

Orbital evolution of asymmetric binaries within accreting environments

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Extreme mass-ratio inspirals embedded in accretion disks provide a natural arena for studying the interplay between relativistic orbital dynamics and environmental effects. In this work, we develop a framework to investigate the secular evolution of compact objects repeatedly crossing an accretion disk around a supermassive black hole. The orbital motion is modeled through Kerr geodesics, while disk interactions are encoded through effective prescriptions for mass accretion and dynamical friction. We find that disk-induced dissipation generically drives a two-stage evolution characterized by rapid alignment of the orbital plane with the disk, followed by slower eccentricity damping. By systematically comparing the dynamics with a purely Keplerian treatment, we show that cumulative relativistic effects produce deviations even at large orbital separations, where the Keplerian approximation would naively be expected to remain accurate. These discrepancies grow through repeated disk crossings and become increasingly pronounced in more relativistic orbital configurations. We further investigate the impact of the accretion-disk model by comparing the Sirko-Goodman and Novikov-Thorne prescriptions. Relativistic disk structures predict systematically lower densities and larger scale heights, leading to weaker orbital dissipation and slower secular evolution. By contrast, the spin of the central black hole has only a minor effect on the overall circularization efficiency. Our results demonstrate the importance of consistently modeling both relativistic orbital dynamics and disk structure when studying compact objects embedded in active galactic nuclei disks, and provide a framework for exploring their long-term evolution, as well as a possible connection to quasiperiodic eruptions.

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I. INTRODUCTION

Compact objects embedded in gaseous environments provide a unique laboratory for studying the interplay between orbital dynamics and astrophysical processes [1,2]. In many realistic scenarios, binaries do not evolve in vacuum, but interact continuously with the surrounding medium through mechanisms such as gas accretion, dynamical friction, hydrodynamic drag, and gravitational

torques. These interactions can modify the orbital evolution over long timescales, altering quantities such as the eccentricity, inclination, and semimajor axis of the system. Understanding such effects is therefore essential for constructing realistic evolutionary models of compact binaries in dense astrophysical environments.

The relevance of these processes is further enhanced by the advent of next-generation gravitational-wave observatories. Future detectors, including the Laser Interferometer Space Antenna (LISA), the Einstein Telescope, and Cosmic Explorer, will probe compact binaries over a broad range of masses, frequencies, and astrophysical environments [3–5]. In this context, environmental effects are increasingly recognized as a potentially important ingredient in the interpretation of future observations, motivating the development of realistic models capable of capturing the interplay between orbital dynamics and the surrounding medium [2,6–8].

Among the environments proposed in the literature, accretion disks surrounding supermassive black holes

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(SMBHs) are particularly interesting. Active galactic nuclei (AGN) provide natural reservoirs of gas in which compact objects may be captured, migrate, and interact repeatedly with the disk material [2,6–8]. In this context, disk-induced dissipation may significantly modify the orbital evolution, potentially affecting the rates, properties, and observational signatures of embedded compact-object populations.

Extreme mass-ratio inspirals (EMRIs) consisting of stellar-mass compact objects orbiting SMBHs represent one of the most promising systems in which to study these phenomena [9,10]. Owing to the large separation of scales between the primary and secondary masses, EMRIs can undergo prolonged interactions with their environment while remaining sensitive to relativistic effects associated with the spacetime geometry of the central black hole (BH). In AGN disks, the evolution of an EMRI is governed by a combination of dissipative mechanisms, including mass accretion, dynamical friction, hydrodynamic drag, and the gravitational influence of the disk itself [11,12]. These effects can drive orbital circularization, alignment with the disk plane, and long-term migration through the gaseous medium.

The efficiency of such processes remains an open question. While early studies suggested rapid alignment and circularization of embedded compact objects, more recent investigations have shown that the outcome depends sensitively on the disk properties, orbital configuration, and the nature of the disk-object interaction [13–16]. In particular, the transition from nearly circular to highly eccentric orbits can substantially modify the dominant dissipative mechanisms, suppressing migration torques and enhancing the role of local dynamical friction.

Despite the growing interest in compact objects embedded in AGN disks, most existing studies rely on Keplerian descriptions of the orbital motion [17]. Such approximations are often well justified at large distances from the SMBH, but they become increasingly questionable as the orbit evolves inward and relativistic corrections accumulate over many orbital periods. Moreover, the interplay between relativistic orbital dynamics and disk-induced dissipation remains poorly understood, particularly for generic eccentric and inclined configurations undergoing repeated disk crossings.

Understanding this interplay is important for several reasons. From a theoretical perspective, it is necessary to establish the validity of commonly adopted approximations and to quantify the impact of relativistic corrections on the secular evolution of embedded compact objects. More broadly, these systems have been proposed in a variety of astrophysical contexts, including AGN-assisted EMRI formation channels and models for quasiperiodic eruptions (QPEs), in which repeated disk crossings may produce recurring electromagnetic flares [18,19].

In this work, we develop a relativistic framework to study the evolution of compact objects embedded in thin

accretion disks around supermassive black holes. Building upon the recent treatment of Ref. [20], where the motion of the secondary is described within a mostly Keplerian approximation, we model the orbital dynamics through bound Kerr geodesics while retaining an effective description of the interaction with the surrounding disk. This hybrid approach allows us to investigate inclined and eccentric orbits undergoing repeated disk crossings, and to assess the impact of relativistic effects, black-hole spin, and different accretion-disk prescriptions on the long-term evolution of the system. In particular, we compare Keplerian and relativistic orbital treatments and explore how the predicted evolution depends on the underlying disk structure. The main features of the evolution uncovered by our simulations are illustrated in Fig. 1.

The remainder of this paper is organized as follows: In Sec. II, we introduce the theoretical framework underlying our study, including the accretion-disk models and the description of disk-secondary interactions. Section III presents the numerical implementation, with particular emphasis on the treatment of Kerr geodesics and disk crossings. In Sec. IV, we discuss the orbital evolution of compact objects embedded in accretion disks, comparing Keplerian and relativistic treatments and assessing the

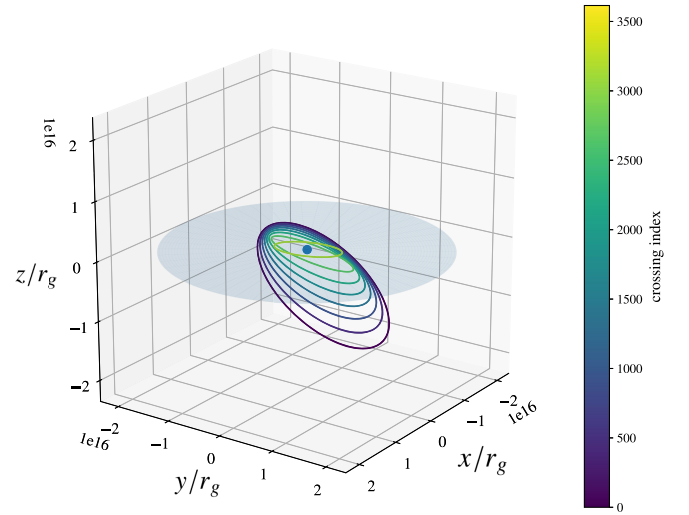


FIG. 1. Orbital dynamics of the binary. Evolution of the orbital configuration of a compact secondary object inspiraling around a spinning SMBH, in the presence of a thin accretion disk. The colors of the trajectories show the quantity of crossings with the disk. This plot summarizes the main results of our work: repeated disk crossings induce hydrodynamic drag, dynamical friction, and mass accretion effects driving a highly nonlinear secular evolution of the binary. The secondary evolves through an arbitrary number of interactions with the disk, moving on inclined and eccentric orbits. The disk effects align the orbit with the disk plane on timescales much shorter than those associated with gravitational-wave emission, and the eccentricity alternates phases of excitation and damping before the system circularizes (yellow orbit).

impact of different disk prescriptions. Finally, in Sec. V, we discuss possible observational implications of our results, including their connection to quasiperiodic eruptions, as well as a summary of our main conclusions and outlines of directions for future work.

II. PRELIMINARIES

In this section, we outline the theoretical framework used to describe accretion-disk environments around an SMBH of mass M (the primary), together with the dynamics of a stellar-mass object of mass m (the secondary), such that the system features a large mass hierarchy, $m/M \ll 1$. We work in the standard thin-disk regime.

A. Thin accreting environments in a nutshell

We aim to describe an axisymmetric, stationary, optically thick, and geometrically thin disk surrounding an SMBH. Adopting cylindrical coordinates (r, ϕ, z) centered on the BH, we assume that $H/r \ll 1$ throughout the radial extent of interest, where H is the thickness of the disk. Under these conditions, the disk structure is governed by mass and angular momentum conservation (see, for instance, [21]), namely,

$$\dot{M} = -2\pi r \Sigma(r) v_r, \quad (1)$$

$$\dot{M}(\ell - \ell_{\text{isco}}) = 2\pi r^2 W_{r\phi}(r), \quad (2)$$

where $\Sigma(r)$ is the surface density

$$\Sigma(r) = \int_{-H/2}^{H/2} \rho(r, z) dz, \quad (3)$$

and $W_{r\phi}(r)$ is the (vertically integrated) viscous stress,

$$W_{r\phi}(r) = \int_{-H/2}^{H/2} T_{r\phi}(r) dz. \quad (4)$$

Here, v_r denotes the radial velocity, and $\ell = r^2\Omega$ is the specific angular momentum of the disk. For a Keplerian disk, the angular velocity is $\Omega = \sqrt{M/r^3}$, and the inner edge is located at the innermost stable circular orbit (ISCO), for which $\ell = \ell_{\text{isco}}$.

