

Collapse of Coulomb bound states of vector bosons

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Charged spin-1 (vector) particles behave very differently from electrons or scalars in a Coulomb field. For an infinitely heavy “pointlike” nucleus, their bound-state wave functions “fall to the center,” and embedding the system in a renormalizable electroweak-type theory does not remedy this short-distance pathology. We therefore solve the pure Coulomb problem for a finite nuclear radius R and recover the point-nucleus limit by letting $R \rightarrow 0$. This approach allows us to include the crucial Υ -term in the wave equations, which for the pointlike nucleus is proportional to $\delta(r)$ and was ignored in the previous calculations of the energy spectrum. Several unusual effects emerge: (i) The Υ -term supports a tower of states located mainly inside the nucleus. As $R \rightarrow 0$ their number diverges, most lying in the negative-energy continuum (energy $\epsilon < -mc^2$). They trigger vacuum breakdown—particle-antiparticle pair creation that ultimately screens the nuclear charge. (ii) Ordinary Sommerfeld-like states (with binding energy smaller mc^2) persist, but a finite fraction of each wave function leaks into the nucleus, even as $R \rightarrow 0$. (iii) Charge density of a negatively charged vector particle changes sign in a vicinity of the nucleus and becomes positive charge density, whereas the Υ -term ensures its density inside the nucleus remains negative. (iv) For weak coupling, $Z\alpha \ll 1$, yet with $mR < Z\alpha$, the “nonrelativistic” solution differs qualitatively from Schrödinger theory despite binding energies that are well below mc^2 ; agreement is recovered only when $Z\alpha \ll mR$. These phenomena highlight the distinctive and subtle behavior of spin-1 particles in the Coulomb field.

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I. INTRODUCTION

The Proca-Maxwell equation [1], which describes the interaction between a massive spin-1 particle and the electromagnetic field, gives unacceptable results to the Coulomb problem, e.g., the scattering cross section for large energy does not decrease with energy but tends to a constant [2–4]. An attempt to amend this situation was made by Corben and Schwinger [5], who introduced to the interaction Lagrangian a new term which has the effect of forcing the particle’s gyromagnetic factor to be $g = 2$ (actually, $g = 2$ is the gyromagnetic factor of the W boson in the standard model). With this value of g , a physically acceptable spectrum was found to be given by the same Sommerfeld formula which describes the spectrum of the atomic electron, with appropriate values for the total angular momentum $j = 0, 1, \dots$ of a spin-1 particle.

The Corben-Schwinger formalism is not, however, free from flaws. In particular, it was found that, for quantum states with $j = 0$ and $j = 1, l = 0$, the charge of the particle near the Coulomb center becomes infinite, signifying the particle’s fall to the center. Therefore, one could conclude that the critical charge for the pointlike nucleus corresponds to $Z\alpha = 1$ for spin-1/2 particles, $Z\alpha = 1/2$ for spin-0 particles, and $Z\alpha = 0$ for spin-1 particles. To resolve this issue, several authors have proposed further refinements [6–8] or completely new equations [9,10] for the vector boson. The problem of the vector particle spectrum has also been considered in Refs. [11–18].

Interesting features also include positive charge density of the negatively charged particle in a vicinity of the nucleus [19]. This behavior, unique to a vector particle, is caused by its electric quadrupole moment and spin-orbit interaction, which give dominating contributions to the charge density in the area of a rapid variation of the wave function. If the wave function is finite everywhere, the net effect of these contributions to the integral of the charge density is zero. However, in the Coulomb case, the divergent integral of this singular wrong sign charge density makes the problem look like an apparent charge nonconservation.

Reference [20] examined the wave function collapse problem within a renormalizable electroweak-type framework

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and showed that renormalizability alone does not ensure well-behaved solutions at short distances for nonperturbative problems such as bound states. In particular, the wave function of a charged vector particle collapses to a radius $r \sim 1/M$, where M is the mass of the heavy Coulomb source—and M can be arbitrarily large.

References [20,21] demonstrated, however, that the Corben-Schwinger formalism gives reasonable results, with the collapse problem disappearing if one takes into account the QED vacuum polarization from the beginning. This is in stark contrast to the situation with scalar and spinor particles, where vacuum polarization acts only as a perturbation to the pure Coulomb field and thus has small effects on the particle's behavior. The difference lies in the fact that the equation governing a spin-1 particle contains a special term (Υ -term) depending on the external current. With QED vacuum polarization taken into account, this term produces a repulsive impenetrable potential near $r = 0$ that prevents the particle from falling to the center. However, other electroweak vacuum polarization corrections, which appear in the standard model (and any asymptotically free model), change the sign of the Υ potential from repulsive to attractive [20]. The collapse can be prevented by addition of light fermions, which switch the ultraviolet behavior of the theory from the asymptotic freedom to the Landau pole (like in pure QED).

To investigate this behavior further, we follow another approach to the Coulomb problem, by considering the Coulomb center, henceforth called the nucleus, not as a pointlike object, but one with a finite size R . Then we recover the pointlike nucleus limit of the problem by taking $R \rightarrow 0$. In this finite R case, the problem with infinite charge near the center does not persist, since at small distances, the pure Coulomb potential with its singularity at the origin is replaced with a regular potential produced by the spread nuclear charge distribution inside the nucleus. It is also of interest to ask what effects this “rounded-off” Coulomb potential has on the spectrum and wave function of the vector boson. Indeed, in atoms, the finite size of the nucleus is responsible for the familiar field isotope shift between energy levels of two atoms with nuclei of the same charge Z but different mass numbers A . These nuclear-size corrections to the energy levels and wave functions of the electron may be calculated accurately with the use of normal perturbation theory, at least in the nonrelativistic limit (the relativistic case indicates some deviation from the simple perturbation theory approach due to the singularity of the Dirac wave function at the origin in the pointlike nucleus case, see, e.g., [22]). In contrast, the finite nuclear radius, even an extremely small one, produces radical changes in the results for a spin-1 particle.

Note that finite nuclear size strongly affects the value of the critical charge for electrons (see, e.g., Refs. [23,24]).

To avoid misunderstanding, we do not consider here any electroweak and finite nuclear mass effects. We are solving

a purely theoretical problem of a light charged vector particle in the field of the Coulomb center of finite size, focusing on the most interesting case $mR \ll Z\alpha \ll 1$. Note that the Coulomb potential inside the nucleus in this case exceeds mass, $Z\alpha/R > m$. This is not a problem for spin-1/2 and spin-0 particles; the Schrödinger equation still gives correct results. However, for spin-1 particles, a nonrelativistic solution for small interaction constant $Z\alpha$, for $mR < Z\alpha \ll 1$ differs from the solution obtained using the Schrödinger equation, in spite of the binding energy $E_b \ll mc^2$ for the Schrödinger-like energy levels (there is agreement with the Schrödinger equation results in the true nonrelativistic case $Z\alpha/R \ll m$).

