturbation frequencies while significantly increasing damping rates, leading to shorter-lived modes. Conversely, an increase in the dark matter halo core radius brings the damping rates of adjacent overtones closer together, counteracting the rapid decay typically associated with higher overtones. The time-domain evolution of these perturbations, illustrated in Fig. 5, further highlights their rapid amplitude decay, suggesting that a logarithmic scaling could better elucidate their duration before reaching equilibrium. However, for small dark matter halo core radii and densities, the impact on frequencies and damping rates remains minimal, resulting in subtle differences among the dark matter models ($\gamma = -1, 0, 1$) in the time domain. Quasinormal modes of these perturbations in the eikonal regime (high-energy limit) are not addressed here, as, in this regime, all scattering effective potentials discussed in this paper converge to the same form, the effective potential of null geodesics. In this case, the real and imaginary parts of the quasinormal modes are expressed as products of the multipole number and the angular velocity of circular null geodesics and half the overtone number and the Lyapunov exponent of these geodesics, respectively, which have been detailed in our previous work [30].

These findings bring some details of the interplay between dark matter properties and black hole perturbation dynamics, demonstrating that while dark matter generally stabilizes perturbations by reducing damping rates, the specific influence depends on both the halo's core radius and the γ parameter. In our future works, we aim to explore additional overtones and alternative dark matter profiles to further refine our understanding of these effects. The impact of dark matter on black hole perturbations may be detectable in the future experiments through advanced astrophysical observations with improved precision and sensitivity. Future gravitational wave detections with LIGO, Virgo, and future missions like LISA could reveal deviations in the ringdown phase of black hole mergers due to the presence of dark matter.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: NONFUNDAMENTAL QUASINORMAL MODES

Here, we present numerical values of the quasinormal modes for scalar, electromagnetic, and gravitational perturbations—similar to those in Tables I–III, but corresponding to nonfundamental modes.

TABLE IV. The nonfundamental quasinormal frequencies for the scalar (first row), electromagnetic (second row), and gravitational (third row) perturbations of the black hole in the dark matter halo with Dehnen-type dark matter density profile $\gamma = -1$ calculated via the WKB (sixth order), AIM (after 40 iterations), and PINNs methods, where we set $\ell = 2$, $\rho_s = 0.5$.

$\gamma = -1$				
n	r_s	WKB	AIM	PINNs
1	0	0.46385–0.29563 i	0.46377–0.29554 i	0.46384–0.29564 i
		0.43653–0.29073 i	0.43654–0.29071 i	0.43653–0.29074 i
		0.34569–0.27396 i	0.34671–0.27391 i	0.34570–0.27398 i
	0.2	0.45882–0.29213 i	0.45881–0.29205 i	0.45883–0.29214 i
		0.43183–0.28730 i	0.43184–0.28728 i	0.43183–0.28732 i
		0.34201–0.27073 i	0.34300–0.27069 i	0.34201–0.27074 i
	0.5	0.40421–0.25168 i	0.40422–0.25169 i	0.40422–0.25170 i
		0.38101–0.24772 i	0.38102–0.24771 i	0.38103–0.24773 i
		0.30243–0.23346 i	0.30313–0.23345 i	0.30244–0.23348 i
	2	0.43039–0.50870 i	0.42872–0.50655 i	0.43041–0.50875 i
		0.40071–0.50198 i	0.40121–0.50153 i	0.40073–0.50203 i
		0.29774–0.47881 i	0.30105–0.47828 i	0.29776–0.47885 i
2	0	0.42581–0.50265 i	0.42839–0.50231 i	0.42584–0.50272 i
		0.39649–0.49602 i	0.39666–0.49556 i	0.39653–0.49612 i
		0.29473–0.47312 i	0.29797–0.47260 i	0.29481–0.47615 i
	0.2	0.37683–0.43243 i	0.37840–0.43414 i	0.37687–0.43249 i
		0.35168–0.42704 i	0.35198–0.42671 i	0.35171–0.42711 i
		0.26350–0.40719 i	0.26586–0.40681 i	0.26365–0.42796 i
	0.5	0.37683–0.43243 i	0.37840–0.43414 i	0.37687–0.43249 i
		0.35168–0.42704 i	0.35198–0.42671 i	0.35171–0.42711 i
		0.26350–0.40719 i	0.26586–0.40681 i	0.26365–0.42796 i
	3	0.39321–0.73989 i	0.40404–0.66245 i	0.40404–0.66245 i
		0.36022–0.73436 i	0.36114–0.73370 i	0.36104–0.73545 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.25236–0.71042 i
3	0	0.38914–0.73103 i	0.42408–0.74886 i	0.39142–0.74385 i
		0.35655–0.72559 i	0.33298–0.70015 i	0.35763–0.72841 i
		0.23950–0.70109 i	0.24906–0.69671 i	0.24190–0.70373 i
	0.2	0.34641–0.62780 i	0.40152–0.62100 i	0.34828–0.63016 i
		0.31844–0.62361 i	0.32064–0.62190 i	0.32103–0.62582 i
		0.21823–0.60215 i	0.22531–0.59850 i	0.22105–0.61057 i
	0.5	0.34641–0.62780 i	0.40152–0.62100 i	0.34828–0.63016 i
		0.31844–0.62361 i	0.32064–0.62190 i	0.32103–0.62582 i
		0.21823–0.60215 i	0.22531–0.59850 i	0.22105–0.61057 i
	3	0.39321–0.73989 i	0.40404–0.66245 i	0.40404–0.66245 i
		0.36022–0.73436 i	0.36114–0.73370 i	0.36104–0.73545 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.25236–0.71042 i

