

Extending the Euler-Heisenberg action to include effects of local Lorentz-symmetry violating backgrounds

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This work sets out to compute the corrections to the Euler-Heisenberg effective action that arise from spacetime-dependent background anisotropies that violate Lorentz symmetry. To accomplish our task, we evaluate the functional determinant of the modified Dirac operator using the spectral regularization method. Within the framework of the Standard Model extension (SME), these Lorentz-violating parameters correspond to the coefficients $m_5(x)$, $a_\mu(x)$, and $b_\mu(x)$. The corrected effective action is attained up to the second-order in the background parameters. Our results indicate a violation of Furry's theorem at second-order for specific C-odd combinations of these parameters, in agreement with previous analyses based on explicit Feynman diagram calculations in scenarios with Lorentz-symmetry violation. In the kinetic (bilinear) piece of the effective action, nondynamical axionlike terms emerge, resembling structures commonly encountered in condensed-matter systems, such as Weyl semimetals. We show how this procedure modifies the Maxwell equations, from which we present both the corresponding (non) conservation of the energy-momentum tensor and the wave equation. We also notice that the vacuum behaves as an inhomogeneous medium, and compute the local dispersion relation by working in the eikonal approximation. As a result of the spacetime-dependence of the background, there appear imaginary contributions in the dispersion relation and, consequently, the wave amplitudes may be amplified or attenuated, which expresses the energy-momentum exchange between the waves and the background.

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I. INTRODUCTION

Lorentz symmetry is one of the fundamental pillars of modern physics, underpinning both special relativity and the framework of quantum field theory, which enables the formulation of the Standard Model of particle physics. This symmetry guarantees the covariance of physical laws under Lorentz transformations, ensuring that relativistic field equations preserve their form in all inertial frames. Nevertheless, several theoretical developments have suggested that this symmetry might not be exact at very high-energy scales [1–3]. Possible deviations from Lorentz invariance may emerge, e.g., in the context of string-inspired models [4–6], quantum gravity approaches [7–11], or in anisotropic spacetime constructions [12]. In such scenarios, Lorentz-symmetry violation gives rise to a

broad range of nontrivial and far-reaching phenomenological repercussions [13–17].

A systematic framework to investigate the effects of spacetime-symmetry breaking is provided by the Standard-Model extension (SME), which describes and classifies all possible Lorentz- and *CPT*-violating terms¹ [5,19–21]. In the Abelian gauge sector, for example, the SME encompasses modifications of Maxwell electrodynamics, including the *CPT*-odd Carroll-Field-Jackiw (CFJ) term [1,22,23] and *CPT*-even extensions involving a rank-four background tensor with the symmetries of the Riemann tensor and a vanishing double trace [24]. These modifications may endow the quantum vacuum with nontrivial electromagnetic properties, such as birefringence or dichroism [19,24].

Within this effective field theory setting, it is also common to consider extensions of electrodynamics beyond the linear Maxwellian regime. In particular, nonlinear electromagnetic theories have been studied in the presence of Lorentz-symmetry violation, either as effective

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¹*CPT* symmetry refers to the combined operation of charge conjugation (*C*), parity transformation (*P*), and time reversal (*T*). A detailed introduction can be found in many quantum field theory textbooks, e.g., in [18].

descriptions arising from spontaneous Lorentz symmetry breaking in Abelian gauge-invariant frameworks [25], or as low-energy manifestations of quantum gravity-inspired scenarios, such as the semiclassical limit of loop quantum gravity [8–10]. Related nonlinear structures also appear in braneworld constructions, where Dirac-Born-Infeld-type actions may emerge dynamically as a consequence of spontaneous Lorentz breaking, even though the primary focus lies in the gravitational and brane sectors [26]. Alternatively, Lorentz symmetry may be explicitly broken in nonlinear electrodynamics. An example is provided by very special relativity, where a Born-Infeld action can be consistently embedded into a gauge-invariant theory with a reduced symmetry group, leading to modified electromagnetic interactions, such as a $\frac{1}{\beta^3}$ correction to the Coulomb potential [27]. More recently, phenomenological aspects of nonlinear electromagnetic theories coupled to axion fields and Lorentz-violating operators have also been investigated in the presence of a CFJ-type anisotropic background [28].

Radiative corrections to the Euler-Heisenberg effective action in the presence of Lorentz-symmetry violation have been analyzed at one loop, typically within setups where the Lorentz-violating backgrounds are taken to be constant (see, e.g., [29], and references therein). In this context, the effects of fermionic operators involving tensorial and axial structures were also calculated using the proper-time method [30,31]. More recently, these investigations were extended to scalar QED, yielding nontrivial *CPT*-even and *CPT*-odd Lorentz-violating contributions in the sector of nonlinear photon self-interactions [32].

At this stage, a natural question concerns whether more general background configurations, in particular those with spacetime dependence, can induce physically relevant corrections to these effective scenarios. As discussed by Kostelecký in Ref. [21], the assumption of constant Lorentz-violating coefficients is not a fundamental requirement, but rather a technical simplification. While constant backgrounds preserve translation invariance and ensure the conservation of the canonical energy-momentum tensor, allowing the coefficients to depend on spacetime generally leads to explicit modifications of the conservation laws [33]. Furthermore, in Riemann-Cartan geometries, constant coefficients impose integrability conditions that are satisfied only in highly restrictive settings, such as parallelizable manifolds, which are of limited physical interest. From this perspective, spacetime-dependent Lorentz-violating backgrounds constitute a more general and physically well-motivated framework, in which small deviations from energy-momentum conservation may arise and potentially provide observable signatures of Lorentz symmetry violation.

In the present work, we shall address this issue by analyzing the additional contributions that arise in the Euler-Heisenberg effective framework when Lorentz-violating backgrounds are allowed to depend on spacetime.

Focusing on the fermionic sector of the SME, we compute, via the proper-time method, the complete set of corrections to photon-photon scattering induced by spacetime-dependent couplings of the form $a_\mu \gamma^\mu$, $im_5 \gamma_5$, and $b_\mu \gamma^\mu \gamma_5$. Our analysis aims to clarify how such generalized backgrounds modify the structure of the effective action and to assess the physical consequences of these corrections.

The paper is organized as follows. In Sec. II, we briefly outline the general structure of the Standard-Model extension and discuss the field redefinitions in the matter sector. Section III is devoted to the calculation of the one-loop effective action, including terms up to fourth order in the field strength and second order in the LSV parameters. In Sec. IV, we analyze the physical implications of the modified effective action. In particular, we study the anisotropic corrections to the quadratic sector and investigate how Maxwell's equations and the conservation of the energy-momentum tensor are modified. Using the eikonal approximation we also derive the corresponding local dispersion relation. Finally, Sec. V contains the concluding remarks.

We adopt natural units such that $c = \hbar = 1$ and use the signature $(+, -, -, -)$ for the Minkowski metric $\eta_{\mu\nu}$. The paper also includes the Appendix, where we discuss the conventions used for the Wick rotation and Euclidean space.

II. THE SME AND FIELD REDEFINITIONS

The renormalizable and gauge-invariant QED sector of the SME is described by the following action in flat spacetime [21]:

$$S_{\text{QED-SME}} = \int d^4x \left[\bar{\psi} (i\Gamma^\mu D_\mu - M) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} (k_F)^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} (k_{AF})^\mu A^\nu F^{\kappa\lambda} \right], \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the standard electromagnetic field strength tensor, and $D_\mu = \partial_\mu + ieA_\mu$ denotes the usual gauge-covariant derivative. The Lorentz violation²

²The SME distinguishes observer Lorentz transformations, under which both fields and background coefficients transform covariantly, from particle Lorentz transformations, which act only on dynamical fields and expose the physical breaking of Lorentz symmetry [21]. Additionally, the *CPT* theorem establishes that *CPT* symmetry follows from Lorentz invariance together with the fundamental assumptions of quantum field theory. Consequently, any violation of *CPT* requires the breakdown of at least one of these conditions, most notably Lorentz invariance. It has been rigorously demonstrated that *CPT* violation necessarily implies Lorentz violation, whereas Lorentz violation does not, in general, entail *CPT* violation [34–36].

in the gauge sector is characterized by the presence of two constant background quantities. The first corresponds to a *CPT*-even contribution, represented by the dimensionless tensor $(k_F)^{\mu\nu\lambda\rho}$, which exhibits the same symmetry properties as the Riemann curvature tensor in the gravitation context. The second is a *CPT*-odd term, described by the vector $(k_{AF})^\mu$, which has canonical mass dimension and generates the well-known CFJ term. In the matter sector, the Lorentz-violating terms are given by

$$\Gamma^\mu = \gamma^\mu + c^{\mu\nu}\gamma_\nu + d^{\mu\nu}\gamma_\nu\gamma_5 + e^\mu + if^\mu\gamma_5 + \frac{1}{4}g^{\lambda\nu\mu}\Sigma_{\lambda\nu} \quad (2)$$

and

$$M = m + im_5\gamma_5 + a_\mu\gamma^\mu + b_\mu\gamma^\mu\gamma_5 + H_{\mu\nu}\Sigma^{\mu\nu}. \quad (3)$$

The Dirac matrices γ^μ satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, with the chirality matrix defined by $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and the generators of the proper orthochronous Lorentz group $SO(1,3)$ in the spinor representation are written as $\Sigma^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Generally, the structures in Eqs. (2) and (3) can be expressed, respectively, as $\Gamma^\mu = \gamma^\mu + \delta\Gamma_{\text{LSV}}^\mu$ and $M = m + \delta M_{\text{LSV}}$, where $\delta\Gamma_{\text{LSV}}^\mu$ and δM_{LSV} denote small and constant Lorentz- and *CPT*-violating corrections.