The considered problem may also be relevant to behavior of a vector particle in the field of a small charged black hole. A singularity of the wave function strongly increases probability of the spin-1 particle capture by the black hole (compare to the spin-0 and spin-1/2 particles).

This paper is organized as follows. In Sec. II, for the purpose of fixing notations, we represent the Corben-Schwinger equation for a vector boson interacting with a static electric field. In Sec. III, we find the energy of the vector boson for the finite size of the nucleus, using the Wentzel-Kramers-Brillouin (WKB) semiclassical method. We also discuss peculiar behavior of the charge density and the effects of the vacuum polarization potentials. A summary of the findings is presented in Sec. IV. For comparison, in the Appendix we present the magnetic polarization case where the wave function does not collapse and a finite nuclear radius R yields only small corrections; the wave equation in this case coincides with the Klein-Gordon equation for spin-0 particles. Throughout the paper, the natural units $\hbar = c = 1$ and $e^2 = 4\pi\alpha$ are used, where $e < 0$.

II. WAVE EQUATIONS FOR VECTOR PARTICLE

The Lagrangian for a vector boson of mass m and charge e propagating in an external electromagnetic field has the form [2]

$$\mathcal{L} = -\frac{1}{2}(\nabla_\mu W_\nu - \nabla_\nu W_\mu)^\dagger(\nabla^\mu W^\nu - \nabla^\nu W^\mu) + ieF^{\mu\nu}W_\mu^\dagger W_\nu + m^2 W_\mu^\dagger W^\mu, \quad (1)$$

where W_μ is the vector boson field, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and $\Delta_\mu \equiv \partial_\mu + ieA_\mu$ is the covariant derivative. Here, A_μ is the electromagnetic four-potential.

The Euler-Lagrange equation resulting from the Lagrangian (1) reads

$$(\nabla^2 + m^2)W^\mu + 2ieF^{\mu\nu}W_\nu - \nabla^\mu \nabla^\nu W_\nu = 0, \quad (2)$$

which, after taking covariant derivative, gives

$$m^2 \nabla_\mu W^\mu + ie j_\mu W^\mu = 0, \quad (3)$$

where $j^\mu \equiv \partial_\nu F^{\mu\nu}$ is the external current that produces the potential A_μ .

In the static case where $j^\mu = (\rho(\mathbf{r}), \mathbf{0})$, one can assume that

$$\nabla_0^2 W_\mu = -(\epsilon - U)^2 W_\mu, \quad (4)$$

where ϵ is the vector boson's energy and $U(\mathbf{r})$ is its potential energy. Equation (3) then becomes

$$iW_0 = (\epsilon - U - \Upsilon)^{-1} \nabla \cdot \mathbf{W}, \quad (5)$$

where W_0 and $\mathbf{W}$ are the temporal and spatial components of W^μ and Υ is defined by

$$\Upsilon \equiv e\rho/m^2 = -\Delta U/m^2. \quad (6)$$

Inserting Eq. (5) into Eq. (2), one finds the equation

$$[(\epsilon - U)^2 - m^2] \mathbf{W} = -\Delta \mathbf{W} - 2w\Delta U - \nabla(\Upsilon w), \quad (7)$$

where $w \equiv iW_0$. For later use, we note that the charge density of the vector boson is given by

$$\begin{aligned} \rho^W = 2e[(\epsilon - U)(\mathbf{W}^\dagger \cdot \mathbf{W} + w^\dagger w) \\ + \mathbf{W}^\dagger \cdot \nabla w + \mathbf{W} \cdot \nabla w^\dagger - \Upsilon w^\dagger w]. \end{aligned} \quad (8)$$

If we assume that the potential U is spherical then the total angular momentum j of the vector particle is conserved and its (vector) wave function $\mathbf{W}$ may be separated into magnetic, electric, and longitudinal components. Also, due to their different parities, the magnetic polarization decouples from the electric and longitudinal polarizations. As a result of this decoupling, we may consider independently the magnetically polarized mode where

$$\mathbf{W} = f(r) \mathbf{Y}_{jm}^{(m)}, \quad (9)$$

and the electrolongitudinally polarized mode where

$$\mathbf{W} = u(r) \mathbf{Y}_{jm}^{(e)} + v(r) \mathbf{Y}_{jm}^{(l)}. \quad (10)$$

Here, $f(r)$, $u(r)$, and $v(r)$ are radial wave functions and $\mathbf{Y}_{jm}^{(m,e,l)}$ are vector spherical harmonics, defined as

$$\mathbf{Y}_{jm}^{(e)} \equiv \nabla_{\mathbf{n}} Y_{jm}(\mathbf{n}) / \sqrt{j(j+1)}, \quad (11a)$$

$$\mathbf{Y}_{jm}^{(m)} \equiv \mathbf{n} \times \mathbf{Y}_{jm}^{(e)}, \quad (11b)$$

$$\mathbf{Y}_{jm}^{(l)} \equiv \mathbf{n} Y_{jm}(\mathbf{n}), \quad (11c)$$

with $\mathbf{n} = \mathbf{r}/r$ being the unit vector in the direction of $\mathbf{r}$, Y_{jm} the spherical harmonics, and $\nabla_{\mathbf{n}}$ the angular part of the gradient operator, i.e., $\nabla F(\mathbf{n}) = \nabla_{\mathbf{n}} F(\mathbf{n})/r$. The magnetic and electrolongitudinal modes, corresponding to the terms $\mathbf{Y}_{jm}^{(l)}$ and $\mathbf{Y}_{jm}^{(e)}$ in Eqs. (9) and (10), are only defined for $j > 0$. For $j = 0$, only the purely longitudinal mode, corresponding to the term $Y_{jm}(\mathbf{n})$ in Eq. (10), exists.

Below, we will solve Eq. (7) in the finite nucleus potential for the purely longitudinal polarizations ($j = 0$), which is the most interesting case.