TABLE V. The same as Table IV, but for $\gamma = 0$.

$\gamma = 0$				
n	r_s	WKB	AIM	PINNs
1	0	0.46385–0.29563 i	0.46387–0.29560 i	0.46386–0.29565 i
		0.43653–0.29073 i	0.43654–0.29071 i	0.43654–0.29070 i
		0.34569–0.27396 i	0.34671–0.27391 i	0.34572–0.27399 i
	0.2	0.45694–0.29092 i	0.45700–0.29089 i	0.45699–0.29103 i
		0.43007–0.28611 i	0.43007–0.28609 i	0.43023–0.28645 i
		0.34061–0.26961 i	0.34160–0.26956 i	0.34072–0.26976 i
	0.5	0.38243–0.23800 i	0.38243–0.23798 i	0.38250–0.23823 i
		0.36049–0.23425 i	0.36050–0.23423 i	0.36052–0.23436 i
		0.28618–0.22075 i	0.28682–0.22075 i	0.28620–0.22083 i
	3	0.39321–0.73989 i	0.40404–0.66245 i	0.40404–0.66245 i
		0.36022–0.73436 i	0.36114–0.73370 i	0.36104–0.73545 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.25236–0.71042 i

(Table continued)

TABLE V. (Continued)

$\gamma = 0$				
n	r_s	WKB	AIM	PINNs
2	0	0.43039–0.50870 i	0.43168–0.50998 i	0.43055–0.50940 i
		0.40071–0.50198 i	0.40111–0.50164 i	0.40096–0.50255 i
		0.29774–0.47881 i	0.30105–0.47828 i	0.29850–0.47896 i
	0.2	0.42408–0.50057 i	0.42792–0.50339 i	0.42539–0.50271 i
		0.39507–0.49372 i	0.39541–0.49340 i	0.39558–0.49417 i
		0.29353–0.47116 i	0.34160–0.26956 i	0.29456–0.47306 i
	0.5	0.35660–0.40891 i	0.35684–0.40817 i	0.35679–0.41045 i
		0.33299–0.40358 i	0.33316–0.40344 i	0.33610–0.41331 i
		0.24947–0.38500 i	0.25166–0.38466 i	0.25087–0.39096 i
	3	0.39321–0.73989 i	0.41192–0.73031 i	0.40335–0.74206 i
		0.36022–0.73436 i	0.36054–0.74358 i	0.36119–0.73994 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.24993–0.71080 i
3	0	0.38756–0.72799 i	0.40273–0.72827 i	0.39671–0.73010 i
		0.35511–0.72258 i	0.35795–0.70913 i	0.35604–0.72087 i
		0.23854–0.69818 i	0.34160–0.26956 i	0.24182–0.70584 i
	0.2	0.32790–0.59362 i	0.33455–0.59683 i	0.33206–0.60412 i
		0.30145–0.58966 i	0.29580–0.58728 i	0.31813–0.59697 i
		0.20681–0.56933 i	0.21344–0.56589 i	0.21272–0.57631 i
	0.5	0.32790–0.59362 i	0.33455–0.59683 i	0.33206–0.60412 i
		0.30145–0.58966 i	0.29580–0.58728 i	0.31813–0.59697 i
		0.20681–0.56933 i	0.21344–0.56589 i	0.21272–0.57631 i
	3	0.39321–0.73989 i	0.41192–0.73031 i	0.40335–0.74206 i
		0.36022–0.73436 i	0.36054–0.74358 i	0.36119–0.73994 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.24993–0.71080 i

