New algorithms for Feynman integral reduction and epsilon-factorized differential equations

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In this paper, we give a detailed account of the algorithm outlined in [I. Bree *et al.* (ε Collaboration), companion Letter, Phys. Rev. Lett. **136**, 241602 (2026).] for Feynman integral reduction and ε-factorized differential equations. The algorithm consists of two steps. In the first step, we use a new geometric order relation in the integration-by-parts reduction to obtain a basis of master integrals, whose differential equations on the maximal cut are of a Laurent polynomial form in the regularization parameter ε and compatible with a filtration. This step works entirely with rational functions. In a second step, we provide a method to ε-factorize the aforementioned Laurent differential equations. The second step may introduce algebraic and transcendental functions. We illustrate the versatility of the algorithm by applying it to different examples with a wide range of complexity.

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I. INTRODUCTION

Feynman integrals play a pivotal role in quantum field theory. Advancements in the experimental program at the Large Hadron Collider require constant progress in pushing the frontier of perturbative calculations further. Besides collider phenomenology, Feynman integrals nowadays play an important role in the context of gravitational wave physics, due to the success of quantum field theory–based methods in the computation of classical corrections in general relativity. In both scenarios, the precision requirements of state-of-the-art computations necessitate the evaluation of Feynman integrals of increasing complexity. Current research focuses, therefore, on advancing the techniques for these calculations [1–18]. At the level of the integrals, the complexity of Feynman integral computations is represented in the number of loops and kinematic invariants involved in the process.

Perturbative calculations are carried out with the help of computer algebra. The available computing resources dictate which higher-order calculations can be done. Almost all perturbative higher-order calculations rely on a few basic algorithms for Feynman integral reduction and the computation of master integrals. These algorithms require significant computing resources in the form of memory and CPU time and are, therefore, often the bottleneck. Improving the efficiency of these basic algorithms will have a direct impact on the higher-order calculations which can be done.

Feynman integral reduction is one of the basic algorithms. Integration-by-parts identities [19,20] allow us to express a Feynman integral from a (large) set of Feynman integrals as a linear combination of Feynman integrals from a smaller set. The Feynman integrals in the smaller set are called master integrals, and we may think of the master integrals as a basis of an (abstract) vector space. The linear system of integration-by-parts identities is usually solved with the Laporta algorithm [21]. The Laporta algorithm is based on Gauß elimination in combination with an

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order relation. The order relation defines which of two Feynman integrals is considered more complicated. In each linear equation, the most complicated Feynman integral is eliminated in favor of simpler ones. Up to now, standard Feynman integral reduction programs [22–26] order Feynman integrals by sectors, but within a sector only an *ad hoc* order is used, usually based on the number of dots and the number of irreducible scalar products. While the number of master integrals is independent of the chosen order relation, the size of the reduction tables may depend on the choice. In this paper, we propose an order relation derived from mathematics, which orders the Feynman integrals within sectors according to their geometric properties. Our pilot studies show that this order relation performs significantly better. This can be traced back to the observation that this order relation eliminates to a large extent the occurrence of spurious polynomials in the denominator.

A second basic algorithm is the computation of master integrals through the method of differential equations [27–31]. For the master integrals, one derives (again by using integration-by-parts identities) differential equations in the external invariants or internal masses. In general, the right-hand side of this differential equation will involve rational functions in the kinematic variables x and in the dimensional regularization parameter ϵ. A differential equation is said to be in ϵ-factorized form if the only explicit occurrence of the dimensional regularization parameter is a simple overall prefactor on the right-hand side of the differential equation [31]. An ϵ-factorized differential equation has the advantage that a solution in terms of iterated integrals [32] is straightforward, provided appropriate boundary values are known. The required boundary values are usually not the bottleneck; in fact, they can be recursively reduced to single-mass vacuum integrals [33–35]. The application of the above-mentioned ϵ-factorized form was successfully employed to compute Feynman integrals evaluating to polylogarithmic functions depending on many kinematic scales; see, e.g., Ref. [36]. Besides this, there have been numerous contributions investigating the role of ϵ factorization in the case of Feynman integrals whose underlying geometry is nontrivial, e.g., elliptic curve(s) [16,37–54], Calabi-Yau manifolds [17,55–63], or higher genus surfaces [11]. One of the distinct features of these cases is that the transformation to the ϵ-factorized form is not restricted to algebraic functions (contrary to the polylogarithmic case). Going beyond case-by-case studies of specific Feynman integrals, there are a few methods which allow the construction of an ϵ-factorized differential equation under specific circumstances, for example, by restricting to the case of multiple polylogarithms [64–69], by requiring a guess of a good initial basis [43], or by requiring advance knowledge of the alphabet [42]. A key observation of Ref. [70] is that the geometric order relation in the Laporta algorithm leads to a differential equation, whose right-hand side is on the maximal cut a Laurent polynomial in the dimensional regularization parameter ϵ and for each entry of the connection matrix the occurring powers of ϵ are compatible with a filtration $F^\bullet$. We call such a differential equation an $F^\bullet$-compatible differential equation. In this paper, we give a constructive proof that we may always transform an $F^\bullet$-compatible differential equation to an ϵ-factorized form.

The improvements of this paper will be beneficial both to analytical approaches for the computation of Feynman integrals [51,54,71,72] as well as to (semi)numerical approaches [33–35,73–76]. In addition, a general algorithm for an ϵ-factorized differential equation offers insight into the underlying mathematical structures, like the singularity structure and the function space. It also provides a playground for some mathematical conjectures [77–81].

In this paper, we give a detailed description of the algorithm outlined in Ref. [70]. Introductions to the main concepts of the method can also be found in Refs. [82,83]. The procedure consists of two steps. The first step consists of Laporta reduction with a new order relation, which orders the Feynman integrals within a sector by their geometric properties. As we do not want to rely on properties of specific geometries like elliptic curves or K3 surfaces, we need one more layer of abstraction, which treats all these geometries in one single framework. Twisted cohomology [84,85] and ideas from classical Hodge theory [86–90] provide this framework. The integrands of Feynman integrals can be viewed as twisted cohomology classes [91,92]. We study the space of differential forms associated with a given family of Feynman integrals in the Baikov representation. For the order relation, we essentially count the number of poles and the number of consecutive nonzero residues. We observe that this order relation always gives a basis of master integrals with an $F^\bullet$-compatible differential equation. At present, we cannot prove this observation in a strict mathematical sense. However, we do not know a counterexample either.

In the second step, we show how to systematically remove the unwanted terms in ϵ through the solution of differential constraints. For the second step, we give a rigorous mathematical proof. These constraints are obtained by decomposing the Laurent polynomial differential equations into different orders of ϵ. For each order, we construct a rotation matrix such that, in the rotated differential equations, this order vanishes. The elements of these rotation matrices are determined by the solution of the aforementioned differential constraints, which are simpler than the original linearly coupled differential system.

The two steps are notably independent of prior knowledge of the underlying geometry of the Feynman integral. We emphasize that the first step involves only rational functions, while the second step may involve algebraic and transcendental functions. This suggests that an $F^\bullet$-compatible differential equation is the best that can be achieved by restricting to rational transformations.

This paper is organized as follows: In Sec. II, we introduce the notation used throughout the paper as well as define relevant concepts. In Sec. III, we construct the mathematical setup of our methodology. In particular, we define pole order and residues associated with a differential form. We use these two objects to define filtrations of differential forms. Furthermore, we establish the underlying geometry corresponding to Feynman integrals as well as the subgeometries associated with different localizations in the twist. In Sec. IV, we present our algorithm. We first outline the steps of the algorithm that lead to a set of candidate master integrals whose differential equation is a Laurent polynomial in ε. Thereafter, we explain the method to systematically ε-factorize the Laurent series differential equations via solutions of a set of differential constraints. In Sec. V, we showcase our method in different examples. These examples include a wide range of possible cases, which underlines the versatility of our algorithm. Finally, Sec. VI contains our conclusions and an outline toward possible future directions.

II. NOTATION, DEFINITIONS, AND REVIEW

A. Summary of the notation

For the convenience of the reader, we summarize in this paragraph the notation used throughout this paper.

(i) $N_F = N_{\text{Fibre}}$. Number of master integrals; master integrals are denoted by $I = (I_1, \ldots, I_{N_F})^T$.
(ii) $N_B = N_{\text{Base}}$. Number of kinematic variables; kinematic variables are denoted by $x = (x_1, \ldots, x_{N_B})^T$.
(iii) $N_E = N_{\text{Edges}}$. Number of Baikov variables; Baikov variables are denoted by $\sigma = (\sigma_1, \ldots, \sigma_{N_E})^T$.
(iv) $n = N_{\text{Variables}}$. Number of Baikov variables on a maximal cut; Baikov variables on a maximal cut are denoted by $z = (z_1, \ldots, z_n)^T$.
(v) $N_D = N_{\text{Divisors}}$. Number of divisors; divisors are denoted by $P = (P_1, \ldots, P_{N_D})^T$.
(vi) $N_L = N_{\text{Letters}}$. Number of letters; differential one-forms are denoted by $\omega = (\omega_1, \ldots, \omega_{N_L})^T$.

By default, all vectors are regarded as column vectors, hence the explicit use of the transpose. We may view the kinematic space as the projective space $\mathbb{CP}^{N_B}$. Homogeneous coordinates are denoted by $[x_0 : x_1 : \cdots : x_{N_B}]$, and $x = (x_1, \ldots, x_{N_B})^T$ are coordinates in the chart $x_0 = 1$. In a similar way, we view the Baikov space on a maximal cut as the projective space $\mathbb{CP}^n$. Homogeneous coordinates are denoted by $[z_0 : z_1 : \cdots : z_n]$, and $z = (z_1, \ldots, z_n)^T$ are coordinates in the chart $z_0 = 1$. The differential with respect to the kinematic variables is denoted by

$$d_B = \sum_{j=0}^{N_B} dx_j \frac{\partial}{\partial x_j}. \tag{1}$$

The differential with respect to the Baikov variables on a maximal cut is denoted by

$$d_F = \sum_{j=0}^{n} dz_j \frac{\partial}{\partial z_j}. \tag{2}$$

For monomials in several variables $z_0, z_1, \ldots, z_n$ we use the following notation:

$$\mathbf{z}^{\boldsymbol{\kappa}} = \prod_{j=0}^{n} z_j^{\kappa_j}. \tag{3}$$

We denote polynomials in the chart $z_0 = 1$ of $\mathbb{CP}^n$ by lowercase letters, e.g., $p(z_1, \ldots, z_n)$. We denote the corresponding homogeneous polynomials on $\mathbb{CP}^n$ (defined as the d-homogenization of p, where $d = \deg\, p$) by capital letters, e.g., $P(z_0, z_1, \ldots, z_n)$.

An ε-factorized differential equation is of the form

$$d_B K = \varepsilon A K, \tag{4}$$

where K is a vector of N_F master integrals and the $(N_F \times N_F)$ matrix A is independent of ε. We write

$$A = \sum_{j=1}^{N_L} C_j \omega_j, \tag{5}$$

where the C_j's are $(N_F \times N_F)$ matrices, whose entries are (rational) numbers and the ω_j's are differential one-forms, called the letters.

B. Feynman integrals

Let G be a Feynman graph with l loops, n_{ext} legs, and N_E propagators. We will consider scalar Feynman integrals associated to this graph. We denote by q_j the momentum flowing through edge e_j (with respect to a chosen orientation of the edge), by m_j the mass of the particle propagating through edge e_j, and by $\nu_j \in \mathbb{Z}$ the power to which this propagator occurs. We set

$$\sigma_j = -q_j^2 + m_j^2. \tag{6}$$

The σ_j's are the inverse propagators. Although we consider in this paper standard Feynman propagators with a quadratic dependence on the loop momenta, the extension toward propagators with a linear dependence on the loop momenta is straightforward. The latter occur, for example, in the eikonal approximation.

Without loss of generality, we assume that the graph G has a Baikov representation. This means that

$$N_E = \frac{1}{2} l(l+1) + el, \tag{7}$$

where e denotes the number of independent external momenta and any scalar product involving a loop momentum can be written as a linear combination of the inverse propagators and a constant term.

We are interested in the family of Feynman integrals

$$I_{\nu_1\ldots\nu_{N_E}}(D_{\text{int}},\varepsilon,x)=e^{l\varepsilon\gamma_{\text{E}}}(M^2)^{\nu-\frac{lD}{2}}\int\prod_{r=1}^{l}\frac{d^Dk_r}{i\pi^{\frac{D}{2}}}\prod_{j=1}^{N_E}\sigma_j^{-\nu_j}, \quad (8)$$

where $D=D_{\text{int}}-2\varepsilon$ (with $D_{\text{int}}\in\mathbb{Z}$ and $\varepsilon\in\mathbb{C}$) denotes the number of space-time dimensions within dimensional regularization, ε is the dimensional regularization parameter, γ_E denotes the Euler-Mascheroni constant, and M is an arbitrary scale introduced to render the Feynman integral dimensionless. Setting M^2 equal to a Lorentz invariant or to a mass squared will turn the Feynman integral into a (in general, multivalued) function defined on projective space $\mathbb{CP}^{N_B}$. We will always assume that such a choice is made. Here, N_B denotes the number of (dimensionless) kinematic variables. Thus, we may view $I_{\nu_1\ldots\nu_{N_E}}$ as a multivalued function:

$$I_{\nu_1\ldots\nu_{N_E}}:\ \mathbb{Z}\times\mathbb{C}\times\mathbb{CP}^{N_B}\to\mathbb{C},\qquad (D_{\text{int}},\varepsilon,x)\mapsto\mathbb{C}. \quad (9)$$

The quantity ν is defined by

$$\nu=\sum_{j=1}^{N_E}\nu_j. \quad (10)$$

The kinematic variables are denoted by $x=(x_1,\ldots,x_{N_B})$ and correspond to affine coordinates of the chart $x_0=1$ of $\mathbb{CP}^{N_B}$. To clarify our conventions, we consider the example of massless $(2\to2)$ scattering. In this case, we have $N_B=1$, and we may choose $x_1=s/t$ as a dimensionless kinematic variable. We may view x_1 as the affine coordinate of the chart $x_0=1$ of $\mathbb{CP}^1$.

For Minkowski spacetime, we have $D_{\text{int}}=4$. We stress that the algorithms we present work for any $D_{\text{int}}\in\mathbb{Z}$. As the value of D_{int} is often clear from the context, we will suppress in the following the dependence on D_{int} and simply write $I_{\nu_1\ldots\nu_{N_E}}(\varepsilon,x)$ for a Feynman integral. For each integral $I_{\nu_1\ldots\nu_{N_E}}(\varepsilon,x)$, we define

$$S=\{j|\nu_j>0\} \quad (11)$$

and the sector id of this integral by

$$N_{\text{id}}=\sum_{j\in S}2^{j-1}=\sum_{j=1}^{N_E}2^{j-1}\Theta(\nu_j). \quad (12)$$

Here, $\Theta(x)$ denotes the Heaviside step function, defined by $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ otherwise. We further define for each integral $I_{\nu_1\ldots\nu_{N_E}}$ the integers

$$N=\sum_{j=1}^{N_E}\Theta(\nu_j),\qquad r=\sum_{j=1}^{N_E}\nu_j\Theta(\nu_j),\qquad s=-\sum_{j=1}^{N_E}\nu_j\Theta(-\nu_j). \quad (13)$$

The quantity N denotes the number of propagators, the quantity $(r-N)$ denotes the number of dots, and s denotes the number of irreducible scalar products in the numerator. It is well known that the members of a family of Feynman integrals decompose into sectors. A sector is uniquely specified either by the set S or the sector id. A sector specified by the set S_1 is called a subsector of a sector S_2, if

$$S_1\subsetneq S_2. \quad (14)$$

We will often work on the maximal cut of a sector S. This means that we work modulo subsectors. In other words, on the maximal cut, we set all subsectors to zero.

C. Integration by parts and the Laporta algorithm

Integration-by-parts identities [19,20] are based on the fact that, within dimensional regularization, the integral of a total derivative vanishes:

$$e^{l\varepsilon\gamma_{\text{E}}}(M^2)^{\nu-\frac{lD}{2}}\int\prod_{r=1}^{l}\frac{d^Dk_r}{i\pi^{\frac{D}{2}}}\frac{\partial}{\partial k_i^\mu}\left(q_{\text{IBP}}^\mu\prod_{j=1}^{N_E}\sigma_j^{-\nu_j}\right)=0; \quad (15)$$

i.e., there are no boundary terms in the loop momentum representation. The vector q_{IBP} can be any linear combination of the external momenta and the loop momenta. Working out the derivatives leads to linear relations among integrals with different sets of indices $(\nu_1,\ldots,\nu_{N_E})$. The Laporta algorithm [21] introduces an order relation among the members of a family of Feynman integrals and eliminates in each linear relation the most complicated Feynman integral. Typical order criteria are the tuples

$$(N,N_{\text{id}},r,s,\ldots)\quad\text{or}\quad(N,N_{\text{id}},s,r,\ldots), \quad (16)$$

together with the lexicographical order. The dots stand for further criteria needed to distinguish inequivalent integrals. The order criteria in Eq. (16) sort Feynman integrals by sectors, but within a sector the criteria are *ad hoc*.

Integration-by-parts identities allow us to express any member from the family $I_{\nu_1\ldots\nu_{N_E}}$ as a finite linear combination of master integrals. We denote the master integrals by $I=(I_1,\ldots,I_{N_F})$ with

$$I_i(\varepsilon,x)=I_{\nu_{i1}\ldots\nu_{iN_E}}(\varepsilon,x),\qquad\nu_{ij}\in\mathbb{Z}. \quad (17)$$

It is important to note that the order relation defines which Feynman integrals are chosen as master integrals; i.e.,

different choices for the order relation lead to different sets of master integrals.

D. The method of differential equations

The method of differential equations [27–31] is a standard tool for the computation of Feynman integrals. The master integrals I satisfy a first-order system of differential equations

$$d_B I(\varepsilon, x) = A(\varepsilon, x) I(\varepsilon, x), \tag{18}$$

where $A(\varepsilon, x)$ is a $N_F \times N_F$ matrix, whose entries are differential one-forms, rational in ε and x. The differential d_B is defined in Eq. (1) and denotes the differential with respect to the kinematic variables. The differential equation is obtained by carrying out the derivative under the integral sign and by using integration-by-parts identities to reexpress the resulting expression in terms of master integrals.

We may change the basis of master integrals by a rotation

$$I(\varepsilon, x) = R(\varepsilon, x) \tilde{I}(\varepsilon, x). \tag{19}$$

In the new basis, we have again a differential equation of the form

$$d_B \tilde{I}(\varepsilon, x) = \tilde{A}(\varepsilon, x) \tilde{I}(\varepsilon, x), \tag{20}$$

where $\tilde{A}(\varepsilon, x)$ is related to $A(\varepsilon, x)$ by

$$\tilde{A}(\varepsilon, x) = R^{-1}(\varepsilon, x) A(\varepsilon, x) R(\varepsilon, x) - R^{-1}(\varepsilon, x) d_B R(\varepsilon, x). \tag{21}$$

If $R(\varepsilon, x)$ is rational in ε and x, then $\tilde{A}(\varepsilon, x)$ is rational in ε and x as well.

E. The Baikov representation

Feynman integrals have a Baikov representation [93]. We may use either the democratic Baikov representation or a loop-by-loop Baikov representation [94]. We require the Baikov representation to be a "good" representation (to be defined below). It will be advantageous to work with the simplest good representation. This is usually a loop-by-loop Baikov representation.