III. LONGITUDINAL POLARIZATION, $j=0$

A. Wave equation for longitudinal polarization

In the case where $j = 0$, the wave function $\mathbf{W}$ is purely longitudinal and we have

$$\mathbf{W} = v(r) \mathbf{n}, \quad (12)$$

which, when substituted into Eq. (7), gives the following equation for v :

$$\frac{d^2 v}{dr^2} + G \frac{dv}{dr} + H v = 0, \quad (13)$$

where

$$\begin{aligned} G &= \frac{2}{r} + \frac{U'}{\epsilon - U} + \frac{U' + \Upsilon'}{\epsilon - U - \Upsilon}, \\ H &= \frac{(\epsilon - U - \Upsilon)[(\epsilon - U)^2 - m^2]}{\epsilon - U} \\ &\quad - \frac{2}{r} \left(\frac{1}{r} - \frac{U'}{\epsilon - U} - \frac{U' + \Upsilon'}{\epsilon - U - \Upsilon} \right). \end{aligned} \quad (14)$$

For the solution outside the nucleus, where $U = -\zeta/r$ ($\zeta = Z\alpha$) and $\Upsilon = 0$, Eq. (13) reduces to

$$\begin{aligned} \frac{d^2 v}{dr^2} + 2 \left(\frac{1}{r} + \frac{U'}{\epsilon + \zeta/r} \right) \frac{dv}{dr} \\ + \left[\left(\epsilon + \frac{\zeta}{r} \right)^2 - m^2 - \frac{2}{r^2} + \frac{4U'}{r(\epsilon + \zeta/r)} \right] v = 0. \end{aligned} \quad (15)$$

We follow Ref. [21] and introduce the dimensionless variable $x \equiv \epsilon r/\zeta$ and perform the transformation

$$v = \frac{1+x}{x^2} \left(\frac{d}{dx} + L \frac{1+x}{x} - \frac{1}{1+x} \right) \tilde{\varphi}, \quad (16)$$

with which Eq. (13) becomes

$$\frac{d^2 \tilde{\varphi}}{dx^2} + \left[\frac{2\zeta^2}{x} - \frac{L(L+1)}{x^2} \right] \tilde{\varphi} = \eta^2 \tilde{\varphi}, \quad (17)$$

where $\eta \equiv \zeta \sqrt{m^2 - \epsilon^2}/\epsilon$ and $L \equiv \gamma + 1/2 = \sqrt{1/4 - \zeta^2} + 1/2$. Equation (17) is a Whittaker equation, which has solution

$$\tilde{\phi} = c_1 M_{\frac{\zeta^2}{\eta}, L+\frac{1}{2}}(2\eta x) + c_2 W_{\frac{\zeta^2}{\eta}, L+\frac{1}{2}}(2\eta x). \quad (18)$$

In the case of a pointlike nucleus, the solution $W_{\frac{\zeta^2}{\eta}, L+\frac{1}{2}}(2\eta x)$ is singular at $r=0$ and must thus be excluded. The requirement that $\tilde{\phi}$ be regular at infinity forces the eigenvalue η to take the form

$$\eta = \frac{\zeta^2}{L + n - 1}, \quad n = 2, 3, \dots, \quad (19)$$

which gives the energy spectrum [5]

$$\epsilon = \epsilon_{n0} = m \left[1 + \frac{\zeta^2}{(\gamma + n - 1/2)^2} \right]^{-1/2}. \quad (20)$$

This result looks similar to the Sommerfeld formula for fermions. The difference is that in our case $j = 0$.

Note that this solution does not include contribution of Υ , which for the pointlike nucleus is proportional to $\delta(r)$ (the contact interaction). As we demonstrate below, incorporating the Υ contribution profoundly alters the result.

B. Semiclassical (WKB) solution for the finite nucleus case

We consider the solution for the longitudinal mode $j = 0$ which includes effects of Υ potential localized inside the nucleus. We can eliminate the first derivative of v using substitution [21]

$$v = \frac{1}{mr} [(\epsilon - U)(\epsilon - U - \Upsilon)]^{1/2} \phi. \quad (21)$$

Then the equation for ϕ gets a Schrödinger-like form

$$-\frac{d^2 \phi}{dr^2} + V \phi = 0, \quad (22)$$

where

$$V = -H + \frac{1}{4} G^2 + \frac{1}{2} \frac{dG}{dr}. \quad (23)$$

Note that, in the nonrelativistic case [$E = \epsilon - m$, $m \gg (E - U)$, $m \gg \Upsilon$], Eq. (21) is reduced to $v = \phi/r$, which is a conventional substitution transforming a radial equation into a Schrödinger one-dimensional equation with the effective potential $U_{\text{eff}} = U + l(l+1)/(2mr^2)$.

The semiclassical WKB solution of a one-dimensional Schrödinger equation is presented as

$$\phi \sim \frac{1}{\sqrt{p}} \sin \left(\int_a^r p dr + \theta_0 \right), \quad (24)$$

where $\theta_0 \lesssim 1$, p is the classical momentum, and a is the classical turning point, where $p = 0$.

Let us first consider the nonrelativistic Coulomb energy levels $E_n = \epsilon_n - m$, based on the Bohr-Sommerfeld quantization rule which follows from the boundary conditions for ϕ in Eq. (24),

$$\int_a^b p dr = \pi n + \theta, \quad (25)$$

where $\theta \lesssim 1$, and a, b are the classical turning points, where $p = 0$; on the distance $r > b$ the wave function decays exponentially.

For an illustration, we first find from this equation the ordinary nonrelativistic Coulomb energy levels for the pointlike nucleus case. We should assume the nonrelativistic energy $E \ll m$ and the potential $U \ll m$. This gives us the conventional Schrödinger equation for ϕ . Semiclassical approximation assumes that momentum is large ($pr \gg 1$) and its derivative is relatively small. Keeping only the dominating term $\frac{d^2 \phi}{dr^2} \propto p^2$ in Eq. (22), we obtain the conventional expression for the classical momentum $p = \sqrt{2m(E - U_{\text{eff}})}$, where $U_{\text{eff}} = -Z\alpha/r + l(l+1)/(2mr^2)$. The orbital angular momentum is $l = 1$ for $j = 0$ and spin-1.

The integration in Eq. (25) gives

$$\pi \left[Z\alpha \sqrt{\frac{m}{2(-E_n)}} - \sqrt{l(l+1)} \right] \approx \pi n_0. \quad (26)$$

This gives Coulomb energy

$$E_n = -\frac{mZ^2\alpha^2}{2n^2}, \quad (27)$$

where the principal quantum number $n \approx n_0 + \sqrt{l(l+1)}$. There is a small difference with accurate expression $n = n_0 + l$. However, the Bohr-Sommerfeld rule gives accurate results for $n \gg 1$, where the WKB approximation is formally applicable. Moreover, for $n \gg l$ we may simply neglect the centrifugal term contribution to the energy.

Now consider contribution of the finite nuclear size to the Bohr-Sommerfeld integral (25),

$$\int_0^R (p_i - p_c) dr, \quad (28)$$

where p_i is the classical momentum within the nucleus and $p_c = \sqrt{2m(E + Z\alpha/r)}$ is the classical momentum in the Coulomb field of the pointlike nucleus, which we should subtract since we have already included its contribution to Eq. (26).