TABLE VI. The same as Tables IV and V, but for $\gamma = 1$.

$\gamma = 1$				
n	r_s	WKB	AIM	PINNs
1	0	0.46385–0.29563 i	0.46383–0.29562 i	0.46385–0.29565 i
		0.43653–0.29073 i	0.43654–0.29071 i	0.43653–0.29074 i
		0.34569–0.27396 i	0.34671–0.27391 i	0.34571–0.27398 i
	0.2	0.45321–0.28853 i	0.45324–0.28849 i	0.45323–0.28856 i
		0.42656–0.28376 i	0.42657–0.28375 i	0.42658–0.28379 i
		0.33842–0.26693 i	0.33882–0.26735 i	0.33845–0.26698 i
	0.5	0.34464–0.21484 i	0.34463–0.21482 i	0.34466–0.21488 i
		0.32482–0.21144 i	0.32483–0.21142 i	0.32485–0.21149 i
		0.25823–0.19894 i	0.25842–0.19924 i	0.25825–0.19901 i
	0	0.43039–0.50870 i	0.42590–0.50870 i	0.43110–0.51106 i
		0.40071–0.50198 i	0.40131–0.50171 i	0.40100–0.50315 i
		0.29774–0.47881 i	0.30105–0.47828 i	0.29890–0.47899 i
2	0	0.42062–0.49646 i	0.42173–0.49674 i	0.42110–0.49660 i
		0.39185–0.48967 i	0.39218–0.48963 i	0.39190–0.48975 i
		0.29191–0.46607 i	0.29435–0.46678 i	0.29301–0.46702 i
	0.2	0.32126–0.36917 i	0.32066–0.36936 i	0.32325–0.37047 i
		0.29994–0.36432 i	0.30007–0.36421 i	0.30119–0.36851 i
		0.22519–0.34667 i	0.22661–0.34724 i	0.22852–0.34968 i
	0.5	0.32126–0.36917 i	0.32066–0.36936 i	0.32325–0.37047 i
		0.29994–0.36432 i	0.30007–0.36421 i	0.30119–0.36851 i
		0.22519–0.34667 i	0.22661–0.34724 i	0.22852–0.34968 i
	3	0.39321–0.73989 i	0.45658–0.76380 i	0.40805–0.74750 i
		0.36022–0.73436 i	0.37587–0.73469 i	0.37091–0.74423 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.24752–0.71184 i
3	0	0.38441–0.72202 i	0.45161–0.72339 i	0.39716–0.73272 i
		0.35314–0.71478 i	0.36186–0.71881 i	0.35759–0.72104 i
		0.23663–0.69240 i	0.24604–0.68813 i	0.24137–0.71386 i
	0.2	0.29531–0.53600 i	0.33763–0.54097 i	0.31056–0.54820 i
		0.27223–0.53081 i	0.27129–0.52897 i	0.28920–0.54955 i
		0.18626–0.51339 i	0.19206–0.51093 i	0.18963–0.51918 i
	0.5	0.29531–0.53600 i	0.33763–0.54097 i	0.31056–0.54820 i
		0.27223–0.53081 i	0.27129–0.52897 i	0.28920–0.54955 i
		0.18626–0.51339 i	0.19206–0.51093 i	0.18963–0.51918 i
	3	0.39321–0.73989 i	0.45658–0.76380 i	0.40805–0.74750 i
		0.36022–0.73436 i	0.37587–0.73469 i	0.37091–0.74423 i
		0.24172–0.70959 i	0.25145–0.70515 i	0.24752–0.71184 i

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