We model the viscous shear stress following the seminal works by Shakura and Sunyaev [22,23], which are assumed to be proportional to the total pressure in the disk midplane; i.e.,

$$T_{r\phi} = -\alpha(P_{\text{gas}} + P_{\text{rad}}), \quad (5)$$

where $0 < \alpha < 1$ is a dimensionless viscosity parameter. Imposing vertical hydrostatic equilibrium, the effective kinematic viscosity can be written as

$$\nu \simeq \frac{\alpha c_s^2}{\Omega}, \quad (6)$$

where $c_s = \sqrt{(P_{\text{gas}} + P_{\text{rad}})/\rho}$ is the isothermal sound speed. The gas and radiation pressure contributions are given by

$$P_{\text{gas}} = nk_B T_c, \quad P_{\text{rad}} = \frac{a}{3} T_c^4, \quad (7)$$

with n the particle number density, T_c the temperature at the equatorial plane, and a the radiation constant.

1. Disk modeling

The formalism introduced so far provides a generic description of thin accretion disks and is common to many of the models considered in the literature [24–26]. The main differences between these prescriptions arise from the treatment of the gravitational field, the role of disk self-gravity, and the boundary conditions imposed near the central BH. Here we consider two disk prescriptions: (i) the classical, nonrelativistic Sirko-Goodman (SG) model and (ii) the relativistic Penna model based on the Novikov-Thorne (NT) thin-disk framework [27].

For both prescriptions, we extract the radial profiles of the disk density $\rho(r)$, sound speed $c_s(r)$, and scale height $H(r)$, which enter directly in the computation of dissipative processes affecting the secondary motion. The two prescriptions predict different density, sound-speed, and scale-height profiles, particularly in the inner regions of the disk where relativistic effects become increasingly important. These profiles are tabulated as functions of the cylindrical radius r and interpolated numerically at each equatorial crossing of the secondary.

The SG model extends the classical Shakura-Sunyaev formalism by including the effects of disk self-gravity. Such contributions may become comparable to the vertical tidal field imposed by the central BH at sufficiently large radii. The onset of self-gravity is usually quantified through the Toomre parameter [28], which for a Keplerian disk is given by

$$\mathcal{Q} = \frac{\Omega c_s}{\pi G \Sigma(r)}. \quad (8)$$

When $\mathcal{Q} \lesssim 1$, gravitational instabilities lead to the growth of density perturbations, potentially resulting in fragmentation and star formation (see [29] and references therein). The central assumption of the SG model is that the feedback associated with these instabilities regulates the disk toward a marginally stable state, $\mathcal{Q} \simeq 1$. This condition provides an additional closure relation for the disk structure. As a consequence, the density, temperature, pressure, and scale height are determined not only by the balance between viscous heating and radiative cooling, but also by

the requirement that self-gravity remains close to the instability threshold [22,23].

The NT formalism, on the other hand, assumes that the gas moves on nearly circular geodesics around the central BH. Since the disk is geometrically thin, radial pressure gradients and advective energy transport are neglected, and the local thermodynamic structure is determined by the balance between viscous heating and radiative cooling [30]. The radial distribution of the emitted flux is then uniquely specified by the accretion rate and the relativistic properties of the spacetime. In its standard form, the model also assumes that the viscous torque vanishes at the ISCO, implying that matter rapidly plunges into the BH once it reaches this radius.

Although the NT model provides the standard theoretical description of thin relativistic accretion disks, general-relativistic magnetohydrodynamic (GRMHD) simulations suggest that magnetic stresses may persist inside the ISCO and modify the dissipation profile of the inner flow [31,32]. To account for these effects, we employ the prescription developed by Penna *et al.* [27], which calibrates thin-disk models against numerical simulations of magnetized accretion flows. In this approach, the NT solution remains the underlying description of the disk structure, while correction factors derived from GRMHD simulations are introduced to model the residual transport of angular momentum and energy near the ISCO.

When comparing the two prescriptions, all global astrophysical parameters of the accreting system are kept fixed. Consequently, any difference in the resulting orbital evolution can be attributed directly to the distinct density, sound-speed, and scale-height profiles predicted by the two disk models, rather than to variations in the astrophysical setup.

B. Orbital evolution *à la* Kepler

In this section, we recall the standard Keplerian description of disk-crossing dynamics, i.e., modeling the interaction between the secondary and the disk. In particular, we follow some of the guidelines developed in [20], which prove especially useful at large distances from the SMBH. In this framework, the secondary follows a Keplerian orbit around the central SMBH, while the interaction with the gaseous environment is modeled through effective dissipative forces. As the orbit approaches the central SMBH from generic configurations, relativistic effects become increasingly important. For this reason, in this work we extend the orbital motion of the secondary to Kerr geodesics, while retaining the same effective disk-interaction prescriptions in a perturbative sense, as shown in the next section.

Following a standard Keplerian description, let us consider a secondary BH orbiting around a central SMBH. The orbit is characterized by the semimajor axis a , the eccentricity e , and the inclination i relative to the

disk midplane. The orbital position $\mathbf{r}(t)$ within the orbital plane is given by

$$\mathbf{r}(t) = \frac{a(1-e^2)}{1+e\cos\vartheta} [\cos\vartheta \hat{\mathbf{x}} + \sin\vartheta \hat{\mathbf{y}}], \quad (9)$$

where ϑ denotes the true anomaly, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthonormal unit vectors spanning the orbital plane. We choose $\hat{\mathbf{x}}$ to point along the direction of the pericenter, while $\hat{\mathbf{y}} = \hat{\mathbf{L}} \times \hat{\mathbf{x}}$, where $\hat{\mathbf{L}}$ is the unit orbital angular-momentum vector. The velocity of the secondary relative to the surrounding gas is

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_{\text{orb}} - \mathbf{v}_{\text{gas}}, \quad (10)$$

with $|\mathbf{v}_{\text{gas}}| \simeq \sqrt{GM/r}$ for a nearly Keplerian disk.

The total acceleration acting on the secondary can be decomposed as [33]

$$\mathbf{a}_{\text{tot}} = -\frac{GM}{r^3} \mathbf{r} + \mathbf{a}_{\text{drag}} + \mathbf{a}_{\text{DF}} + \mathbf{a}_{\text{disk}}. \quad (11)$$

As expected, in addition to the central gravitational force, three disk-induced effects contribute as well: hydrodynamic drag due to motion through the gas, dynamical friction associated with the gravitational wake induced in the medium, and the direct gravitational force arising from the disk potential. In typical AGN disks, the first two effects dominate for $r \lesssim 10^3 r_g$, while the contribution from the disk potential is generally subdominant [14].

For inclined orbits, the secondary crosses the disk twice per orbital period. At each crossing, it experiences a hydrodynamic drag force

$$\mathbf{F}_{\text{drag}} \simeq -\pi R_2^2 \rho(r, z) v_{\text{rel}} \mathbf{v}_{\text{rel}}, \quad (12)$$

where πR_2^2 is the cross-sectional area of the secondary, and a Gaussian vertical density profile is assumed, namely,

$$\rho(r, z) = \frac{\Sigma(r)}{\sqrt{2\pi}H} \exp\left[-\frac{z^2}{2H^2}\right]. \quad (13)$$

The duration of each crossing can be estimated as

$$\Delta t_{\text{cross}} \simeq \frac{2H}{v_{\text{orb}} \sin i}, \quad (14)$$

leading to an impulse in the orbital momentum

$$\Delta \mathbf{p}_{\text{drag}} \simeq -\frac{2\pi R_2^2 \Sigma(r)}{\sin i} \mathbf{v}_{\text{rel}}. \quad (15)$$

At this point, it is necessary to estimate the relative importance of the different effects induced by the disk, with respect to the central gravitational force $a_{\text{grav}} = GM/r^2$.

For an effective drag acceleration over a disk crossing, it holds [34]

$$\frac{a_{\text{drag}}}{a_{\text{grav}}} \sim \frac{3\kappa_d \rho R_2^2 v_{\text{rel}}^2}{8\pi m} \frac{r^2}{GM}, \quad (16)$$

where $\kappa_d \sim \mathcal{O}(1)$ is the drag coefficient. For instance, stellar-mass secondaries embedded in AGN disks satisfy $a_{\text{drag}}/a_{\text{grav}} \sim 10^{-8}-10^{-5}$.

The gravitational wake generated in the gaseous medium also gives rise to dynamical friction. Following Ostriker's treatment of a collisional gaseous medium [35], the corresponding force acting on the secondary can be written as

$$\mathbf{F}_{\text{DF}} = -4\pi\rho(r, z) \left(\frac{Gm}{v_{\text{rel}}} \right)^2 \mathcal{I}(\mathcal{M}) \hat{\mathbf{v}}_{\text{rel}}, \quad (17)$$

where $\mathcal{M} = v_{\text{rel}}/c_s$ is the Mach number and

$$\mathcal{I}(\mathcal{M}) = \begin{cases} \frac{1}{2} \ln \left(\frac{1+\mathcal{M}}{1-\mathcal{M}} \right) - \mathcal{M}, & \mathcal{M} < 1, \\ \ln \Lambda + \frac{1}{2} \ln \left(1 - \frac{1}{\mathcal{M}^2} \right), & \mathcal{M} > 1. \end{cases} \quad (18)$$

In AGN disks, the motion is typically supersonic ($\mathcal{M} \gg 1$), yielding $F_{\text{DF}} \propto \ln \Lambda / v_{\text{rel}}^2$ (see, e.g., [36]).