We will consider the most interesting case $mR \ll Z\alpha \ll 1$. This condition gives us $m \ll U \ll \Upsilon$ inside the nucleus. The case of the nonrelativistic Coulomb energies $E_n \ll m \ll U \ll \Upsilon$, which we are considering in this subsection, gives us an additional energy-independent contribution

$$\int_0^R (p_i - p_c) dr \equiv \pi\Delta, \quad (29)$$

to the Bohr-Sommerfeld equation

$$\pi \left(\Delta + Z\alpha \sqrt{\frac{m}{2(-E_n)}} \right) = \pi n. \quad (30)$$

This gives a corrected energy

$$E_n = -\frac{mZ^2\alpha^2}{2\nu^2}, \quad (31)$$

where $\nu = n - \Delta$ is the effective principal quantum number. We may say that the levels are renumbered. The magnitude of $\Delta = n - \nu$ indicates the number of orbitals with the same angular quantum numbers ($j = 0$ for the longitudinal polarization) inside the nucleus, $n_{\text{in}} \sim \Delta$.

Parameter Δ and the effective principal quantum number $\nu = n - \Delta$ are not expected to be integer numbers, so the energy levels do not coincide with the energy levels in the Coulomb potential with a pointlike nucleus. However, the centrifugal barrier for $l = 1$ separates two areas of the classical radial motion, outside the nucleus and inside the nucleus, so there are two nearly independent contributions to the phase. This may reduce the difference with the pure Coulomb spectrum after the renumbering of the levels.

This formula (31) is similar to the Rydberg formula for the energy of the external electron traveling far from atomic core in the Coulomb field of the remaining ion. The explanation is similar. The energy of the external electron E_n is negligible within the electron core, $E_n \ll U$. Therefore, the phase shift $\pi\Delta$ is energy independent. Taking into account that the potential for the external electron outside the core is $U = -(Z - N + 1)\alpha/r$, we obtain the Rydberg formula for external electron energy

$$E_n^{\text{Rydberg}} = -\frac{m(Z - N + 1)^2\alpha^2}{2\nu^2}, \quad (32)$$

where N is the total number of electrons (for a neutral atom $N = Z$), and $\nu = n - \Delta$ is the effective principal quantum number. The magnitude of $\Delta = n - \nu$ indicates the number of orbitals with the same angular quantum numbers inside the electron core, $n_{\text{core}} \sim \Delta - n_{\text{min}}$, where $n_{\text{min}} = l + 1$ is the minimal principal quantum number for a given orbital angular momentum l . Typical value of ν for the lowest external electron level in a neutral atom is given by $1 < \nu < 2$.

We may say that we, in fact, shifted the boundary condition for the external electron from $r = 0$ to the boundary of the electron core $R_c \sim a_B$ (a_B is the Bohr radius) and then solved the ordinary Coulomb problem. In our spin-1 particle case, we may provide a similar qualitative explanation for the Rydberg-type equation (31) for energy levels: we effectively shifted the boundary condition from $r = 0$ to the nuclear surface $r = R$ and then solved the Coulomb problem in the potential $U = -Z\alpha/r$ with this new boundary condition at $r = R$.

Because of the condition $E \ll m \ll U \ll \Upsilon$, we may neglect E and m inside the nucleus and estimate classical momentum using Eqs. (22) and (23): $p^2 \approx U\Upsilon$. Assuming constant nuclear charge density $\rho = 3Z|e|/(4\pi R^3)$, we obtain

$$\Upsilon = -\frac{3Z\alpha}{m^2 R^3}, \quad U = -\frac{Z\alpha}{R} \left(\frac{3}{2} - \frac{r^2}{2R^2} \right). \quad (33)$$

Integration in Eq. (28) gives the principal quantum number shift

$$\Delta \approx 0.6 \frac{Z\alpha}{mR} \gg 1. \quad (34)$$

This means that there are many orbitals confined to the nucleus, on the distance $r \lesssim R$. For $R \rightarrow 0$ the number of such states tends to infinity. This result agrees with the conclusion of Ref. [25].

C. Vector particle states inside the nucleus

Condition $mR \ll Z\alpha \ll 1$ gives us $m \ll U \ll \Upsilon$ inside the nucleus. However, the uncertainty relation for the localized states, $p \gtrsim 1/R$, indicates that we cannot neglect ϵ . Then the substitution of the semiclassical wave function (24) to Eqs. (22) and (23) gives $p^2 \approx \Upsilon(U - \epsilon)$, and the Bohr-Sommerfeld quantization equation

$$\int_0^{r_t} [\Upsilon(U - \epsilon)]^{1/2} dr \approx \pi n. \quad (35)$$

This equation is valid if the classical turning point $r_t < R$, equivalent to $\epsilon < -Z\alpha/R$. Calculation of this integral gives

$$\epsilon \approx -\frac{3Z\alpha}{2R} + \sqrt{\frac{8}{3}} mn. \quad (36)$$

The condition $\epsilon < -Z\alpha/R$ gives

$$n < 0.3 \frac{Z\alpha}{mR}. \quad (37)$$

Levels with higher n may have a tail outside the nucleus. For $R \rightarrow 0$, the number of levels inside the nucleus tends to infinity.

At large distance, when potential U may be neglected, the wave equation depends on ϵ^2 , so both $\epsilon > m$ and $\epsilon < -m$ solutions belong to the continuum spectrum. This means that all states described by Eq. (36) are not true eigenstates with real values of energy, but they are resonances with a large width corresponding to complex energy poles in the W^+ scattering amplitude (compare with similar complex energy electron states with $\epsilon < -m$ in nuclei with charge $Z > 170$ described in the review [26]). In the formal solution of the wave equations, wave functions of such resonance states at real energies, Eq. (36), have an oscillating long-distance tail instead of exponential decay. Near the resonance positions described by Eq. (36), the ratio of the squared wave function at small distance divided by the squared wave function amplitude at large distance has the Breit-Wigner-type enhancement. Note that, for states with $\epsilon < -m$, the potential U presents a potential barrier since it appears in the wave equations as a combination $\epsilon - U$.

Existence of states with $\text{Re}(\epsilon) < -m$ corresponds to creation of particle-antiparticle pairs, the vacuum breakdown. In a dynamical picture, negatively charged particles, produced this way, neutralize nuclear charge, stopping further pair production. In other words, the nuclear charge is screened completely by the particles from vacuum.