1. The democratic Baikov representation

Let us start with the democratic Baikov representation. The democratic Baikov representation is, from a notational perspective, the simplest (there is only one polynomial), but it is not the simplest from a computational point of view. We assume that we have N_E Baikov variables and N internal edges. Without loss of generality, we may assume that the first N Baikov variables correspond to the internal edges. The integral measure can be written as

$$e^{l\varepsilon\gamma_E} (M^2)^{-\frac{lD}{2}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} = C' b(\sigma_1, \ldots, \sigma_{N_E})^{\frac{D-l-e-1}{2}} \frac{d^{N_E}\sigma}{(2\pi i)^{N_E}}, \tag{22}$$

where C' is an irrelevant prefactor, e is the number of independent external momenta, and $b(\sigma)$ is the Baikov polynomial. The Baikov polynomial is given as the Gram determinant of the independent loop momenta and the independent external momenta. Let $\mathcal{C}_{\text{maxcut}}$ be a contour which corresponds for the first N variables to small circles around $\sigma_i = 0$ (for $1 \le i \le N$). This amounts to taking the maximal cut. On the maximal cut, we have

$$\begin{aligned} &e^{l\varepsilon\gamma_E} (M^2)^{-\frac{lD}{2}} \int_{\mathcal{C}_{\text{maxcut}}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{(M^2)^N}{\prod_{j=1}^{N} \sigma_j} \\ &= C'' \int \frac{d^{N_E-N}\sigma}{(2\pi i)^{N_E-N}} b_{\text{maxcut}}(\sigma_{N+1}, \ldots, \sigma_{N_E})^{\alpha}, \end{aligned} \tag{23}$$

with

$$b_{\text{maxcut}}(\sigma_{N+1}, \ldots, \sigma_{N_E}) = b(0, \ldots, 0, \sigma_{N+1}, \ldots, \sigma_{N_E}) \tag{24}$$

and

$$C'' = (M^2)^N C', \qquad \alpha = \frac{D - l - e - 1}{2}. \tag{25}$$

Note that, by discussing the maximal cut, we already performed N integrations with the help of the residue theorem. We introduce dimensionless variables $(z_1, \ldots, z_{N_E-N})$ for the remaining Baikov variables, i.e., $z_j = \sigma_{N+j}/M^2$.

The exponent of the Baikov polynomial b_{maxcut} is of the form

$$\alpha = \frac{1}{2}(a + b\varepsilon), \quad \text{with} \quad a, b \in \mathbb{Z}. \tag{26}$$

In the democratic Baikov representation, we always have $b = -2$ for $D = D_{\text{int}} - 2\varepsilon$.

2. The loop-by-loop Baikov representation

Let us now turn to the loop-by-loop Baikov representation [94]. We set n to be the number of Baikov variables on the maximal cut. We denote these (dimensionless) Baikov variables by $z = (z_1, \ldots, z_n)$. We have $n \le N_E - N$. On the maximal cut, we have

$$\begin{aligned} &e^{l\varepsilon\gamma_E} (M^2)^{-\frac{lD}{2}} \int_{\mathcal{C}_{\text{maxcut}}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{(M^2)^N}{\prod_{j=1}^{N} \sigma_j} \\ &= C \int \frac{d^n z}{(2\pi i)^n} \prod_{i \in I_{\text{all}}} [p_i(z)]^{\alpha_i}, \end{aligned} \tag{27}$$

where the p_i's are irreducible polynomials in the n remaining Baikov variables. I_{all} is a finite index set.

The exponents α_i in Eq. (27) are of the form as in Eq. (26):

$$\alpha_i = \frac{1}{2}(a_i + b_i\varepsilon), \quad \text{with} \quad a_i, b_i \in \mathbb{Z}. \tag{28}$$

We define I_{odd} as the set of indices for which a_i is odd and I_{even} as the set of indices for which a_i is even. Clearly, we have

$$I_{\text{odd}} \cup I_{\text{even}} = I_{\text{all}}, \qquad I_{\text{odd}} \cap I_{\text{even}} = \varnothing. \tag{29}$$

We are, in particular, interested in the "minimal" case, where $a_i \in \{-1, 0\}$ for all i. Let $r(\sigma)$ be an ε-independent function, rational in σ and of mass dimension zero, leading to a minimal Baikov representation

$$\begin{aligned} &e^{l\varepsilon\gamma_{\text{E}}}(M^2)^{-\frac{lD}{2}} \int_{\mathcal{C}_{\text{maxcut}}} \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} r(\sigma) \\ &= C_{\text{Baikov}} \int \frac{d^n z}{(2\pi i)^n} \prod_{i\in I_{\text{odd}}} [p_i(z)]^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{i\in I_{\text{even}}} [p_i(z)]^{\frac{1}{2}b_i\varepsilon}. \end{aligned} \tag{30}$$

The prefactor C_{Baikov} is defined by this equation.

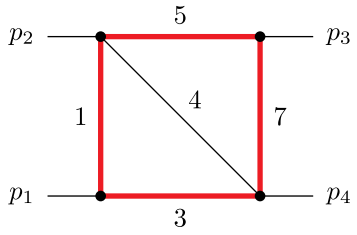


FIG. 1. The Feynman graph for sector 93. Massive propagators are indicated by red lines.

3. Good Baikov representations

We call a Baikov representation a "good" Baikov representation for a particular sector, if all singularities are regulated by the twist. This implies that all exponents α_i in Eq. (26) or Eq. (28) satisfy $\alpha_i \notin \mathbb{Z}$. This ensures that, on the maximal cut, we may always work with twisted cohomology and are not forced to use relative twisted cohomology. We note that the democratic Baikov representation is always a good representation; however, not every loop-by-loop representation is good.

Example 1. A counterexample is given by a two-loop four-point integral. We consider sector 93 for the inverse propagators defined by

$$\begin{aligned} &\sigma_1 = -(k_1 - p_1)^2 + m^2, && \sigma_2 = -(k_1 - p_{12})^2, && \sigma_3 = -k_1^2 + m^2, \\ &\sigma_4 = -(k_1 + k_2)^2, && \sigma_5 = -(k_2 + p_{12})^2 + m^2, && \sigma_6 = -k_2^2, \\ &\sigma_7 = -(k_2 + p_{123})^2 + m^2, && \sigma_8 = -(k_1 - p_{13})^2, && \sigma_9 = -(k_2 + p_{13})^2. \end{aligned} \tag{31}$$

The external particles are massless $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$. The Mandelstam variables are denoted by

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2, \tag{32}$$

and we use the notation $p_{ij} = p_i + p_j$, $p_{ijk} = p_i + p_j + p_k$. The Feynman graph for sector 93 is shown in Fig. 1 and corresponds to the propagator set $S = \{1, 3, 4, 5, 7\}$. We consider the loop-by-loop Baikov representation, where we take as the inner loop the one with inverse propagators σ_4, σ_5, and σ_7. This will lead to a Baikov polynomial (or a divisor in the mathematical language)

$$m^2 - t - \sigma_8, \tag{33}$$

which is not regulated by the twist. In this case, we have to rely on the democratic Baikov representation or an alternative loop-by-loop Baikov representation.

We conclude this subsection with a side remark: If one uses a loop-by-loop Baikov representation, it is not directly evident that there is a basis of master integrals for this sector, such that the integrand of each master integral can be written in this Baikov representation. In particular, one might be worried about Feynman integrals involving irreducible scalar products in the numerator, which do not appear in the chosen loop-by-loop Baikov representation ("missing ISPs"). One has to show that any Feynman integral involving a product of irreducible scalar products σ_j with $j \in \{N + n + 1, \ldots, N_E\}$ in the numerator can be expressed in terms of Feynman integrals involving only the Baikov variables σ_j with $j \in \{1, \ldots, N + n\}$. This is indeed the case, and Ref. [95] provides an algorithmic solution for this task.

F. Projective space

In order to capture possible singularities at infinity, we extend the affine space with coordinates $(z_1, \ldots, z_n)$ to projective space $\mathbb{CP}^n$ with homogeneous coordinates $[z_0 : z_1 : \cdots : z_n]$. Let d_i be the degree of p_i and denote by P_i the d_i homogenization

$$P_i(z_0, z_1, \ldots, z_n) = z_0^{d_i} p_i\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right). \tag{34}$$

We further set $P_0(z_0, z_1, \ldots, z_n) = z_0$ and define the exponent

$$\alpha_0 = \frac{1}{2}(a_0 + b_0\varepsilon) \tag{35}$$

by

$$a_0 = \begin{cases} 0 & \text{if } \sum_{i \in I_{\text{odd}}} d_i \text{ even}, \\ -1 & \text{if } \sum_{i \in I_{\text{odd}}} d_i \text{ odd}, \end{cases}$$
$$b_0 = -\sum_{i \in I_{\text{all}}} b_i d_i. \tag{36}$$

We extend the definition of a "good" Baikov representation to include the requirement $\alpha_0 \neq \mathbb{Z}$ for $n > 0$. The case $n = 0$ is special and discussed in detail in Sec. III F 4. In the case $n = 0$, we have $\alpha_0 = 0$, as I_{all} is the empty set.

We can unify the notation by including the index 0 in I_{even} or I_{odd}, depending on a_0 being zero or (-1), respectively. We denote the resulting index sets by I^0_{even}, I^0_{odd}, and I^0_{all}. The definition of a_0 ensures that the sum of the degrees of the polynomials in I^0_{odd} is even. The definition of b_0 follows from homogeneity requirements. We set

$$N_D = |I_{\text{all}}|. \tag{37}$$

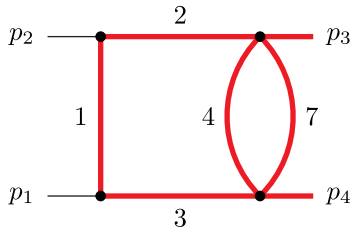


FIG. 2. The Feynman graph for sector 79. Massive propagators are indicated by red lines.

Example 2. As an example, we consider a two-loop four-point integral from Ref. [40]. We consider sector 79 for the inverse propagators defined by

$$\begin{aligned} &\sigma_1 = -(k_1 + p_2)^2 + m^2, \quad \sigma_2 = -k_1^2 + m^2, \quad \sigma_3 = -(k_1 + p_1 + p_2)^2 + m^2, \\ &\sigma_4 = -(k_1 + k_2)^2 + m^2, \quad \sigma_5 = -k_2^2, \quad \sigma_6 = -(k_2 + p_3 + p_4)^2, \\ &\sigma_7 = -(k_2 + p_3)^2 + m^2, \quad \sigma_8 = -(k_1 + p_2 - p_3)^2 + m^2, \quad \sigma_9 = -(k_2 - p_2 + p_3)^2. \end{aligned} \tag{38}$$

Sector 79 corresponds to the propagator set $S = \{1, 2, 3, 4, 7\}$. The Feynman graph for sector 79 is shown in Fig. 2. We set $M = m$, $x_1 = s/m^2$, and $x_2 = t/m^2$. On the maximal cut, we obtain, using the loop-by-loop approach, a minimal one-dimensional Baikov representation from an integrand with a dot on either propagator 4 or 7. Equation (30) specializes with $D = 4 - 2\varepsilon$ and $z_1 = \sigma_8/m^2$ to

$$\begin{aligned} &e^{2\varepsilon\gamma_{\text{E}}} \int_{\mathcal{C}_{\text{maxcut}}} \prod_{r=1}^{2} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{(m^2)^{2+2\varepsilon}}{\sigma_1\sigma_2\sigma_3\sigma_4^2\sigma_7} \\ &= C_{\text{Baikov}} \int \frac{dz_1}{2\pi i} [p_1(z)]^{-\frac{1}{2}} [p_2(z)]^{-\frac{1}{2}-\varepsilon} [p_3(z)]^{-\frac{1}{2}-\varepsilon}. \end{aligned}$$

The prefactor is given by

$$C_{\text{Baikov}} = \frac{2^{4+4\varepsilon}\pi^4 e^{2\varepsilon\gamma_{\text{E}}}}{[\Gamma(\frac{1}{2} - \varepsilon)]^2 x_1^{1+\varepsilon}} [(1 - x_2)^2 + x_1 x_2]^{\varepsilon}. \tag{39}$$

The Baikov polynomials read

$$\begin{aligned} &p_1 = z_1 - x_2, \quad p_2 = z_1 + 4 - x_2, \\ &p_3 = (z_1 + 1)^2 - 4\left[x_2 + \frac{(1 - x_2)^2}{x_1}\right]. \end{aligned} \tag{40}$$

We have $I_{\text{even}} = \varnothing$ and $I_{\text{odd}} = \{1, 2, 3\}$. The equation

$$y^2 = p_1(z_1) p_2(z_1) p_3(z_1) \tag{41}$$

defines an elliptic curve, where the four branch points are at finite distance. We expect two master integrals associated with this elliptic curve. The sector 79 has three master integrals. To see the third master integral, we need to go to projective space and include the point at infinity. The homogenizations of p_1, p_2, and p_3 are

$$\begin{aligned} &P_1 = z_1 - x_2 z_0, \quad P_2 = z_1 + (4 - x_2) z_0, \\ &P_3 = (z_1 + z_0)^2 - 4\left[x_2 + \frac{(1 - x_2)^2}{x_1}\right] z_0^2. \end{aligned} \tag{42}$$

We further introduce $P_0 = z_0$ with $a_0 = 0$ and $b_0 = 6\varepsilon$, according to Eq. (36). We have $I^0_{\text{even}} = \{0\}$ and

$I^0_{\text{odd}} = \{1, 2, 3\}$. The third master integral is associated with the residue at $z_0 = 0$.

G. Twisted cohomology

We define the minimal twist U as the homogenization of the twist in Eq. (30) by

$$U(z_0, z_1, \ldots, z_n) = \prod_{i \in I^0_{\text{odd}}} P_i^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j \in I^0_{\text{even}}} P_j^{\frac{1}{2}b_j\varepsilon}. \quad (43)$$

One can see that $U(z)$ is homogeneous of degree

$$d_U = -\frac{1}{2} \sum_{i \in I^0_{\text{odd}}} d_i. \quad (44)$$

The quantity b_0 has been defined in Eq. (36) such that d_U is independent of ε.

Each polynomial in the twist defines a hypersurface

$$D_i = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n | P_i(z_0, z_1, \ldots, z_n) = 0\}, \quad (45)$$

and we set D to be the union of these hypersurfaces:

$$D = \bigcup_{i \in I^0_{\text{all}}} D_i. \quad (46)$$

It is customary to set

$$\omega = d_F \ln U(z_0, z_1, \ldots, z_n), \quad (47)$$

where d_F is defined in Eq. (2) and denotes the differential with respect to the Baikov variables $[z_0 : z_1 : \cdots : z_n]$. We further introduce the covariant derivative

$$\nabla_F = d_F + \omega. \quad (48)$$

As we are interested in only differential forms, which depend on the holomorphic coordinates $[z_0 : \cdots : z_n]$ but not on the antiholomorphic coordinates $[\bar{z}_0 : \cdots : \bar{z}_n]$, we may restrict to the holomorphic derivatives appearing in d_F and ignore the antiholomorphic derivatives. We are interested in the twisted cohomology group

$$H^n_\omega, \quad (49)$$

which is defined as the set of equivalence classes of the ∇_F-closed differential $(n, 0)$-forms holomorphic on $\mathbb{CP}^n$-D modulo the ∇_F-exact ones.

For later purposes, we also define the $(\varepsilon = 0)$ part U_0 of the minimal twist by

$$U_0(z_0, z_1, \ldots, z_n) = \prod_{i \in I^0_{\text{odd}}} P_i^{-\frac{1}{2}}. \quad (50)$$

III. THE SETUP

In this section, we introduce the framework for our approach. Our aim is to order the Feynman integrals within a sector by their geometric properties. We therefore focus on the maximal cut.[1] We start with the definition of the objects which we will be studying. Instead of integrals, we study integrands and, more precisely, differential n-forms on projective space $\mathbb{CP}^n$. These will be defined in Sec. III A. In Sec. III B, we discuss the relation between the vector space of integrals and the vector space of integrands. We are, in particular, interested in linear relations between the integrands. These are discussed in Sec. III C. For the order relation, we count poles and residues. These concepts are introduced in Sec. III D. In Sec. III E, we define three filtrations, based on the number of residues, the pole order, and the sum of the indices. The algorithm proceeds recursively by considering all possible localizations (i.e., places where we can take a residue). Polynomials of higher degree might introduce algebraic extensions. In Sec. III F, we show how this can be avoided. We present a method which works entirely with rational functions. In Sec. III G, we discuss the specific geometries associated with the maximal cut of Feynman integrals.

A. Objects

For our purposes, it is slightly more convenient to include the twist in the differential forms, as the twist affects the definition of a prefactor. The central objects of our study are differential forms, which can be written as

$$\Psi_{\mu_0 \ldots \mu_{N_D}}[Q] = C_{\text{Baikov}} C_\varepsilon U(z) \hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[Q] \eta. \quad (51)$$

The integrands in Eq. (51) are all what we need to study. Up to prefactors, any integrand can be written in this form. We explain the ingredients right to left. η is the standard n-form on $\mathbb{CP}^n$ defined by

$$\eta = \sum_{j=0}^{n} (-1)^j z_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n, \quad (52)$$

where the hat indicates that the corresponding term is omitted. $\hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[Q]$ is an ε-independent meromorphic function in z, holomorphic on $\mathbb{CP}^n$-D and given by

$$\hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[Q] = \frac{Q}{\prod_{i \in I^0_{\text{all}}} P_i^{\mu_i}}, \quad (53)$$

where $\mu_j \in \mathbb{N}_0$ and Q is a homogeneous polynomial of degree

[1] Subtleties related to the Feynman integrals within one sector, Feynman integrals on the maximal cut, and magic relations will be discussed in a separate forthcoming publication.

$$d_Q = \sum_{i \in I^0_{\text{all}}} \mu_i d_i - d_U - n - 1. \tag{54}$$

We set

$$|\mu| = \sum_{i \in I^0_{\text{all}}} \mu_i. \tag{55}$$

The twist function $U(z)$ has been defined in Eq. (43).

The ε-dependent prefactor C_ε is independent of z (but may depend on x) and given as a product:

$$C_\varepsilon = C_{\text{abs}} \cdot C_{\text{rel}} \cdot C_{\text{clutch}}. \tag{56}$$

The prefactor C_{Baikov} has been defined in Eq. (30). The factor C_{abs} is defined such that $C_{\text{abs}} \cdot C_{\text{Baikov}}$ is pure of transcendental weight zero.

The relative prefactor C_{rel} depends on $(\mu_0, \mu_1, \ldots)$ and is given by

$$C_{\text{rel}} = \prod_{i \in I^0_{\text{all}}} (\alpha_i)_{\mu_i}, \tag{57}$$

with $(a)_n = \Gamma(a+1)/\Gamma(a+1-n)$ being the falling factorial. The α_i's are the exponents appearing in Eqs. (28) and (35). For the minimal twist function U defined by Eq. (43), the relative prefactor reduces to

$$C_{\text{rel}} = \prod_{i \in I^0_{\text{odd}}} \left(-\frac{1}{2} + \frac{1}{2} b_i \varepsilon\right)_{\mu_i} \prod_{i \in I^0_{\text{even}}} \left(\frac{1}{2} b_i \varepsilon\right)_{\mu_i}. \tag{58}$$

The inclusion of the relative prefactor is motivated by the structure of the integration-by-parts identities within twisted cohomology. The relative prefactor is chosen such that it trivializes the ε dependence of the integration-by-parts identities. The integration-by-parts identities within twisted cohomology will be derived in Sec. III C 1.

The clutch prefactor C_{clutch} is given by

$$C_{\text{clutch}} = \varepsilon^{-|\mu|}, \tag{59}$$

where $|\mu|$ is defined in Eq. (55). The clutch factor follows from known examples involving nontrivial geometries, where an ε-factorized differential equation has been constructed. In the case of the equal-mass l-loop banana integrals [37,55], the first master integral starts at the order of ε^l, and this factor is contributed by C_{abs}. The following master integrals start each one power of ε earlier compared to the previous one. These powers of ε are provided by C_{clutch}.

Example 3. We continue with Example 2: The prefactor C_{Baikov} was given in Eq. (39) by

$$C_{\text{Baikov}} = \frac{2^{4+4\varepsilon} \pi^4 e^{2\varepsilon\gamma_{\text{E}}}}{[\Gamma(\frac{1}{2} - \varepsilon)]^2 x_1^{1+\varepsilon}} [(1-x_2)^2 + x_1 x_2]^\varepsilon. \tag{60}$$

It is easy to check that with $C_{\text{abs}} = \varepsilon^3 x_1$ the product $C_{\text{abs}} C_{\text{Baikov}}$ is pure of transcendental weight zero:

$$\begin{aligned} C_{\text{abs}} \cdot C_{\text{Baikov}} &= 16\pi^3 \varepsilon^3 - 16\pi^3 (L_1 - L_2)\varepsilon^4 \\ &\quad + 8\pi^3 [(L_1 - L_2)^2 - \pi^2] \varepsilon^5 + \mathcal{O}(\varepsilon^6), \end{aligned} \tag{61}$$

where $L_1 = \ln(x_1)$ and $L_2 = \ln((1-x_2)^2 + x_1 x_2)$.