Requiring the Lagrangian to be Hermitian implies that all LSV coefficients must be real. The parameters a_μ , b_μ , e_μ , and $g_{\mu\nu\lambda}$ are *CPT*-odd, whereas f_μ is *CPT*-even. This latter property follows from the fact that f_μ can be absorbed into the coefficients $c_{\mu\nu}$ by means of an appropriate spinor redefinition, so that f_μ contributes to physical observables only through bilinear combinations $f_\mu f_\nu$ [37]. It is also worth emphasizing that, in certain cases, a field redefinition can be performed to eliminate some of the background terms appearing in Eqs. (2) and (3). In other words, some of the LSV-terms can be absorbed into a reparametrization of the spinor field ψ . A typical example is the a_μ term, which can be removed by the transformation $\psi(x) = e^{-ia_\mu x} \chi(x)$. Similarly, the b_μ term could also be eliminated through a suitable redefinition, but only in the absence of the fermionic mass term, m . Another LSV-term that can be straightforwardly removed is m_5 , through a chiral rotation of the form $\psi = e^{i\alpha\gamma_5} \psi'$, with a global parameter given by $\alpha = -\frac{1}{2}\tan^{-1}(\frac{m_5}{m})$, and a corresponding redefinition of the mass term: $m \rightarrow m' = m \cos(2\alpha) - m_5 \sin(2\alpha)$ [37] (see also [38], for a rigorous analysis of field redefinitions that can eliminate redundant background choices).

Regarding the Lorentz and *CPT* invariance of the action (1), it is important to recall that these symmetries

are defined with respect to the free field theory, $\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$. From this perspective, the distinction between observer and particle transformations becomes essential. In a certain sense, Lorentz symmetry remains preserved: the theory behaves normally under rotations and boosts as long as one considers passive Lorentz transformations, i.e., those associated with changes of coordinates in the laboratory frame. Within the framework of the SME, LSV emerges exclusively under active transformations, when particle transformations modify the orientation of the fields relative to fixed background tensor structures that characterize the vacuum. In this formulation, the LSV terms, such as $(k_F)^{\mu\nu\lambda\rho}$, $(k_{AF})^\mu$, $\delta\Gamma_{\text{LSV}}^\mu$, and δM_{LSV} , transform as tensors, vectors, or scalars under observer transformations, but remain invariant (i.e., behave effectively as scalars) under particle transformations. This feature enables these terms to be consistently treated as constant background fields [5]. Furthermore, it is worth noting that the pseudoscalar mass parameter m_5 does not introduce an apparent violation of Lorentz symmetry, since the combined presence of m and m_5 still yields a Lorentz-invariant mass term at the level of the dispersion relation [21,33].

In what follows, we shall restrict our analysis only to the Lorentz symmetry-violating corrections in the fermionic mass sector, δM_{LSV} , setting all $H_{\mu\nu}$, $(k_F)^{\mu\nu\lambda\rho}$, $(k_{AF})^\mu$, and $\delta\Gamma_{\text{LSV}}^\mu$ equal to zero. Furthermore, we assume that the background responsible for Lorentz violation is spacetime-dependent. Under these assumptions, the modified covariant QED action in Minkowski spacetime, incorporating Lorentz-symmetry breaking contributions, reads

$$S_1 = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m - im_5\gamma_5 - a_\mu\gamma^\mu - b_\mu\gamma^\mu\gamma_5)\psi \right], \quad (4)$$

where now $a_\mu = a_\mu(x)$, $b_\mu = b_\mu(x)$ and $m_5 = m_5(x)$. In this case, the spinor field redefinitions discussed above are no longer sufficient to eliminate $a_\mu(x)$ and $m_5(x)$. The a_μ -term cannot be removed, since the derivative operator acts on both x_μ and a_μ in the exponential factor $e^{-ia_\mu x}$, thereby generating additional derivative couplings. On the other hand, the elimination of the m_5 -term through a chiral rotation leads to a spacetime-dependent mass, which complicates the interpretation of the resulting parameter as the physical fermion mass. Therefore, in the setting (4), all parameters m_5 , a_μ , and b_μ are expected to provide nontrivial contributions to Lorentz symmetry breaking and to play a significant role in the analysis presented throughout this work.

III. ONE-LOOP EFFECTIVE ACTION

In this section, based on the spectral regularization method [39], we employ a procedure similar to that adopted in Refs. [30,31] to evaluate the functional determinant of the modified Dirac operator. Since our interest is to integrate out only the matter fields (fermions, in this case), while keeping the gauge field as a classical background, the starting point is the fermionic functional trace associated with the action (4),

$$\begin{aligned} -i\text{Tr} \ln(i\mathcal{D} - \mathcal{M}) &= -\frac{i}{2} \text{Tr} [\ln(i\mathcal{D} - \mathcal{M}) + \ln(-i\mathcal{D} - \mathcal{M}^*)] \\ &= -\frac{i}{2} \text{Tr} \ln \mathcal{O}, \end{aligned} \quad (5)$$

where Tr denotes the functional trace, including integration over spacetime coordinates and trace over spinor indices. For convenience, we introduce the compact notation

$$\begin{aligned} \mathcal{M} &\equiv m + im_5\gamma_5 + a_\mu\gamma^\mu + b_\mu\gamma^\mu\gamma_5, \\ \mathcal{M}^* &\equiv m + im_5\gamma_5 - a_\mu\gamma^\mu - b_\mu\gamma^\mu\gamma_5, \end{aligned} \quad (6)$$

and adopt the Feynman slash notation $\mathcal{D} \equiv \gamma^\mu D_\mu$. The squared operator $\mathcal{O}$ can be written in the form

$$\mathcal{O} = \mathcal{K}[1 + \mathcal{K}^{-1}\mathcal{X}], \quad (7)$$

with $\mathcal{K} \equiv D^2 + m^2 + e\Sigma^{\mu\nu}F_{\mu\nu}$ representing the minimal second-order differential operator, and the Lorentz-violating part given by

$$\begin{aligned} \mathcal{X} &\equiv -a^2 + b^2 - m_5^2 + i(\partial \cdot a + 2a \cdot D) \\ &+ i\gamma_5(\partial \cdot b + 2m_5m) + 2im_5\cancel{b} \\ &- \gamma_5(\cancel{\partial}m_5 + 2m_5\cancel{\partial} + 2im_5\cancel{a}) \\ &+ 4\Sigma^{\mu\nu}(2ia_{[\mu}b_{\nu]}\gamma_5 - 2\gamma_5b_{[\mu}D_{\nu]} - \partial_{[\nu}a_{\mu]} - \gamma_5\partial_{[\nu}b_{\mu]}). \end{aligned} \quad (8)$$

Since the Lorentz-violating parameters are expected to be small (see, e.g., Ref. [5] for theoretical discussions), the decomposition (7) proves particularly useful, as it allows for a systematic expansion of the effective action in which $\mathcal{X}$ can be treated as a small perturbation around the minimal operator $\mathcal{K}$,

$$\begin{aligned} \Gamma &= -\frac{i}{2} \text{Tr} \ln \mathcal{O} = -\frac{i}{2} \text{Tr} \ln \mathcal{K} - \frac{i}{2} \text{Tr}(\mathcal{K}^{-1}\mathcal{X}) \\ &+ \frac{i}{4} \text{Tr}(\mathcal{K}^{-1}\mathcal{X}\mathcal{K}^{-1}\mathcal{X}) + (\dots), \end{aligned} \quad (9)$$

where (...) denotes higher-order terms in $\mathcal{X}$. In the present work, we restrict our analysis to contributions up to second order in the Lorentz-violating background parameters. In this regard, it is important to note that the third term in Eq. (9) contains cubic and quartic powers of the Lorentz-violating structures and, hence, these contributions will be neglected in what follows.