The vacuum breakdown problem in the case of spin-1/2 fermions and a possibility of its experimental observation in collision of heavy ions (e.g., uranium ions), where the sum of the ion charges exceeds the critical charge, has been intensively considered for many years (see, e.g., Refs. [26–28] and references therein). There is also a number of works considering vacuum breakdown in supercritical field for spin-0 bosons (see, e.g., [29]). An interesting phenomenon in this case is Bose condensation of scalar particles in the supercritical field and the effect of stabilization [30,31].

Our estimates are based on the inequality $\Upsilon \gg U$. However, the accuracy of our semiclassical estimate of the energy levels may be affected by the singularity $\sim 1/(\epsilon - U)$ in the wave equations (22), (23), and (14), which is close to the classical turning point $\epsilon - m = U$. However, this singular term only slightly shifts the classical turning point $p = 0$. The interval near the turning point, where neglected terms are significant, is small, $x \sim (mR/Z\alpha)^{2/3}R$, and excluding this interval does not affect the Bohr-Sommerfeld integral significantly, if $n \gg 1$. A similar argument justifies neglecting the centrifugal term $\sim 1/r^2$.

For further analysis, it is convenient to present the wave equations in the two-component form (v, w) , which looks similar to the radial Dirac equations for electrons,

$$\begin{aligned} [(\epsilon - U)^2 - m^2]v &= -(\epsilon - U)w' - U'w, \\ (\epsilon - U - \Upsilon)w &= \frac{2v}{r} + v'. \end{aligned} \quad (38)$$

From these equations, we see that v and w have no singularities at $(\epsilon - U) = 0$; they pass this point smoothly. Similar to the Dirac equation, such singularity appears only when we substitute the lower component w into the equation for the upper component v and divide equation by $(\epsilon - U)$, to have coefficient before v'' equal to 1.

Note that the lower component w in the nonrelativistic limit $\epsilon \approx m \gg U$ at large distance r is proportional to $(p/m)v$, similar to the lower component in the Dirac equation.

D. Vacuum polarization

In the case of a pointlike nucleus, the vacuum polarization potential at small distance $rm_z \ll 1$ (m_z is the maximal mass of the vacuum particle) may be presented as

$$U_v(r) = [\alpha\beta \ln(m_z r)] \frac{Z\alpha}{r}, \quad (39)$$

where β is the lowest-order coefficient of the Gell-Mann–Low β function. This gives for $r > 0$,

$$\Upsilon_v = \frac{e\rho_v}{m^2} = -\frac{\Delta U_v}{m^2} = \frac{Z\alpha^2\beta}{m^2 r^3}. \quad (40)$$

In QED, $\beta > 0$ and Υ produces an impenetrable barrier which prevents collapse of the vector particle wave function [21]. However, in asymptotically free theories with $\beta < 0$ there is no such barrier.

In addition to the polarization charge distribution outside the nucleus, there is a $\delta(r)$ -term [from $\Delta(1/r)$ factor in Eqs. (39) and (40)] bringing the total vacuum polarization charge to zero. The integral of the polarization charge outside the nucleus is logarithmically divergent as $\int dr/r$, so the coefficient before the $\delta(r)$ is infinite. One can also see this from the divergent at $r \rightarrow 0$ factor $\ln(m_z r)$ in Eqs. (39) and (40).

Finite nuclear radius R eliminates this divergency and produces Υ_v inside the nucleus, which has opposite sign to that of Υ_v outside the nucleus. In the QED case with $\beta > 0$, the particle penetrates the Υ_v barrier and falls to the deep potential well inside the nucleus, while in the asymptotically free case $\beta < 0$, the vacuum polarization barrier appears inside the nucleus.

E. Charge density

Using Eqs. (8) and (12), we write the charge density as

$$\rho^W = 2e \left[(\epsilon - U)(v^2 + w^2) + 2v \frac{dw}{dr} - \Upsilon w^2 \right], \quad (41)$$

where w is defined as

$$w \equiv \frac{1}{\epsilon - U - \Upsilon} \left(\frac{dv}{dr} + \frac{2v}{r} \right). \quad (42)$$

For a pointlike nucleus, in the relativistic region $r \ll \zeta/m$, the wave function v behaves as $v \sim r^{L-2}$ so

$$w \sim Lr^{L-2}, \quad (43)$$

and, as a result, in this regime, the charge density (41) behaves as (recall that $L = \sqrt{1/4 - \zeta^2} + 1/2$)

$$\rho^W \sim -2eL(3 - 2L)r^{2L-5}/\zeta. \quad (44)$$

This charge density diverges so badly as $r \rightarrow 0$ that the total charge $Q = \int \rho^W d^3r$ localized inside any small sphere around the nucleus is infinite. It is also of a “wrong” sign (positive for a negatively charged vector particle), as pointed out in Ref. [19]. This looks like the integral of the charge density has opposite sign to the particle charge e .

Let us now consider our case of the finite nucleus. As we have shown above, there are states localized mainly inside the nucleus. However, we will show that even for the states having a Schrödinger-like spectrum, a finite fraction of their charge is located inside the nucleus for $R \rightarrow 0$.

Let us first estimate the contribution of the charge density without the Υ -term. A singular part of the vector particle charge density outside may be presented as $\rho^W(r) = \rho^W(R)(r/R)^{2\gamma-4}$. Charge density inside is $\rho^{no\Upsilon}(r) \approx \rho^W(R)$. Integrating charge density inside from 0 to R and charge density outside from R to infinity with $r^2 dr$, we obtain

$$\frac{Q_{in}^{no\Upsilon}}{Q_{out}} \approx \frac{1 - 2\gamma}{3} \approx \frac{2Z^2\alpha^2}{3}. \quad (45)$$

We see that this ratio does not depend on R , i.e., the vector particle has a finite probability to be inside the nucleus even in the case when the nucleus has zero size. Also, both charges inside and near the nucleus have the wrong sign.

Now include the $\Upsilon = -\frac{3Z\alpha}{m^2 R^3} \gg U$ contribution to w . This gives the ratio of the charge inside to the charge outside as

$$\frac{Q_{in}}{Q_{out}} \sim -\frac{2(Z\alpha)^4}{3}, \quad (46)$$

which stays finite for $R \rightarrow 0$. This means that even for the states with the Schrödinger-type spectrum their charge has finite fraction inside the nucleus, if $mR \ll Z\alpha$.

Note that while the particle charge density changes sign in a vicinity of the nucleus and ρ_W becomes positive, ρ_W stays negative inside the nucleus due to the large Υ -term in the denominator of Eq. (42) suppressing w inside the nucleus.