The corresponding acceleration can be compared with the central gravitational attraction, yielding [37]

$$\frac{a_{\text{DF}}}{a_{\text{grav}}} \sim 4\pi \ln \Lambda \frac{Gm\rho r^2}{Mv_{\text{rel}}^2}, \quad (19)$$

where $\ln \Lambda$ is the Coulomb logarithm. Since $a_{\text{DF}} \propto m$ whereas $a_{\text{drag}} \propto R_2^2$, dynamical friction generally dominates over hydrodynamic drag for compact objects such as BHs.

Lastly, the direct gravitational acceleration generated by the disk itself simply leads to

$$\frac{a_{\text{disk}}}{a_{\text{grav}}} \sim \frac{M_{\text{disk}}(< r)}{M} \ll 1. \quad (20)$$

Since AGN disks are typically much less massive than the central SMBH, this contribution is usually subdominant compared to both hydrodynamic drag and dynamical friction at $r \lesssim 10^3 r_g$ [14]. Nonetheless, the cumulative action of the dominant dissipative effects over $\sim 10^4-10^6$ orbital periods may drive a significant secular evolution of the orbital parameters.

In addition, the secondary accretes mass from the disk during each crossing, further modifying its momentum. Modeling the interaction as an inelastic collision, momentum conservation implies

$$(m + \Delta m) \mathbf{v}' = m \mathbf{v} + \Delta m \mathbf{v}_{\text{gas}}. \quad (21)$$

The accreted mass Δm is estimated using the Bondi-Hoyle-Lyttleton prescription [38]:

$$\Delta m = \frac{8\pi G^2 H \rho m^2}{v_z (v_{\text{rel}}^2 + c_s^2)^{3/2}}, \quad (22)$$

where v_z is the vertical component of the orbital velocity, and c_s is the local sound speed [consistently defined in Eq. (6)]. The corresponding velocity change is then

$$\Delta \mathbf{v} = \frac{\Delta m}{m + \Delta m} (\mathbf{v}_{\text{gas}} - \mathbf{v}). \quad (23)$$

The hierarchy among the various contributions provides a useful guide for the numerical implementation. In particular, the estimates above suggest that only a subset of the disk-induced effects needs to be retained in order to capture the dominant orbital evolution.

In view of the above estimates, throughout this work we treat mass accretion and dynamical friction as the dominant mechanisms governing the interaction between the secondary and the disk, while neglecting the comparatively weaker effects discussed above. We further adopt a fixed value $\ln \Lambda = 3$ in all simulations.

C. Particle motion in Kerr geometry

The Keplerian treatment discussed so far provides a useful description of the local forces acting on the secondary. However, the conservative component of the orbital dynamics is more accurately described by geodesic motion in the spacetime generated by the central SMBH. For this reason, throughout this work we replace Keplerian trajectories by bound timelike geodesics in Kerr spacetime, while retaining the same effective prescriptions for the interaction between the secondary and the disk.

This approximation should be regarded as a hybrid scheme. Between two successive disk crossings, the secondary follows an exact Kerr geodesic, whereas the momentum exchange with the gaseous medium is computed using the Keplerian prescriptions introduced in the previous section. Such an approach allows us to incorporate the dominant relativistic corrections to the orbital motion while keeping the environmental coupling computationally tractable. A fully relativistic treatment of the disk-secondary interaction would require modeling the hydrodynamic and accretion processes directly in the Kerr geometry, which is beyond the scope of the present work and will be addressed in future investigations. Finally, we neglect higher-order post-Keplerian terms such as gravitational-wave (GW) emission because they are insignificant, as we show in the Appendix.

1. Orbital motion between crossings

We follow the analytical framework developed in [39], which provides closed-form solutions for bound timelike

geodesics in Kerr spacetime in terms of elliptic functions, whose main ingredients are briefly summarized here.

The starting point is the separability of the Hamilton-Jacobi equation in Boyer-Lindquist coordinates (t, r, θ, ϕ) , which implies conservation of the energy E , the axial component of the angular momentum L , and the Carter constant Q [40], along each geodesic. Introducing the Mino time λ through

$$d\lambda = \frac{d\tau}{r^2 + \chi^2 \cos^2 \theta},$$

where χ denotes the Kerr spin parameter, the radial and polar equations of motion decouple in the form

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 &= R(r(\lambda)), \\ \left(\frac{d\theta}{d\lambda}\right)^2 &= \Theta(\theta(\lambda)), \end{aligned} \quad (24)$$

with

$$R(r) = [E(r^2 + \chi^2) - \chi L]^2 - \Delta[r^2 + (L - \chi E)^2 + Q], \quad (25)$$

$$\Theta(\theta) = Q - \cos^2 \theta \left[\chi^2 (1 - E^2) + \frac{L^2}{\sin^2 \theta} \right], \quad (26)$$

and $\Delta = r^2 - 2Mr + \chi^2$. The functions $R(r)$ and $\Theta(\theta)$ determine the allowed regions of motion, fully characterizing the radial and polar dynamics. In particular, when restricting the analysis to bound orbits, the radial potential can be factorized as

$$R(r) = (1 - E^2)(r_1 - r)(r - r_2)(r - r_3)(r - r_4), \quad (27)$$

with $r_1 \geq r_2 \geq r_3 \geq r_4$. The radial motion can then be expressed analytically as

$$r(\lambda) = \frac{r_3(r_1 - r_2)\text{sn}^2(u_r(\lambda), k_r) - r_2(r_1 - r_3)}{(r_1 - r_2)\text{sn}^2(u_r(\lambda), k_r) - (r_1 - r_3)}, \quad (28)$$

where $\text{sn}(u, k)$ denotes the Jacobi elliptic function and

$$u_r(\lambda) = \frac{\sqrt{(1 - E^2)(r_1 - r_3)(r_2 - r_4)}}{2} \lambda, \quad (29)$$

$$k_r = \sqrt{\frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}}. \quad (30)$$

A similar construction applies to the polar motion. In fact, by defining $z = \cos^2 \theta$, the polar potential can be parametrized in terms of two turning points, z_+ and z_- . The solution of the polar equation yields

$$\cos \theta(\lambda) = \sqrt{z_-} \text{sn}(u_\theta, k_\theta), \quad (31)$$

with

$$u_\theta = \sqrt{\chi^2(1 - E^2)} z_+ \lambda, \quad k_\theta = \sqrt{\frac{z_-}{z_+}}. \quad (32)$$

This representation is particularly useful, as it makes explicit the periodic nature of the polar motion in Mino time. Once the radial and polar trajectories are known, the azimuthal and temporal components follow from the first-order equations

$$\begin{aligned} \frac{d\phi}{d\lambda} &= \Phi_r(r(\lambda)) + \Phi_\theta(\theta(\lambda)), \\ \frac{dt}{d\lambda} &= T_r(r(\lambda)) + T_\theta(\theta(\lambda)), \end{aligned} \quad (33)$$

where Φ_r and T_r are functions only of r , while Φ_θ and T_θ depend exclusively on θ . Explicit expressions for these functions follow directly from the separated Kerr-geodesic equations [39] (see also [41]). The solutions of Eq. (33) can then be written schematically as¹

$$\phi(\lambda) = \phi_r(\lambda) + \phi_\theta(\lambda), \quad t(\lambda) = t_r(\lambda) + t_\theta(\lambda). \quad (34)$$

III. NUMERICAL IMPLEMENTATION

In this section, we describe the numerical setup employed throughout the simulations. In particular, we discuss the initialization procedure, the orbital updates applied after each interaction with the disk, and the set of quantities used to characterize the orbital evolution.

A. Initialization and evolution updates

We initialize each orbit by specifying the triplet (a_0, e_0, x_0) corresponding to the initial values of the semimajor axis, eccentricity, and orbital inclination i_0 with respect to the equatorial plane, through $x_0 = \cos i_0$. This parametrization is in one-to-one correspondence with the set (E, L, Q) , together with the normalization condition for the four-velocity. The explicit relation between both sets of parameters is detailed in Appendix A.

In addition, we fix the initial argument of pericenter to $\omega_0 = -\pi/3$ in all our simulations. We adopt the initial true anomaly $\vartheta_0 = \pi/3$, such that $\omega_0 + \vartheta_0 = 0$. Thus, the initial position of the secondary lies at the intersection of the orbital and equatorial planes corresponding to a disk crossing. This choice follows the convention adopted in Ref. [20], allowing for a direct comparison with the results presented there. In general, disk crossings occur at true

¹Each contribution can be evaluated analytically and expressed in terms of elliptic integrals of the first, second, and third kinds.

anomalies $\vartheta_{\text{cross}} = -\omega$ and $\vartheta_{\text{cross}} = \pi - \omega$, and the corresponding crossing radii are

$$r_{\pm} = \frac{a(1 - e^2)}{1 \pm e \cos \omega}. \quad (35)$$

For our choice of $\omega_0 = -\pi/3$, this gives

$$r_{\pm} = \frac{a(1 - e^2)}{1 \pm e/2}. \quad (36)$$

The two disk crossings therefore occur at distinct radii, with their separation increasing with eccentricity. Consequently, the secondary encounters the disk at different local densities, sound speeds, and relative velocities at the two crossings, resulting in different accretion and dynamical-friction kicks during each orbit.²

We initialize the orbital velocity using the standard Keplerian relations,

$$v_r = \sqrt{\frac{M}{a(1 - e^2)}} e \sin \vartheta, \quad (37)$$

$$v_{\theta} = \sqrt{\frac{M}{a(1 - e^2)}} (1 + e \cos \vartheta), \quad (38)$$

while $v_{\phi} = 0$ for a planar orbit. These expressions follow from $v_r = \dot{r}$ and $v_{\theta} = r\dot{\vartheta}$, with $r = |\mathbf{r}(t)|$ given by Eq. (9). In a purely Keplerian treatment, such as that developed in [20], once the orbital parameters after a disk crossing are known, the location of the next crossing can be determined analytically from the intersection of the Keplerian orbit with the disk plane.