In order to perform the analytic continuation from Minkowski to Euclidean space, we implement a Wick rotation of the time coordinate, $x^0 \rightarrow -i\hat{x}_4$, in the squared operator $\mathcal{O}$. Consequently, the gauge-covariant derivative in Euclidean space takes the form³

$$\hat{D}_\mu = \hat{\partial}_\mu - ie\hat{A}_\mu, \quad (10)$$

and the effective action becomes

$$\hat{\Gamma} = -i\Gamma. \quad (11)$$

Then, by collecting only the first- and second-order terms in the Lorentz-violating parameters, and expressing the inverse of the minimal operator via the Schwinger proper-time representation,

$$\frac{1}{\hat{\mathcal{K}}} = \int_0^\infty ds e^{-s\hat{\mathcal{K}}}, \quad (12)$$

we arrive at

$$\hat{\Gamma} = \hat{\Gamma}_0 + \hat{\Gamma}_{\text{LSV}}, \quad (13)$$

where $\hat{\Gamma}_0$ denotes the Lorentz-invariant part of the effective action, corresponding to the standard QED contribution [40], and $\hat{\Gamma}_{\text{LSV}}$ encodes the contributions induced by the Lorentz-violating background. In what follows, for notational simplicity, we omit the hats on quantities defined in Euclidean space. In this way, at first-order in the Lorentz-violating parameters we have

$$\begin{aligned} \Gamma_{\text{LSV}}^{(1)} &= -\frac{i}{2\mu^2} \text{Tr} \int_0^\infty ds \left\{ e^{\frac{s}{\mu^2}(D^2 - m^2 - e\Sigma^{\lambda\sigma}F_{\lambda\sigma})} \right. \\ &\times [\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b + 2m_5m) \\ &\left. + 2i\Sigma^{\mu\nu}(\partial_\nu a_\mu + \gamma_5\partial_\nu b_\mu + 2\gamma_5b_\mu D_\nu)] \right\}, \end{aligned} \quad (14)$$

while the second-order terms are

³Additional details on the conventions employed for the Wick rotation of the remaining operator structures can be found in the Appendix.

$$\begin{aligned}
\Gamma_{\text{LSV}}|^{(2)} = & -\frac{1}{2\mu^2} \text{Tr} \int_0^\infty ds \left\{ e^{\frac{s}{\mu^2}(D^2 - m^2 - e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} [a^2 - b^2 + m_5^2 - 4i\Sigma^{\mu\nu} \gamma_5 a_\mu b_\nu] \right\} \\
& - \frac{1}{4\mu^4} \text{Tr} \int_0^\infty ds \int_0^\infty d\tau \left\{ e^{\frac{s}{\mu^2}(D^2 - m^2 - e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} \left[\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b \right. \right. \\
& \left. \left. + 2m_5 m) + 2i\Sigma^{\mu\nu}(\partial_\nu a_\mu + \gamma_5 \partial_\nu b_\mu + 2\gamma_5 b_\mu D_\nu) + i\gamma_5(\partial \mathcal{M}_5 + 2m_5 \mathcal{D}) \right] e^{\frac{\tau}{\mu^2} D^2} \right. \\
& \left. \times e^{-\frac{\tau}{\mu^2}(m^2 + e\Sigma^{\rho\sigma} F_{\rho\sigma})} \left[\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b + 2m_5 m) + 2i\Sigma^{\alpha\beta}(\partial_\beta a_\alpha \right. \right. \\
& \left. \left. + \gamma_5 \partial_\beta b_\alpha + 2\gamma_5 b_\alpha D_\beta) + i\gamma_5(\partial \mathcal{M}_5 + 2m_5 \mathcal{D}) \right] \right\}. \tag{15}
\end{aligned}$$

Here, μ denotes a mass scale introduced to render the proper-time parameter dimensionless. Notice that in Eqs. (14) and (15) we have already discarded terms containing an odd number of γ -matrices, whose traces vanish identically.

In what follows, we evaluate these contributions separately, since the expressions are lengthy and technically involved, particularly in the second-order case.

A. First-order LSV corrections

Considering the functional trace, defined as $\text{Tr} = \text{tr} \int d^4x \delta^4(x - x')|_{x=x'}$, and making use of the trace cyclic property to change $e^{-\frac{s}{\mu^2} D^2}$ position, the expression (14) reads

$$\begin{aligned}
\Gamma_{\text{LSV}}|^{(1)} = & -\frac{i}{2\mu^2} \int d^4x \int_0^\infty ds \left\{ e^{-\frac{sm^2}{\mu^2}} \text{tr} e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} \left[\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b + 2m_5 m) \right. \right. \\
& \left. \left. + 2i\Sigma^{\mu\nu}(\partial_\nu a_\mu + \gamma_5 \partial_\nu b_\mu + 2\gamma_5 b_\mu D_\nu) \right] e^{\frac{s}{\mu^2} D^2} \delta^4(x - x')|_{x=x'} \right\}. \tag{16}
\end{aligned}$$

The kernel can be expressed as [39]

$$K(s) = e^{\frac{s}{\mu^2} D^2} \delta^4(x - x')|_{x=x'} = \frac{\mu^4}{16\pi^2 s^2} \det^{1/2} \left[\frac{iesF}{\mu^2 \sinh(\frac{ies}{\mu^2} F)} \right]. \tag{17}$$

It is worth noting that the terms containing a single gauge-covariant derivative acting on the kernel vanish, since [39]

$$D^\mu e^{\frac{s}{\mu^2} D^2} \delta^4(x - x')|_{x=x'} = 0. \tag{18}$$

Then,

$$\begin{aligned}
\Gamma_{\text{LSV}}|^{(1)} = & -\frac{i\mu^2}{32\pi^2} \int d^4x \int_0^\infty ds \left\{ \frac{1}{s^2} e^{-\frac{sm^2}{\mu^2}} \text{tr} e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} \left[\partial \cdot a - \gamma_5(\partial \cdot b - 2m_5 m) \right. \right. \\
& \left. \left. + 2i\Sigma^{\mu\nu}(\partial_\nu a_\mu - \gamma_5 \partial_\nu b_\mu) \right] \det^{1/2} \left[\frac{iesF}{\mu^2 \sinh(\frac{ies}{\mu^2} F)} \right] \right\}. \tag{19}
\end{aligned}$$

For weak background electromagnetic fields, we can proceed with the following expansion:

$$\det^{1/2} \left[\frac{iesF}{\mu^2 \sinh(\frac{ies}{\mu^2} F)} \right] = 1 + \frac{e^2 s^2}{12\mu^4} \text{Tr}(F^2) + \frac{e^4 s^4}{72\mu^8} \left[\frac{1}{4} F^4 + \frac{1}{5} \text{Tr}(F^4) \right] + O(F^6). \tag{20}$$

In this way, the traces of the exponentials are explicitly given by

$$\begin{aligned}
\text{tr} \left[e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} \right] &= \text{tr} \left[\delta_\mu^\mu - \frac{es}{\mu^2} \Sigma^{\lambda\sigma} F_{\lambda\sigma} + \frac{e^2 s^2}{2\mu^4} (\Sigma^{\lambda\sigma} F_{\lambda\sigma})^2 - \frac{e^3 s^3}{6\mu^6} (\Sigma^{\lambda\sigma} F_{\lambda\sigma})^3 + \frac{e^4 s^4}{24\mu^8} (\Sigma^{\lambda\sigma} F_{\lambda\sigma})^4 + \dots \right] \\
&= 4 + \frac{e^2 s^2}{\mu^4} F^2 + \frac{e^4 s^4}{2\mu^8} \left(\frac{1}{4} F^4 - \frac{1}{3} F^{\mu\nu} F_{\nu\alpha} F^{\alpha\beta} F_{\beta\mu} \right) + O(F^6); \tag{21}
\end{aligned}$$

$$\text{tr}[e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma}F_{\lambda\sigma})}\gamma_5] = -\frac{e^2s^2}{\mu^4}(\tilde{F}^{\mu\nu}F_{\mu\nu}) - \frac{e^4s^4}{12\mu^8}F^2(\tilde{F}^{\mu\nu}F_{\mu\nu}) + O(F^6); \quad (22)$$

$$\text{tr}[e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma}F_{\lambda\sigma})}\Sigma^{\mu\nu}] = -\frac{2es}{\mu^2}F^{\mu\nu} - \frac{e^3s^3}{\mu^6}\left[\frac{1}{2}F^2F^{\mu\nu} + \frac{2}{3}F^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu}\right] + O(F^5); \quad (23)$$

and

$$\begin{aligned} \text{tr}[e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma}F_{\lambda\sigma})}\Sigma^{\mu\nu}\gamma_5] &= \frac{2es}{\mu^2}\tilde{F}^{\mu\nu} + \frac{e^3s^3}{12\mu^6}[F^2\tilde{F}^{\mu\nu} + \tilde{F}_{\lambda\sigma}F^{\lambda\sigma}F^{\mu\nu} + 2\tilde{F}_{\lambda\sigma}F^{\mu\lambda}F^{\sigma\nu} \\ &\quad - F^{\mu\lambda}F_{\lambda\sigma}\tilde{F}^{\sigma\nu} - \tilde{F}^{\mu\lambda}F_{\lambda\sigma}F^{\sigma\nu}] + O(F^5), \end{aligned} \quad (24)$$

with $\tilde{F}_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ denoting the dual field strength tensor in the Euclidean space.