IV. SUMMARY

The Coulomb bound-state problem for a charged vector particle is pathological at short distances: its wave function collapses onto the Coulomb center already for infinitesimal coupling ($\zeta = Z\alpha \rightarrow 0$) [5]. By contrast, the critical charge for the pointlike nucleus is $Z\alpha = 1/2$ for a spin-0 particle and $Z\alpha = 1$ for a spin-1/2 particle.

To investigate this behavior further, we have solved the problem for a light charged vector particle in the field of a finite-size, infinitely heavy nucleus, neglecting electroweak corrections and nuclear recoil. The pointlike limit is recovered by taking $R \rightarrow 0$. This approach allows us to include a critically important Υ -term in the wave equations, which for the pointlike nucleus is proportional to $\delta(r)$ and was ignored in the previous calculations of the energy spectrum.

We focus on the most interesting case of the longitudinal polarization with the angular momentum $j = 0$ and a small nuclear radius, $mR \ll Z\alpha \ll 1$. Note that the Coulomb potential inside the nucleus in this case exceeds mass, $Z\alpha/R > m$. This is a typical situation for atomic electrons and does not create any problem; the Schrödinger equation still gives correct results. However, for spin-1 particles the difference with the Schrödinger equation solution is profound, even for a very small interaction constant $Z\alpha$ (there is agreement with the Schrödinger equation results in the true nonrelativistic case $Z\alpha/R \ll m$, when $\Upsilon \ll U \ll m$).

The spectrum splits naturally into two parts. A set of “deep” states lies predominantly inside the nucleus. Their number grows as $N \sim 0.6Z\alpha/mR$, so $N \rightarrow \infty$ as the nuclear radius R shrinks to zero. The energies of these intranuclear states (in Gaussian units) are

$$\epsilon \approx -\frac{3Ze^2}{2R} + \sqrt{\frac{8}{3}}mc^2n, \quad (47)$$

which places them deep in the negative-energy continuum, $\epsilon \sim -Ze^2/R < -mc^2$.¹ Physically, this means the Coulomb field becomes strong enough to pull particle-antiparticle pairs out of the vacuum (“vacuum breakdown”). The negative charges produced in this way screen the nuclear charge, halting further pair creation and driving the system toward complete charge neutralization.²

¹Wave functions of states with $\epsilon < -mc^2$ do not vanish at large distance. These states are not true bound states with real values of energy, but they are resonances with a finite width corresponding to complex energy poles in the W^+ scattering amplitude.

²The negative-energy states and vacuum breakdown may be eliminated by the additional Compton-like diagram, which appears in a renormalizable model, in which the heavy particle producing Coulomb potential has a neutral partner. This diagram produces repulsive potential on the distance $r \lesssim 1/M$, where M is the mass of the partner [20]. Nevertheless, the wave function still exhibits pathological behavior, collapsing to $r \sim 1/M$, where M may be arbitrary large.

The second part of the spectrum (in Gaussian units) is given by

$$E_n = -\frac{mZ^2e^4}{2\nu^2}, \quad (48)$$

where $E_n = \epsilon - mc^2$ is the nonrelativistic energy, $\nu = n - \Delta$ is the effective principal quantum number, and

$$\Delta \approx 0.6 \frac{Z\alpha}{mR} \gg 1. \quad (49)$$

The magnitude of $\Delta = n - \nu$ indicates the number of levels with $j = 0$ inside the nucleus. Parameter Δ and the effective principal quantum number $\nu = n - \Delta$ are not integer numbers, so the energy levels do not coincide with the (renumbered) energy levels in the Coulomb potential with a pointlike nucleus. This formula is similar to the Rydberg formula for energy of an external electron traveling far from atomic core in the Coulomb field of the remaining ion.

The ratio of the charge inside the nucleus to the charge outside the nucleus for such Sommerfeld-type levels is estimated as

$$\frac{Q_{\text{in}}}{Q_{\text{out}}} \sim -\frac{2(Z\alpha)^4}{3}, \quad (50)$$

which stays finite for $R \rightarrow 0$. This means that the charge of the states with “ordinary” Sommerfeld-type spectrum has a finite fraction inside the nucleus, even for an infinitely small nucleus.

Reference [19] found that the charge density ρ_W for a negatively charged vector particle near the nucleus changes sign and becomes positive. This effect is due to the dominating contribution of the vector boson’s electric quadrupole moment and spin-orbit interaction in the area of rapidly varying wave function. Since the integral of this wrong sign density is divergent for a pointlike nucleus, this looks like a paradox; the charge of a vector particle in the Coulomb field changes sign. However, we found that for finite R , the charge density ρ_W is negative inside the nucleus (due to the Υ -term contribution).

Vacuum polarization modifies the Coulomb field at $r \ll 1/m_z$ as $U_v(r) = [\alpha\beta \ln(m_z r)]Z\alpha/r$, inducing $\Upsilon_v(r) = Z\alpha^2\beta/m^2 r^3$ plus a compensating $\delta(r)$ -term so the total induced charge vanishes. For QED ($\beta > 0$) this gives a repulsive $1/r^3$ barrier outside the nucleus that suppresses short-distance collapse; a finite nuclear radius R regularizes the divergence and produces an opposite-sign Υ_v inside, effectively creating a deep inner well accessible by penetration. In asymptotically free theories ($\beta < 0$) the exterior barrier is absent and a barrier appears inside the nucleus.

These results show that the Coulomb problem for spin-1 particles requires special regularization: the probability

density exhibits an unavoidable short-distance collapse unless stabilized by the finite nuclear mass, finite nuclear size, or additional short-range interactions. Note that the Higgs boson exchange produces an attractive potential that cannot provide a cure.

Further investigation has shown that there is no collapse of the W wave function in either the Lorentz-scalar Yukawa potential or in the gravitational field.³ We therefore conclude that wave function collapse is specific to vector-mediated interactions (Coulomb photon exchange and neutral weak Z exchange) and does not occur for Lorentz-scalar or gravitational interactions.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: MAGNETIC POLARIZATION, $j \geq 1$

In this appendix, we consider the magnetically polarized mode. Substituting Eq. (9) into Eq. (7), we obtain the following equation for the radial wave function f :

$$[\Delta_j + (\epsilon - U)^2 - m^2]f = 0, \quad (\text{A1})$$

where

$$\Delta_j \equiv \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{j(j+1)}{r^2}. \quad (\text{A2})$$

One recognizes Eq. (A1) as the Klein-Gordon equation.