In the present work, however, the orbital motion between successive interactions is described by Kerr geodesics. As a consequence, the trajectory is specified through the functions $r(\lambda)$ and $\theta(\lambda)$ rather than by a fixed Keplerian ellipse, making the location of the next equatorial crossing a numerical problem. We therefore determine the next crossing by searching for the roots of

$$\theta(\lambda) - \frac{\pi}{2} = 0 \quad (39)$$

corresponding to intersections of the orbit with the equatorial plane. Our implementation samples $\theta(\lambda)$ on a numerical grid, identifies sign changes of $\theta(\lambda) - \pi/2$, and refines the root using a Brent root-finding algorithm.

For the disk interactions, we follow the framework described in the previous section. The density $\rho(r)$, sound speed $c_s(r)$, and scale height $H(r)$ are interpolated from the

²We do not explore the dependence of the orbital evolution on ω in the present work and leave such a systematic study for future investigation.

selected disk model and evaluated at each equatorial crossing. Similar to the treatment adopted in [20], the velocity is modified by dynamical friction according to

$$\Delta \vec{v} = \frac{\vec{F}_{\text{DF}} \Delta t}{m}, \quad (40)$$

where $\vec{F}_{\text{DF}}$ is given by Eq. (17), and the interaction time is estimated as

$$\Delta t = \frac{2H}{v_{\theta}}. \quad (41)$$

At each disk crossing, the orbital parameters are recorded. For the large semimajor axes considered in our simulations, the orbital inclination evolves slowly, and the parameter $x = \cos i$ remains a useful proxy for tracking the orientation of the orbit. Following the interaction with the disk, we reconstruct the four-velocity of the secondary and compute the corresponding constants of motion (E, L, Q), or equivalently, the updated set of orbital parameters. The geodesic is then reinitialized and evolved until the next disk-crossing event.

The orbital evolution between successive disk crossings is computed using `KerrGeoPy`, a Python package for the accurate calculation of bound timelike geodesics in Kerr spacetime [42]. The package implements the analytical formalism based on Mino-time parametrization and elliptic integrals, enabling a fast and accurate evaluation of generic bound trajectories together with their associated fundamental frequencies and spacetime coordinates.

Throughout the evolution, we record the orbital parameters and constants of motion after each disk crossing. In particular, we monitor the semimajor axis, eccentricity, inclination, and the corresponding set of Kerr integrals (E, L, Q), together with the mass growth of the secondary.

Finally, the simulation is terminated if the eccentricity falls below a prescribed threshold e_{min} (in our case, $e_{\text{min}} = 10^{-3}$), indicating a quasicircular orbit,³ or when the orbit becomes unbound or plunging, usually signaled by the failure of the conversion from constants of motion to orbital parameters.

IV. RESULTS

We now present the main results of our work. We begin by investigating the orbital evolution of the secondary at large distance from the primary BH, highlighting the differences between relativistic and Keplerian treatments. We then extend the study to the intermediate-separation

³By the time $e < e_{\text{min}}$ is reached, the orbit has already undergone the faster alignment phase and is effectively coplanar with the disk. The secondary is then embedded in the disk, so discrete crossings cease to be well defined and quantities such as Δt diverge.

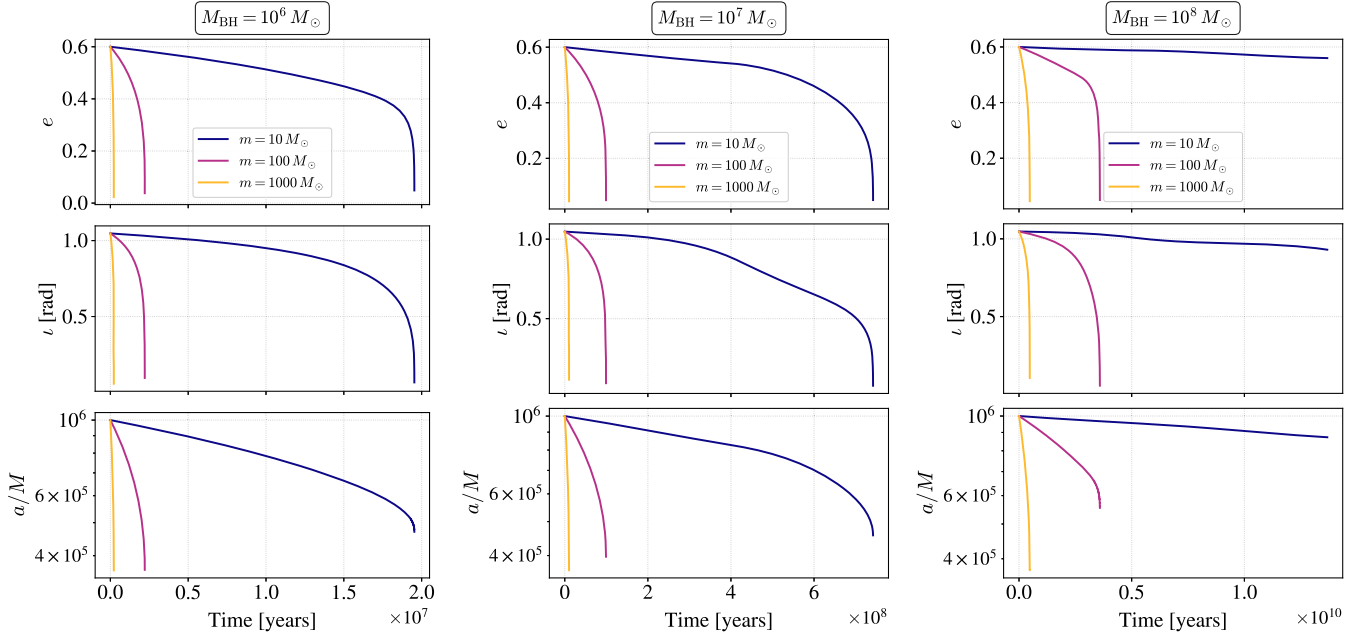


FIG. 2. Orbital evolution of the secondary. Time evolution of the orbital parameters (eccentricity e , orbital inclination i , and normalized semimajor axis a/M), for a set of nine configurations. Each column (from left to right) corresponds to different central BH masses $M_{\text{BH}} = 10^6, 10^7, 10^8 M_\odot$, respectively, while colors distinguish the secondary masses $m = 10, 10^2, 10^3 M_\odot$. All systems are initialized at a separation $a_0 = 10^6 M$ from the central BH and evolved in a Schwarzschild spacetime including disk-induced accretion and dynamical-friction effects. The dynamics exhibit a two-stage behavior: a rapid decrease of the orbital inclination, driving the system toward a nearly coplanar configuration with the disk, followed by a slower circularization driven by dissipative interactions during repeated disk crossings. The characteristic circularization timescale is $\tau \sim 10^7$ yr, on average.

regime, where relativistic effects become important. We also examine the dependence on the disk model, as well as with the spin of the central BH, assessing their impact on the efficiency of orbital circularization and alignment.

A. Evolution at large separation

We start by studying the evolution of the secondary orbit around a nonrotating Schwarzschild BH, assuming an SG disk model. This setup facilitates a direct comparison with previous Keplerian studies [20]. We consider different values of the primary and secondary masses, namely, $M_{\text{BH}} = \{10^6, 10^7, 10^8\} M_\odot$ and $m = \{10, 10^2, 10^3\} M_\odot$, respectively.

We follow the evolution of the eccentricity e , inclination i , and semimajor axis a/M of the secondary orbit throughout the evolution, starting from an initial separation $a_0 = 10^6 M$. For all configurations considered, the orbit exhibits a clear tendency toward circularization. Local interactions with the disk, together with the exchange of mass and linear momentum between the secondary and the surrounding gas, drive a gradual dissipation of the initial eccentricity. For the range of primary and secondary masses displayed in Fig. 2, we report a characteristic circularization timescale of order $\tau \sim 10^7$ years. We also find that increasing the mass of the secondary generally leads to longer circularization timescales.

In addition to the eccentricity damping, the orbital plane gradually aligns with the accretion disk. Interestingly, the alignment timescale is typically about an order of magnitude shorter than the circularization timescale, as can also be seen in Fig. 2. Here, the quoted alignment timescale refers specifically to the initial fast-relaxation stage, during which the inclination decreases rapidly while the eccentricity remains of the same order as its initial value. This naturally leads to a two-stage evolution. The orbit first evolves toward a nearly coplanar configuration while retaining a significant eccentricity, although exact alignment is generally not achieved during this initial stage. Only at later times, once the orbit is nearly coplanar with the disk, it undergoes a more gradual circularization driven by dissipative interactions with the surrounding gas. For the present initial conditions, the ratio $e_0/e_{\text{min}} \sim 10^2\text{--}10^3$ implies that the subsequent slow evolution drives the inclination to values far below those reached during the initial relaxation, rendering the orbit effectively coplanar by the time the circularization threshold $e_{\text{min}} = 10^{-3}$ is reached. This also motivates our choice of $e < e_{\text{min}}$ as the termination criterion, as although the inclination does not reach exactly zero in finite time, by the time this eccentricity threshold is reached the secondary is already effectively embedded within the disk, and the assumption of discrete, localized disk crossings ceases to be physically meaningful. A geometrical argument underlying this