By inserting the expressions derived above into (19), evaluating the proper-time integrals, and returning to the Minkowski spacetime, we arrive at the following result

$$\Gamma_{\text{LSV}}^{(1)} = -\frac{e^2}{16\pi^2 m} \int d^4x m_5 \tilde{F}^{\mu\nu} F_{\mu\nu}, \quad (25)$$

where we neglect contributions that, in the covariantly constant background field approximation, reduce to total derivative terms after integration by parts.

One notes that the axial mass parameter $m_5(x)$ gives rise to a nontrivial contribution to the effective action at first-order in LSV, with the same local structure as an axion-photon coupling. In contrast, for constant m_5 , this contribution reduces to a surface term, up to integration by parts, and hence does not lead to physical modifications of the electromagnetic dynamics. Nevertheless, even in the spacetime-dependent case, $m_5(x)$ should be regarded as an external pseudoscalar background rather than a propagating axionlike field, since no kinetic or potential terms for m_5 appear at this order.

An interesting point concerns the possible relevance of terms analogous to the action (25) in condensed-matter systems. Nonpropagating excitations, in some cases

spacetime-dependent and effectively described by axion-like parameters, have been reported in several low-energy scenarios (see, e.g., [41] for the review). Particularly noteworthy are systems in which such excitations arise in materials exhibiting emergent Lorentz symmetry (and its violation [42]), as discussed in Refs. [43–45]. In this context, Euler-Heisenberg-type nonlinear excitations have also been reported in these materials [46]. Therefore, approaches such as the one developed in the present work may provide a suitable framework for describing these systems, in which multiple effects can coexist.

Since the action (25) is the only term contributing at first-order in LSV, we show in what follows that additional axionlike terms induced by Lorentz breaking emerge at second-order in LSV corrections.

B. Second-order LSV corrections

We now turn to the evaluation of the second-order contributions to Lorentz violation, encoded in $\Gamma_{\text{LSV}}^{(2)}$. The first line of (15) contains only a proper-time integral and can thus be computed in a straightforward way, by applying the same procedures used in the derivation of the first-order LSV corrections. In particular, one makes use of the cyclic property of the trace and the expansions given in Eqs. (20)–(22), leading to

$$\begin{aligned} \Gamma_{\text{LSV}}^{(2)} &= -\frac{1}{2\mu^2} \int d^4x \int_0^\infty ds \left\{ e^{-\frac{sm^2}{\mu^2}} \text{tr} e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma}F_{\lambda\sigma})} \left[a^2 - b^2 + m_5^2 - 4i\Sigma^{\mu\nu}\gamma_5 a_\mu b_\nu \right] K(s) \right\} \\ &\quad - \frac{1}{4\mu^4} \int d^4x \int_0^\infty ds \int_0^\infty d\tau \left\{ e^{-\frac{(s+\tau)m^2}{\mu^2}} \text{tr} e^{-\frac{s}{\mu^2}(e\Sigma^{\lambda\sigma}F_{\lambda\sigma})} \left[\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b + 2m_5 m) \right. \right. \\ &\quad \left. \left. + 2i\Sigma^{\mu\nu}(\partial_\nu a_\mu + \gamma_5 \partial_\nu b_\mu + 2\gamma_5 b_\mu D_\nu) + i\gamma_5(\partial m_5 + 2m_5 D) \right] e^{-\frac{\tau}{\mu^2}(e\Sigma^{\rho\sigma}F_{\rho\sigma})} \right. \\ &\quad \left. \times e^{\frac{\tau}{\mu^2}D^2} \left[\partial \cdot a + 2a \cdot D + \gamma_5(\partial \cdot b + 2m_5 m) + 2i\Sigma^{\alpha\beta}(\partial_\beta a_\alpha + \gamma_5 \partial_\beta b_\alpha + 2\gamma_5 b_\alpha D_\beta) \right. \right. \\ &\quad \left. \left. + i\gamma_5(\partial m_5 + 2m_5 D) \right] e^{-\frac{\tau}{\mu^2}D^2} e^{\frac{(s+\tau)D^2}{\mu^2}} \delta^4(x-x') \right\}. \end{aligned} \quad (26)$$

On the other hand, the part of Eq. (26) containing two proper-time integrals involves more complicated operator structures. Their treatment requires the use of nontrivial commutation relations between derivative operators and

background fields. To evaluate such commutators, we employ the identities derived from the Baker-Campbell-Hausdorff series

$$\begin{aligned} e^{\frac{\tau}{\mu^2} D^2} (\partial_\mu b^\mu + 2b^\mu D_\mu) e^{-\frac{\tau}{\mu^2} D^2} &= \partial_\mu b^\mu + 2b^\mu D_\mu + \frac{\tau}{\mu^2} [D^2, \partial_\mu b^\mu + 2b^\mu D_\mu] + \frac{\tau^2}{2\mu^4} [D^2, [D^2, \partial_\mu b^\mu + 2b^\mu D_\mu]] + \dots \\ &= \left(e^{\frac{2ie\tau F}{\mu^2}} \right)_\mu^\nu (\partial_\nu b^\mu + 2b^\mu D_\nu) + (\partial_\nu b^\mu) \Pi_\mu^\nu, \end{aligned} \quad (27)$$

with

$$\begin{aligned} \left(e^{\frac{2ie\tau F}{\mu^2}} \right)_\mu^\nu (\partial_\nu b^\mu + 2b^\mu D_\nu) &= \left[\delta_\mu^\nu + \frac{2i\tau e}{\mu^2} F_\mu^\nu - \frac{2\tau^2 e^2}{\mu^4} F_{\mu\lambda} F^{\lambda\nu} - \frac{4i\tau^3 e^3}{3\mu^6} F_{\mu\lambda} F^{\lambda\gamma} F_\gamma^\nu \right. \\ &\quad \left. + \frac{2\tau^4 e^4}{3\mu^8} F_{\mu\lambda} F^{\lambda\gamma} F_{\gamma\zeta} F^{\zeta\nu} + O(F^5) \right] (\partial_\nu b^\mu + 2b^\mu D_\nu), \end{aligned} \quad (28)$$

and

$$\begin{aligned} \Pi_\mu^\nu &= \frac{2\tau}{\mu^2} \left\{ 2D_\mu D^\nu + ieF_\mu^\nu + \frac{i\tau}{\mu^2} [4F_{\mu\lambda} D^\lambda D^\nu + 2F^{\nu\lambda} D_\lambda D_\mu + 3ieF_{\mu\lambda} F^{\lambda\nu}] \right. \\ &\quad - \frac{2e^2\tau^2}{3\mu^4} [6F_{\mu\lambda} F^{\lambda\gamma} D_\gamma D^\nu + 2F^{\nu\lambda} F_{\lambda\gamma} D^\gamma D_\mu + 6F_{\mu\lambda} F^{\nu\gamma} D_\gamma D^\lambda - 5ieF_{\mu\lambda} F^{\lambda\gamma} F_\gamma^\nu] \\ &\quad - \frac{ie^3\tau^3}{3\mu^6} [8F_{\mu\lambda} F^{\lambda\gamma} F_{\gamma\zeta} D^\zeta D^\nu + 2F^{\nu\lambda} F_{\lambda\gamma} F^{\gamma\zeta} D_\zeta D_\mu - 5ieF_{\mu\lambda} F^{\lambda\gamma} F_{\gamma\zeta} F^{\zeta\nu} \\ &\quad + 4(3F_\mu^\lambda F^\nu{}_\gamma + 2F_{\mu\gamma} F^{\nu\lambda}) F_{\lambda\zeta} D^\zeta D^\gamma] + \frac{4e^4\tau^4}{15\mu^8} [5F_{\mu\lambda} F^{\lambda\gamma} F_{\gamma\zeta} F^{\zeta\xi} D_\xi D^\nu \\ &\quad \left. + F^{\nu\lambda} F_{\lambda\gamma} F^{\gamma\zeta} F_{\zeta\xi} D^\xi D_\mu + 5(2F_\mu^\lambda F^\nu{}_\gamma F_{\lambda\zeta}^\zeta + F_{\mu\gamma} F^{\nu\lambda} F_{\lambda\zeta}^\zeta + 2F_\mu^\zeta F^{\nu\lambda} F_{\lambda\gamma}) F_{\zeta\xi} D^\xi D^\gamma] + O(F^5) \right\}. \end{aligned} \quad (29)$$

Another useful relation is

$$\begin{aligned} e^{\frac{\tau}{\mu^2} D^2} b^\mu e^{-\frac{\tau}{\mu^2} D^2} &= b^\mu + \frac{2\tau}{\mu^2} (\partial_\gamma b^\mu) \left[\delta_\rho^\gamma + \frac{i\tau}{\mu^2} F_\rho^\gamma - \frac{2e^2\tau^2}{3\mu^4} F^{\gamma\sigma} F_{\sigma\rho} - \frac{ie^3\tau^3}{3\mu^6} F^{\gamma\sigma} F_{\sigma\zeta} F_\rho^\zeta + \frac{2e^4\tau^4}{15\mu^8} F^{\gamma\sigma} F_{\sigma\zeta} F^{\zeta\xi} F_{\xi\rho} + O(F^5) \right] D^\rho \\ &= b^\mu + (\partial_\gamma b^\mu) \bar{\Pi}^\gamma{}_\rho D^\rho. \end{aligned} \quad (30)$$