B. Vector spaces

We denote the vector space spanned by the Feynman integrals on the maximal cut (viewed as abstract symbols) by A^n. This is an infinite-dimensional vector space of countable dimension. We denote the vector space spanned by the differential forms of Eq. (51) by Ω^n_ω. This is again an infinite-dimensional vector space of countable dimension. For a good Baikov representation, we denote the linear map, which associates to each Feynman integral its integrand on the maximal cut by

$$\hat{\iota}: A^n \to \Omega^n_\omega. \tag{62}$$

We are interested in finite-dimensional vector spaces. We denote the vector space of Feynman integrals on the maximal cut modulo linear relations by V^n and the twisted cohomology group related to the integrands in Eq. (51) by H^n_ω. Both are finite-dimensional vector spaces. Their dimensions can be computed with the Laporta algorithm, and this will be the method used throughout this paper. In addition, we will sometimes also report the number based on the counting of critical points [96].

There is an injective linear map

$$\iota: V^n \rightarrowtail H^n_\omega, \tag{63}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} A^n & \xrightarrow{\hat{\iota}} & \Omega^n_\omega \\ {\scriptstyle\sigma}\uparrow & & \downarrow{\scriptstyle\pi_\omega}, \\ V^n & \xrightarrow{\iota} & H^n_\omega \end{array} \tag{64}$$

where the map $\sigma: V^n \to A^n$ is trivial, once a basis of V^n has been chosen (it sends each master integral to itself). The map $\pi_\omega: \Omega^n_\omega \to H^n_\omega$ sends each differential form to its cohomology class. In other words, ι is defined by $\iota = \pi_\omega \circ \hat{\iota} \circ \sigma$. The map ι is injective: If the integrand of the Baikov representation is zero in H^n_ω, then the maximal cut integral is zero in V^n. Hence, we have

$$\dim V^n \leq \dim H^n_\omega. \tag{65}$$

The map is surjective, if dim $V^n = \dim H^n_\omega$. In general, the map will not be surjective. There are two reasons for this: symmetries and supersectors.

1. Symmetries

Integration can lead to symmetries among Feynman integrals (elements in V^n), which are not symmetries of the integrands (elements in H^n_ω). As a very simple example, we have the equality of the integrals

$$\int_{[0,1]^2} z_1 dz_1 \wedge dz_2 = \int_{[0,1]^2} z_2 dz_1 \wedge dz_2; \tag{66}$$

however, this does not imply the equality of the integrands

$$z_1 dz_1 \wedge dz_2 \neq z_2 dz_1 \wedge dz_2. \tag{67}$$

An example from the context of Feynman integrals is the following.

Example 4. We consider the equal-mass sunrise integral in $D = 2 - 2\varepsilon$ dimensions. The inverse propagators are defined by

$$\begin{aligned} &\sigma_1 = -k_1^2 + m^2, \qquad \sigma_2 = -k_2^2 + m^2, \\ &\sigma_3 = -(k_1 + k_2 - p)^2 + m^2, \\ &\sigma_4 = -(k_1 - p)^2, \qquad \sigma_5 = -(k_1 + k_2)^2. \end{aligned} \tag{68}$$

The sunrise integral has sector id 7, corresponding to the propagator set $S = \{1, 2, 3\}$. We consider the loop-by-loop Baikov representation, where we take as the inner loop the one with inverse propagators σ_2 and σ_3. This gives us a one-dimensional Baikov representation with σ_4 as the remaining Baikov variable. With $z_1 = \sigma_4/(-p^2)$ and $x = m^2/(-p^2)$, the twist function reads

$$\begin{aligned} U(z) = z_0^{3\varepsilon} z_1^{-\frac{1}{2}} \{ &(z_1 + 4x z_0)[z_1^2 + 2(x-1) z_0 z_1 \\ &+ (x+1)^2 z_0^2]\}^{-\frac{1}{2}-\varepsilon}. \end{aligned} \tag{69}$$

On the Feynman integral side, we have two master integrals for sector 7; hence, dim $V^1 = 2$. However, in twisted cohomology, we find for the twist function of Eq. (69) that dim $H^1_\omega = 3$. This is an example where the map from V^1 to H^1_ω is not surjective. We have two symmetry relations on the Feynman integral side, which can be taken as

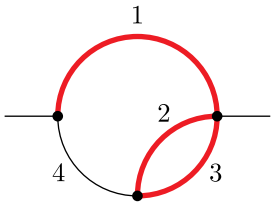


FIG. 3. The Feynman graph for sector 15. Massive propagators are indicated by red lines.

$$\begin{aligned} I_{12100} - I_{11200} &= 0, \\ 2I_{21100} - I_{12100} - I_{11200} &= 0. \end{aligned} \tag{70}$$

Our loop-by-loop Baikov representation is shown pictorially in Fig. 3 and treats the inverse propagators σ_2 and σ_3 symmetrically; hence, the left-hand side of the first symmetry relation maps to zero in twisted cohomology. However, the left-hand side of the second symmetry relation does not map to zero in twisted cohomology. Moding out the image

$$\hat{\imath}(2I_{21100} - I_{12100} - I_{11200}) \tag{71}$$

in H^1_ω gives a two-dimensional space. This corresponds to the two master integrals in the top sector.

2. Supersectors

A sector S_1 is called a supersector of the sector S_2, if S_2 can be obtained from S_1 by contracting some of the edges of S_1.

In order to motivate the relevance of supersectors for the twisted cohomology group H^n_ω, consider the situation where a polynomial $p_j(z)$ with $j \in I_{\text{even}}$ is simply a factor $z_r = \sigma_l/M^2$, where σ_l is an uncut inverse propagator. In this case, H^n_ω will also contain the integrands of the sector where the exponent of this inverse propagator is positive. If this sector has additional master integrals, they will also appear in H^n_ω. Such a sector is a supersector, and we include it in the analysis.

Example 5. An example is given by the three-loop unequal-mass banana integral [63]. The inverse propagators are given by

$$\begin{aligned} &\sigma_1 = -k_1^2 + m_1^2, && \sigma_2 = -k_2^2 + m_2^2, && \sigma_3 = -k_3^2 + m_3^2, \\ &\sigma_4 = -(k_1 + k_2 + k_3 - p)^2 + m_4^2, && \sigma_5 = -(k_1 + k_2 - p)^2, && \sigma_6 = -(k_1 - p)^2, \\ &\sigma_7 = -(k_1 + k_2)^2, && \sigma_8 = -(k_1 + k_3)^2, && \sigma_9 = -(k_2 + k_3)^2, \end{aligned} \tag{72}$$

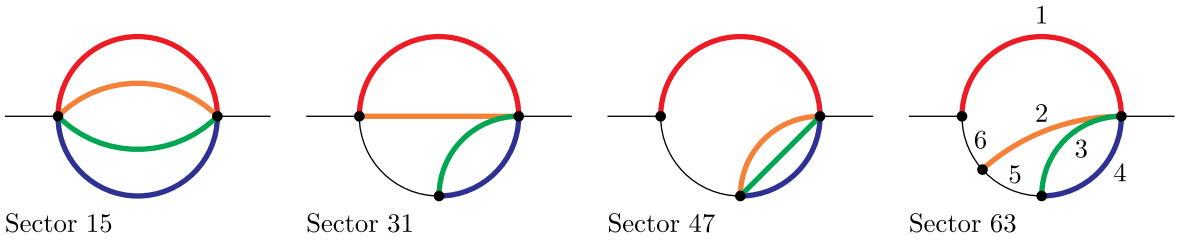


FIG. 4. The sector of the banana integral together with the relevant supersectors.

and we consider the loop-by-loop Baikov representation, where the innermost loop is formed by the inverse propagators σ_3 and σ_4, followed by the loop formed by the inverse propagators σ_2 and σ_5, and finally the loop formed by the inverse propagators σ_1 and σ_6. The three-loop banana integral has sector id 15, corresponding to the propagator set $S = \{1, 2, 3, 4\}$. With $z_1 = \sigma_5/(-p^2)$ and $z_2 = \sigma_6/(-p^2)$, the twist function is given by

$$U = z_0^{4\varepsilon} z_1^{\varepsilon} z_2^{\varepsilon} P_3^{-\frac{1}{2}-\varepsilon} P_4^{-\frac{1}{2}-\varepsilon} P_5^{-\frac{1}{2}-\varepsilon}. \tag{73}$$

The definition of the polynomials $P_3 - P_5$ is not relevant here (the definition can be found in Ref. [63]); what is relevant is the fact that z_1 and z_2 appear as even polynomials in the twist function. Hence, we have to include the supersectors 31 (with propagator set $S = \{1, 2, 3, 4, 5\}$), 47 (with propagator set $S = \{1, 2, 3, 4, 6\}$), and 63 (with propagator set $S = \{1, 2, 3, 4, 5, 6\}$). These are shown in Fig. 4. The supersectors 31 and 47 have one master integral each. A possible basis for these supersectors on the Feynman integral side is

$$I_{111110000}, \qquad I_{111101000}. \tag{74}$$

The supersector 63 is reducible and does not have any master integral. These two additional master integrals make up for the difference in dimensions: We have

$$\dim V^2 = 11, \qquad \dim H^2_{\omega} = 13. \tag{75}$$

Our starting point is always a graph G, which has the property that any scalar product involving a loop momenta can be expressed as a linear combination of inverse propagators (see Sec. II B). This does not imply that all relevant supersectors are subgraphs of the graph G. We provide a counterexample.

Example 6. We start from a nonplanar double-box integral with inverse propagators

$$\begin{aligned}
&\sigma_1 = -(k_1 - p_1)^2 + m^2, && \sigma_2 = -(k_1 - p_{12})^2, && \sigma_3 = -k_1^2,\\
&\sigma_4 = -k_{12}^2 + m^2, && \sigma_5 = -(k_{12} + p_3)^2, && \sigma_6 = -k_2^2,\\
&\sigma_7 = -(k_2 + p_{123})^2 + m^2, && \sigma_8 = -(k_1 - p_{13})^2, && \sigma_9 = -(k_2 + p_{13})^2.
\end{aligned} \tag{76}$$

The auxiliary graph G with nine propagators has sector id 511, and the nonplanar double-box integral has sector id 127. We are interested in sector 123 with propagator set $S = \{1, 2, 4, 5, 6, 7\}$. The sectors are shown in Fig. 5. For sector 123, we consider the loop-by-loop Baikov representation, where the inner loop is given by σ_4, σ_5, σ_6, σ_7. We obtain a two-dimensional Baikov representation with twist

$$U = z_0^{4\varepsilon} P_1^{\varepsilon} P_2^{\varepsilon} P_3^{-\frac{1}{2}-\varepsilon} P_4^{-\frac{1}{2}-\varepsilon}, \tag{77}$$

with

$$P_1 = \frac{\sigma_8 + (m^2 - t) z_0}{s}, \qquad P_2 = \frac{\sigma_8 - \sigma_3 + (m^2 - t - s) z_0}{s}. \tag{78}$$

The explicit expressions for P_3 and P_4 are not relevant here. P_1 and P_2 are not proportional to any of the inverse propagators appearing in Eq. (76). We have

$$\dim V^2 = 8, \qquad \dim H^2_{\omega} = 12. \tag{79}$$

The difference in dimensions is due to supersectors, where P_1 or P_2 appear in the denominator. These supersectors are not subsectors of sector 511 of the graph G. We may, however, construct a new auxiliary graph $\tilde{G}$, with inverse propagators

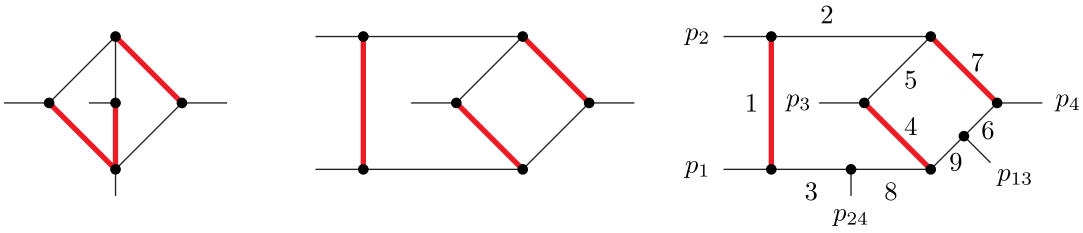


FIG. 5. Sector 123 (left), sector 127 (middle), and sector 511 (right).

$$\begin{aligned}
&\sigma_1 = -(k_1 - p_1)^2 + m^2, && \sigma_2 = -(k_1 - p_{12})^2, && \tilde{\sigma}_3 = -(k_1 - p_{123})^2,\\
&\sigma_4 = -k_{12}^2 + m^2, && \sigma_5 = -(k_{12} + p_3)^2, && \sigma_6 = -k_2^2,\\
&\sigma_7 = -(k_2 + p_{123})^2 + m^2, && \tilde{\sigma}_8 = -(k_1 - 2p_{12} - p_3)^2, && \sigma_9 = -(k_2 + p_{13})^2,
\end{aligned} \tag{80}$$

such that sector 123 and the supersectors are subgraphs of $\tilde{G}$. The supersectors 127 and 251 with respect to the inverse propagators defined in Eq. (80) contribute two master integrals each, explaining the difference in dimensions in Eq. (79).

3. Conversion between V^n and H^n_ω

As already mentioned at the beginning of Sec. III B, the map $\iota: V^n \hookrightarrow H^n_\omega$ goes from the vector space of Feynman integrals on the maximal cut modulo linear relations to the twisted cohomology classes. We write each master integral on the maximal cut in the Baikov representation and take the integrand in this representation as a representative of a twisted cohomology class. However, this assumes that we already have a basis of master integrals on V^n, which is required to define the map $\sigma: V^n \to A^n$ in Eq. (64).

What we would like to have is a map $j: H^n_\omega \twoheadrightarrow V^n$ and use this map to define master integrals on V^n from the master integrands on H^n_ω. This can be done as follows: We denote by A^n_{all} the infinite-dimensional vector space of Feynman integrals on the maximal cut including the supersectors and by A^n_{super} the infinite-dimensional vector space of Feynman integrals of the supersectors only. We denote by V^n_{all} and V^n_{super} the corresponding finite-dimensional vector spaces, obtained by moding out linear relations. We have

$$V^n_{\text{all}} = V^n + V^n_{\text{super}}. \tag{81}$$

The map $\hat{\iota}$ extends trivially to $\hat{\iota}: A^n_{\text{all}} \to \Omega^n_\omega$. Note that the map $\hat{\iota}$ is not necessarily injective, due to the "missing ISPs" discussed at the end of Sec. II E 3. There is a surjective map

$$j: H^n_\omega \twoheadrightarrow V^n, \tag{82}$$

which is obtained from the following sequence of maps:

$$H^n_\omega \xrightarrow{\pi_{\text{integration}}} V^n_{\text{all}} \xrightarrow{\pi_{\text{sector}}} V^n. \tag{83}$$

The first map projects out symmetry relations due to integration, and the second map subtracts out contributions from supersectors. This allows us to define master integrals on V^n and to decompose H^n_ω as

$$H^n_\omega = \text{Im}(\iota) + \text{Ker}(j). \tag{84}$$

In detail, we proceed as follows: A symmetry relation is a linear relation among Feynman integrals, i.e.,

$$\sum_i c_i I_i = 0. \tag{85}$$

A nontrivial symmetry relation defines a linear relation

$$\hat{\iota}\left(\sum_i c_i I_i\right) = 0 \tag{86}$$

in Ω^n_ω. We denote by H^n_{symm} the vector space H^n_ω modulo these relations. The cohomology classes defined by the left-hand side of Eq. (86) span the kernel of $\pi_{\text{integration}}$. A basis element of H^n_{symm} is represented by $\Psi \in \Omega^n_\omega$. This differential form has one or more preimages in A^n_{all}, which can be found by linear algebra. We define an order relation on the space of Feynman integrals and, hence, on A^n_{all}. We select the simplest preimage according to this order relation. As a side remark, we point out that this order relation will appear again in Eq. (200) as the lowest-order criterion. Note that the order relation on A^n_{all} is used only to select among a set of possible preimages the simplest preimage. All possible preimages are equivalent, once integration-by-parts identities and symmetry relations have been taken into account. Note, further, that we do not need to know all preimages; we are interested in only the simplest one. In practice, this involves, therefore, only a rather small linear system of equations. This procedure defines a basis on V^n_{all}. We may now define the map $\sigma: V^n_{\text{all}} \to A^n_{\text{all}}$ and, in turn, $\iota = \pi_\omega \circ \hat{\iota} \circ \sigma$. With a slight abuse of notation, we use σ

to denote the maps $\sigma\colon V^n \to A^n$ and $\sigma\colon V^n_{\text{all}} \to A^n_{\text{all}}$. In the same spirit, we use ι to denote the maps $V^n \to H^n_\omega$ and $V^n_{\text{all}} \to H^n_\omega$.

Among the basis elements of V^n_{all}, we have the ones which span V^n_{super}. Let us assume that these are labeled $I_{N_F+1}, I_{N_F+2}, \ldots$. The kernel of π_{sector} is given by these integrals. The preimages of $I_{N_F+1}, I_{N_F+2}, \ldots$ in H^n_ω are given by $\iota(I_{N_F+1}), \iota(I_{N_F+2}), \ldots$.

We may decompose H^n_ω into $\text{Im}(\iota)$, the kernel of $\pi_{\text{integration}}$, and the preimage of the kernel of π_{sector}. The latter two span the kernel of j. There is an isomorphism between V^n and $\text{Im}(\iota)$.

C. Linear relations

For the integrands defined in Eq. (51), we have three types of linear relations: integration-by-parts identities, distribution identities, and cancellation identities.

1. Integration-by-parts identities

The integration-by-parts identities read

$$\frac{1}{\varepsilon}\Psi_{\mu_0\ldots\mu_i\ldots\mu_{N_D}}[\partial_{z_j} Q_+] + \sum_{i\in I^0_{\text{all}}} \Psi_{\mu_0\ldots(\mu_i+1)\ldots\mu_{N_D}}[Q_+ \cdot (\partial_{z_j} P_i)] = 0, \tag{87}$$

where Q_+ is an ε-independent homogeneous polynomial of degree $\deg Q_+ = d_Q + 1$. The integration-by-parts identities are derived as follows: We start from the fact that, within twisted cohomology, we may always add a ∇_F-exact form to a twisted cohomology class:

$$\nabla_F \Xi = 0, \tag{88}$$

where Ξ is a $(n-1)$-form, holomorphic on $\mathbb{CP}^n - D$. We call Ξ the seed. Ξ may have singularities on D. Following Griffiths [97], we may write Ξ as

$$\Xi = \frac{Q_+}{\prod_{i\in I^0_{\text{all}}} P_i^{\mu_i}} \left(\sum_{k<j} (-1)^k z_k dz_0 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n - \sum_{k>j} (-1)^k z_k dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n \right), \tag{89}$$

with

$$\mu_i \in \mathbb{N}_0, \qquad i \in \{0, 1, \ldots, N_D\}, \tag{90}$$

subject to the homogeneity condition

$$\deg Q_+ = \sum_{i\in I^0_{\text{all}}} \mu_i d_i - d_U - n. \tag{91}$$

Note that the degree of Q_+ is one higher than the degree of Q given in Eq. (54) (hence the notation "Q_+"). The possible seeds are specified by the (N_D+2)-tuple $(\mu_0, \mu_1, \ldots, \mu_{N_D}, j)$ and the polynomial Q_+, subject to the constraint given in Eq. (91). We have

$$\nabla_F \Xi = \eta \nabla_{F,z_j} \left(\frac{Q_+}{\prod_{i\in I^0_{\text{all}}} P_i^{\mu_i}} \right) = 0, \tag{92}$$

where ∇_{F,z_j} is defined by

$$\nabla_F = \sum_{j=0}^{n} dz_j \nabla_{F,z_j}. \tag{93}$$

Thus, if we strip off η, the integration-by-parts identities are

$$\nabla_{F,z_j} \left(\frac{Q_+}{\prod_{i\in I^0_{\text{all}}} P_i^{\mu_i}} \right) = 0. \tag{94}$$

In this form, each integration-by-parts identity is linear in ε, where the part proportional to ε comes from the twist. Including the twist and the prefactors, we have

$$\partial_{z_j} \left(C_{\text{Baikov}} C_\varepsilon U \frac{Q_+}{\prod_{i\in I^0_{\text{all}}} P_i^{\mu_i}} \right) = 0. \tag{95}$$

Working this out yields Eq. (87).