For simplicity, we assume a model in which U is a Coulomb potential with a sharp cutoff at the nuclear radius R ,

³Nonstationary quasibound states of vector particles in the Schwarzschild black hole metric were found in Ref. [32]. To consider stationary states, we studied a near-black-hole metric in which the radius of the central body R slightly exceeds the gravitational radius r_g and then took the limit $R \rightarrow r_g$. This approach parallels that of Refs. [33–36], where the cases of massless vector particles (photons) as well as massive fermions and scalars were examined. These works reported a dense spectrum of resonances and—for massive particles—bound states inside the body (with the spectrum becoming infinitely dense as $R \rightarrow r_g$). However, this spectrum is unrelated to the collapse of the wave function observed for vector particles in the Coulomb field. In particular, we find no qualitative difference between a scalar field and the $j = 0$ component of a vector field in this respect.

$$U = -\frac{\zeta}{R}\theta(R-r) - \frac{\zeta}{r}\theta(r-R), \quad (\text{A3})$$

where $\zeta \equiv Z\alpha$. This potential, which is produced by a uniform distribution of the nuclear charge on the surface of the nucleus, allows for an analytical solution.

Inside the nucleus, where the potential is constant, $U = -\zeta/R$, the solution to Eq. (A1) which is regular at $r = 0$ reads

$$f_{\text{in}} = j_j(kr), \quad (\text{A4})$$

where $k \equiv \sqrt{(\epsilon + \zeta/R)^2 - m^2}$ and j_j is the spherical Bessel wave function of the first kind.

Outside the nucleus, where $U = -\zeta/r$, a general solution to Eq. (A1) has the form

$$f = [c_1 M_{\lambda,\gamma}(2\beta r) + c_2 W_{\lambda,\gamma}(2\beta r)]/r, \quad (\text{A5})$$

where $\beta \equiv \sqrt{m^2 - \epsilon^2}$, $\lambda \equiv \zeta\epsilon/\sqrt{m^2 - \epsilon^2}$, and $\gamma \equiv \sqrt{(j+1/2)^2 - \zeta^2}$. Here, $M_{\lambda,\gamma}$ and $W_{\lambda,\gamma}$ are Whittaker functions of the first and second kind, respectively.

In the pointlike case, since $W_{\lambda,\gamma}(2\beta r)$ is singular at $r = 0$, it must be excluded from the solution (A5). Requiring the function $M_{\lambda,\gamma}(2\beta r)$ to be finite at $r = \infty$ then results in the Sommerfeld formula for the energy eigenvalue

$$\epsilon_{nj} = m \left[1 + \frac{\zeta^2}{(\gamma + n - j - 1/2)^2} \right]^{-1/2}, \quad (\text{A6})$$

where $n = 1, 2, \dots$ is the principle quantum number.

In the case of a nonpointlike nucleus, the solution (A5) is valid only for the region outside the nucleus, where $r \neq 0$, so $W_{\lambda,\gamma}(2\beta r)$ is admissible and automatically satisfies the finiteness condition at infinity. On the other hand, since the energy ϵ receives corrections due to the finite size of the nucleus, it no longer satisfies the Sommerfeld condition (A6). As a result, $M_{\lambda,\gamma}(2\beta r)$ is now irregular at $r = \infty$ and must thus be excluded from Eq. (A5), giving

$$f_{\text{out}} = W_{\lambda,\gamma}(2\beta r)/r. \quad (\text{A7})$$

By matching the logarithmic derivatives of the two solutions (A4) and (A7) at the nuclear boundary $r = R$, one is able to find the nuclear-size correction to the Sommerfeld energy (A6). Since we assume that $mR, \zeta \ll 1$, the parameter kR is always small⁴ and we may find the logarithmic derivative of the “inside” wave function as

⁴For later use, we note in passing that in the relativistic limit $\zeta/R \gg m$, $kR \approx \zeta$, whereas in the nonrelativistic limit $\zeta/R \ll m$, $kR \approx \sqrt{2\zeta m R}$.

$$\left. \frac{f'_{\text{in}}}{f_{\text{in}}} \right|_{r=R} = \frac{\xi_j(R)}{R}, \quad (\text{A8})$$

where

$$\begin{aligned} \xi_j(R) &\equiv \frac{kR[j_{j-1}(kR) - j_{j+1}(kR)]}{2j_j(kR)} - \frac{1}{2} \\ &\approx j - \frac{(kR)^2}{2j+3}. \end{aligned} \quad (\text{A9})$$

On the other hand, outside the nucleus, since $2\beta R = 2\sqrt{m^2 - \epsilon^2}R < 2mR \ll 1$, for $r \approx R$, we can make use of the following expansion of the Whittaker function:

$$\begin{aligned} W_{\lambda,\gamma}(x) &\approx \frac{\Gamma(-2\gamma)x^{\gamma+1/2}}{\Gamma(-\gamma+1/2-\lambda)} \left(1 - \frac{\lambda x}{2\gamma+1} \right) \\ &\quad + \frac{\Gamma(2\gamma)x^{-\gamma+1/2}}{\Gamma(\gamma+1/2-\lambda)} \left(1 + \frac{\lambda x}{2\gamma-1} \right), \end{aligned} \quad (\text{A10})$$

and write

$$\begin{aligned} f_{\text{out}} &\approx r^{\gamma-1/2} \left(1 - \frac{2\zeta\epsilon r}{2\gamma+1} \right) \\ &\quad + \frac{\Gamma(2\gamma)\Gamma(-\gamma+1/2-\lambda)}{\Gamma(-2\gamma)\Gamma(\gamma+1/2-\lambda)} \frac{r^{-\gamma-1/2}}{(2\beta)^{2\gamma}} \left(1 + \frac{2\zeta\epsilon r}{2\gamma-1} \right), \end{aligned} \quad (\text{A11})$$

where we have taken out the common factor $\Gamma(-2\gamma)(2\beta)^\gamma/\Gamma(-\gamma+1/2-\lambda)$.

At a first glance, it appears that by setting $r = R$ and taking the limit $R \rightarrow 0$, the second term in Eq. (A11) is singular. However, we point out that as $R \rightarrow 0$, the energy ϵ assumes the Sommerfeld form (A6), which is equivalent to $\gamma + 1/2 - \lambda = -n + j + 1 = 0, -1, \dots$, so the function $\Gamma(\gamma + 1/2 - \lambda) = \Gamma(-n + j + 1)$ becomes infinite. This singularity of the Gamma function necessarily cancels that of $R^{-\gamma-1/2}$, making the wave function f finite. In fact, as $R \rightarrow 0$, the solution $W_{\lambda,\gamma}(\rho)$ becomes $M_{\lambda,\gamma}(\rho)$.