It is worth emphasizing that these relations are valid for a background with constant field strength, i.e., $[D_\mu, D_\nu] = -ieF_{\mu\nu} = \text{constant}$. Furthermore, for simplicity, we restrict our analysis to Lorentz-violating backgrounds $m_5(x)$, $a_\mu(x)$, and $b_\mu(x)$ that depend linearly on the spacetime coordinates, as adopted for the gauge field $A_\mu(x)$ in the constant-field-strength approximation. This choice corresponds to the simplest nontrivial class of spacetime-dependent Lorentz-violating backgrounds.⁴ Using these results, we can proceed to compute the relevant contributions to the effective action, which are given by

⁴Consequently, all terms involving second and higher-order derivatives of the Lorentz-violating backgrounds vanish identically. In this framework, one may also introduce field-strengthlike quantities associated with the Lorentz-violating parameters, which are likewise constant.

$$\begin{aligned}
\Gamma_{\text{LSV}}^{(2)} = & -\frac{\mu^2}{32\pi^2} \int d^4x \int_0^\infty ds \frac{1}{s^2} e^{-\frac{sm^2}{\mu^2}} \left\{ \left[4 + \frac{e^2 s^2}{\mu^4} F^2 + \frac{e^4 s^4}{2\mu^8} \left(\frac{1}{4} F^4 - \frac{1}{3} F^{\mu\nu} F_{\nu\alpha} F^{\alpha\beta} F_{\beta\mu} \right) \right] \right. \\
& \times (a^2 + b^2 - m_5^2) - i \frac{8es}{\mu^2} \tilde{F}^{\mu\nu} a_\mu b_\nu - i \frac{e^3 s^3}{3\mu^6} \left[F^2 \tilde{F}^{\mu\nu} + \tilde{F}_{\lambda\sigma} F^{\lambda\sigma} F^{\mu\nu} + 2\tilde{F}_{\lambda\sigma} F^{\mu\lambda} F^{\sigma\nu} \right. \\
& \left. \left. - F^{\mu\lambda} F_{\lambda\sigma} \tilde{F}^{\sigma\nu} - \tilde{F}^{\mu\lambda} F_{\lambda\sigma} F^{\sigma\nu} \right] a_\mu b_\nu \right\} - \frac{1}{4\mu^4} \int d^4x \int_0^\infty ds \int_0^\infty d\tau \left\{ e^{-\frac{(s+\tau)m^2}{\mu^2}} \text{tr} e^{-\frac{s}{\mu^2} (e\Sigma^{\lambda\sigma} F_{\lambda\sigma})} \right. \\
& \times \left[2i\Sigma^{\mu\nu} (\partial_\nu a_\mu - \gamma_5 \partial_\nu b_\mu) - \gamma_5 (\partial \cdot b + 2b \cdot D - 2m_5 m) + \partial \cdot a \right. \\
& \left. + 2a \cdot D - \gamma_5 (\partial m_5 + 2m_5 D - 2m b) \right] e^{-\frac{\tau}{\mu^2} (e\Sigma^{\rho\eta} F_{\rho\eta})} \left[2i\Sigma^{\mu\nu} (\partial_\nu a_\mu - \gamma_5 \partial_\nu b_\mu) \right. \\
& \left. + \left(e^{\frac{2ie\tau F}{\mu^2}} \right)_\mu^\nu (\partial_\nu a^\mu + 2a^\mu D_\nu - \gamma_5 \partial_\nu b^\mu - 2\gamma_5 b^\mu D_\nu - \gamma_5 \gamma^\mu \partial_\nu m_5 - 2\gamma_5 \gamma^\mu m_5 D_\nu) \right. \\
& \left. + (\partial_\nu a^\mu - \gamma_5 \partial_\nu b^\mu - \gamma_5 \gamma^\mu \partial_\nu m_5) \Pi_\mu^\nu + 2\gamma_5 m (m_5 + b) + 2\gamma_5 m (\partial_\gamma m_5 + \partial_\gamma b) \bar{\Pi}^\gamma_\rho D^\rho \right] e^{\frac{(s+\tau)}{\mu^2} D^2} \delta^4(x - x') \right\}, \quad (31)
\end{aligned}$$

where the result of the action of two gauge-covariant derivatives on the kernel can be expressed as [39]

$$\begin{aligned}
D^\mu D^\nu e^{\frac{(s+\tau)}{\mu^2} D^2} \delta^4(x - x')|_{x=x'} &= \left[\frac{-ieF}{e^{2ie(s+\tau)F/\mu^2} - 1} \right]^{\nu\mu} K(s + \tau) \\
&= \left[-\frac{\mu^2}{2(s+\tau)} \delta^{\mu\nu} - \frac{ie}{2} F^{\mu\nu} + \frac{e^2(s+\tau)}{6\mu^2} F^{\mu\kappa} F_\kappa^\nu + \frac{e^4(s+\tau)^3}{90\mu^6} F^{\mu\kappa} F_{\kappa\theta} F^{\theta\eta} F_\eta^\nu \right] K(s + \tau). \quad (32)
\end{aligned}$$

The terms involving two proper-time integrals require extensive Clifford algebra manipulations, in particular due to the presence of products of several Lorentz generators in combination with γ_5 . For this reason, the computation of these structures was carried out separately, order by order in the field strength, and independently cross-checked using the `xAct` package in *Mathematica* [47]. After evaluating the corresponding traces, performing the integrations over the proper times s and τ , and returning to Minkowski spacetime, one arrives at

$$\Gamma_{\text{LSV}}^{(2)} = \sum_{n=1}^4 \Gamma_{F^n}, \quad (33)$$

where n labels the power of the field strength appearing in each term of the effective action. For $n = 1$, we then obtain corrections linear in F and second-order in the LSV background, given by

$$\Gamma_{F^1} = \frac{e}{16\pi^2 m} \int d^4x \left\{ \left[\frac{1}{\epsilon} - \gamma_E + \ln\left(\frac{\mu^2}{m^2}\right) \right] 2a_\mu b_\nu \tilde{F}^{\mu\nu} + 2m_5 (\partial_\mu a_\nu) \tilde{F}^{\mu\nu} \right\}, \quad (34)$$

where γ_E is the Euler-Mascheroni constant. The term $a_\mu b_\nu \tilde{F}^{\mu\nu}$ in the action (34) originates from a purely divergent contribution. This corresponds to shifting the vacuum, since this infinity contribution appears in the tadpole of the photon field. In contrast, the term proportional to $m_5 (\partial_\mu a_\nu) \tilde{F}^{\mu\nu}$ is finite and suggests that LSV background can effectively act as a source term in Maxwell equations.

For $n = 2$, the contribution takes the following form

$$\Gamma_{F^2} = \frac{e^2}{16\pi^2 m^2} \int d^4x \left[\frac{1}{3} (m_5^2 + b^2) F^2 + \frac{1}{6} (a \cdot b) F_{\mu\nu} \tilde{F}^{\mu\nu} \right]. \quad (35)$$

In this quadratic sector of the effective action, the second-order corrections in LSV contribute to the Maxwell equations in a nontrivial way. It is important to note that, as in the case of the contribution to the photon kinetic sector given by Eq. (25), the new terms appearing in Eq. (35) also contain axionlike contributions. However, it should be emphasized that these are not dynamical axions, but rather axion-type parameters that may be related to possible phenomenological applications.

Now, for $n = 3$, we obtain that

$$\Gamma_{F^3} = \frac{e^3}{16\pi^2 m^4} \int d^4x \left\{ \frac{37}{96} a^\mu b^\nu F_{\mu\nu} (F \cdot \tilde{F}) - \frac{7m_5}{5m} (\partial^\mu a^\nu) F_{\mu\nu} \left[F^2 - \frac{1}{2} (F \cdot \tilde{F}) \right] \right\}. \quad (36)$$

Finally, but not least, we obtain the contribution for $n = 4$, given by

$$\begin{aligned} \Gamma_{F^4} = & -\frac{e^4}{16\pi^2 m^6} \int d^4x \left\{ \frac{m_5^2}{4} \left[\frac{1}{9} F^4 + \frac{7}{45} (F \cdot \tilde{F}) \right] + \frac{523}{5472} (a \cdot b) F^2 (F \cdot \tilde{F}) \right. \\ & + \frac{b^2}{45} [F^4 + 2(F \cdot \tilde{F})^2] + \frac{2}{45} b^\mu b_\nu F_{\mu\kappa} F^{\kappa\nu} F^2 - \frac{817}{720} a^\mu b_\nu F_{\mu\kappa} F^{\kappa\nu} (F \cdot \tilde{F}) \\ & \left. - \frac{m_5}{m} \left[\frac{31}{60} (\partial^\mu a_\nu) F_{\mu\kappa} F^{\kappa\nu} (F \cdot \tilde{F}) + \frac{2}{25} (\partial \cdot b) (F \cdot \tilde{F})^2 - 12 (\partial^\mu b_\nu) F_{\mu\kappa} F^{\kappa\nu} F^2 \right] \right\}. \end{aligned} \quad (37)$$

The results (34) and (36) point to a violation of Furry's theorem. This theorem states that any Feynman diagram containing a closed fermion loop with an odd number of external photon lines gives a vanishing contribution to physical amplitudes [48]. This occurs because the QED interaction is invariant under charge conjugation, while the photon field changes sign under this transformation. As a consequence, diagrams related by reversing the direction of the fermion loop cancel each other exactly when the number of photon vertices is odd. Furry's theorem plays an important role in simplifying perturbative calculations in QED and explains, e.g., why processes such as single-photon or three-photon amplitudes generated by a fermion loop do not contribute to the effective action.