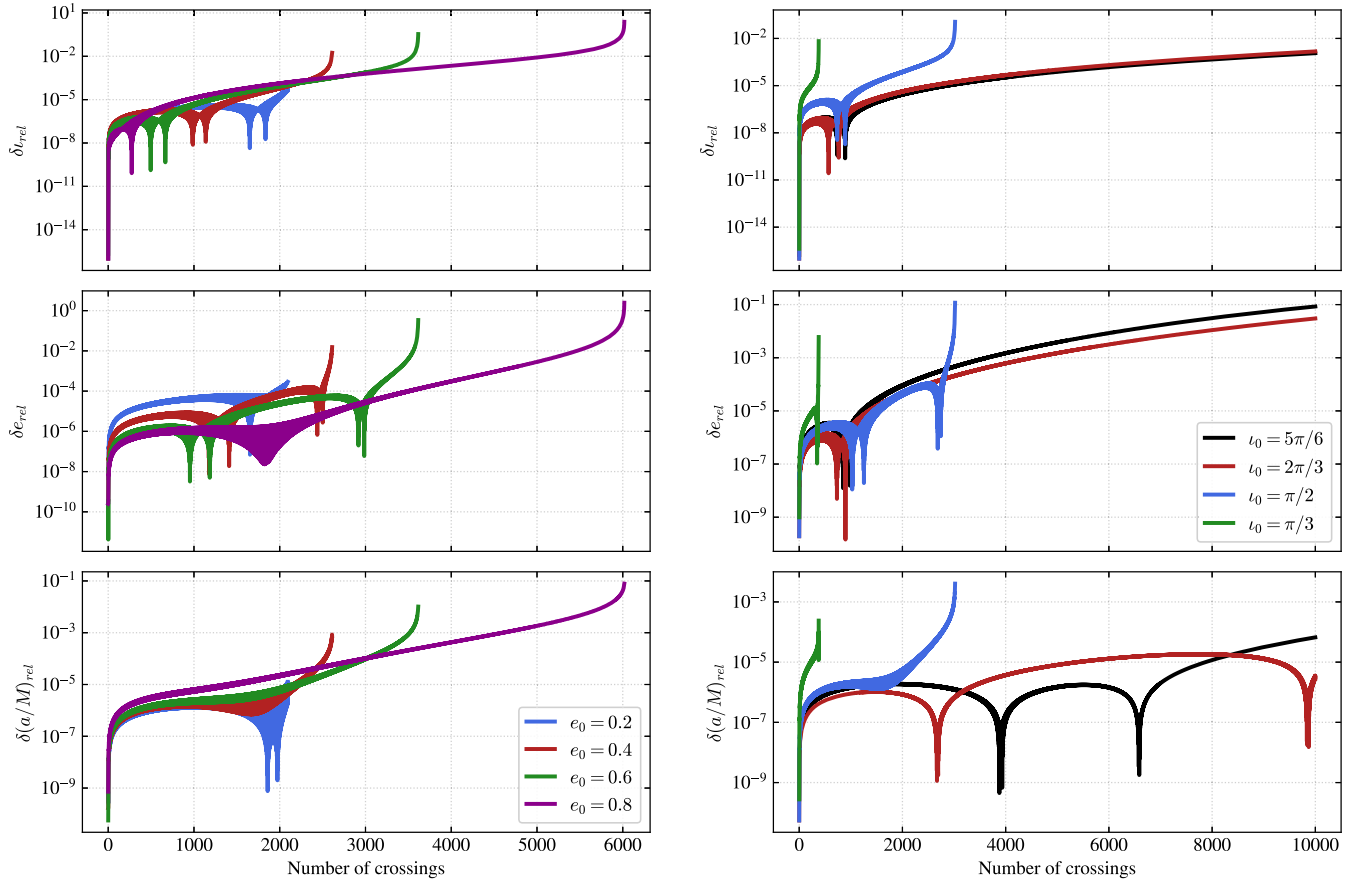


FIG. 3. Comparison with a Keplerian treatment. Relative logarithmic difference between the Keplerian and relativistic evolutions of binaries embedded in an SG disk. Two sets of initial conditions are considered. In the left panel, the initial eccentricity is varied over $e_0 \in \{0.2, 0.4, 0.6, 0.8\}$ while keeping the inclination fixed at $\iota_0 = \pi/3$. In the right panel, the initial inclination ι_0 is varied while the eccentricity is fixed to $e_0 = 0.5$. In all cases, the initial separation is $a_0 = 10^6 M$, and the initial true anomaly is set to $\theta_0 = \pi/2$. The relative discrepancy grows throughout the evolution, revealing systematic differences between the Keplerian evolution and the Kerr-based treatment even for configurations initialized at large orbital separations.

behavior is discussed in Appendix C, where we show that the orbital shrinkage accompanying circularization naturally leads to the embedded regime near the termination of the evolution.

The configurations shown in Fig. 2 correspond to relatively large orbital separations, where one might expect a Keplerian description to provide an adequate approximation to the BH-disk-secondary dynamics. Indeed, the gravitational field is weak enough that relativistic corrections are often assumed to be negligible. Nevertheless, our results show that even mild relativistic effects associated with the spacetime geometry of the central BH accumulate over the long evolution timescales considered here, producing systematic deviations with respect to the Keplerian description.

This behavior is illustrated in Fig. 3, which shows the relative difference between the Keplerian and relativistic evolutions. For this analysis, we consider two sets of initial conditions. In the first one, the eccentricity is varied over the range $e_0 \in \{0.2, 0.4, 0.6, 0.8\}$ while keeping the

inclination fixed at $\iota_0 = \pi/3$ (left panel). In the second, the initial inclination ι_0 is varied while the eccentricity is fixed to $e_0 = 0.5$ (right panel). All simulations are initialized with an orbital separation $a_0 = 10^6 M$ and true anomaly $\theta_0 = \pi/2$. These configurations encompass the region of parameter space previously explored in [20] within a purely Keplerian framework.

We find that the relative differences do not remain bounded during the evolution, but instead grow systematically as the number of crossings between the secondary and the accretion disk increases. This behavior suggests that even small relativistic corrections can accumulate over long timescales, producing progressively larger deviations in both the orbital geometry and the dissipative processes associated with the environment. Consequently, the accuracy of a purely Keplerian description appears to deteriorate well before the system enters a strongly relativistic regime.

Furthermore, both the magnitude of the discrepancy and its growth rate depend sensitively on the initial orbital configuration. In particular, systems with larger

eccentricities exhibit more pronounced deviations, since the secondary experiences stronger variations in both orbital velocity and radial distance during each revolution. This enhances the relative impact of relativistic corrections during passages closer to the central BH.

A similar trend is observed when varying the initial inclination. Changes in ι_0 modify both the frequency and geometry of the disk crossings, thereby altering the cumulative mismatch between the Keplerian and relativistic descriptions. Taken together, these results indicate that, even for initially distant configurations with $a_0 = 10^6 M$, a purely Keplerian treatment may be insufficient to accurately capture the secular evolution of systems undergoing repeated disk-object interactions over long integration times.

B. Evolution at intermediate separations

In light of the discrepancies identified at large orbital separations, we now move to configurations that probe a more relativistic region of the orbital parameter space. Specifically, we consider systems initialized at $a_0 = 10^4 M$, which we refer to as the intermediate regime. At these separations, the interaction with the disk remains significant, while relativistic effects associated with the spacetime geometry of the central BH exert a more noticeable influence on the orbital evolution.

Figure 4 shows the time evolution of the inclination (top panel), eccentricity (middle panel), and semimajor axis (bottom panel) of the secondary orbit due to its interaction with the accretion disk.

(bottom panel), comparing trajectories obtained from a Kerr-geodesic evolution (solid lines) with those derived from Kepler's laws (dashed lines). Two sets of initial conditions are considered once again. The left panel explores variations in the initial eccentricity ($e_0 \in \{0.2, 0.4, 0.6, 0.8\}$) while keeping the inclination fixed at $\iota_0 = \pi/3$, whereas the right panel examines different initial inclinations $\iota_0 \in \{\pi/3, \pi/2, 2\pi/3, 5\pi/6\}$ for a fixed eccentricity $e_0 = 0.6$.

This regime is particularly interesting because it combines repeated interactions with the accretion disk within a stronger gravitational field. As a result, geometric effects arising from the curved spacetime around the central BH begin to noticeably influence the orbital dynamics, as already suggested by the trajectories shown in Fig. 3.

In contrast to the large-separation case, the discrepancies between the relativistic and Keplerian evolutions emerge considerably earlier in the intermediate regime and grow more rapidly throughout the evolution. The orbital variables exhibit systematic differences in both the dissipation timescales and the geometry of the trajectories, indicating that relativistic effects play a progressively more important role in shaping the dynamics of the system.