2. Distribution identities

The distribution identities are rather trivial and originate from writing a polynomial $Q = Q_1 + Q_2$ as a sum of two other polynomials:

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q] = \Psi_{\mu_0\ldots\mu_{N_D}}[Q_1] + \Psi_{\mu_0\ldots\mu_{N_D}}[Q_2]. \tag{96}$$

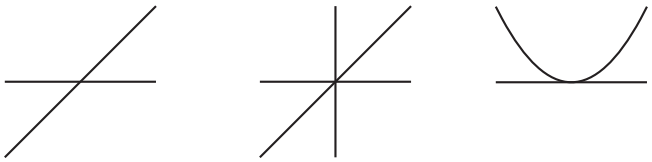


FIG. 6. Crossings in a plane: the left figure shows a normal crossing; the middle figure and the right figure show non-normal crossings.

3. Cancellation identities

The cancellation identities originate from a cancellation of P_j in the numerator and the denominator:

$$\frac{P_j Q}{P_j^{\mu_j+1}\prod_{\substack{i\in I^0_{\text{all}}\\ i\neq j}} P_i^{\mu_i}} = \frac{Q}{\prod_{i\in I^0_{\text{all}}} P_i^{\mu_i}}. \tag{97}$$

They read

$$\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}[P_j\cdot Q] = \frac{1}{\varepsilon}\frac{C^{(j)}_{\text{rel}}}{C_{\text{rel}}}\Psi_{\mu_0\ldots\mu_j\ldots\mu_{N_D}}[Q], \tag{98}$$

where C_{rel} denotes the relative prefactor of $\Psi_{\ldots\mu_j\ldots}[Q]$ and $C^{(j)}_{\text{rel}}$ denotes the relative prefactor of $\Psi_{\ldots(\mu_j+1)\ldots}[P_j\cdot Q]$. The ratio of the relative prefactors depends on a_j, b_j, μ_j, and ε and is always linear in ε:

$$\frac{C^{(j)}_{\text{rel}}}{C_{\text{rel}}} = \frac{1}{2}a_j - \mu_j + \frac{b_j}{2}\varepsilon. \tag{99}$$

D. Poles and residues

We will count the pole order and the number of nonzero consecutive residues of the integrands $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ defined in Eq. (51). The regulator ε in the twist function will not be relevant for this counting, and we define $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ by replacing $U(z)$ by $U_0(z)$:

$$\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q] = C_{\text{Baikov}}C_\varepsilon U_0(z)\hat{\Phi}_{\mu_0\ldots\mu_{N_D}}[Q]\eta. \tag{100}$$

The differential form $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ is algebraic in z; the algebraic part comes entirely from the square roots of the odd polynomials in U_0. The differential forms $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ and $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ are holomorphic on $\mathbb{CP}^n - D$, where the divisor D is defined in Eq. (46). They may have singularities on the divisor D.

For a differential form $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ (and $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$), we define the active divisor as follows: We set

$$D_{\text{active}} = \bigcup_{i\in I^0_{\text{active}}} D_i,$$
$$\text{where } I^0_{\text{active}} = \{i | i\in I^0_{\text{even}} \text{ and } \mu_i > 0\}\cup I^0_{\text{odd}}. \tag{101}$$

In Sec. II, we divided the polynomials defining the divisors into even and odd, depending on whether the ε^0 term of the corresponding exponent in the twist function is zero or $(-\frac{1}{2})$. The even and the odd polynomials will play different roles in the following. If an even polynomial is present in the denominator, we may take a residue and reduce to a simpler problem. The odd polynomials define a geometry, which—to a first approximation—is associated with the Feynman integral. We will elaborate on this point in Sec. III G.

1. Blowups

Before we start to count the pole order, we need to discuss a technical detail: In general, the active divisor D_{active} (and the divisor D) will have self-intersections. We distinguish normal crossings and non-normal crossings. Figure 6 shows an example of a normal crossing in a plane and two examples of non-normal crossings in a plane. In order to determine the pole order correctly, we would like to have normal crossings. A theorem by Hironaka [98] guarantees that by a (finite) sequence of blowups we may always convert a non-normal crossing to a normal crossing. Within the physics community, this procedure is known as sector decomposition [99–102]. Whenever we encounter a non-normal crossing, we will always perform a blowup.

Example 7. Consider the twist function (which corresponds to a Feynman integral contributing to the three-loop electron self-energy, which is shown in Fig. 15)

$$U(z_0,z_1,z_2) = P_0^{4\varepsilon}P_1^{\varepsilon}P_2^{-2\varepsilon}P_3^{-\frac{1}{2}}P_4^{-\frac{1}{2}-\varepsilon}P_5^{-\frac{1}{2}-\varepsilon} \tag{102}$$

with

$$\begin{aligned} &P_0 = z_0, \qquad P_3 = z_1,\\ &P_1 = z_2, \qquad P_4 = z_1 + 4xz_0,\\ &P_2 = z_2 + z_0, \qquad P_5 = (z_2-z_1)^2 + 2xz_0(z_1+z_2) + x^2z_0^2. \end{aligned} \tag{103}$$

P_3 and P_5 have a non-normal intersection at $[z_0 : z_1 : z_2] = [1:0:-x]$. In the chart $z_0 = 1$, we blow up the point $(z_1,z_2) = (0,-x)$. The linear transformation $\tilde{z}_2 = z_2 - z_1 + x$ shifts the point to $(z_1,\tilde{z}_2) = (0,0)$. The blowup gives us three charts:

$$\begin{aligned} &\text{Chart 1}: \; z_1 = z_1', \qquad \tilde{z}_2 = z_1'z_2';\\ &\text{Chart 2}: \; z_1 = z_1''(z_2'')^2, \qquad \tilde{z}_2 = z_2'';\\ &\text{Chart 3}: \; z_1 = (z_1''')^2 z_2''', \qquad \tilde{z}_2 = z_1'''z_2'''. \end{aligned} \tag{104}$$

2. Poles

The differential forms $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ and $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ may have singularities on D. We now define an integer o, called the pole order of the differential form, which measures how singular the differential form is. We define the pole order o of $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ as follows: If the active divisors have non-normal crossings, we first perform a blowup. After this step, we may assume that we have only normal crossings. The pole order is the maximum of the pole orders at individual points. In the univariate case, we define the pole order of $z^{-\alpha}dz$ at $z=0$ for $\alpha>0$ to be $\lfloor\alpha\rfloor$, where $\lfloor x\rfloor$ denotes the floor function. Hence, the pole order of dz/z at $z=0$ is 1, and the pole order of $dz/\sqrt{z}$ at $z=0$ is 0. In the multivariate case, we define the pole order for normal-crossing singularities to be additive; i.e., the pole order of $dz_1/z_1\wedge dz_2/z_2^2$ at $(z_1,z_2)=(0,0)$ is 3. We define the pole order of $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ to be the pole order of $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$.

Example 8. The differential form

$$\frac{z_0dz_1\wedge dz_2-z_1dz_0\wedge dz_2+z_2dz_0\wedge dz_1}{z_1z_2\sqrt{(z_2-z_1)^2+2xz_0(z_1+z_2)+x^2z_0^2}} \tag{105}$$

has pole order $o=2$. The pole order 2 is attained at the points $[1:0:0]$, $[1:0:-x]$, and $[1:-x:0]$. The differential form

$$\frac{z_0dz_1\wedge dz_2-z_1dz_0\wedge dz_2+z_2dz_0\wedge dz_1}{z_1[(z_2-z_1)^2+2xz_0(z_1+z_2)+x^2z_0^2]} \tag{106}$$

has pole order $o=3$. The pole order 3 is attained at the point $[z_0:z_1:z_2]=[1:0:-x]$. At the points $[1:0:-x]$ and $[1:-x:0]$, we have non-normal crossings and a blowup, as in Example 7 is required.

3. Residues

If an even polynomial appears in the denominator of a differential form, we may take a residue. If more than one even polynomial appears in the denominator, we may iterate this procedure. We denote by r the largest number such that the r-fold residue of $\Psi^0_{\mu_0\ldots\mu_{N_D}}[Q]$ is nonzero. We define the number of nonzero consecutive residues of $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ to be r.

Example 9. The differential form

$$\frac{z_0dz_1\wedge dz_2-z_1dz_0\wedge dz_2+z_2dz_0\wedge dz_1}{z_1z_2\sqrt{(z_2-z_1)^2+2xz_0(z_1+z_2)+x^2z_0^2}} \tag{107}$$

has $r=2$. We may take a twofold residue at $[z_0:z_1:z_2]=[1:0:0]$, at $[z_0:z_1:z_2]=[1:0:-x]$, and at $[z_0:z_1:z_2]=[1:-x:0]$.

E. Filtrations

For each object $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]\in\Omega^n_\omega$, we defined three integers r, o, and $|\mu|$. The number r of nonzero consecutive residues has been defined in Sec. III D 3, the pole order o has been defined in Sec. III D 2, and the quantity $|\mu|$ has been defined in Eq. (55). These numbers define three filtrations $W_\bullet$, $F^\bullet_{\text{geom}}$, and $F^\bullet_{\text{comb}}$ on the space Ω^n_ω.

(i) The weight filtration $W_\bullet$ is defined by

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q]\in W_w\Omega^n_\omega \quad \text{if } n+r\le w. \tag{108}$$

(ii) The filtration $F^\bullet_{\text{geom}}$ is defined by

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q]\in F^p_{\text{geom}}\Omega^n_\omega \quad \text{if } n+r-o\ge p. \tag{109}$$

(iii) The filtration $F^\bullet_{\text{comb}}$ is defined by

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q]\in F^{p'}_{\text{comb}}\Omega^n_\omega \quad \text{if } n-|\mu|\ge p'. \tag{110}$$

These filtrations are inspired by Hodge theory. The weight filtration is the standard weight filtration from Hodge theory. As n is fixed, it is essentially a filtration by the number of nonzero consecutive residues. At fixed weight w, the filtration $F^\bullet_{\text{geom}}$ is a filtration by the pole order o. The combinatorial filtration $F^\bullet_{\text{comb}}$ is a filtration by the quantity $|\mu|$. The general idea is that we always work modulo simpler terms, i.e., modulo terms with fewer residues, lower pole order, or a smaller sum of indices $|\mu|$.

With the filtrations $F^\bullet_{\text{comb}}$, $F^\bullet_{\text{geom}}$, and $W_\bullet$, we may, in principle, decompose Ω^n_ω into subspaces $\Omega^{|\mu|,o,r}_\omega$. A projection onto two indices is easier to display, and we define two coarse-grained decompositions: We set

$$\Omega^{p,q}_{\text{geom}}=\text{Gr}^p_{F_{\text{geom}}}\text{Gr}^W_{p+q}\Omega^n_\omega \quad\text{and}\quad \Omega^{p',q'}_{\text{comb}}=\text{Gr}^{p'}_{F_{\text{comb}}}\text{Gr}^W_{p'+q'}\Omega^n_\omega, \tag{111}$$

where the graded parts are defined by

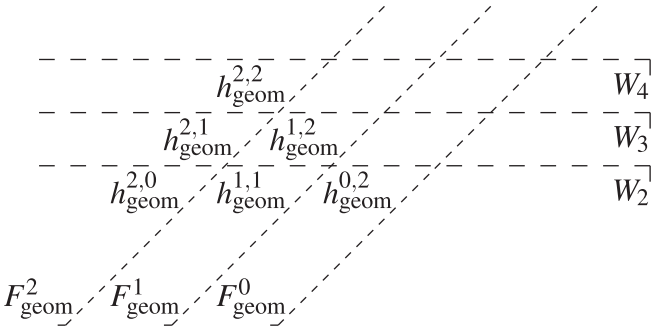
$$\text{Gr}^W_wX=W_wX/W_{w-1}X \quad\text{and}\quad \text{Gr}^p_FX=F^pX/F^{p+1}X. \tag{112}$$

We carry over the filtrations to H^n_ω as follows: Cohomology classes in H^n_ω are cosets, and different elements of a coset may have different values of $(|\mu|,o,r)$. In Sec. IV, we define a total order on the elements $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ of Ω^n_ω based on the values $(|\mu|,o,r)$. We define the standard representative of a cohomology class to be the object $\Psi_{\mu_0\ldots\mu_{N_D}}[Q]$ in this cohomology class, which is minimal with respect to the order relation. For a fixed total order, this element is unique. We call this standard representative a master integrand. We let $H^{p,q}_{\text{geom}}$ be generated by all master integrands $\Psi\in\Omega^{p,q}_{\text{geom}}$ and $H^{p',q'}_{\text{comb}}$ be generated by all master

integrands $\Psi \in \Omega^{p',q'}_{\text{comb}}$. We set

$$h^{p,q}_{\text{geom}} = \dim H^{p,q}_{\text{geom}} \quad \text{and} \quad h^{p',q'}_{\text{comb}} = \dim H^{p',q'}_{\text{comb}}. \tag{113}$$

We display this information in the form of a Hodge-like diagram, for example, for $n = 2$,



F. Localizations

If a polynomial P_i with $i \in I^0_{\text{even}}$ appears in the denominator, we may take a residue or, in other words, localize on $P_i = 0$. By taking a residue, we go from a variety of dimension n to a subvariety of dimension $(n-1)$. The equation $P_i = 0$ with $i \in I^0_{\text{even}}$ defines a distinguished subvariety, which plays an important role in our algorithm. Clearly, this process can be iterated. On a subvariety, we may take a further residue and obtain a subsubvariety of codimension two. In general, we obtain a set of nested subvarieties as shown in Fig. 7. Ultimately, these distinguished subvarieties determine the decomposition of the vector space V^n of Feynman integrals on the maximal cut into pieces associated with specific geometries.

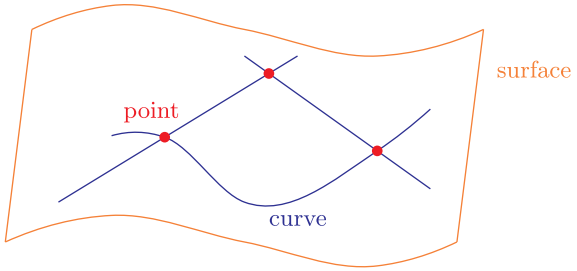


FIG. 7. A mixed geometry: inside a surface of dimension two, there can be curves of dimension one and points of dimension zero.

Our algorithm will first recursively treat all possible localizations. In the simplest case, where P equals one Baikov variable (e.g., $P_i = z_r$), the localization eliminates this Baikov variable, and we are left with a subproblem with one Baikov variable less. Localizations are rather straightforward if each divisor we localize on is linear in at least one Baikov variable. Below, we present a method which avoids algebraic extensions even if this does not hold. The philosophy is not to eliminate variables but to work with the original variables modulo an ideal on a restricted set of objects with a restricted set of linear relations. As a localization defines a subvariety of $\mathbb{CP}^n$, we inevitably need to work with charts.

Example 10. In general, we need to consider all charts. Integration-by-parts identities in a single chart might not give the full set of linear relations. To see this, consider in $\mathbb{CP}^2$ the twist

$$U = P_0^{2\epsilon} P_1^{2\epsilon} P_2^{-\epsilon} P_3^{-\frac{1}{2}-\epsilon}, \tag{114}$$

where the polynomials are given by

$$\begin{aligned} P_0 &= z_0, & P_2 &= z_1 z_2 + x_1 z_0 z_2 - z_1^2, \\ P_1 &= z_1, & P_3 &= [x_2 z_1 - (1+x_2) z_2 + (x_1 - x_2) z_0]^2 + 4 x_1 (1+x_2) z_0 z_2. \end{aligned} \tag{115}$$

This example has six master integrands. However, if we use only the integration-by-parts identities in the chart $z_2 = 1$ [by excluding $j = 2$ in Eq. (15)], we will be able only to reduce the integrands to seven free integrands. This example will be continued in Example 14 and discussed in detail in Sec. V D.

In general, we denote localized integrands by

$$\text{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r} \Psi_{\mu_0,\ldots,\mu_{N_D}} [Q, I, \mathcal{P}_s], \tag{116}$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_r$ denotes an ordered sequence, specifying the order in which the residues are taken (the first residue is taken with respect to $\mathcal{P}_1$, and the last residue is taken with respect to $\mathcal{P}_r$). $I = \langle \mathcal{P}_1, \ldots, \mathcal{P}_r \rangle$ denotes the ideal we localize on, and $\mathcal{P}_s \notin I$ denotes a scale polynomial, which we need to introduce in order to keep the integrand homogeneous. We will be considering only differential forms $\Psi_{\mu_0,\ldots,\mu_{N_D}}$ which have simple poles along I. Let G be a Gröbner basis of the ideal I. With the help of the Gröbner basis, we may reduce the numerator polynomial Q modulo the ideal I. If

$$Q_1 \mod I = Q_2 \mod I, \tag{117}$$

we have

$$\begin{aligned}&\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0,\ldots,\mu_{N_D}}[Q_1,I,\mathcal{P}_s]\\&=\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0,\ldots,\mu_{N_D}}[Q_2,I,\mathcal{P}_s],\end{aligned}\tag{118}$$

since we assumed simple poles and terms regular on I will vanish under the residue map. Among the preimages of the residue map, we may single out the one where the numerator polynomial Q is reduced modulo I:

$$\Psi_{\mu_0,\ldots,\mu_{N_D}}[Q],\tag{119}$$

with $Q = Q \bmod I$. There is a one-to-one map between the residues and these special preimages. When there is no risk of confusion, we will use the shorter notation of the preimage of Eq. (119) instead of the longer notation of Eq. (116).

We treat cases with increasing complexity. We start in Sec. III F 1 with the simplest case, where P equals one Baikov variable. In Sec. III F 2, we discuss the case of a single divisor, not necessarily equal to one Baikov variable. In Sec. III F 3, we treat the general case, where we take multiple residues and localize on the intersection of multiple divisors. In Sec. III F 4, we discuss the special case where the localization on multiple divisors leads to a localization on a point.

1. Localization on $z_n = 0$

We start from the twist function

$$U=\prod_{k=0}^{N_D}P_k^{\alpha_k}.\tag{120}$$

We first consider the case where the divisor defining the localization equals one Baikov variable. Without loss of generality, we may assume

$$P_{N_D}=z_n.\tag{121}$$

We assume that P_{N_D} is an even polynomial; hence,

$$\alpha_{N_D}=\frac{b_{N_D}\varepsilon}{2}.\tag{122}$$

We denote by $V(\langle z_n\rangle)$ the subvariety in $\mathbb{CP}^n$ defined by $z_n = 0$. Although $V(\langle z_n\rangle)\sim\mathbb{CP}^{n-1}$, the situation for the differential forms is a little bit trickier: We would like to study the "residue" of $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q]$ on $V(\langle z_n\rangle)$. We first have to remove

$$z_n^{\frac{b_{N_D}\varepsilon}{2}}\tag{123}$$

from the twist function in order to be able to take the residue. However, we cannot just remove this factor alone, as the remainder would no longer be homogeneous and well defined on $\mathbb{CP}^n$. We choose a homogeneous polynomial P_s of degree $\deg P_s = d_s$, which we call the scale polynomial, and define $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s]$ by

$$\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q]=\frac{C_{\mathrm{rel}}}{C_{\mathrm{rel}}^{\mathrm{loc}}}\left(\frac{z_n^{d_s}}{P_s}\right)^{\frac{b_{N_D}\varepsilon}{2d_s}}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s],\tag{124}$$

where C_{rel} is the relative prefactor of $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q]$ and $C_{\mathrm{rel}}^{\mathrm{loc}}$ is the relative prefactor of $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s]$, computed from the twist function

$$U^{\mathrm{loc}}=P_s^{(\alpha_s+\frac{b_{N_D}\varepsilon}{2d_s})}\prod_{\substack{k=0\\k\neq s}}^{N_D-1}P_k^{\alpha_k}.\tag{125}$$

The exponents of the divisors in the original twist function U and in the modified twist function U^{loc} differ for $k\in\{s,N_D\}$. By construction, the exponent of P_{N_D} in U^{loc} will be zero. In order to compute the relative prefactor $C_{\mathrm{rel}}^{\mathrm{loc}}$, we have to define the falling factorial for this case. As $\mu_{N_D}=1$ is constant, the falling factorial will also be constant for all cases of interest. We can define it to be any nonzero finite value. In accordance with Sec. III F 4, we will use the convention that

$$(0)_1=-\varepsilon.\tag{126}$$

In order to avoid introducing new spurious singularities, it is advantageous to choose

$$P_s\in\{P_0,P_1,\ldots,P_{N_D-1}\}.\tag{127}$$

$\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s]$ is a well-defined differential n-form on $\mathbb{CP}^n$ with a simple pole along $z_n = 0$. Hence, we may take the residue at $z_n = 0$, which we denote by

$$\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s].\tag{128}$$

$\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s]$ is a differential $(n-1)$-form on $V(\langle z_n\rangle)\sim\mathbb{CP}^{n-1}$. Let us define $\tilde{Q}$ by

$$\tilde{Q}=Q|_{z_n=0}=Q\mod\langle z_n\rangle.\tag{129}$$

It is clear that

$$\begin{aligned}&\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[\tilde{Q},\langle z_n\rangle,P_s]\\&=\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle z_n\rangle,P_s],\end{aligned}\tag{130}$$

as terms regular on $z_n = 0$ have no residue. From now on,

we will always assume that the numerator polynomial is reduced to $\tilde{Q}$ according to Eq. (129).