The seminal work [49] studies the validity of Furry's theorem in the presence of first-order corrections in LSV and shows that only C-odd fermionic sectors are expected to yield contributions that violate the theorem. In our case, where the LSV backgrounds are spacetime dependent, the terms responsible for the violation of Furry's theorem arise from the mixing of two distinct LSV sectors and appear only at second-order. This indicates that the results on the violation of Furry's theorem obtained in Ref. [49] can be extended to second-order contributions in LSV.

Finally, we obtain the second-order LSV contribution to photon-photon scattering, given by Eq. (37). Actually, this set of quartic terms in the field strengths of the photon can be discussed in the context of the anomalous vertices in the electroweak theory. Moreover, the Z^0 field also couples, through weak coupling constant, to electron-positron field so that as similar neutral four-vertices can also appear associated Z^0 field [50,51]. This means that they may be used to constrain the corresponding parameters through photon-photon scattering data from the ATLAS and CMS experiments [52–54]. Also, we should point out that quartic terms cast in Eq. (37) yield tree-level contributions to the photon-photon scattering.

At this point, it is worth commenting on the absence of CFJ-type terms, of the form $\sim A_\mu \tilde{F}^{\mu\nu}$. In Ref. [55], it was shown that the calculation of the vacuum polarization with first-order insertions of the term $b_\mu \gamma^\mu \gamma_5$ and constant b_μ , leads to a CFJ-type correction to the effective action at second-order in the photon field, with $(k_{AF})^\mu = \frac{3}{16\pi^2} b^\mu$. However, it was emphasized that this result depends on the

regularization scheme employed. In our calculation, the electromagnetic background is frozen, $F_{\mu\nu} = \text{constant}$, hence avoiding contributions involving higher-derivative and CFJ-type terms. Once we freeze the field strength, it is equivalent to adopting the Fock-Schwinger gauge, $A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu$, and consequently it becomes clear that the effective action is written entirely in terms of $F_{\mu\nu}$. There is no way the gauge potential explicitly appears.

A additional remark concerns the fact that, if we restrict ourselves to the regime in which the coefficients are constant, in addition to discarding terms involving derivatives acting on the LSV background, we must also neglect the contributions depending on a_μ and m_5 at all orders in the field strength of the corrected effective action. This follows from the fact that one can perform a field redefinition from the outset to eliminate these terms, indicating that they cannot generate physical effects, as discussed in Sec. II.

In this regard, by setting the parameters a_μ and m_5 to zero and assuming b_μ to be constant in Eq. (35), one obtains

$$\Gamma_{F^2} = \frac{e^2}{48\pi^2 m^2} \int d^4x b^2 F^2. \quad (38)$$

In contrast to the result presented in work [31], when considering in Eq. (38) the particular case in which b_μ is lightlike, we find that the quadratic photon sector does not receive contributions from second-order in LSV. This discrepancy stems from the use of different parametrizations in the construction of the quadratic operator given in Eq. (5). In the present work, we make use of the properties of the γ_5 matrix together with the cyclicity of the trace, whereas Ref. [31] adopts a parametrization that effectively changes the sign of the mass term m and of $b_\mu \gamma^\mu \gamma_5$. As a consequence, a relative sign difference in the mass term m inside of $\mathcal{M}^*$, Eq. (6), appears when comparing the two approaches.

This ambiguity in the parametrization of the quadratic Dirac operator is related to the so-called multiplicative anomaly (see, e.g., [56–58] for the details). In a work currently in preparation [59], we revisit the calculation of the effective action with contributions from two constant Lorentz-violating parameters, b_μ and $H_{\mu\nu}$, and analyze this issue in a fully general scenario, with arbitrary

parametrization. In the present work, we choose to use the parametrization implemented through the γ_5 matrix, since it is the only one consistent with all results available in the literature obtained via explicit Feynman-diagram loop calculations in presence of LSV, such as those reported in [49,60].

In the next section, we analyze how the quadratic sector, composed of the results (25) and (35), modifies Maxwell's equations and the conservation of the energy-momentum tensor. We also derive the local dispersion relation associated with this corrections.

IV. MODIFIED MAXWELL EQUATIONS

The kinetic sector that contributes to vacuum Maxwell equations is given by

$$\Gamma_{\text{kinetic}} = \Gamma_{\text{Maxwell}} + \Gamma_{\text{LSV}}^{(1)} + \Gamma_{F^2} \\ = -\frac{1}{4} \int d^4x [(1 + c_1)F^2 + (c_2 + c_3)\tilde{F}^{\mu\nu}F_{\mu\nu}], \quad (39)$$

where we have used the definition of the fine-structure constant $\alpha = \frac{e^2}{4\pi}$ and the following compact notations:

$$c_1 = -\frac{\alpha}{3\pi m^2}(m_5^2 + b^2), \quad (40)$$

$$c_2 = -\frac{\alpha}{6\pi m^2}(a \cdot b), \quad (41)$$

$$c_3 = \frac{\alpha}{\pi m} m_5. \quad (42)$$

From the variational principle applied to the photon field, we obtain the covariant Maxwell equations in the form

$$(1 + c_1)\partial_\mu F^{\mu\nu} + (\partial_\mu c_1)F^{\mu\nu} + [\partial_\mu(c_2 + c_3)]\tilde{F}^{\mu\nu} = 0, \quad (43)$$

where the Bianchi identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$ holds. Since LSV is expected to be small, Eq. (43) can be rewritten as $\partial_\mu F^{\mu\nu} = (1 + c_1)^{-1}(\dots)$, allowing for a perturbative expansion of the right-hand side in powers of c_1 . Keeping terms up to second-order in the LSV parameters, Eq. (43) reduces to

$$\partial_\mu F^{\mu\nu} = (\partial_\mu c_1)F^{\mu\nu} + [\partial_\mu(c_2 + c_3)]\tilde{F}^{\mu\nu}. \quad (44)$$

The energy-momentum tensor conservation can be explicitly verified by contracting the right-hand side of Eq. (44) with $F_{\nu\rho}$ and subsequently constructing a continuity equation of the form $\partial_\mu(\dots)$. This procedure leads to

$$\partial_\mu[(1 + c_1)T^\mu{}_\kappa] = (1 + c_1)J^\mu F_{\mu\kappa} + \frac{1}{4}(\partial_\kappa c_1)F_{\alpha\beta}F^{\alpha\beta} \\ + \frac{1}{4}[\partial_\kappa(c_2 + c_3)]F_{\alpha\beta}\tilde{F}^{\alpha\beta}, \quad (45)$$

where $T^\mu{}_\kappa = F^{\mu\nu}F_{\nu\kappa} + \frac{1}{4}\delta^\mu_\kappa F_{\alpha\beta}F^{\alpha\beta}$ is the standard energy-momentum tensor of Maxwell electrodynamics. In Eq. (45), the source term J^ν may arise either from the coupling of the photon field to matter or from a purely LSV-induced source term originating from the effective action (34), when it is divergence-free. We observe that allowing the LSV background to depend on the spacetime coordinates leads to a violation of the conservation of the energy-momentum tensor, which is restored in the limit of a constant background.

This nonconservation of the energy-momentum tensor requires a careful assessment of the domain of validity of the effective description. It is well known that spacetime-dependent Lorentz-violating backgrounds generally lead to the nonconservation of the energy-momentum tensor [21] and, in certain cases, obstruct the consistent definition of a symmetric energy-momentum tensor [33,61]. In the present case, this feature arises from treating the coefficients $m_5(x)$, $a_\mu(x)$, and $b_\mu(x)$ as nondynamical external backgrounds. Consequently, Lorentz symmetry is broken explicitly, and the apparent nonconservation of the electromagnetic energy-momentum tensor can be interpreted as an exchange of energy and momentum with the background sector.

On the other hand, as discussed in Refs. [4,5,21], energy-momentum conservation is recovered once the complete dynamical system is taken into account. In scenarios with spontaneous Lorentz-symmetry breaking, the Lorentz-violating coefficients arise as vacuum expectation values of underlying dynamical fields, so that the apparent nonconservation in the observable sector reflects an exchange of energy and momentum with the fields responsible for the symmetry breaking, while the total energy-momentum tensor remains conserved.