In particular, the most eccentric configurations display a faster evolution and a more pronounced divergence between the two descriptions. This behavior can be attributed to the repeated close passages near the central BH, which enhance the impact of relativistic corrections on

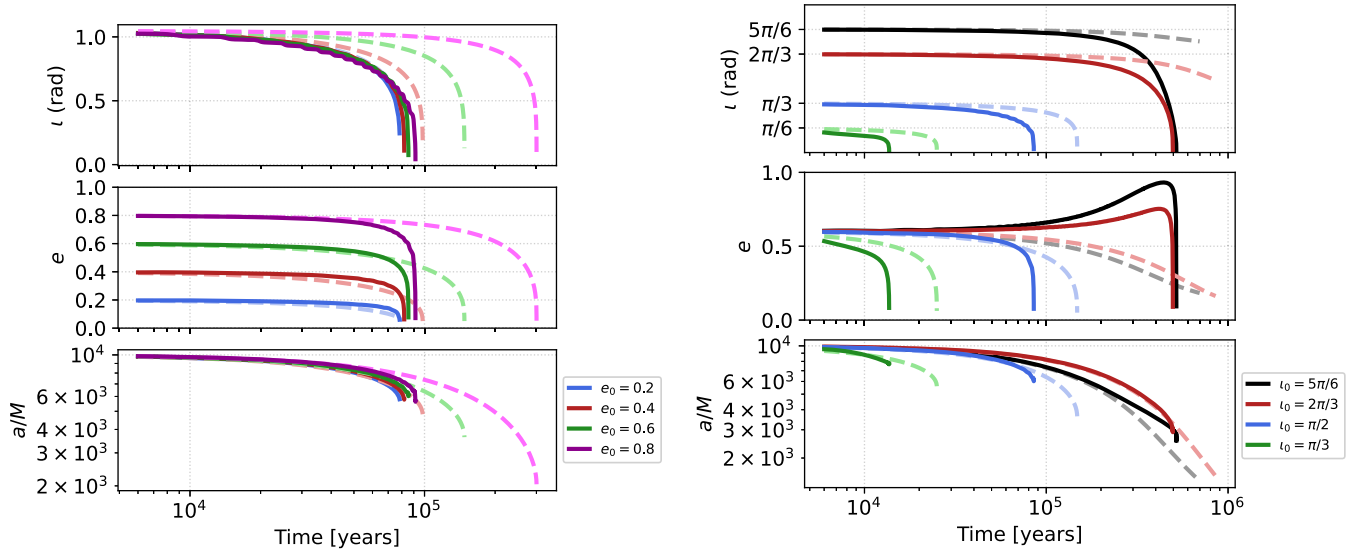


FIG. 4. Dynamics in the intermediate region. Time evolution of the inclination ι (top), eccentricity e (middle), and semimajor axis a (bottom) of the secondary orbit due to its interaction with the accretion disk. The mass of the secondary is set to $m = 10M_\odot$, while the initial separation is $a_0 = 10^4 M$. Solid curves correspond to evolutions computed on a Kerr background, while dashed curves denote the corresponding Keplerian evolution. The left column shows sequences with fixed initial inclination $\iota_0 = \pi/3$ and varying eccentricities $e_0 \in \{0.2, 0.4, 0.6, 0.8\}$, whereas the right column displays configurations with fixed eccentricity $e_0 = 0.6$ and initial inclinations $\iota_0 \in \{\pi/3, \pi/2, 2\pi/3, 5\pi/6\}$. Noticeable deviations from the Keplerian evolution emerge already at early times, revealing systematic differences in the dissipation rates and secular evolution of the orbit. The effect becomes increasingly pronounced for highly eccentric and highly inclined configurations, where the geometry and frequency of the disk crossings are more strongly affected by relativistic corrections.

the exchange of energy and angular momentum. Similarly, variations in the initial inclination modify both the frequency and geometry of the disk crossings, leading to appreciable differences in the orbital damping rate and in the inclination evolution.

Overall, these results suggest that the intermediate regime marks the onset of a transition in which relativistic effects become increasingly relevant for the long-term evolution of the system. Consequently, relativistic corrections must be properly accounted for well before the secondary reaches the innermost regions of the BH gravitational potential.

C. Comparison between disk prescriptions

Having established the impact of relativistic corrections on the orbital dynamics, we now investigate the dependence of the evolution on the adopted accretion-disk model. To this end, we compare simulations performed using the SG and NT/Penna disk prescriptions introduced in Sec. II. Since both models are initialized using the same global astrophysical parameters, any difference between the resulting trajectories can be attributed directly to differences in the underlying disk structure. In particular, the distinct radial profiles of the density, sound speed, and scale height modify both the strength of the disk-secondary interaction and the cumulative exchange of mass and momentum throughout the evolution.

The top panel of Fig. 5 compares the radial structure of the two disk prescriptions through the midplane density ρ and geometric aspect ratio H/R as functions of the dimensionless radius R/M . The dashed curves correspond to the SG model, whereas the solid blue curves represent the Penna solution computed for different values of the BH spin parameter.

The two prescriptions exhibit systematic variations throughout the radial domain of interest. In particular, the Penna model predicts lower midplane densities in the intermediate region $10^2 M \lesssim R \lesssim 10^4 M$, together with a geometrically thicker disk, as reflected by the larger values of H/R . These differences arise from the relativistic corrections incorporated in the disk structure, which modify the radial distribution of pressure, temperature, and viscous dissipation with respect to the purely Keplerian SG model.

The influence of the BH spin becomes appreciable only at radii $R \lesssim 10^2 M$, where strong-field effects associated with the Kerr geometry begin to alter the disk properties. At larger distances, the various Penna solutions progressively converge toward a common behavior, although differences with respect to the SG model persist in both the density and thickness profiles.

These structural differences are particularly relevant for the evolution of the secondary, since the efficiency of disk-object interactions depends sensitively on both the local gas density and the vertical structure of the disk. Consequently,

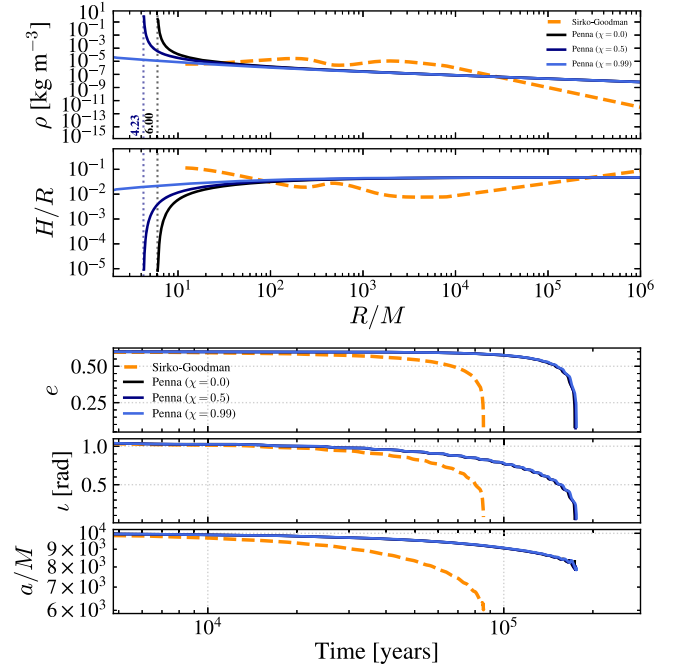


FIG. 5. Impact of disk models on the orbital dynamics. Comparison between the SG and Penna disk prescriptions and their impact on the evolution of an embedded secondary. Top panel: midplane density ρ and aspect ratio H/R as functions of R/M . Dashed curves correspond to the SG model, while solid blue curves show the Penna solution for different values of the BH spin parameter χ . The Penna model predicts lower densities and a thicker disk in the region $10^2 M \lesssim R \lesssim 10^4 M$, whereas spin-dependent effects become appreciable only for $R \lesssim 10^2 M$. Bottom panel: evolution of the orbital parameters for a secondary initialized at $a_0 = 10^4 M$ with eccentricity $e_0 = 0.6$. The Penna disk (solid blue) produces a slower orbital decay and eccentricity damping than the SG model (dashed orange), consistent with its lower densities and larger scale heights.

even modest changes in the underlying disk prescription can lead to appreciable differences in the secular evolution of the orbit.

The bottom panel of Fig. 5 illustrates the dynamical consequences of these structural differences on the evolution of a secondary initially located at $a_0 = 10^4 M$ with eccentricity $e_0 = 0.6$. We show the time evolution of the main orbital parameters under the action of the dissipative effects generated by each disk model. The dashed orange curves correspond to the SG prescription, while the solid blue curves represent the evolution obtained using the relativistic NT/Penna disk structure.

The two models lead to quantitatively different evolutionary tracks. The SG prescription predicts a more efficient extraction of orbital energy and angular momentum, resulting in a faster circularization of the orbit and a more rapid decrease of the semimajor axis. By contrast, the Penna model produces a slower evolution of all orbital parameters.

This difference can be traced back to the lower densities and larger scale heights predicted by the relativistic disk model. Since the efficiency of the disk-secondary interaction depends sensitively on the local gas density and on the vertical structure of the medium crossed by the compact object, even moderate differences in the underlying disk properties accumulate over long timescales and produce appreciable changes in the orbital evolution.

Our results indicate that the adopted disk prescription can have a significant impact on the predicted evolution of compact objects embedded in AGN disks, particularly when repeated disk crossings are involved.

Finally, we investigate the dependence of the circularization efficiency on the spin parameter χ of the central BH for both disk prescriptions considered in this work. For each configuration, we compute the number of disk crossings N_{cross} required to reduce the orbital eccentricity below the threshold $e_f < 0.05$, thereby reaching a quasicircular state. The results are reported in Fig. 6, where the color scale indicates the final semimajor axis, in units of the central BH mass, associated with the circularized orbit.

For the SG model (top panel), the number of crossings required to achieve circularization is found to be remarkably insensitive to the value of χ . Although the physical timescale associated with the evolution depends on the spin through the orbital frequencies and disk properties, the total dispersion in N_{cross} remains below 0.3%. This indicates that the overall efficiency of the circularization process is largely unaffected by the rotation of the central BH.

A similar behavior is observed for the Penna model (bottom panel). Despite the fact that the disk structure itself depends on the Kerr geometry, the number of crossings required to reach the circularized state varies only weakly across the range of spins considered. The impact of the spin therefore remains modest even when relativistic corrections significantly affect the properties of the accretion flow.

Taken together, these results indicate that the structure of the accretion disk plays a substantially larger role in determining the circularization efficiency than the spin of the central BH. The process is primarily controlled by the cumulative interaction between the secondary and the surrounding gaseous medium, while spin-dependent relativistic effects provide only a secondary correction.