We are interested in the integration-by-parts identities of $\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[\tilde{Q},\langle z_n\rangle,P_s]$ on $V(\langle z_n\rangle)\sim\mathbb{CP}^{n-1}$. From now on, we will assume that we always choose the scale polynomial from the set of the remaining divisors, i.e., according to Eq. (127). We will further assume that for $\dim V(\langle z_n\rangle)>0$ the choice is generic; i.e., the exponent of $\tilde{P}_s$ in U^{loc} is not an integer. The case $\dim V(\langle z_n\rangle)=0$ is special and will be treated in Sec. III F 4. With this assumption, the integration-by-parts identities are then given for $j\in\{0,\ldots,n-1\}$ by

$$0=\frac{1}{\varepsilon}\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_i,\ldots,\mu_{(N_D-1)},1}[\partial_{z_j}\tilde{Q}_+,\langle z_n\rangle,P_s]+\sum_{i=0}^{N_D-1}\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_i+1,\ldots,\mu_{(N_D-1)},1}[\tilde{Q}_+\cdot(\partial_{z_j}\tilde{P}_i),\langle z_n\rangle,P_s],\tag{131}$$

where $\tilde{Q}_+=Q_+|_{z_n=0}$. A few remarks are in order.

(1) Note that we exclude $j=n$ in Eq. (131). We are interested only in derivatives within the hyperplane $V(\langle z_n\rangle)$, not in derivatives normal to it.

In the full system (without localization), we have additional integration-by-parts identities (namely, the ones with $j=n$), for example,

$$\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q_+]=-\frac{1}{\varepsilon}\Psi_{\mu_0,\ldots,\mu_i,\ldots,\mu_{(N_D-1)},0}[\partial_{z_n}Q_+]-\sum_{i=0}^{N_D-1}\Psi_{\mu_0,\ldots,\mu_i+1,\ldots,\mu_{(N_D-1)},0}[Q_+\cdot(\partial_{z_n}P_i)].\tag{132}$$

(2) Note that, even if we start with a minimal twist function U and irreducible polynomials P_k, the modified twist function U^{loc} need not be minimal and the polynomials $\tilde{P}_k$ need not be irreducible.

(3) The integral reduction tables for $\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}\times[\tilde{Q},\langle z_n\rangle,P_s]$ on $V(\langle z_n\rangle)$ depend on the choice for the scale polynomial P_s.

(4) In general, the integral reductions for $\mathrm{Res}_{z_n}\times\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[\tilde{Q},\langle z_n\rangle,P_s]$ on $V(\langle z_n\rangle)$ are not related to the ones for $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q]$ on $\mathbb{CP}^n$, as they correspond to different twist functions U and U^{loc}. The only information which we carry from $V(\langle z_n\rangle)$ to $\mathbb{CP}^n$ is the following: If $\mathrm{Res}_{z_n}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}\times[\tilde{Q},\langle z_n\rangle,P_s]$ is a master integrand on $V(\langle z_n\rangle)$, then we take $\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[\tilde{Q}]$ as a preferred master candidate on $\mathbb{CP}^n$.

(5) On the localization $z_n=0$, we consider only differential forms which have a simple pole at $z_n=0$, i.e., the differential forms with $\mu_{N_D}=1$. We do not consider differential forms which have higher poles at $z_n=0$; i.e., we exclude differential forms with $\mu_{N_D}>1$. Note that differential forms with higher poles are not generated by the integration-by-parts identities given in Eq. (131).

(6) On the localization $z_n=0$, differential forms with $\mu_{N_D}=0$ are automatically zero due to the residue operation. This implies that we may restrict all numerator polynomials Q and all seeds Q_+ to

$$\tilde{Q}=Q\mod\langle z_n\rangle,\qquad\tilde{Q}_+=Q_+\mod\langle z_n\rangle.\tag{133}$$

2. Localization on a single divisor

We now consider the case that we localize on $P_{N_D}=0$ but do not assume that P_{N_D} equals one Baikov variable. For the derivation of the integration-by-parts identities on the divisor $P_{N_D}=0$, we will assume that

$$\frac{\partial P_{N_D}}{\partial z_n}\neq 0;\tag{134}$$

if P_{N_D} does not depend on z_n, we replace z_n by a Baikov variable it depends on. We consider again the objects

$$\Psi_{\mu_0,\ldots,\mu_{N_D-1},1}[Q,\langle P_{N_D}\rangle,P_s],\tag{135}$$

now defined by

$$\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q]=\frac{C_{\mathrm{rel}}}{C_{\mathrm{rel}}^{\mathrm{loc}}}\left(\frac{P_{N_D}^{d_s}}{P_s^{d_{N_D}}}\right)^{\frac{b_{N_D}\varepsilon}{2d_s}}\Psi_{\mu_0,\ldots,\mu_{(N_D-1)},1}[Q,\langle P_{N_D}\rangle,P_s].\tag{136}$$

The modified twist function U^{loc} is given by

$$U^{\mathrm{loc}}=P_s^{(\alpha_s+\frac{d_{N_D}b_{N_D}\varepsilon}{2d_s})}\prod_{\substack{k=0\\k\neq N_D}}^{N_D-1}P_k^{\alpha_k}.\tag{137}$$

We need to work out the integration-by-parts identities on the localization. These are the ones where we take a derivative within the hypersurface defined by $P_{N_D}=0$ but not the one transversal to it. We work temporarily in the

chart $z_0 = 1$. The equation $P_{N_D} = 0$ defines implicitly a function $z_n = f(z_1, \ldots, z_{n-1})$, and we use $(z_1, \ldots, z_{n-1})$ as coordinates on the hypersurface. With the help of the implicit function theorem, we obtain for the derivatives within the hypersurface

$$\frac{\partial}{\partial z_j} + \frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_n} = \frac{\partial}{\partial z_j} - \left(\frac{\partial P_{N_D}}{\partial z_n}\right)^{-1}\frac{\partial P_{N_D}}{\partial z_j}\frac{\partial}{\partial z_n}, \qquad j \in \{1, \ldots, n-1\}. \tag{138}$$

Hence, the integration-by-parts identities in the chart $z_0 = 1$ of the localization $P_{N_D} = 0$ read

$$\begin{aligned} 0 = {} & \frac{1}{\varepsilon}\mathrm{Res}_{P_{N_D}}\Psi_{\mu_0\ldots\mu_i\ldots\mu_{N_D-1},1}[(\partial_{z_n}P_{N_D})(\partial_{z_j}Q_+) - (\partial_{z_j}P_{N_D})(\partial_{z_n}Q_+), \langle P_{N_D}\rangle, P_s] \\ & + \sum_{i=0}^{N_D-1}\mathrm{Res}_{P_{N_D}}\Psi_{\mu_0\ldots(\mu_i+1)\ldots\mu_{N_D-1},1}[Q_+((\partial_{z_n}P_{N_D})(\partial_{z_j}P_i) - (\partial_{z_j}P_{N_D})(\partial_{z_n}P_i)), \langle P_{N_D}\rangle, P_s], \end{aligned} \tag{139}$$

where

$$\deg Q_+ = \sum_{i=0}^{N_D-1}\mu_i d_i - d_U - n + 1 \tag{140}$$

and $j \in \{1, \ldots, n-1\}$. Integration-by-parts identities in other charts may be worked out along the same lines.

Again, we may reduce all numerator polynomials Q modulo P_{N_D}, and it is sufficient to restrict the numerators Q_+ of the seeds to polynomials, which are not reducible by P_{N_D}. The cancellation identities take the following form:

$$\begin{aligned} & \mathrm{Res}_{P_{N_D}}\Psi_{\mu_0,\ldots,(\mu_j+1),\ldots,\mu_{N_D-1},1}[(P_j\cdot Q) \mod \langle P_{N_D}\rangle, \langle P_{N_D}\rangle, P_s] \\ & = \frac{1}{\varepsilon}\frac{C^{\mathrm{loc},(j)}_{\mathrm{rel}}}{C^{\mathrm{loc}}_{\mathrm{rel}}}\mathrm{Res}_{P_{N_D}}\Psi_{\mu_0,\ldots,\mu_j,\ldots,\mu_{N_D-1},1}[Q \mod \langle P_{N_D}\rangle, \langle P_{N_D}\rangle, P_s]. \end{aligned} \tag{141}$$

We may look at Eq. (138) also from a slightly different perspective: This equation actually defines a logarithmic vector field [103] along $P_{N_D} = 0$, since

$$\left(\frac{\partial P_{N_D}}{\partial z_n}\frac{\partial}{\partial z_j} - \frac{\partial P_{N_D}}{\partial z_j}\frac{\partial}{\partial z_n}\right)P_{N_D} = 0. \tag{142}$$

Therefore, we can define the following $(n-1)$-form:

$$\Xi = \frac{Q_+}{(n-1)!P_{N_D}\prod_{k=1}^{N_D-1}P_k^{\mu_k}}\sum_{j_0,j_1,\ldots,j_n} z_{j_0}\frac{\partial P_{N_D}}{\partial z_{j_1}}\varepsilon_{j_0 j_1\ldots j_n}dz_{j_2}\wedge\cdots\wedge dz_{j_n}. \tag{143}$$

The covariant derivative of Ξ gives the integration-by-parts identities of Eq. (139).

3. Localization on multiple divisors

We now consider the general case, where we take r consecutive residues. We follow the same ideas as in the previous two subsections. The case of r consecutive residues is more cumbersome only from a notational perspective. After we have taken the first residue, an odd polynomial may become a perfect square, thus turning into an even polynomial. For this reason, we rename the divisors where we take the residues to $\mathcal{P}_1, \ldots, \mathcal{P}_r$. We localize on the ideal

$$\mathcal{I} = \langle \mathcal{P}_1, \ldots, \mathcal{P}_r\rangle. \tag{144}$$

We temporarily work in a chart, and without loss of generality we consider the chart $z_0 = 1$. We will assume that the determinant of the $(r \times r)$ matrix

$$\mathcal{J}_{ij} = \frac{\partial \mathcal{P}_i}{\partial z_{n-r+j}}, \qquad i,j \in \{1,\ldots,r\}, \tag{145}$$

is nonzero. If this is not the case, we consider a different chart. With this assumption, we may again use the implicit function theorem and obtain for the derivatives within the variety defined by the ideal $\mathcal{I}$:

$$\det(\mathcal{J})\frac{\partial}{\partial z_j} - \sum_{k,l=1}^{r} (\mathrm{adj}\mathcal{J})_{kl}\frac{\partial \mathcal{P}_l}{\partial z_j}\frac{\partial}{\partial z_{n-r+k}}, \quad j \in \{1,\ldots,n-r\}, \tag{146}$$

where $\mathrm{adj}\mathcal{J}$ denotes the adjoint matrix of $\mathcal{J}$. Applying these differential operators to seeds gives us the integration-by-parts identities in the chart $z_0 = 1$ of the localization defined by the ideal $\mathcal{I}$:

$$0 = \frac{1}{\varepsilon}\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0\ldots\mu_i\ldots\mu_{N_D}}[\hat{Q}, I, P_s] + \sum_{i=0}^{N_D}\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0\ldots(\mu_i+1)\ldots\mu_{N_D}}[\hat{Q}_i, I, P_s], \tag{147}$$

with

$$\hat{Q} = \det(\mathcal{J})(\partial_{z_j}Q_+) - \sum_{k,l=1}^{r}(\mathrm{adj}\mathcal{J})_{kl}(\partial_{z_j}\mathcal{P}_l)(\partial_{z_{n-r+k}}Q_+),$$
$$\hat{Q}_i = Q_+\left[\det(\mathcal{J})(\partial_{z_j}P_i) - \sum_{k,l=1}^{r}(\mathrm{adj}\mathcal{J})_{kl}(\partial_{z_j}\mathcal{P}_l)(\partial_{z_{n-r+k}}P_i)\right], \tag{148}$$

where

$$\deg\, Q_+ = \sum_{i=0}^{N_D}\mu_i d_i - d_U - n + r - \sum_{j=1}^{r}\deg\,\mathcal{P}_j \tag{149}$$

and $j \in \{1,\ldots,n-r\}$. Integration-by-parts identities in other charts may be worked out along the same lines.

We restrict to differential forms with simple poles along $\mathcal{P}_1,\ldots,\mathcal{P}_r$; i.e., in any chart, we have that

$$\mathcal{P}_1\cdots\mathcal{P}_r\Psi_{\mu_0\ldots\mu_{N_D}}[Q] \tag{150}$$

has no residue on the r-fold localization defined by the ideal $\mathcal{I}$.

In addition, we may reduce the polynomials in the numerator modulo the ideal $\mathcal{I}$. For this task, we compute a Gröbner basis of $\mathcal{I}$. The cancellation identities then take the following form:

$$\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}[(P_j\cdot Q) \mod I, I, P_s] = \frac{1}{\varepsilon}\frac{C_{\mathrm{rel}}^{\mathrm{loc},(j)}}{C_{\mathrm{rel}}^{\mathrm{loc}}}\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0\ldots\mu_j\ldots\mu_{N_D}}[Q \mod I, I, P_s]. \tag{151}$$

4. Localization on a point

The recursion starts with integrands localized on a point. It is worth examining this case in detail.

We first investigate the case where we start from a zero-dimensional Baikov representation. This corresponds to $\mathbb{CP}^0$, which is a point. In homogeneous coordinates, we denote this point by $[z_0]$ with $z_0 \neq 0$. The 0-form η is given by

$$\eta = z_0. \tag{152}$$

There is only one nontrivial irreducible homogeneous polynomial:

$$P_0 = z_0. \tag{153}$$

The twist is simply

$$U = 1. \tag{154}$$

We may write the twist as

$$U = P_0^{\alpha_0} \quad \text{with} \quad \alpha_0 = 0. \tag{155}$$

As already briefly mentioned in Sec. II F, in zero dimensions we no longer have $\alpha_0 \notin \mathbb{Z}$. In this subsection, we carefully investigate the implications of this fact.

The rational function $\hat{\Phi}$ has to be of degree (-1). The differential 0-forms are of the form

$$\Psi_{\mu_0}[Q] = C_{\mathrm{Baikov}}C_\varepsilon\frac{Q}{P^{\mu_0}}\eta, \tag{156}$$

with $\deg Q = \mu_0 - 1$ and $\mu_0 \in \mathbb{N}$. Note that $\mu_0 = 0$ is not allowed, since $\deg\hat{\Phi} = -1$. The clutch prefactor is given by

$$C_{\mathrm{clutch}} = \frac{1}{\varepsilon^{\mu_0}}. \tag{157}$$

The relative prefactor is a little bit trickier: We cannot simply take the limit $\alpha \to 0$ of the standard definition, as this would give zero. We determine the relative prefactor from the limit

$$\lim_{\varepsilon\to 0}\left(-\frac{1}{2}b_0\varepsilon\right)^{-1}C_{\mathrm{rel}} = (-1)^{\mu_0}(\mu_0-1)!. \tag{158}$$

The factor $-\frac{1}{2}b_0$ is not essential, and we set

$$C_{\text{rel}} = \varepsilon(-1)^{\mu_0}(\mu_0 - 1)!. \tag{159}$$

With this definition, the integration-by-parts identities and the cancellation identities coincide and read

$$\frac{\mu_0}{\varepsilon}\Psi_{\mu_0}[z_0^{\mu_0-1}] + \Psi_{\mu_0+1}[z_0^{\mu_0}] = 0. \tag{160}$$

Thus, there is one master integrand, the one with minimal $|\mu|$, which is given by

$$\Psi_1[1]. \tag{161}$$

Note that, although $\mathbb{CP}^0$ is a point, there is still an integration-by-parts identity, the one given by Eq. (160). This can be understood as follows: $\mathbb{CP}^0$ equals $\mathbb{C}^\times/\sim$, where $z \sim z'$ if $z = \lambda z'$. The integration-by-parts identity corresponds to the direction which is moded out by the equivalence relation $\sim$.

We now turn back to a general n-dimensional Baikov representation. Localizing on an ideal

$$\mathcal{I} = \langle \mathcal{P}_1, \ldots, \mathcal{P}_n \rangle \tag{162}$$

corresponds to localization on N points. From the remarks above, it is clear that in the case $N = 1$, i.e., on the localization of a single point, we have one master integrand. We take this integrand to be the simplest among the ones which have a nonzero residue at this point. For the localization on a zero-dimensional variety consisting of N points, we take the N simplest integrands such that the $(N \times N)$-period matrix (i.e., the matrix of residues) is nondegenerate.

Example 11. We illustrate this with a simple example. We consider in $\mathbb{CP}^1$ the twist

$$U(z_0, z_1) = z_0^{-2\varepsilon}(z_1^2 + z_0 z_1 + z_0^2)^{\varepsilon}, \tag{163}$$

together with $C_{\text{Baikov}} = C_{\text{abs}} = 1$. We have

$$P_0 = z_0, \qquad P_1 = z_1^2 + z_0 z_1 + z_0^2. \tag{164}$$

The objects on $\mathbb{CP}^1$ are

$$\begin{aligned}\Psi_{\mu_0\mu_1}[Q] &= C_{\text{clutch}} C_{\text{rel}} U(z_0, z_1)\hat{\Phi}_{\mu_0\mu_1}[Q]\eta,\\ \hat{\Phi}_{\mu_0\mu_1}[Q] &= \frac{Q}{P_0^{\mu_0} P_1^{\mu_1}},\end{aligned} \tag{165}$$

with $\deg Q = \mu_0 + 2\mu_1 - 2$. From an analysis of the critical points, we expect $\dim H^1_\omega = 1$.

Let us first look at the localization $P_0 = 0$. We consider the differential one-forms $\Psi_{1\mu_1}[Q]$ and the corresponding zero-forms $\text{Res}_{z_0}\Psi_{1\mu_1}[\tilde{Q}, \langle z_0\rangle, P_1]$. From the homogeneity condition, we need $\mu_1 \geq 1$. The simplest integrand with a nonvanishing residue at $P_0 = 0$ is

$$\Psi_{11}[z_1, \langle z_0\rangle, P_1]. \tag{166}$$

Let us now look at the localization $P_1 = 0$. We consider the differential one-forms $\Psi_{\mu_0 1}[Q]$ and the corresponding zero-forms $\text{Res}_{P_1}\Psi_{\mu_0 1}[\tilde{Q}, \langle P_1\rangle, z_0]$. The simplest integrands with a nondegenerate residue matrix at the two points of $P_1 = 0$ are

$$\Psi_{01}[1, \langle P_1\rangle, z_0], \qquad \Psi_{11}[z_1, \langle P_1\rangle, z_0]. \tag{167}$$

After having done all subproblems, we go back to the original problem and consider the integrands on $\mathbb{CP}^1$. From Eqs. (166) and (167), we infer that

$$\Psi_{01}[1], \qquad \Psi_{11}[z_1] \tag{168}$$

are preferred candidates for master integrands on $\mathbb{CP}^1$. We express this preference by assigning a number $a = -w < 0$ to the preferred candidates and $a = 0$ to all other objects. In this case, we assign $a = -2$ to $\Psi_{11}[z_1]$ and $\Psi_{01}[1]$.