Furthermore, Refs. [21,62,63] establish a series of no-go results showing that explicit local Lorentz-symmetry breaking is incompatible with the geometric framework of gravitational theories formulated on Riemann and Riemann-Cartan manifolds. In such geometries, diffeomorphism invariance together with the Bianchi identities impose constraints on the gravitational field equations that require the covariant conservation of the energy-momentum tensor, a condition naturally satisfied in the case of spontaneous symmetry breaking. Consequently, spacetime-dependent nondynamical backgrounds may still be consistently employed in flat spacetime as an effective description, provided that the theory is not regarded as a closed system. Therefore, the no-go results constrain the gravitational extension of such constructions rather than their use as effective field theories in Minkowski spacetime.

In the following, we demonstrate that the spacetime-dependent coefficients $m_5(x)$, $a_\mu(x)$, and $b_\mu(x)$ produce nontrivial corrections to the photon local dispersion relation. These modifications cause the vacuum to behave as an optically active medium, supporting nonpropagating modes

typically associated with dichroism, inducing the possibility of absorption and attenuation effects in electromagnetic wave propagation. To investigate these phenomena, we start from the $\nu = 0$ component of Eq. (44), which allows us to write Gauss's law for the electric field as

$$\nabla \cdot \mathbf{E} = (\nabla c_1) \cdot \mathbf{E} + [\nabla(c_2 + c_3)] \cdot \mathbf{B}. \quad (46)$$

Similarly, from $\nu = i$ component of Eq. (44) the Ampère-Maxwell equation in vacuum becomes

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = (\nabla c_1) \times \mathbf{B} - (\partial_t c_1) \mathbf{E} - [\nabla(c_2 + c_3)] \times \mathbf{E} - [\partial_t(c_2 + c_3)] \mathbf{B}. \quad (47)$$

As a consequence of the Bianchi identity, Gauss's law for magnetism and Faraday-Lenz law remain unchanged:

$$\nabla \cdot \mathbf{B} = 0, \quad (48)$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0. \quad (49)$$

By taking the curl of Faraday-Lenz law and using the Ampère-Maxwell equation, the wave equation for the electric field takes the form

$$\{[\partial_t^2 - \nabla^2 - (\nabla c_1) \cdot \nabla - (\partial_t c_1) \partial_t] \delta_{ij} + \nabla_i \nabla_j + (\nabla_j c_1) \nabla_i + \varepsilon_{ijl} [\nabla_l(c_2 + c_3) \partial_t - \partial_t(c_2 + c_3) \nabla_l]\} E_j = 0. \quad (50)$$

There is a central complication in the modified wave equation (50) related to the fact that the plane-wave ansatz (or the usual Fourier transform) is no longer the appropriate procedure to obtain the dispersion relation of the model, due to the presence of a background that depends explicitly on spacetime coordinates. To proceed, we adopt a method commonly used in the plasma physics literature for nonhomogeneous media, namely the eikonal approximation for electromagnetic waves [64], which allows one to define a local dispersion relation.

The eikonal approximation consists in the introduction of a small parameter ϵ^n multiplying each derivative term in the wave equation, where n denotes the order of the derivative operator. This allows us to systematically track the different orders in a perturbative expansion that can be carried out afterward. At the end of the calculation, the limit $\epsilon \rightarrow 1$ is taken. Under these considerations, the eikonal approximation for the electric field is written as

$$E_i(t, \mathbf{x}) = A_i(t, \mathbf{x}) \exp \left[\frac{i}{\epsilon} S(t, \mathbf{x}) \right], \quad (51)$$

where the amplitude $A_i(t, \mathbf{x})$ and the phase $S(t, \mathbf{x})$ are both assumed to be real (see, e.g., Ref. [64] for more details). Thus, at zeroth order in ϵ , the wave equation is given by

$$M_{ij}(t, \mathbf{x}) E_j = 0, \quad (52)$$

with

$$\begin{aligned} M_{ij}(t, \mathbf{x}) = & [\omega^2 - \mathbf{k}^2 - i(\partial_t c_1) \omega + i(\nabla c_1) \cdot \mathbf{k}] \delta_{ij} \\ & + k_i k_j - i(\nabla_j c_1) k_i \\ & + i \varepsilon_{ijl} [\nabla_l(c_2 + c_3) \omega + \partial_t(c_2 + c_3) k_l]. \end{aligned} \quad (53)$$

and taking into account the definitions

$$\nabla_i S(t, \mathbf{x}) \equiv k_i(t, \mathbf{x}) \quad \text{and} \quad \partial_t S(t, \mathbf{x}) \equiv -\omega(t, \mathbf{x}) \quad (54)$$

as the local wave vector and the local frequency, respectively.

Therefore, for a nontrivial electric field, the local dispersion equation (or eikonal equation) is obtained by $\mathcal{D}(\omega = -\partial_t S, \mathbf{k} = \nabla S, t, \mathbf{x}) \equiv \det M_{ij}(t, \mathbf{x}) = 0$.

Performing this determinant calculation and considering contributions up to second-order in LSV, one finds

$$\begin{aligned} \mathcal{D}(\omega, \mathbf{k}, t, \mathbf{x}) = & 4\omega^6 - 12i(\partial_t c_1) \omega^5 - [9\mathbf{k}^2 - 9i\mathbf{k} \cdot (\nabla c_1) + 2(\nabla c_3)^2] \omega^4 + 2[9i\mathbf{k}^2(\partial_t c_1) \\ & - 2(\partial_t c_3) \mathbf{k} \cdot (\nabla c_3)] \omega^3 + \{2\mathbf{k}^2[3\mathbf{k}^2 - (\partial_t c_3)^2 + (\nabla c_3)^2 - 12i(\nabla c_1) \cdot \mathbf{k}] \\ & - [(\nabla c_3) \cdot \mathbf{k}]^2\} \omega^2 - 2\mathbf{k}^2[3i\mathbf{k}^2(\partial_t c_1) - (\partial_t c_3)(\nabla c_3) \cdot \mathbf{k}] \omega - \mathbf{k}^4[\mathbf{k}^2 \\ & - (\partial_t c_3)^2 - 3i(\nabla c_1) \cdot \mathbf{k}]. \end{aligned} \quad (55)$$

The modified dispersion relation (55) shows that the spacetime-dependent LSV contributions lead to a complex local frequency. The real part governs wave propagation, while the imaginary part, proportional to spacetime derivatives of c_1 , may induce local amplification or attenuation of the wave amplitude. Indeed, by considering nonconstant parameters means that the energy-momentum of the wave

is not conserved, by virtue of the exchange with the background, as discussed above. The wave may, therefore, acquire or lose energy. When restricting the LSV sector to lightlike structures, the dispersion relation becomes real, and only the parameter m_5 yields nontrivial contributions.

To close this section, we wish to call the reader's attention to the fact that a spacetime-dependent frequency,

as a consequence of nonconstant/nonhomogeneous LSV parameters, may lead to a red or a blueshift under specific conditions on the way the local parameters depend on time and space coordinates. We believe this a relevant issue to be inspected. The reason being that the LSV parameters are expected to originate from some more fundamental physics, valid at (high) energies close to the scale where the vacuum fluctuations are very strong. It is then reasonable to expect that the LSV entities should emerge as local parameters. Considering constant parameters is a very particular situation and, even though it is a particular case, a broad class of interesting effects beyond Standard-Model physics may emerge.

V. CONCLUDING COMMENTS AND PROSPECTS

In this work, we present an explicit calculation of the effective photonic action up to fourth order in the electromagnetic field strength, taking into account Lorentz symmetry violation effects up to second-order in the SME spacetime dependent parameters m_5 , a_μ , and b_μ . The calculation is carried out within the proper-time formalism, with the heat-kernel results obtained from the evaluation of the fermionic determinant via zeta-function regularization, following Ref. [39].

At first-order in the electromagnetic field strength, an ultraviolet divergent contribution is generated, yielding a shift in the vacuum by virtue of the tadpole-type contribution to the effective action. At this order, we also obtain a nontrivial finite contribution that may act as an external source to the Maxwell equations. In the case of a constant Lorentz symmetry-violating background, all these terms reduce to surface terms and can therefore be dropped.

At second-order in the electromagnetic field strength, the corrections to the effective action are entirely finite. The quadratic order is the only one that yields a contribution at first-order in LSV arising from the m_5 term. This correction leads to modified Maxwell equations in vacuum, which in turn result in a nonconserved energy-momentum tensor for the photon field and a local dispersion relation obtained within the eikonal approximation, which may encode potentially interesting phenomenology to be explored in a subsequent work.

The third order correction to the effective action in the field strength, when combined with the first-order contribution, points to a violation of Furry's theorem by C-odd combinations at second-order in LSV. A similar result was previously reported in Ref. [49] based on a diagrammatic analysis at first-order in LSV.