D. Comparison with previous studies

We now compare the results here presented with recent studies of EMRIs interacting with accretion disks, particularly with Refs. [16,17]. We find this comparison very useful as it provides a consistency check of our results and, at the same time, allows us to identify the effects that arise from extending the standard disk-crossing picture to relativistic orbital dynamics. Reference [16] presents a fully relativistic treatment of EMRIs repeatedly crossing an accretion disk, in which both the orbital dynamics and the disk-secondary interaction are modeled relativistically.

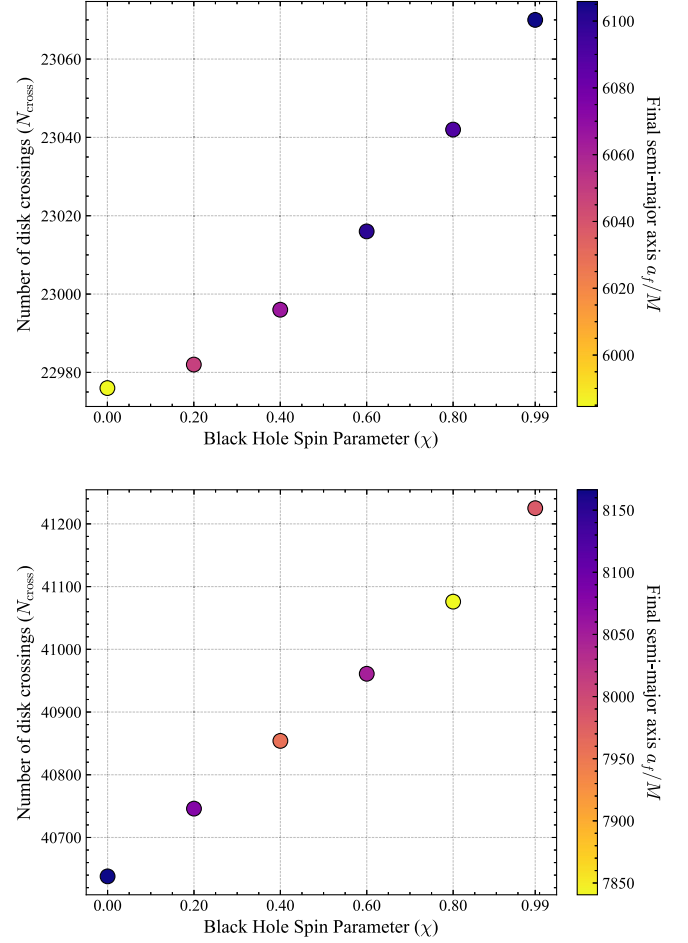


FIG. 6. Spin influence on the circularization efficiency. Number of disk crossings N_{cross} required for the eccentricity to decrease below the threshold $e_f < 0.05$, starting from identical initial orbital configurations for the SG (top) and Penna (Novikov-Thorne) (bottom) disk models. The color scale indicates the final semimajor axis, in units of the central BH mass, of the circularized orbit. In both disk prescriptions, the circularization efficiency exhibits only a weak dependence on the spin parameter χ , with variations in N_{cross} remaining below the percent level. This indicates that the circularization efficiency depends more strongly on the adopted disk structure than on the spin of the central BH.

Under very similar disk-EMRI configurations, the authors report that the a disk-driven alignment of the secondary's orbital plane, irrespective of the initial inclination. Also, they justify that the final eccentricity of a captured object is generically low, but that it can undergo a transient growth for initially highly inclined orbits when the secondary is a stellar-mass BH, and that only a modest fraction of nuclear-cluster black holes are captured within a typical AGN-disk lifetime through disk crossings alone. These results are complementary to ours in several respects. The transient eccentricity growth at large inclination is qualitatively consistent with the nonmonotonic, oscillatory behavior we report since Fig. 1, where the eccentricity dumps before

the system circularizes. In our analysis, this behavior arises within a hybrid framework in which the orbital motion is described by Kerr geodesics, while the disk-secondary interaction is incorporated through an effective interaction model, rather than through a fully relativistic treatment of the disk interaction itself. The qualitative agreement between these two independent approaches therefore provides a useful cross-check of the eccentricity evolution in the highly inclined regime.

We also analyzed Ref. [17], in which analytic and semianalytic estimates of capture and alignment timescales for nuclear-cluster orbiters were provided. The latter are assumed to interact with an AGN disk through geometric, drag-dominated crossings, within essentially the same physical picture that we adopt for the disk-secondary interaction. This means that disk crossings can be treated as discrete/quasi-instantaneous events of momentum exchange. The authors likewise find behavior consistent with the two-stage evolution observed in our simulations: (i) an initial phase in which the semimajor axis remains approximately constant while the inclination decreases slowly, followed by (ii) a more rapid orbital evolution once some critical relative velocity or inclination is reached. On top of this, the results reported by the authors (together with the subsequent literature on AGN-disk capture and alignment that builds on this framework), is closely related to the physical picture underlying Ref. [20], on which our Keplerian treatment introduced in Sec. II is based. Moreover, our results provide a relativistic extension of this picture. At large orbital separations, where relativistic corrections to the orbital dynamics are small, our alignment timescales are consistent, at the order-of-magnitude level, with the nonrelativistic estimates obtained in [17] (see Sec. IV A). At smaller separations, however, relativistic corrections accumulate, and the orbital evolution departs from the corresponding nonrelativistic behavior, as discussed in Sec. IV B. Thus, the agreement with the previous estimates at large separation provides the expected Newtonian-limit consistency check, while the deviations at smaller separation quantify the regime in which relativistic orbital dynamics becomes essential.

V. DISCUSSION AND OUTLOOK

A. Implications for quasiperiodic eruptions

In this section, we discuss a possible observational application of the results presented in this work, namely, QPEs. These events consist of recurring soft x-ray flares observed in time-domain light curves by instruments such as *XMM-Newton* and *Chandra*. To date, a small but growing sample of QPE sources has been identified, including GSN 069 (an ~ 1 hr soft x-ray flare recurring every ~ 9 hr) and RX J1301.9, together with several additional candidates in the local Universe [43].

The quality of the available observations has motivated a wide range of theoretical interpretations. Proposed explanations include partial tidal disruption events [44–47] and accretion-disk instabilities [48–50]. Another possibility involves a compact object orbiting an SMBH and repeatedly interacting with a surrounding accretion disk [18,51]. In particular, Ref. [18] considered the case of a star orbiting an SMBH, while Ref. [19] showed that an EMRI embedded in an accretion disk can reproduce several features of observed QPE light curves when the disk angular momentum is misaligned with the spin of the central BH. Within this picture, x-ray flares are generated whenever the orbiting object crosses the disk.

The orbital periods relevant for QPE observations fall within the parameter space explored in this work. Indeed, for a secondary orbiting an SMBH of mass M , the orbital period can be estimated from Kepler’s law as

$$T \sim 9 \text{ hr} \left(\frac{M}{10^8 M_\odot} \right) \left(\frac{a}{5M} \right)^{3/2}, \quad (42)$$

where a denotes the semimajor axis in units of the gravitational radius of the primary.

Within the EMRI-disk scenario, an eccentric orbit naturally produces two alternating timescales between successive bursts. These correspond to disk crossings occurring near apocenter and pericenter, yielding a longer recurrence time T_1 and a shorter one T_2 , respectively. Both timescales scale inversely with the local orbital velocity of the secondary. Using the Keplerian expressions for the orbital speed at the turning points, one finds

$$\frac{T_1}{T_2} \sim \frac{1+e}{1-e}, \quad (43)$$

where e is the orbital eccentricity. For example, an orbit with $e \sim 0.6$ predicts $T_1/T_2 \sim 4$. As the orbit circularizes, the two recurrence times progressively converge.

This behavior is particularly interesting in light of our results. We have shown that repeated disk crossings efficiently damp the orbital eccentricity while simultaneously driving the system toward coplanarity with the disk. Consequently, if disk-driven dissipation operates with an efficiency comparable to that found in our simulations, QPE systems associated with compact objects embedded in AGN disks should evolve toward configurations characterized by nearly equal recurrence times, $T_1 \sim T_2$. Measurements of alternating QPE periods may therefore provide an indirect probe of the orbital eccentricity and, more generally, of the strength of disk-driven dissipation in compact-object systems embedded in AGN environments.

B. Conclusions

In this work, we developed a relativistic framework to study the secular evolution of EMRIs embedded in an

accretion disk, with a hydrodynamic prescription for repeated disk crossings. This allowed us to consistently track the coupled evolution of the eccentricity, inclination, and semimajor axis across a wide range of orbital configurations.

Our analysis shows that disk-induced dissipation generically drives a robust two-stage evolutionary scenario, in which orbital-plane alignment occurs on (slightly) shorter timescales than eccentricity damping. This hierarchy naturally leads to a prolonged phase in which systems remain significantly eccentric while becoming coplanar with the disk, thereby increasing the likelihood of measurable eccentricity in the LISA band. Our results suggest a significant dependence of the eccentricity damping rate on the flow three-velocity, substantially weakening circularization and, in some cases, producing damping timescales comparable to (or exceeding) the inspiral timescale.

By systematically comparing Keplerian and relativistic orbital treatments, we demonstrate that cumulative relativistic effects become important even at large orbital separations, where one would naively expect Keplerian gravity to provide an accurate description. These effects are progressively amplified by repeated disk crossings, leading to secular deviations in both orbital geometry and dissipation rates. In the intermediate regime, relativistic corrections become dynamically dominant, significantly altering migration and damping timescales and highlighting the breakdown of purely Keplerian approximations well before the strong-field region is reached.