Among the set of integration-by-parts identities on $\mathbb{CP}^1$, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\Psi_{01}[1] + \Psi_{11}[2z_1 + z_0] + \Psi_{02}[(2z_0 + z_1)(2z_1 + z_0)] = 0,\\ &\frac{1}{\varepsilon}\Psi_{01}[1] + \Psi_{02}[(2z_1 + z_0)(2z_0 + z_1)] = 0.\end{aligned} \tag{169}$$

These two equations imply

$$\Psi_{11}[2z_1 + z_0] = 0, \tag{170}$$

and together with the cancellation identity

$$\Psi_{11}[z_0] = -2\Psi_{01}[1] \tag{171}$$

it follows that $\Psi_{11}[z_1] = \Psi_{01}[1]$. The algorithm will eliminate $\Psi_{11}[z_1]$ (with $|\mu| = 2$) in favor of $\Psi_{01}[1]$ (with $|\mu| = 1$), and we obtain one master integrand given by

$$\Psi_{01}[1]. \tag{172}$$

Remark. This example also shows that twisted cohomology "sees" the internal structure of $z_1^2 + z_1 + 1$. The dlog-form given in the chart $z_0 = 1$ by

$$\frac{(2z_1 + 1)dz_1}{z_1^2 + z_1 + 1} \tag{173}$$

is in the cohomology class of zero.

G. Geometries

The algorithms we present are agnostic to the specific geometry associated with the Feynman integrals under consideration. This does not mean that the algorithms are

not based on geometric properties. By using methods from twisted cohomology and Hodge theory, we are using methods which treat all specific geometries in an (abstract) unified setting. Although not strictly needed, it can be helpful to distill the specific geometries associated with Feynman integrals. We are mainly interested in the geometries associated with the maximal cut of Feynman integrals. We distinguish the twisted geometry, the nodal geometry, and the square-free geometry. The twisted geometry contains the full information and is determined by the twist function U. For the nodal geometry and the square-free geometry, only the $(\varepsilon = 0)$ part of the twist function is relevant. The nodal geometry is determined by all polynomials entering the twist function, even and odd. The square-free geometry is defined by the odd polynomials only. A statement that a certain Feynman integral corresponds to a certain geometry usually refers to the square-free geometry. We would like to stress that this neglects subgeometries due to localizations on even polynomials as well as the ε-dependent part in the twist function.

1. The twisted geometry

The twisted geometry carries the most information. It is defined as in Sec. II G by the Baikov representation. We consider the space

$$\mathbb{CP}^n - D, \tag{174}$$

where the divisor D is defined in Eq. (46), together with the connection

$$\nabla_F = d_F + \omega, \tag{175}$$

defined in Eq. (48). We are interested in the twisted cohomology group

$$H^n_\omega, \tag{176}$$

which is defined as the set of equivalence classes of the ∇_F-closed differential $(n,0)$-forms holomorphic on $\mathbb{CP}^n - D$ modulo the ∇_F-exact ones.

2. The nodal geometry

The nodal geometry carries less information. We remove the multivaluedness entering through the twist function U and the dependence on the dimensional regulator ε. However, we keep the information on all polynomials entering the twist function, even and odd.

We define the nodal geometry Y_{nodal} as a hypersurface in a weighted projective space $X_{\text{nodal}} = \mathbb{CP}^{n+1}_{1,\ldots,1,h}$ by the equation

$$y^2 - F(z_0, z_1, \ldots, z_n) = 0, \tag{177}$$

where

$$F(z_0, z_1, \ldots, z_n) = \prod_{i \in I^0_{\text{even}}} [P_i(z)]^2 \prod_{i \in I^0_{\text{odd}}} P_i(z). \tag{178}$$

The weight h of the coordinate y is given by

$$h = \prod_{i \in I^0_{\text{even}}} d_i + \frac{1}{2} \prod_{i \in I^0_{\text{odd}}} d_i. \tag{179}$$

Note that $\prod_{i \in I^0_{\text{odd}}} d_i$ is always even; this follows from the definition of a_0 in Eq. (36). The nodal geometry Y_{nodal} carries a mixed Hodge structure.

Example 12. We consider again Example 2: In $\mathbb{CP}^1$, we had the polynomials

$$P_0 = z_0, \qquad P_1 = z_1 - x_2 z_0, \qquad P_2 = z_1 + (4 - x_2) z_0,$$
$$P_3 = (z_1 + z_0)^2 - 4\left[x_2 + \frac{(1 - x_2)^2}{x_1}\right] z_0^2, \tag{180}$$

together with $I^0_{\text{even}} = \{0\}$ and $I^0_{\text{odd}} = \{1, 2, 3\}$. The nodal geometry Y_{nodal} is given by the hypersurface

$$y^2 = P_0^2 P_1 P_2 P_3 \tag{181}$$

in $\mathbb{CP}^2_{113}$. This is a nodal curve of arithmetic genus 2 and geometric genus 1.

3. The square-free geometry

The square-free geometry carries even less information. We remove the information on the even polynomials. We define the square-free geometry $Y_{\text{square-free}}$ as a hypersurface in a weighted projective space $X_{\text{square-free}} = \mathbb{CP}^{n+1}_{1,\ldots,1,h}$ by the equation

$$y^2 - F(z_0, z_1, \ldots, z_n) = 0, \tag{182}$$

where

$$F(z_0, z_1, \ldots, z_n) = \prod_{i \in I^0_{\text{odd}}} P_i(z). \tag{183}$$

The weight h of the coordinate y is now given by

$$h = \frac{1}{2} \prod_{i \in I^0_{\text{odd}}} d_i. \tag{184}$$

Example 13. We continue with Examples 2 and 11: The square-free geometry $Y_{\text{square-free}}$ is given by the hypersurface

$$y^2 = P_1 P_2 P_3 \tag{185}$$

in $\mathbb{CP}^2_{112}$. This is an elliptic curve.

4. Square-free subgeometries

By first considering all possible localizations and then, for each localization, the square-free geometry on this localization, we obtain the square-free subgeometries. These can be nontrivial, as the following example shows.

Example 14. We continue with Example 10. We consider in $\mathbb{CP}^2$ the polynomials

$$\begin{aligned} P_0 &= z_0, \qquad & P_2 &= z_1 z_2 + x_1 z_0 z_2 - z_1^2, \\ P_1 &= z_1, \qquad & P_3 &= [x_2 z_1 - (1+x_2) z_2 + (x_1 - x_2) z_0]^2 + 4 x_1 (1+x_2) z_0 z_2, \end{aligned} \tag{186}$$

together with $I^0_{\text{even}} = \{0,1,2\}$ and $I^0_{\text{odd}} = \{3\}$. On the localization $P_2 = 0$, we obtain the square-free subgeometry defined by the hypersurface

$$\begin{aligned} y^2 = {} & z_1^4 + 2(x_1 + x_2 + x_1 x_2) z_0 z_1^3 + (3x_1^2 + x_2^2 + 6 x_2 x_1^2 + x_2^2 x_1^2 - 2 x_2^2 x_1) z_0^2 z_1^2 \\ & + 2 x_1 (x_1 + x_1 x_2 - x_2)(x_1 - x_2) z_0^3 z_1 + x_1^2 (x_1 - x_2)^2 z_0^4 \end{aligned} \tag{187}$$

in $\mathbb{CP}^2_{112}$. This is an elliptic curve. This example will be discussed in more detail in Sec. V D.

IV. THE METHOD

In this section, we give a detailed description of the algorithm. We start in Sec. IV A with the modifications to the Laporta algorithm. The two main modifications are that we use a new geometric order relation and that we first recursively consider all possible localizations, in order to get preferred candidates for the master integrands. In Sec. IV B, we turn to the system of differential equations. We introduce the concept of an $F^\bullet$-compatible differential equation. We observe that the reduction algorithm from Sec. IV A gives us an intermediate basis J with an $F^\bullet$-compatible differential equation. In Sec. IV C, we prove that we can always transform an $F^\bullet$-compatible differential equation to an ε-factorized form.

A. The Laporta algorithm and the order relation

We first construct a basis of master integrands for the vector space H^n_ω. We do this by first considering all possible localizations, starting with localizations on points, followed by localizations on curves, followed by surfaces, etc., until we reach the full system. The essential idea is that master integrands on a localization will define preferred candidates for master integrands in the next higher dimension.

Technically, we do this as follows: Each localization is defined by an ordered sequence $\mathcal{P}_1, \ldots, \mathcal{P}_{r'}$, which, in turn, defines the ideal

$$\mathcal{I} = \langle \mathcal{P}_1, \ldots, \mathcal{P}_{r'} \rangle. \tag{188}$$

We denote by $V(\mathcal{I})$ the variety defined by the ideal $\mathcal{I}$. The dimension of the variety is given by

$$\dim V(\mathcal{I}) = n - r'. \tag{189}$$

We define the n'-dimensional skeleton of $V(\mathcal{I})$ (or n'-skeleton for short) to be the union of the varieties $V(\mathcal{I}')$ with $\mathcal{I} \subset \mathcal{I}'$ (where $\mathcal{I}'$ defines a sublocalization) and dimension $\dim V(\mathcal{I}') \leq n'$. As an example, we show in Fig. 8 the 1-skeleton of the geometry shown in Fig. 7.

For each localization, we consider integrands $\Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, I, P_s]$, which have a nonzero residue on $V(\mathcal{I})$ and simple poles along $\mathcal{P}_1, \ldots, \mathcal{P}_{r'}$, together with the linear relations given by the integration-by-parts identities given for each chart of $V(\mathcal{I})$ in Eq. (147), the distribution identities Eq. (96), and the cancellation identities Eq. (151). Note that we are considering fewer integrands [as the integrands are required to have simple poles and a nonzero residue on $V(\mathcal{I})$] and fewer relations [as we consider only the integration-by-parts identities in $V(\mathcal{I})$ but not the ones orthogonal to it]. We may map the differential form $\Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, I, P_s]$ (associated with the modified twist function U^{loc}) to the differential form $\Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}]$ (associated to the original twist function U) by simply forgetting about I and P_s. As before, we denote by r the number of residues of $\Psi_{\mu_0 \ldots \mu_{N_D}}[Q]$ and by o its pole order in the full space $\mathbb{CP}^n$.

For the Laporta algorithm, on each localization, we use the order relation

$$(a, w, o, |\mu|, \ldots), \tag{190}$$

where the dots stand for further criteria needed to distinguish inequivalent integrands. In practice, the dots stand for

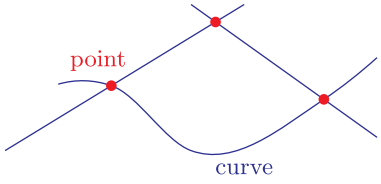


FIG. 8. The 1-skeleton of the example from Fig. 7.

ALGORITHM 1: Construction of the masters on a localization.

Input: The ordered sequence $\mathcal{P}_1, \ldots, \mathcal{P}_{r'}$.
Output: Assign a value $a < 0$ to all master integrands.
Implementation:

(1) If $\dim V(\mathcal{I}) = 0$, return the master integrands as described in Sec. III F 4 and assign $a = -2n$ to those.
(2) If $\dim V(\mathcal{I}) > 0$: Let $n' = \dim V(\mathcal{I})$ and $w' = n + r' = 2n - n'$.
  (a) Recursion: Consider first all sublocalizations of $V(\mathcal{I})$ of dimension $(n' - 1)$.
  (b) Merge procedure: Merge the master integrands from the $(n' - 1)$-skeleton of $V(\mathcal{I})$.
  (c) Find additional masters at weight w'.

an order on the indices $(\mu_0 \ldots \mu_{N_D})$ (where the sum of the indices is fixed to $|\mu|$) followed by an order on the numerator polynomial Q [where the degree of the polynomial Q is fixed according to Eq. (54)]. We use the string of integers in Eq. (190) together with lexicographic ordering; i.e., the relation $a_1 < a_2$ implies $\Psi_1 < \Psi_2$, with ties broken by w, etc. On the localization $V(I)$ with $\dim V(I) = n'$, the variable a is given by

$$a = \begin{cases} -r - n, & \Psi_{\mu_0 \ldots \mu_{N_D}}[Q] \text{ is a preferred candidate from the } (n' - 1)\text{-skeleton of } V(I), \\ 0, & \text{otherwise.} \end{cases} \tag{191}$$

The purpose of the variable a is to give preference to the master integrands from the subproblems as master integrands of the current problem. This does not imply that all master integrands from the subproblems will be master integrands of the current problem. As we have more integration-by-parts relations in the current problem, it may occur that some of the master integrands from the subproblems are eliminated.

The variable w is defined by $w = n + r$. It corresponds to the Hodge weight of the differential form. The purpose of the variable w is the following: If we run the Laporta algorithm at weight w, we already have a basis of master integrands at weight $w' > w$. These are the ones to which we have assigned $a < 0$. Therefore, the basis at higher weights w' is already complete, and we do not want to add any new master integrands with weights $w' > w$. The use of the variable w as the second-order criterion prohibits this and forces the algorithm to choose the additional master integrands from the integrands of weight w.

As the third-order criterion, we use the pole order o. We are now at a fixed weight w, and we are considering a square-free geometry. The algorithm will start to fill up the remaining master integrands by increasing pole order; i.e., the algorithm will first try to pick holomorphic differential forms, followed by the ones with pole order 1, etc. This pattern is compatible with the previously known example of the equal-mass l-loop banana integral [55]. In this example, the square-free geometry is a Calabi-Yau $(l - 1)$-fold. The algorithm will start with picking the holomorphic integrand as a master integrand. There will be one master integrand for each pole order, up to pole order $(l - 1)$.

As the fourth-order criterion, we use $|\mu|$, giving preference to objects with smaller $|\mu|$.

The top-level algorithm to determine the master integrands on a localization specified by the ideal $\mathcal{I}$ is as follows.[2]

In general, the variety $V(\mathcal{I})$ of dimension n' may have more than one sublocalization of dimension $(n' - 1)$. In step 2(b), we merge the master integrands from the sublocalizations of dimension $(n' - 1)$. This is necessary because we cannot simply take the union of the master integrands from the sublocalizations of dimension $(n' - 1)$. To see this, consider the following situation: Assume that in one sublocalization we get one master integrand Ψ_1 and that Ψ_2 can be reduced to Ψ_1 on this sublocalization. Assume further that in a second sublocalization we obtain Ψ_2 as a master integrand. In this situation, the merge of the two sublocalizations should give us only one master integrand Ψ_1, as Ψ_1 is simpler than Ψ_2 according to the ordering criteria, and Ψ_2 can be expressed in terms of Ψ_1 on the first sublocalization. Note that, in this situation, Ψ_2 can be localized on the intersection of the two sublocalizations.

Technically, we do this as follows: Let us assume that the $(n' - 1)$-skeleton of $V(\mathcal{I})$ is given by

$$V(\mathcal{I}_1) \cup V(\mathcal{I}_2) \cup \cdots \cup V(\mathcal{I}_t), \tag{192}$$

where each ideal I_i is defined by an ordered sequence $\mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'}^{(i)}, \mathcal{P}_{r'+1}^{(i)}$ of $(r' + 1)$ polynomials

[2]The case $\mathcal{I} = \varnothing$ corresponds to the full system on the maximal cut.

$$I_i = \langle \mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'}^{(i)}, \mathcal{P}_{r'+1}^{(i)} \rangle. \tag{193}$$

It may happen that different ordered sequences define the same ideal. As an example, we have

$$\begin{aligned}\langle z_1, z_2 \rangle &= \langle z_2, z_1 \rangle = \langle z_1, z_1 + z_2 \rangle = \langle z_1 + z_2, z_1 \rangle \\ &= \langle z_2, z_1 + z_2 \rangle = \langle z_1 + z_2, z_2 \rangle.\end{aligned} \tag{194}$$

If the intersection of $V(\mathcal{I}_i)$ and $V(\mathcal{I}_j)$ is nonempty and of dimension $(n' - 2)$, we denote this intersection by

$$V(\mathcal{I}_{ij}) = V(\mathcal{I}_i) \cap V(\mathcal{I}_j). \tag{195}$$

The $(n' - 2)$-skeleton of $V(\mathcal{I})$ contains the union of these intersections, which we write as a union of connected irreducible components C_k:

$$\bigcup_{(ij)} V(\mathcal{I}_{ij}) = \bigcup_k C_k. \tag{196}$$

We set

$$S_k = \{ i | C_k \subset V(I_i) \}. \tag{197}$$

S_k is the set of indices i of varieties $V(I_i)$, which intersect at C_k. As at least two varieties intersect at C_k, we have $|S_k| \geq 2$. We emphasize that we can have $|S_k| > 2$. At this stage, we have already determined a basis of C_k, coming from a localization specified by an ordered sequence $\mathcal{P}_1^{(S_k)}, \ldots, \mathcal{P}_{r'}^{(S_k)}, \mathcal{P}_{r'+1}^{(S_k)}, \mathcal{P}_{r'+2}^{(S_k)}$ of $(r' + 2)$ polynomials. We set

$$I_{S_k} = \langle \mathcal{P}_1^{(S_k)}, \ldots, \mathcal{P}_{r'}^{(S_k)}, \mathcal{P}_{r'+1}^{(S_k)}, \mathcal{P}_{r'+2}^{(S_k)} \rangle. \tag{198}$$

A basis element Ψ of C_k will define an integrand on $V(\mathcal{I}_i)$ for at least one index $i \in S_k$, but it may happen that Ψ does not define an integrand for all $i \in S_k$. It may happen that there are $j \in S_k$ such that Ψ does not have a residue on $V(\mathcal{I}_j)$. We will later see examples where this happens in Secs. V D and V E.

The master integrands of C_k provide for $i, j \in S_k$ the communication between $V(\mathcal{I}_i)$ and $V(\mathcal{I}_j)$, provided they have nonzero residues on $V(\mathcal{I}_i)$ and $V(\mathcal{I}_j)$. We obtain the following algorithm. An implicit assumption for this

ALGORITHM 2: Merge procedure.

Input: The ordered sequence $\mathcal{P}_1, \ldots, \mathcal{P}_{r'}$.
Output: Let $w' = n + r'$. Assign a value $a = 0$ to all integrands of weight $w'' > w'$, which are not masters at weight w'.
Implementation:

(1) Loop over all connected components C_k of dimension $(n' - 2)$ (as defined above).
   (a) If $\mathrm{Res}_{\mathcal{P}_1^{(S_k)}, \ldots, \mathcal{P}_{r'+2}^{(S_k)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_{S_k}, P_{s_{S_k}}]$ is a master integrand of C_k, $i, j \in S_k$ and both $\mathrm{Res}_{\mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'+1}^{(i)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_i, P_{s_i}]$ and $\mathrm{Res}_{\mathcal{P}_1^{(j)}, \ldots, \mathcal{P}_{r'+1}^{(j)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_j, P_{s_j}]$ are nonzero, add the equation

$$\mathrm{Res}_{\mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'+1}^{(i)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_i, P_{s_i}] = \mathrm{Res}_{\mathcal{P}_1^{(j)}, \ldots, \mathcal{P}_{r'+1}^{(j)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_j, P_{s_j}]. \tag{199}$$

   (b) If $\mathrm{Res}_{\mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'+1}^{(i)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_i, P_{s_i}]$ is not a master integrand of $V(\mathcal{I}_i)$, add the equation from the reductions on $V(\mathcal{I}_i)$, which expresses this differential form in terms of the masters of $V(\mathcal{I}_i)$.
(2) Run the Laporta algorithm on this linear system.
(3) Substitute $\mathrm{Res}_{\mathcal{P}_1^{(i)}, \ldots, \mathcal{P}_{r'+1}^{(i)}} \Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}, \mathcal{I}_i, P_{s_i}]$ by $\Psi_{\mu_0 \ldots \mu_{N_D}}[\tilde{Q}]$.
(4) Set $a = 0$ for all nonmasters.

ALGORITHM 3: Find additional masters.

Input: The ordered sequence $\mathcal{P}_1, \ldots, \mathcal{P}_{r'}$.
Output: Let $w' = n + r'$. Assign the value $a = -w'$ to the additional masters at weight w'.
Implementation:

(1) Generate the linear relations for this localization.
(2) Perform the Gauß elimination.
(3) Set $a = -w'$ for all masters, which have $a = 0$.
(4) Set $a = 0$ for all nonmasters.

algorithm is that the equations added in step 1(b) are generic; i.e., the system has full rank. A rank drop should throw an exception.

Remark. If $n' = \dim V(I) = 1$, the $(n'-2)$-skeleton of $V(I)$ is empty, and the merge procedure does nothing in this case.

Finally, we specify the subroutine "Find additional masters" appearing as step 2(c) in Algorithm 1.

Running Algorithm 1 with the order criterion of Eq. (190) will give a basis of the vector space H^n_ω. Using the homomorphism between the vector spaces V^n and H^n_ω from Sec. III B 3, we then obtain a basis of master integrals on the maximal cut. We may do this for all sectors on the maximal cut bottom up.