We have also worked out the contribution to the photonic action quartic in the electromagnetic field strengths and at the second-order in LSV parameters, as presented in Eq. (37). The resulting structures can be relevant in connection with the anomalous gauge boson interactions

in the electroweak theory. In this regard, it is worth emphasizing that the neutral gauge boson, Z^0 , also couples to fermionic currents via the weak charge, which yield, analogously, the emergence of neutral quartic vertices involving the Z^0 field [50,51]. From a phenomenological perspective, such interactions open up the possibility of constraining the associated parameters through photon-photon scattering measurements, particularly those reported by the ATLAS and CMS collaborations [52–54]. Furthermore, we stress that the quartic operators appearing in Eq. (37) correspond to contributions to light-by-light scattering already at the tree level.

Additionally, the attainment of an effective photonic action of the Euler-Heisenberg type including the space-time-dependent parameters may be of interest if we identify the four-vector, $a_\mu(x)$, and the pseudovector, $b_\mu(x)$, respectively with the gradients of the saxion (the CP -even axion field—the supersymmetric scalar partner of the axion [65,66]) and the axion field itself, in the context of Eq. (35). This does not mean that these identifications reproduce interactions dictated by supersymmetry. We are merely borrowing the saxion field from the supermultiplet that accommodates the axion in a supersymmetric scenario. This could be a way to introduce dark matter candidates in interaction with the photon in a sort of extended Euler-Heisenberg action. By upgrading these variable LSV parameters to the status of dynamical fields, one should also introduce their respective kinetic and mass terms. If we take this path, an effective axionic Euler-Heisenberg emerges which may be further exploited. Differently from simply adding the (purely photonic) Euler-Heisenberg action to the axionic electrodynamics, such a construction would incorporate the saxion and axion fields into the classes of terms with powers of the photon field-strength and its dual.

To conclude, we emphasize, upon comparison with the results of Ref. [31], that there is an explicit dependence on the parametrization adopted to construct the quadratic Dirac operator in Eq. (5), which affects the final form of the effective action. We briefly argue that this ambiguity may be understood in terms of the multiplicative anomaly, which reflects the breakdown of the naive factorization property of functional determinants under a regularization procedure. This issue will be addressed in detail in a forthcoming work, where we consider constant b_μ and $H_{\mu\nu}$ terms. In that analysis, we shall clarify the origin of this ambiguity and argue in favor of the parametrization involving the γ_5 matrix as the most reliable one, showing that it is consistent with results obtained through Feynman diagram calculations.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: WICK ROTATION CONVENTIONS

The calculation of the effective action performed in Sec. III is well defined in Euclidean space, $\mathbb{E}^4$. In this appendix, we summarize the conventions adopted and the corresponding relations used to implement the Wick rotation from Minkowski spacetime, $\mathbb{M}^{1,3}$, with metric tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, to Euclidean space $\mathbb{E}^4$, endowed with the metric $\hat{g}_{\mu\nu} = -\delta_{\mu\nu}$. In $\mathbb{M}^{1,3}$, Greek indices run over $\mu, \nu = (0, 1, 2, 3)$, whereas in $\mathbb{E}^4$ they run over $\mu, \nu = (1, 2, 3, 4)$.

At the outset, the conventions adopted for the Wick rotation affect the coordinates x^μ , the partial derivatives ∂_μ , and the four-vectors v^μ defined in $\mathbb{M}^{1,3}$. The correspondence between these quantities in $\mathbb{M}^{1,3}$ and their counterparts in $\mathbb{E}^4$ is given, respectively, by

$$\{x^0 \rightarrow -i\hat{x}_4, \quad x^i \rightarrow \hat{x}_i\}, \quad (\text{A1})$$

$$\{\partial_0 \rightarrow i\hat{\partial}_4, \quad \nabla_i \rightarrow \hat{\nabla}_i\}, \quad (\text{A2})$$

$$\{v^0 \rightarrow -i\hat{v}_4, \quad v^i \rightarrow \hat{v}_i\}, \quad (\text{A3})$$

where the bar denotes quantities defined in Euclidean space. These definitions imply that the scalar product of four-vectors is transformed in the following manner

$$u_\mu v^\mu \rightarrow -\hat{u}_\mu \hat{v}^\mu, \quad (\text{A4})$$

$$\partial_\mu v^\mu \rightarrow \hat{\partial}_\mu \hat{v}^\mu. \quad (\text{A5})$$

For the Dirac gamma matrices, we adopt the following conventions:

$$\{\gamma^i \rightarrow i\hat{\gamma}^i; \quad \gamma^0 \rightarrow \hat{\gamma}^4\}, \quad (\text{A6})$$

with the Euclidean gamma matrices taken to be Hermitian, i.e., $\hat{\gamma}^{i\dagger} = \hat{\gamma}^i$ and $\hat{\gamma}^{4\dagger} = \hat{\gamma}^4$. The consequences of these conventions for the Clifford algebra are

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \rightarrow \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu} = -2\delta^{\mu\nu}. \quad (\text{A7})$$

In this case, the matrix γ_5 transforms accordingly as

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \rightarrow \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4 = \hat{\gamma}_5. \quad (\text{A8})$$

It is straightforward to verify that $\hat{\gamma}_5$ is Hermitian, $\hat{\gamma}_5^\dagger = \hat{\gamma}_5$. The contraction of gamma matrices with the derivative operator and an arbitrary four-vector follows the rules

$$\gamma^\mu \partial_\mu \rightarrow i\hat{\gamma}^\mu \hat{\partial}_\mu, \quad (\text{A9})$$

$$v_\mu \gamma^\mu \rightarrow -i\hat{v}_\mu \hat{\gamma}^\mu. \quad (\text{A10})$$

Finally, we show how the components of the Lorentz group generator $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ transform according to the prescriptions above. This yields

$$\Sigma^{0i} \rightarrow i\hat{\Sigma}^{4i}, \quad \Sigma^{ij} \rightarrow -\hat{\Sigma}^{ij}. \quad (\text{A11})$$

Thus, the contraction of two arbitrary four-vectors u_μ and v_ν with $\Sigma^{\mu\nu}$ transforms as

$$u_\mu v_\nu \Sigma^{\mu\nu} \rightarrow -\hat{u}_\mu \hat{v}_\nu \hat{\Sigma}^{\mu\nu}. \quad (\text{A12})$$

For the electromagnetic field-strength tensor, we have

$$F_{0i} \rightarrow -i\hat{F}_{4i}, \quad F^{ij} \rightarrow -\hat{F}^{ij}, \quad (\text{A13})$$

which implies that $F_{\mu\nu}^2 = \hat{F}_{\mu\nu}^2$.

To determine the correspondence for terms involving the dual electromagnetic field-strength tensor, it is necessary to examine how the four-dimensional Levi-Civita symbol transforms. We adopt the conventions $\epsilon^{0123} = +1$ and $\epsilon_{0123} = -1$ for the Levi-Civita symbol in Minkowski spacetime. Under the Wick rotation, the corresponding relations in Euclidean space are

$$\epsilon^{0123} = +1 \rightarrow i\hat{\epsilon}^{1234} = +i, \quad (\text{A14})$$

$$\epsilon_{0123} = -1 \rightarrow -i\hat{\epsilon}_{1234} = -i. \quad (\text{A15})$$

With these definitions of the Levi-Civita symbol, we can establish the mapping of the dual field-strength tensor, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$, from Minkowski spacetime $\mathbb{M}^{1,3}$ to Euclidean space $\mathbb{E}^4$. In particular, the components transform as

$$\tilde{F}_{0i} \rightarrow -\hat{\tilde{F}}_{4i}, \quad \tilde{F}_{ij} \rightarrow -i\hat{\tilde{F}}_{ji}. \quad (\text{A16})$$

From these relations, it follows that $\tilde{F}_{\mu\nu}F^{\mu\nu} \rightarrow i\hat{\tilde{F}}_{\mu\nu}\hat{F}^{\mu\nu}$. We can also determine the correspondence of contractions involving the Lorentz generators $\Sigma^{\mu\nu}$ and the tensors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$:

$$\Sigma^{\mu\nu}F_{\mu\nu} \rightarrow \hat{\Sigma}^{\mu\nu}\hat{F}_{\mu\nu}, \quad (\text{A17})$$

$$\Sigma^{\mu\nu} \tilde{F}_{\mu\nu} \rightarrow -i \hat{\Sigma}^{\mu\nu} \hat{\tilde{F}}_{\mu\nu}. \quad (\text{A18})$$

Finally, according to these conventions, the components of the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ transforms as

$$\{D_0 \rightarrow i\hat{D}_4, \quad D_i \rightarrow \hat{D}_i\}. \quad (\text{A19})$$

Thus, contractions involving covariant derivatives take the form

$$\gamma^\mu D_\mu \rightarrow i\hat{\gamma}^\mu \hat{D}_\mu, \quad (\text{A20})$$

$$D_\mu v^\mu \rightarrow \hat{D}_\mu \hat{v}^\mu. \quad (\text{A21})$$

These are the main conversions adopted in the present work. Obviously, the same relations apply in the inverse Wick rotation to express the final results of Sec. III in Minkowski spacetime.

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