We have also investigated the impact of the accretion-disk model by comparing the widely used SG prescription with a relativistic NT disk constructed numerically via the Penna formalism. We find that relativistic disks exhibit systematically lower midplane densities and larger scale heights in the region $10^2 M \lesssim R \lesssim 10^4 M$, which leads to a substantially reduced efficiency of hydrodynamic drag and, therefore, slower orbital evolution. Thus, the choice of the disk model is not a secondary detail, but a critical ingredient in determining the long-term dynamics of compact objects embedded, for instance, in strong AGN disks. In particular, Keplerian disk models may systematically overestimate the efficiency of orbital capture, alignment, and migration in realistic relativistic environments.

Additionally, we have quantified the impact of the BH spin on the secular evolution of the system. We find that, despite its strong influence on the relativistic spacetime structure and on the inner disk region, the spin parameter has only a subdominant effect on the overall circularization efficiency, as measured by the number of disk crossings required to reach a quasicircular state. This robustness suggests that eccentricity damping is primarily governed by cumulative hydrodynamic interactions with the disk rather than by spin-induced relativistic corrections.

The results reported in this work open several directions for future investigation. A natural extension of the present framework consists in developing a fully relativistic description of the interaction between the secondary and the surrounding disk. While the orbital motion is modeled here through exact Kerr geodesics, the dissipative processes associated with accretion and dynamical friction are still treated using Keplerian prescriptions. A self-consistent relativistic treatment of both the dynamics and the disk interaction would provide a more accurate description of the evolution in the strong-field regime.

Another important question concerns the long-term astrophysical consequences of disk-driven orbital evolution. In particular, it would be interesting to investigate whether the eccentricity and inclination distributions predicted by our simulations leave observable imprints on EMRI populations entering the LISA band. Such a study could help establish whether gravitational-wave observations retain memory of an earlier phase of interaction with an AGN disk.

Finally, the framework developed here may prove useful in the context of quasiperiodic eruptions. The orbital trajectories generated by our simulations naturally determine the sequence and geometry of disk crossings, providing a basis for constructing synthetic flare light curves. Comparing such predictions with QPE observations may offer a complementary way of testing compact-object scenarios in AGN environments.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CONVERSION BETWEEN KEPLERIAN ORBITAL PARAMETERS AND KERR CONSTANTS OF MOTION

In this Appendix, we describe the procedure used to convert the Keplerian orbital parameters (a, e, ι) , or equivalently (a, e, x) with $x = \cos \iota$, into the corresponding constants of motion (E, L, Q) of a bound Kerr geodesic introduced in Sec. III. This conversion is required to initialize the geodesic evolution and to reconstruct the orbital parameters after each disk-crossing interaction.

The radial and polar potentials entering the Kerr-geodesic equations are given in Eqs. (25) and (26) of Sec. III, respectively. For a bound orbit, the conversion from the orbital parameters to the Kerr constants is obtained by requiring that the radial motion have turning points at the apoapsis $r_a = a(1 + e)$ and periapsis $r_p = a(1 - e)$, while the polar motion has a turning point at the maximum orbital latitude $\theta_{\min} = \frac{\pi}{2} - \iota$. We therefore impose

$$R(r_a) = R(r_p) = \Theta(\theta_{\min}) = 0. \quad (\text{A1})$$

This is equivalent, in terms of $x = \cos \iota$, to $\cos \theta_{\min} = x$ and $\sin^2 \theta_{\min} = 1 - x^2$. Then, substituting into the polar turning-point condition yields

$$Q = x^2 \left[\chi^2 (1 - E^2) + \frac{L^2}{1 - x^2} \right]. \quad (\text{A2})$$

The remaining two conditions in Eq. (A1), with the radial potential given by Eq. (25), constitute now two algebraic equations for E and L . Together with Eq. (A2), they provide a closed system for the three constants of motion (E, L, Q) corresponding to a specified set (a, e, ι) .

In practice, we solve this system numerically and select the physical branch corresponding to the specified bound orbit. In the weak-field limit, this branch is continuously connected to the corresponding Newtonian Keplerian orbit. This procedure is equivalent to the standard conversion between orbital parameters and Kerr constants of motion described by Schmidt [41] and employed in the (p, e, x) parametrization of Fujita and Hikida [39]. The same conversion underlies the `KerrGeoPy` implementation used in our simulations.

APPENDIX B: TIMESCALE OF GRAVITATIONAL-WAVE EMISSION DURING THE INSPIRAL

In this Appendix, we analyze the importance of dissipative GW effects, justifying why we can safely neglect them during the analysis of the binary evolution.

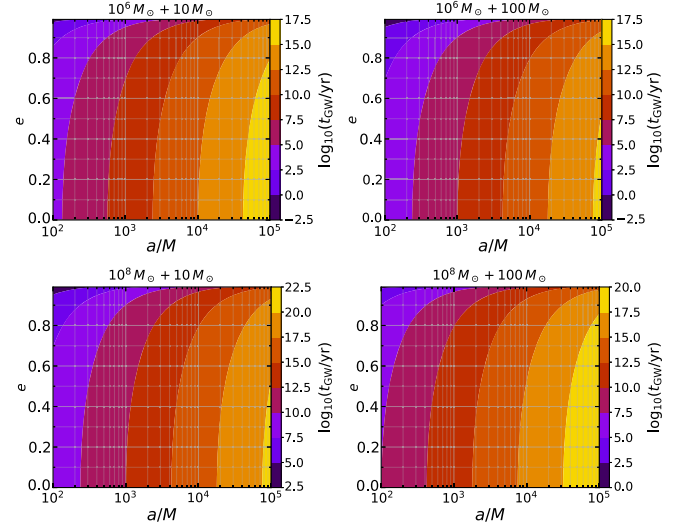


FIG. 7. Gravitational-wave emission. Timescale of gravitational-wave merging as a function of the initial semimajor axis and eccentricity for an EMRI evolving only through GW energy emission. The top (bottom) panels correspond to a primary mass of $10^6 M_\odot$ ($10^8 M_\odot$), while the left (right) panels assume a secondary mass of $10 M_\odot$ ($100 M_\odot$). The long merger timescales obtained throughout the region of parameter space relevant to this work show that GW emission, and therefore the associated 2.5PN contributions, remain subdominant to disk-driven effects.

Figure 7 shows the merger timescale of an EMRI driven solely by GW emission, which we model using a 2.5 post-Newtonian (PN) expression, as a function of the secondary's initial semimajor axis and eccentricity, following [52]. The merger time is computed using the semianalytic fit of Ref. [53], which reproduces the exact result to within 3% for eccentricities up to 0.99999. This comparison allows us to identify the region of the parameter space in which GW energy losses, and therefore the associated 2.5PN dissipative contributions, can be safely neglected.

We find that, within the parameters for the EMRIs considered in this work, the evolution is dominated by disk crossings rather than by GW emission. Indeed, for the typical initial conditions considered here, namely, $a \sim 10^4 M$ and eccentricities up to $e \sim 0.8$, the inspiral timescale driven by disk interactions is $\tau \lesssim 10^6$ yr, whereas the merger timescale due solely to GW energy loss exceeds 10^9 yr. Gravitational-wave dissipation becomes important only once the binary reaches substantially more compact and eccentric configurations, approximately $a \lesssim 10^3 M$ and $e \gtrsim 0.9$. Since the EMRIs studied here remain well outside this regime during the phase of interest, the contribution of the 2.5PN terms is negligible, thereby justifying their omission from our study. Modeling the late-time evolution once GW dissipation becomes dominant is beyond the scope of the present work and is deferred to future studies.

APPENDIX C: ECCENTRICITY VS INCLINATION CRITERION

In this Appendix, we provide a simple geometrical argument justifying why the eccentricity-based termination criterion used in our simulations corresponds, in practice, to the onset of the embedded regime in which discrete disk crossings cease to be well defined.

The argument is geometrical. By the time the eccentricity has decayed close to $e_{\min} = 10^{-3}$, the semimajor axis has also shrunk substantially relative to its initial value $a_0 = 10^6 M$ (see Fig. 2). Indeed, dynamical friction and accretion act cumulatively over many crossings, and by the late stages of the evolution the object is confined to the innermost portion of its original orbital range. From the top panel of Fig. 5, the disk aspect ratio H/R is a strongly increasing function of decreasing R/M in the inner region reached during the late stages of the evolution (both for the SG and the Penna/NT models). The vertical excursion of a nearly circular, residually inclined orbit of semimajor axis a and inclination i_{res} is $z_{\max} \simeq a \sin i_{\text{res}}$; the orbit is effectively embedded in the disk, in the sense that it never leaves the region where the density is non-negligible, once $z_{\max} \lesssim H(a)$, i.e.,

$$\sin i_{\text{res}} \lesssim \frac{H(a)}{a}. \quad (\text{C1})$$

Because H/a grows precisely as a shrinks toward the values reached once $e \rightarrow e_{\min}$, this embedding condition becomes progressively easier to satisfy as circularization proceeds, independent of how small i_{res} actually is. That is to say that the shrinkage of the orbit that accompanies circularization and the thickening of the disk at small radii act together to bring the system into the embedded regime by the time e reaches $e_{\min}$, even though i_{res} itself need not be arbitrarily small. This is the geometrical content behind the statement in footnote 3 that the eccentricity threshold corresponds to the onset of effective embedding in the disk, rather than to exact coplanarity.

Finally, we stress that this is a statement about the late-time, near-termination behavior of the system, and is logically compatible with the separate statement in Sec. III that the (early-time) alignment timescale is an order of magnitude shorter than the circularization timescale. In this sense, the eccentricity threshold does not imply exact coplanarity; rather, it identifies the point at which the orbit has become effectively embedded in the disk, and the discrete-crossing description ceases to be applicable.

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