At the level of Feynman integrals and including subsectors, we then run for the reduction to master integrals the Laporta algorithm on the full system with the order relation

$$(\hat{N}, \hat{N}_{\mathrm{id}}, a, w, o, |\mu|, \ldots), \tag{200}$$

where

$$(\hat{N}, \hat{N}_{\mathrm{id}}) = \begin{cases} (N', N'_{\mathrm{id}}) & \text{if } N'_{\mathrm{id}} \text{ is the smallest sector such that } N_{\mathrm{id}} \text{ is a supersector of } N'_{\mathrm{id}}, \\ (N, N_{\mathrm{id}}) & \text{otherwise.} \end{cases}$$

This will yield a basis of master integrals $J = (J_1, J_2, \ldots)^T$. The integers a, w, o, and $|\mu|$ for Feynman integrals are determined from the images under the homomorphism $\hat{\imath}: A^n_{\mathrm{all}} \to \Omega^n_\omega$. This treatment completely avoids relative twisted cohomology. Note that the determination of the integers a, w, o, and $|\mu|$ needs to be done only once at initialization time and only on the maximal cut. No computation beyond the maximal cut is required. The dots stand for further criteria needed to distinguish inequivalent integrals in the full system. More concretely, the dots stand for an order on the indices $(\mu_0 \ldots \mu_{N_D})$, followed by an order on the numerator polynomial Q, followed by an order relation on $I_{\nu_1 \ldots \nu_{N_E}}$. With the exception of the last criterion, all other order variables are obtained from the differential form $\Psi = \hat{\imath}(I_{\nu_1 \ldots \nu_{N_E}})$. The last order criterion is needed to distinguish inequivalent integrals, which have the same image $\Psi = \hat{\imath}(I_{\nu_1 \ldots \nu_{N_E}})$. This can be due to subsectors (which are not seen on the maximal cut) or missing ISPs. The last order criterion enters also as an order criterion on A^n_{all} in Sec. III B 3. For the last criterion, a standard order relation on Feynman integrals can be used.

This order relation performs significantly better than the standard order relation. We observe that it eliminates to a large extent the occurrence of spurious polynomials in the denominator. This holds beyond the maximal cut, and the most drastic improvements are in the coefficients of the simplest master integrals in the reduction of the most complicated sectors.

B. Differential equations, step 1: The basis J

In this section, we consider a sector of the basis J, obtained as described in Sec. IV A, on the maximal cut. We are, in particular, interested in the differential equation on the maximal cut. We denote by $\Psi = (\Psi_1, \ldots, \Psi_{N_F})^T$ the basis of master integrands in H^n_ω. In all examples we tested, we observed that if $\Psi_i \in \mathrm{Gr}^{n-|\mu|_i}_{F_{\mathrm{comb}}} \Omega^n_\omega$ and $\Psi_j \in \mathrm{Gr}^{n-|\mu|_j}_{F_{\mathrm{comb}}} \Omega^n_\omega$, then

$$d_B J(\varepsilon, x) = A(\varepsilon, x) J(\varepsilon, x), \tag{201}$$

with

$$A_{ij}(\varepsilon, x) = \sum_{k=-(|\mu|_i - |\mu|_j)}^{1} \varepsilon^k A^{(k)}_{ij}(x). \tag{202}$$

We call a differential equation which satisfies Eq. (202) an $F^\bullet$-compatible differential equation for the filtration $F^\bullet_{\mathrm{comb}}$. An $F^\bullet$-compatible differential equation implies Griffiths transversality. It is, however, a stronger statement, as it requires the differential equation to be in Laurent polynomial form with restrictions on the occurring powers of ε. We recall that Griffiths transversality [104,105] is the statement that $A_{ij} = 0$ for $|\mu|_j > |\mu|_i + 1$.

It is worth elaborating on this observation: We may split the linear relations among the objects $\Psi_{\mu_0 \ldots \mu_{N_D}}[Q]$ into a set A_1, consisting of the integration-by-parts identities and the distribution identities, and a set A_2, consisting of the cancellation identities. We may run the Laporta algorithm first on the set A_1. For this task, we may set $\varepsilon = 1$. This is a significant efficiency improvement, as we have one variable less. This is possible, because in the integration-by-parts identities Eq. (87) and the distribution identities Eq. (96) the explicit ε factors are synchronized with $|\mu|$. At the end, we may restore the ε dependence from $|\mu|$ as follows: Suppose we get for $\varepsilon = 1$ a relation like

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q] = c_1\Psi_{\mu'_0\ldots\mu'_{N_D}}[Q'] + c_2\Psi_{\mu''_0\ldots\mu''_{N_D}}[Q'']. \tag{203}$$

Restoring ε, we obtain

$$\Psi_{\mu_0\ldots\mu_{N_D}}[Q] = c_1\varepsilon^{|\mu'|-|\mu|}\Psi_{\mu'_0\ldots\mu'_{N_D}}[Q'] + c_2\varepsilon^{|\mu''|-|\mu|}\Psi_{\mu''_0\ldots\mu''_{N_D}}[Q'']. \tag{204}$$

The same argument also shows that, in reducing the system A_1, the ε dependence of the coefficients is always monomial.

We then merge the reduced system A_1 with the system A_2. The system A_2 consists entirely of cancellation identities, i.e., reduction identities for objects $\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}[P_j\cdot Q]$. Some of these objects are masters in the reduced system A_1. Substituting the cancellation identities for these objects will reduce them. The ε dependence of the coefficients in the linear system will no longer be monomial but a Laurent polynomial in ε. This is due to the factor in Eq. (99).

Other objects $\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}[P_j\cdot Q]$ will not be masters in the reduced system A_1. In this case, there are two possibilities: Either the integration-by-parts identities reduce the entire cancellation identity for $\Psi_{\mu_0\ldots(\mu_j+1)\ldots\mu_{N_D}}[P_j\cdot Q]$ to $0=0$—this is unproblematic—or the second possibility is that this does not happen. In this case, the cancellation identity will eliminate a lower object. However, as the lower objects are multiplied by the linear factor of Eq. (99), we no longer can guarantee the Laurent polynomial form of the ε dependence of the coefficients, and, in general, the ε dependence of the coefficients will be rational in ε.

For the differential equation related to Eq. (202), we observe that all reductions required to express d_BJ in terms of the masters J are Laurent polynomials in ε. The reductions of other objects may (and actually do) involve rational functions in ε.

C. Differential equations, step 2: The basis K

In this section, we show that we may always construct an ε-factorized differential equation from an $F^\bullet$-compatible differential equation. We remark that, in this section, we require the differential equation only to be compatible with some filtration $F^\bullet$; it does not have to be the $F^\bullet_{\text{comb}}$ filtration.

Let $J=(J_1,\ldots,J_{N_F})^T$ be a basis with an $F^\bullet$-compatible differential equation and assume that J is ordered according to the $F^\bullet$ filtration, i.e., $J_1\in F^{p_{\max}}V^n$ and $J_{N_F}\in F^{p_{\min}}V^n$. The assumptions imply that the differential equation for the basis J reads

$$d_BJ(\varepsilon,x) = A(\varepsilon,x)J(\varepsilon,x), \qquad A(\varepsilon,x)=\sum_{k=-n}^{1}\varepsilon^kA^{(k)}(x). \tag{205}$$

The matrix A has then a block structure induced by the $F^\bullet$ filtration. It will be convenient to organize the matrix A as

$$A=\sum_{k=-n}^{1}B^{(k)}(x), \tag{206}$$

with $B^{(1)}(x)=\varepsilon A^{(1)}(x)$. For $k<1$, the matrices $B^{(k)}(x)$ are lower block-triangular. The blocks on the lower jth block subdiagonal are given by the terms of the order of ε^{n+k-j} of the corresponding blocks of A. We say that a term is of B-order k if the term appears in $B^{(k)}$.

To give an example, consider the matrix A with three 1×1 blocks:

$$A=\begin{pmatrix} A_{11}^{(0)}+\varepsilon A_{11}^{(1)} & \varepsilon A_{12}^{(1)} & 0 \\ \frac{1}{\varepsilon}A_{21}^{(-1)}+A_{21}^{(0)}+\varepsilon A_{21}^{(1)} & A_{22}^{(0)}+\varepsilon A_{22}^{(1)} & \varepsilon A_{23}^{(1)} \\ \frac{1}{\varepsilon^2}A_{31}^{(-2)}+\frac{1}{\varepsilon}A_{31}^{(-1)}+A_{31}^{(0)}+\varepsilon A_{31}^{(1)} & \frac{1}{\varepsilon}A_{32}^{(-1)}+A_{32}^{(0)}+\varepsilon A_{32}^{(1)} & A_{33}^{(0)}+\varepsilon A_{33}^{(1)} \end{pmatrix}. \tag{207}$$

Then,

$$B^{(1)} = \begin{pmatrix} \varepsilon A_{13}^{(1)} & \varepsilon A_{12}^{(1)} & 0 \\ \varepsilon A_{21}^{(1)} & \varepsilon A_{22}^{(1)} & \varepsilon A_{23}^{(1)} \\ \varepsilon A_{31}^{(1)} & \varepsilon A_{32}^{(1)} & \varepsilon A_{33}^{(1)} \end{pmatrix},$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{31}^{(0)} & 0 & 0 \end{pmatrix},$$

$$B^{(-1)} = \begin{pmatrix} 0 & 0 & 0 \\ A_{21}^{(0)} & 0 & 0 \\ \frac{1}{\varepsilon} A_{31}^{(-1)} & A_{32}^{(0)} & 0 \end{pmatrix},$$

$$B^{(-2)} = \begin{pmatrix} A_{13}^{(0)} & 0 & 0 \\ \frac{1}{\varepsilon} A_{21}^{(-1)} & A_{22}^{(0)} & 0 \\ \frac{1}{\varepsilon^2} A_{31}^{(-2)} & \frac{1}{\varepsilon} A_{32}^{(-1)} & A_{33}^{(0)} \end{pmatrix}. \tag{208}$$

The differential equation is in ε-factorized form if all terms of B-order $k < 1$ are absent. Below, we present an algorithm which iteratively first removes terms of B-order $(-n)$, then terms of B-order $(-n+1)$, etc., until we arrive at an ε-factorized form. We consider a transformation

$$K = R_2^{-1} J, \tag{209}$$

such that the new basis K satisfies an ε-factorized differential equation (on the maximal cut). It is convenient to use the inverse matrix R_2^{-1} in Eq. (209). Obviously, we have $J = R_2 K$. The matrix R_2 is given as

$$R_2 = R_2^{(-n)} R_2^{(-n+1)} \cdots R_2^{(-1)} R_2^{(0)}, \tag{210}$$

where $R_2^{(k)}$ removes terms of B-order k. All matrices $R_2^{(k)}$ are lower block-triangular. The matrix $R_2^{(-n)}$ is of B-order $(-n)$; the matrices $R_2^{(k)}$ with $-n < k \leq 0$ are given by

$$R_2^{(k)} = \mathbf{1} + T_2^{(k)}, \tag{211}$$

where $T_2^{(k)}$ is of B-order k and $\mathbf{1}$ denotes the $N_F \times N_F$ unit matrix.

Lemma 1 (compatibility of the B order with matrix multiplication). Let A be a matrix with terms of B-order $\{k, k+1, \ldots, 1\}$ (with $k \geq -n$) and $R_2^{(l)}$ with $l \in \{k, k+1, \ldots, 0\}$ as above. Then, the products $AR_2^{(l)}$ and $R_2^{(l)}A$ have again only terms of B-order $\{k, k+1, \ldots, 1\}$.

Proof. The multiplication with the unit matrix 1 does not change the B order of A; hence, it is sufficient to consider the multiplication of A with $R_2^{(-n)}, T_2^{(-n+1)}, \ldots, T_2^{(0)}$. Thus, we need to show that the multiplication of a matrix A with terms of B-order $\{k, k+1, \ldots, 1\}$ with a matrix X of B-order l (with $l \in \{k, k+1, \ldots, 0\}$) results again in a matrix with terms of B-order $\{k, k+1, \ldots, 1\}$. This follows directly from matrix multiplication. We index the blocks of the matrices A and X by i and j. The ε order of the blocks is

$$A_{ij} \in \begin{cases} \{\mathcal{O}(\varepsilon^{\min(1, n+k+j-i)}), \ldots, \mathcal{O}(\varepsilon^1)\}, & j - i \leq 1, \\ 0, & j - i > 1, \end{cases}$$
$$X_{ij} = \begin{cases} \mathcal{O}(\varepsilon^{n+l+j-i}), & n + l + j - i \leq 0, \\ 0, & n + l + j - i > 0. \end{cases} \tag{212}$$

We have

$$\sum_{m=1}^{n} A_{im} X_{mj} = \sum_{m=\max(1, n+l+j)}^{\min(n, i+1)} A_{im} X_{mj} \in \{\mathcal{O}(\varepsilon^{\min(1, n+l+n+k+j-i)}), \ldots, \mathcal{O}(\varepsilon^1)\}. \tag{213}$$

As $n + l \geq 0$, we have shown the claim for $AR_2^{(l)}$. In a similar way, we have

$$\sum_{m=1}^{n} X_{im} A_{mj} = \sum_{m=\max(1, j-1)}^{\min(n, i-n-l)} X_{im} A_{mj} \in \{\mathcal{O}(\varepsilon^{\min(1, n+l+n+k+j-i)}), \ldots, \mathcal{O}(\varepsilon^1)\}. \tag{214}$$

This shows the claim for $R_2^{(l)}A$. ∎

Corollary 1 (multiplication of matrices of B-order $k \leq 0$). Let S be a matrix of B-order k_1 and let T be a matrix of B-order k_2. If $k_1, k_2 \in \{-n, \ldots, 0\}$, then the product $S \cdot T$ has B-order $(n + k_1 + k_2)$.

Proof. The proof follows along the lines of Lemma 1. ∎

Lemma 2 (inverse matrix). Let $R_2^{(k)}$ with $k \in \{-n, \ldots, 0\}$ as above. If $k = -n$, the inverse matrix $(R_2^{(-n)})^{-1}$ is again of B-order $(-n)$; otherwise, for $k \in \{-n+1, \ldots, 0\}$ the inverse matrix can be written as

$$\left(R_2^{(k)}\right)^{-1} = \mathbf{1} + \sum_{j=k}^{0} S_2^{(j)}, \tag{215}$$

with $S_2^{(j)}$ of B-order j.

Proof. We first consider the case $k = -n$. We write $R_2^{(-n)} = D_2^{(-n)} + N_2^{(-n)}$, where $D_2^{(-n)}$ is block-diagonal and invertible. The matrix $N_2^{(-n)}$ is nilpotent, with $(N_2^{(-n)})^{n+1} = \mathbf{0}$. We have

$$\left(R_2^{(-n)}\right)^{-1} = \sum_{j=0}^{n} \left(-\left(D_2^{(-n)}\right)^{-1} N_2^{(-n)}\right)^j \left(D_2^{(-n)}\right)^{-1}. \tag{216}$$

Multiplication with the block-diagonal matrix $(D_2^{(-n)})^{-1}$ does not change the B order. The expression $(-(D_2^{(-n)})^{-1} N_2^{(-n)})^j$ is again of B-order $(-n)$.

We now consider the case $k \in \{-n+1, \ldots, 0\}$. We have $R_2^{(k)} = \mathbf{1} + T_2^{(k)}$, where $T_2^{(k)}$ is nilpotent. The inverse is given by

$$\left(R_2^{(k)}\right)^{-1} = \mathbf{1} + \sum_{j=1}^{\lfloor \frac{n}{n+k} \rfloor} \left(-T_2^{(k)}\right)^j. \tag{217}$$

The B order of $(-T_2^{(k)})^j$ is $j(n+k) - n$. ∎

Corollary 2 (transformation by $R_2^{(k)}$). Let A be a matrix with terms of B-order $\{k, k+1, \ldots, 1\}$ and $R_2^{(k)}$ as above. Then,

$$\left(R_2^{(k)}\right)^{-1} A R_2^{(k)} - \left(R_2^{(k)}\right)^{-1} d_B R_2^{(k)} \tag{218}$$

has only terms of B-order $\{k, k+1, \ldots, 1\}$.

Proof. This is a direct consequence of Lemmas 1 and 2. ∎

Theorem 1 (ε-factorization). Let $J = (J_1, \ldots, J_{N_F})^T$ be a basis for a given sector, which satisfies (on the maximal cut) a differential equation compatible with a filtration $F^\bullet$. Then there exists a transformation $K = R_2^{-1} J$ with $R_2 = R_2^{(-n)} R_2^{(-n+1)} \cdots R_2^{(-1)} R_2^{(0)}$, where the matrices $R_2^{(k)}$ are in the form as above, such that the differential equation is in ε-factorized form.

Proof. We construct the matrices $R_2^{(k)}$ iteratively, starting from $k = -n$ and ending with $k = 0$. We set $\tilde{A}^{(-n)} = A$ and

$$\tilde{A}^{(k+1)} = (R_2^{(k)})^{-1} \tilde{A}^{(k)} R_2^{(k)} - (R_2^{(k)})^{-1} d_B R_2^{(k)}. \tag{219}$$

The matrix $R_2^{(k)}$ is determined by

$$[(R_2^{(k)})^{-1} \tilde{A}^{(k)} R_2^{(k)} - (R_2^{(k)})^{-1} d_B R_2^{(k)}]|_k = 0, \tag{220}$$

where $|_k$ indicates that only terms of B-order k are taken. Equation (220) defines an ε-independent system of first-order differential equations for the unknown functions in the ansatz for $R_2^{(k)}$. There are as many equations as there are unknown functions. Equation (220) also ensures that $\tilde{A}^{(k+1)}$ has only terms of B-order $\{k+1, \ldots, 1\}$. Thus, in every iteration step, we improve the B order. After transformation with $R_2^{(0)}$, the matrix $\tilde{A}^{(1)}$ has only terms of B order 1. ∎

A few comments are in order: At first sight, it may seem that Eq. (220) yields a nonlinear system of differential equations. This is not the case. In fact, Eq. (220) decomposes into a block-triangular system of first-order linear inhomogeneous differential equations. In practice, we solve this system as follows: We first determine the functions appearing in $R_2^{(-n)}$, followed by the functions appearing in $R_2^{(-n+1)}$. We continue in this way until we reach $R_2^{(0)}$. The differential equations for the functions appearing in $R_2^{(k)}$ will not depend on the functions appearing in $R_2^{(j)}$ with $j > k$. Furthermore, we may assume that all functions appearing in $R_2^{(j)}$ with $j < k$ are already known at this stage. In addition, the differential equations for the functions appearing in $R_2^{(k)}$ will form a block-triangular system according to the block columns. The functions appearing in block column l will not be influenced by functions from the block columns $l' > l$, and functions from block columns $l' < l$ can be considered to be known at this stage. The net result is that the system decomposes into smaller systems of first-order linear inhomogeneous differential equations.

Note that we cannot expect that the system uncouples into single first-order linear inhomogeneous differential equations. For example, in the case of an elliptic Feynman integral, we will obtain the differential equations describing the variation of the periods of the elliptic curve with the kinematic variables. This will always be a coupled system of two first-order differential equations. Of course, the variation along a given path can be converted (if desired) into a single second-order differential equation.

We may simplify Eq. (220), depending on whether we consider the case $k = -n$ or the case $k \in \{-n+1, \ldots, 0\}$. We have the following two lemmas.