In order to account for this behavior of the wave function f_{out} , we expand the Gamma function around its pole as

$$\begin{aligned} \Gamma(\gamma + 1/2 - \lambda) &\approx \Gamma\left(\gamma + 1/2 - \lambda(\epsilon_{nj}) - \frac{d\lambda}{d\epsilon}\Delta\epsilon\right) \\ &= \Gamma\left(-n + j + 1 - \frac{\zeta m^2 \Delta\epsilon}{(m^2 - \epsilon_{nj}^2)^{3/2}}\right) \\ &\approx \frac{(-1)^{n-j}}{(n-j-1)!} \frac{(m^2 - \epsilon_{nj}^2)^{3/2}}{\zeta m^2 \Delta\epsilon}, \end{aligned} \quad (\text{A12})$$

where we have made use of the approximation

$$\Gamma(-n+x) \approx \frac{(-1)^n}{n!x}. \quad (\text{A13})$$

Substituting Eq. (A12) into Eq. (A11) and setting $\epsilon \approx \epsilon_{nj}$ everywhere, one obtains

$$f_{\text{out}} \approx r^{\gamma-\frac{1}{2}} \left(1 - \frac{2\zeta\epsilon_{nj}r}{2\gamma+1} \right) + \frac{\Gamma(-2\gamma-n+j+1)}{\Gamma(-2\gamma)} \frac{(n-j-1)!\Gamma(2\gamma)}{(-1)^{n-j}} \frac{\zeta m^2 \Delta\epsilon}{\beta_{nj}^3} \frac{r^{-\gamma-\frac{1}{2}}}{(2\beta_{nj})^{2\gamma}} \left(1 + \frac{2\zeta\epsilon_{nj}r}{2\gamma-1} \right), \quad (\text{A14})$$

where $\beta_{nj} \equiv \beta(\epsilon_{nj}) = \sqrt{m^2 - \epsilon_{nj}^2}$. It is clear from Eq. (A14) that, for f_{out} to remain finite as $R \rightarrow 0$, $\Delta\epsilon$ must vanish at least as fast as $R^{2\gamma}$. Using Eq. (A14), one finds

$$\left. \frac{f'_{\text{out}}}{f_{\text{out}}} \right|_{r=R} \approx \frac{\gamma - \frac{1}{2} - \zeta\epsilon_{nj}R - \frac{\Gamma(-2\gamma-n+j+1)}{\Gamma(-2\gamma)} \frac{(n-j-1)!\Gamma(2\gamma)}{(-1)^{n-j}} \frac{\zeta m^2 \Delta\epsilon}{\beta_{nj}^3} \frac{R^{-2\gamma}}{(2\beta_{nj})^{2\gamma}} \left(\gamma + \frac{1}{2} + \zeta\epsilon_{nj}R \right)}{R \left[1 - \frac{2\zeta\epsilon_{nj}R}{2\gamma+1} + \frac{\Gamma(-2\gamma-n+j+1)}{\Gamma(-2\gamma)} \frac{(n-j-1)!\Gamma(2\gamma)}{(-1)^{n-j}} \frac{\zeta m^2 \Delta\epsilon}{\beta_{nj}^3} \frac{R^{-2\gamma}}{(2\beta_{nj})^{2\gamma}} \left(1 + \frac{2\zeta\epsilon_{nj}R}{2\gamma-1} \right) \right]}. \quad (\text{A15})$$

Equating the logarithmic derivatives of the “inside” and “outside” solutions, Eqs. (A8) and (A15), one may solve for the energy correction $\Delta\epsilon$ as

$$\Delta\epsilon \approx \frac{(-1)^{n-j}\Gamma(-2\gamma)(2\beta_{nj}R)^{2\gamma}}{(n-j-1)!\Gamma(2\gamma)\Gamma(-2\gamma-n+j+1)} \left[\frac{\gamma - \xi_j - \frac{1}{2}}{\gamma + \xi_j + \frac{1}{2}} + \frac{(2\xi_j - 2\gamma - 1)\zeta\epsilon_{nj}R}{(2\gamma+1)(\gamma + \xi_j + \frac{1}{2})} \right] \frac{\beta_{nj}^3}{\zeta m^2}. \quad (\text{A16})$$

We observe that the leading term in Eq. (A16) scales as $R^{2\gamma}$, in agreement with the requirement that the outside wave function is regular at $r = R \rightarrow 0$.

We point out, however, that this leading behavior of the energy correction is strictly relativistic. Indeed, it may be verified that, in the nonrelativistic limit where $\zeta \ll mR \ll 1$, the term in Eq. (A16), which is proportional to $R^{2\gamma}$, vanishes and the remaining term reduces to

$$\Delta\epsilon \approx \frac{(n+l)!}{n^{2l+1}(n-l-1)!} \frac{(2\zeta mR)^{2l+2}}{(2l+3)!(2l+1)!} \frac{\zeta^2 m}{n^3}, \quad (\text{A17})$$

where $l = 1, 2, \dots$ is the vector boson’s orbital angular momentum (for the magnetic mode, $j = l$). Equation (A17) agrees with the nonrelativistic result (B3) obtained by applying first-order perturbation theory to the Schrödinger equation.

APPENDIX B: ENERGY CORRECTION BY PERTURBATION THEORY

Assuming the model of a hollow nuclear shell, Eq. (A3), the perturbation to the Coulomb potential in the Schrödinger equation may be written as

$$V = \left(\frac{\zeta}{R} - \frac{\zeta}{r} \right) \theta(R-r), \quad (\text{B1})$$

where θ is the Heaviside step function.

Since the perturbation (B1) is localized inside the nucleus, in the limit where the nuclear radius R is much smaller than the Bohr radius a_B , the unperturbed wave function corresponding to the principal quantum number n and angular momentum l may be written as

$$\varphi_{r \ll a_B} \approx \sqrt{\left(\frac{2Z}{na_B} \right)^3 \frac{(n+l)!}{2n(n-l-1)!(2l+1)!}} \frac{1}{\left(\frac{2Zr}{na_B} \right)^l}. \quad (\text{B2})$$

Using Eqs. (B1) and (B2), we find the energy correction as

$$\begin{aligned} \Delta\epsilon &= \langle \varphi | V | \varphi \rangle \approx \left(\frac{2Z}{na_B} \right)^3 \frac{(n+l)!}{2n(n-l-1)!} \\ &\quad \times \frac{1}{[(2l+1)!]^2} \int_0^R \left(\frac{2Zr}{na_B} \right)^{2l} \left(\frac{\zeta}{r} - \frac{\zeta}{R} \right) r^2 dr \\ &= \frac{(n+l)!}{n^{2l+1}(n-l-1)!(2l+3)!(2l+1)!} \frac{\zeta^2 m}{n^3}. \end{aligned} \quad (\text{B3})$$

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