Lemma 3 (differential equation for the rotation matrix $R_2^{(-n)}$). The rotation matrix $R_2^{(-n)}$ is determined from the differential equation

$$d_B R_2^{(-n)} = B^{(-n)} R_2^{(-n)} + R_2^{(-n)} \left[\left(R_2^{(-n)}\right)^{-1} B^{(1)} R_2^{(-n)}\right]\Big|_{(-n)}. \tag{221}$$

Proof. The matrices $(R_2^{(-n)})^{-1}$ and $dR_2^{(-n)}$ are both of B-order $(-n)$; hence, we may drop the $|_{-n}$ prescription in the second term of Eq. (220). We have

$$\tilde{A}^{(-n)} = A = \sum_{k=-n}^{1} B^{(k)}. \tag{222}$$

Only the terms with $B^{(-n)}$ and $B^{(1)}$ will contribute. In the term with $B^{(-n)}$, we may again drop the $|_{-n}$ prescription. ∎

Lemma 4 (differential equation for the rotation matrix $R_2^{(k)}$ for $k > -n$). The rotation matrix $R_2^{(k)}$ for $k \in \{-n+1, \ldots, 0\}$ is determined from the differential equation

$$d_B R_2^{(k)} = \left[(R_2^{(k)})^{-1} \tilde{A}^{(k)} R_2^{(k)} \right] \Big|_k. \tag{223}$$

Proof. We consider the term $(R_2^{(k)})^{-1} d_B R_2^{(k)}|_k$. The matrix $d_B R_2^{(k)}$ is of B-order k. The inverse matrix $(R_2^{(k)})^{-1}$ has the structure as given in Eq. (215). With Corollary 2, it follows that only the identity matrix in Eq. (215) contributes to the final B-order k. ∎

Example 15. We illustrate the block-triangular structure of the differential equations for an example. We consider the case where we have two nontrivial parts in the filtration

$$\varnothing = F^2V \subseteq F^1V \subseteq F^0V = V. \tag{224}$$

We set

$$d_1 = \dim \mathrm{Gr}_F^1 V, \qquad d_2 = \dim \mathrm{Gr}_F^0 V. \tag{225}$$

The matrix A appearing in the differential equation $d_B J = AJ$ is then of the form

$$A = \begin{pmatrix} A_{11}^{(0)} + \varepsilon A_{11}^{(1)} & \varepsilon A_{12}^{(1)} \\ \frac{1}{\varepsilon} A_{21}^{(-1)} + A_{21}^{(0)} + \varepsilon A_{21}^{(1)} & A_{22}^{(0)} + \varepsilon A_{22}^{(1)} \end{pmatrix}, \tag{226}$$

where the entries are, in general, matrices. We have $\dim A_{11}^{(j)} = d_1$ and $\dim A_{22}^{(j)} = d_2$. The off-diagonal entries have the dimensions dictated by the diagonal blocks. We construct a transformation

$$R_2 = R_2^{(-1)} R_2^{(0)}, \tag{227}$$

leading to the ε-factorized basis $K = R_2^{-1} J$. The ansatz for $R_2^{(-1)}$ reads

$$R_2^{(-1)} = \begin{pmatrix} R_{11}^{(0)} & 0 \\ \frac{1}{\varepsilon} R_{21}^{(-1)} & R_{22}^{(0)} \end{pmatrix}. \tag{228}$$

For a basis transformation, the determinant of $R_2^{(-1)}$ has to be nonzero; this implies that $R_{11}^{(0)}$ and $R_{22}^{(0)}$ are invertible matrices.

Requiring that terms of B-order (-1) vanish gives us three equations. These equations group into $3 = 2 + 1$ as follows—the first group of two equations involves only $R_{11}^{(0)}$ and $R_{21}^{(0)}$:

$$\begin{aligned} d_B R_{11}^{(0)} &= A_{11}^{(0)} R_{11}^{(0)} + A_{12}^{(1)} R_{21}^{(-1)}, \\ d_B R_{21}^{(-1)} &= A_{21}^{(-1)} R_{11}^{(0)} + A_{22}^{(0)} R_{21}^{(-1)}. \end{aligned} \tag{229}$$

The second set involves, in addition, $R_{22}^{(0)}$:

$$d_B R_{22}^{(0)} = A_{22}^{(0)} R_{22}^{(0)} - R_{21}^{(-1)} \left(R_{11}^{(0)} \right)^{-1} A_{12}^{(1)} R_{22}^{(0)}. \tag{230}$$

We then turn to $R_2^{(0)}$. The ansatz for $R_2^{(0)}$ reads

$$R_2^{(0)} = \begin{pmatrix} 1 & 0 \\ R_{21}^{(0)} & 1 \end{pmatrix}. \tag{231}$$

Requiring that terms of B-order 0 vanish gives us one equation:

$$\begin{aligned} d_B R_{21}^{(0)} = \left(R_{22}^{(0)} \right)^{-1} \Big(& A_{21}^{(0)} R_{11}^{(0)} + A_{22}^{(1)} R_{21}^{(-1)} \\ & - R_{21}^{(-1)} (R_{11}^{(0)})^{-1} A_{11}^{(1)} R_{11}^{(0)} \Big). \end{aligned} \tag{232}$$

A similar, but slightly more complicated, example with three nontrivial parts in the filtration can be found in Appendix B in Ref. [63].

This completes the construction of the ε-factorized differential equation on the maximal cut. The extension beyond the maximal cut is straightforward: Any offending term is strictly lower block-triangular and can be removed with an ansatz similar to R_2. We do not need to assume that the ε dependence of the terms in the nondiagonal blocks is given by a Laurent polynomial in ε. The algorithm can handle a rational dependence in ε: One first performs a partial fraction decomposition in ε and then removes any term which is not proportional to ε^1.

Example 16. We illustrate this with a simple example consisting of two sectors. Assume that the differential equation $d_B J = AJ$ is of the form

$$A = \begin{pmatrix} \varepsilon A_{11}^{(1)} & 0 \\ \frac{1}{1+\varepsilon} X_{21} + A_{21}^{(0)} + \varepsilon A_{21}^{(1)} & \varepsilon A_{22}^{(1)} \end{pmatrix}. \tag{233}$$

We first remove X_{21} with the transformation

$$R = \begin{pmatrix} 1 & 0 \\ \frac{1}{1+\varepsilon} R_{21} & 1 \end{pmatrix}. \tag{234}$$

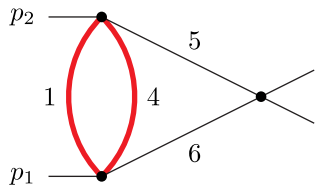


FIG. 9. The Feynman graph for sector 57. Massive propagators are indicated by red lines.

R_{21} is determined by the ε-independent differential equation

$$d_B R_{21} = X_{21} - A^{(1)}_{22} R_{21} + R_{21} A^{(1)}_{11}. \tag{235}$$

After this transformation, the lower-left entry is linear in ε. The ε^0 term is then removed as in Example 15.

D. The top weight master integrals

The master integrals in $V^{(n,n)}$ are particularly simple. They have Hodge weight $w = 2n$ and correspond to zero-dimensional geometries (i.e., a set of points). In particular, all Feynman integrals which evaluate to multiple polylogarithms are of this type. The leading singularities [106,107] of the top weight master integrals are algebraic. Normalizing these master integrals by their leading singularities reduces the number of auxiliary functions in step 2 of our algorithm.

Example 17. We consider sector 57 in the family of Feynman integrals defined by the inverse propagators

$$\begin{aligned}
&\sigma_1 = -(k_1 - p_1)^2 + m^2, && \sigma_2 = -(k_1 - p_{12})^2, && \sigma_3 = -k_1^2,\\
&\sigma_4 = -(k_1 + k_2)^2 + m^2, && \sigma_5 = -(k_2 + p_{12})^2, && \sigma_6 = -k_2^2,\\
&\sigma_7 = -(k_2 + p_{123})^2 + m^2, && \sigma_8 = -(k_1 - p_{13})^2, && \sigma_9 = -(k_2 + p_{13})^2.
\end{aligned} \tag{236}$$

The Feynman graph for sector 57 is shown in Fig. 9. With $x_1 = m^2/s$ and $z_1 = (\sigma_9 - \sigma_7 - t + m^2)/s$, we obtain the twist function

$$U = P_0^{2\varepsilon} P_1^{-\varepsilon} P_2^{-\frac{1}{2}-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon}, \tag{237}$$

where the Baikov polynomials are given by

$$P_0 = z_0, \quad P_1 = z_1 - z_0, \quad P_2 = z_1, \quad P_3 = z_1 + 4x_1 z_0. \tag{238}$$

We further have

$$C_{\text{Baikov}} = -\frac{8 i e^{2\varepsilon\gamma_{\text{E}}} \pi^3}{\Gamma(1-2\varepsilon)}, \qquad C_{\text{abs}} = \varepsilon^3. \tag{239}$$

This sector has two master integrals, which reside in $V^{(1,1)}$. We set

$$J_1 = 2\varepsilon^3 I_{200111000}, \qquad J_2 = -\varepsilon^2 I_{200112000}. \tag{240}$$

We have

$$\iota(J_1) = \Psi_{1000}[1], \qquad \iota(J_2) = \Psi_{0100}[1]. \tag{241}$$

The differential equation in the basis $J = (J_1, J_2)$ is of the form

$$d_B J = (A^{(0)} + \varepsilon A^{(1)}) J, \qquad A^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & A^{(0)}_{22} \end{pmatrix}. \tag{242}$$

The differential form $\Psi^0_{1000}[1]$ has constant leading singularities

$$\text{Res}_{P_0} \Psi^0_{1000}[1] = C, \qquad C = -16 i \frac{e^{2\varepsilon\gamma_{\text{E}}} \pi^3 \varepsilon^3}{\Gamma(1-2\varepsilon)}, \tag{243}$$

where C is a constant of uniform transcendental weight zero. However, the differential form $\Psi^0_{0100}[1]$ does not have constant leading singularities. We find

$$\text{Res}_{P_1} \Psi^0_{0100}[1] = -\frac{1}{2\sqrt{1+4x_1}} C. \tag{244}$$

It is well known that an ε-factorized form can be achieved by normalizing J_2 with $\sqrt{1+4x_1}$; e.g., setting

$$J_1 = K_1, \qquad J_2 = \frac{1}{\sqrt{1+4x_1}} K_2 \tag{245}$$

will lead on the maximal cut to an ε-factorized differential equation in the basis K. The same is achieved by our algorithm. We start from the ansatz

$$J = R^{(0)}_2 K, \qquad R^{(0)}_2 = \begin{pmatrix} R^{(0)}_{11} & R^{(0)}_{12} \\ R^{(0)}_{21} & R^{(0)}_{22} \end{pmatrix}. \tag{246}$$

The functions in $R^{(0)}_2$ are determined by the differential equation

$$d_B R^{(0)}_2 = A^{(0)} R^{(0)}_2. \tag{247}$$

We immediately see that $R^{(0)}_{11}$ and $R^{(0)}_{12}$ are constant, and we may choose $R^{(0)}_{11} = 1$ and $R^{(0)}_{12} = 0$. $R^{(0)}_{21}$ and $R^{(0)}_{22}$ satisfy the differential equation

$$d_B R^{(0)}_{2i} = A^{(0)}_{22} R^{(0)}_{2i}, \qquad i \in \{1,2\}. \tag{248}$$

For $R^{(0)}_{21}$, we may pick the trivial solution $R^{(0)}_{21} = 0$; however, we cannot do this for $R^{(0)}_{22}$, as we must have $\det R^{(0)}_2 \neq 0$. Thus, we seek a nontrivial solution to the differential equation

$$\frac{d}{dx_1} R^{(0)}_{22} = -\frac{2}{1+4x_1} R^{(0)}_{22}. \tag{249}$$

This differential equation has the solution

$$R^{(0)}_{22} = \frac{C'}{\sqrt{1+4x_1}}, \tag{250}$$

where C' is an integration constant. We see that we recover exactly the basis change of Eq. (245).

V. EXAMPLES

In this section, we illustrate the algorithm with several examples. We start with a trivial example, the one-loop massless box integral in Sec. V A. All one-loop integrals correspond on the maximal cut to differential 0-forms in $\mathbb{CP}^0$ and are, therefore, trivial. Starting from two loops, we get on the maximal cut Baikov representations of dimension larger than zero. We first discuss in Sec. V B the massless planar double-box integral, followed by the massless planar pentabox integral in Sec. V C. These are still rather simple integrals, as they both evaluate to multiple polylogarithms. In the language of our algorithm, this means that all integrands can be localized on points. In order to appreciate the capabilities of our algorithm, we have to go beyond multiple polylogarithms. In Secs. V D and V E, we discuss two examples of Feynman integrals associated with elliptic curves. In both examples, we start with differential forms on $\mathbb{CP}^2$, and, in both cases, the differential one-forms associated with the elliptic curves live on a localization. In Sec. V F, we discuss an example associated to a Calabi-Yau threefold.

In the examples, we mainly focus on step 1 of the algorithm, as this part contains most of the new features. Part 2 of the algorithm is mechanical—we provide for this part one example in Sec. V E, and other examples can be found in Refs. [63,70].

A. The one-loop box

We start with a trivial example, the massless one-loop box integral. There is one master integral in the top sector. The inverse propagators are

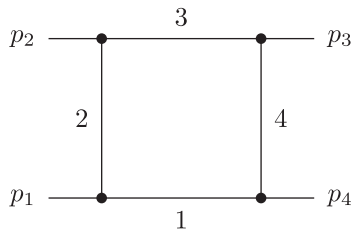


FIG. 10. The one-loop box graph.

$$\sigma_1 = -k_1^2, \qquad \sigma_2 = -(k_1 - p_1)^2,$$
$$\sigma_3 = -(k_1 - p_{12})^2, \qquad \sigma_4 = -(k_1 - p_{123})^2, \tag{251}$$

with $p_{ij} = p_i + p_j$, $p_{ijk} = p_i + p_j + p_k$, and likewise for p_{ijkl} if any, which will be used for later examples as well. We set $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, $x = s/t$, and the arbitrary scale $M^2 = t$. The graph is shown in Fig. 10. The Baikov representation on the maximal cut in $D = 4 - 2\epsilon$ space-time dimensions reads

$$e^{\epsilon\gamma_E} t^{2+\epsilon} \int_{C_{\text{maxcut}}} \frac{d^D k_1}{i\pi^{2-\epsilon}} \frac{1}{\prod_{j=1}^4 \sigma_j} = \frac{2^{3+2\epsilon}\pi^{\frac{5}{2}} e^{\epsilon\gamma_E}}{\Gamma(\frac{1}{2}-\epsilon)} x^{-1-\epsilon}(1+x)^{\epsilon}. \tag{252}$$

This is a zero-dimensional Baikov representation. We have

$$C_{\text{Baikov}} = \frac{2^{3+2\epsilon}\pi^{\frac{5}{2}} e^{\epsilon\gamma_E}}{\Gamma(\frac{1}{2}-\epsilon)} x^{-1-\epsilon}(1+x)^{\epsilon}. \tag{253}$$

Setting

$$C_{\text{abs}} = \epsilon^2 x, \tag{254}$$

one easily verifies that $C_{\text{Baikov}} \cdot C_{\text{abs}}$ is pure of transcendental weight zero.

In twisted cohomology, we consider $\mathbb{CP}^0$. This is a point. We have only one polynomial $P_0 = z_0$, and the twist is simply $U = 1$. Following Sec. III F 4, there is one master integrand

$$\Psi_1[1] = -\frac{2^{3+2\epsilon}\pi^{\frac{5}{2}} e^{\epsilon\gamma_E}\epsilon^2}{\Gamma(\frac{1}{2}-\epsilon)} x^{-\epsilon}(1+x)^{\epsilon}. \tag{255}$$

In this example, the integrand is a 0-form. The differential equation for $\Psi_1[1]$ is already in ϵ-factorized form:

$$\frac{d}{dx}\Psi_1[1] = \epsilon\left(\frac{1}{1+x} - \frac{1}{x}\right)\Psi_1[1]. \tag{256}$$

For this example, we have an isomorphism $\iota\colon V^0 \to H^0_\omega$, given by

$$\iota(-\epsilon^2 x I_{1111}) = \Psi_1[1]. \tag{257}$$

It is easily verified that $(-\varepsilon^2 x I_{1111})$ is a pure master integral of transcendental weight zero and satisfies an ε-factorized differential equation, including subsectors.

B. The two-loop double box

We continue with a simple example, the two-loop planar massless double-box integral. This is now an example with two master integrals in the top sector. The inverse propagators are

$$\begin{aligned}
&\sigma_1 = -(k_1 - p_1)^2, && \sigma_2 = -(k_1 - p_{12})^2, && \sigma_3 = -k_1^2,\\
&\sigma_4 = -(k_1 + k_2)^2, && \sigma_5 = -(k_2 + p_{12})^2, && \sigma_6 = -k_2^2,\\
&\sigma_7 = -(k_2 + p_{123})^2, && \sigma_8 = -(k_1 - p_{13})^2, && \sigma_9 = -(k_2 + p_{13})^2.
\end{aligned} \tag{258}$$

The kinematics is as in the previous example. The graph is shown in Fig. 11. We consider the loop-by-loop Baikov representation, where the inner loop is formed by the inverse propagators σ_4, σ_5, σ_6, and σ_7. With

$$r = \frac{t^5 \sigma_8 (\sigma_8 - t)}{\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7} \tag{259}$$

and $z_1 = \sigma_8/t$, we obtain a Baikov representation with the minimal twist function:

$$e^{2\varepsilon\gamma_E} t^{-4+2\varepsilon} \int_{\mathcal{C}_{\text{maxcut}}} \frac{d^D k_1}{i\pi^{2-\varepsilon}} \frac{d^D k_2}{i\pi^{2-\varepsilon}} r(\sigma) = \frac{2^{6+4\varepsilon}\pi^5 e^{2\varepsilon\gamma_E}}{\Gamma(\frac{1}{2}-\varepsilon)^2} x^{-2-2\varepsilon}(1+x)^{\varepsilon} \int \frac{dz_1}{(2\pi i)} z_1^{-2\varepsilon}(z_1-1)^{-\varepsilon}(z_1-x-1)^{\varepsilon}. \tag{260}$$

We read off

$$C_{\text{Baikov}} = \frac{2^{6+4\varepsilon}\pi^5 e^{2\varepsilon\gamma_E}}{\Gamma(\frac{1}{2}-\varepsilon)^2} x^{-2-2\varepsilon}(1+x)^{\varepsilon}. \tag{261}$$

Setting

$$C_{\text{abs}} = \varepsilon^4 x^2 \tag{262}$$

ensures that $C_{\text{Baikov}} \cdot C_{\text{abs}}$ is pure of transcendental weight zero. In projective space, the minimal twist function reads

$$U = z_0^{2\varepsilon} z_1^{-2\varepsilon}(z_1 - z_0)^{-\varepsilon}[z_1 - (x+1)z_0]^{\varepsilon}. \tag{263}$$

We set

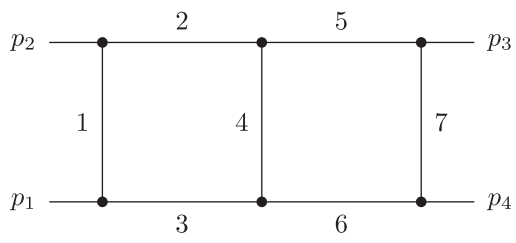


FIG. 11. The two-loop double-box graph.

$$P_0 = z_0, \quad P_1 = z_1, \quad P_2 = z_1 - z_0, \quad P_3 = z_1 - (x+1)z_0. \tag{264}$$

The sets I^0_{even} and I^0_{odd} are given by

$$I^0_{\text{even}} = \{0,1,2,3\}, \qquad I^0_{\text{odd}} = \varnothing. \tag{265}$$

We have

$$\dim V^1 = \dim H^1_{\omega} = 2; \tag{266}$$

hence, there is an isomorphism between V^1 and H^1_{ω}, and we do not need to worry about symmetries or supersectors. The rational function $\hat{\Phi}_{\mu_0\mu_1\mu_2\mu_3}[Q]$ has to be homogeneous of degree (-2), as $\eta = z_0 dz_1 - z_1 dz_0$ is homogeneous of degree 2. As we have a one-dimensional Baikov representation, we work in projective space $\mathbb{CP}^1$. The algorithm will first consider all possible localizations. As we have four even polynomials, we may localize on the points

$$[z_0 : z_1] \in \{[0:1], [1:0], [1:1], [1:1+x]\}. \tag{267}$$

From each localization, we obtain one master integrand. We may obtain the same master integrand from different localizations. An example is given by

$$\Psi_{1100}[1] = -4\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_0 z_1}, \tag{268}$$

which has residues at $P_0 = 0$ and $P_1 = 0$. Note that the number of consecutive nonzero residues equals one for this differential form. After we have taken a residue at either $P_0 = 0$ or $P_1 = 0$, we localize to a point, and we may no longer take another residue at another point. Note further that the differential forms

$$\Psi_{1100}[1], \qquad \Psi_{1010}[1], \qquad \Psi_{1001}[1] \tag{269}$$

are up to trivial prefactors equal on the localization $P_0 = 0$. It will depend on the unspecified dots in the order relation $(a, w, o, |\mu|, \ldots)$ which differential form is picked. After considering all possible localizations, we will have up to four differential forms with $a = -2$. Running the Laporta algorithm on the full sector will reduce them to two master integrands. A possible choice for the master integrands is

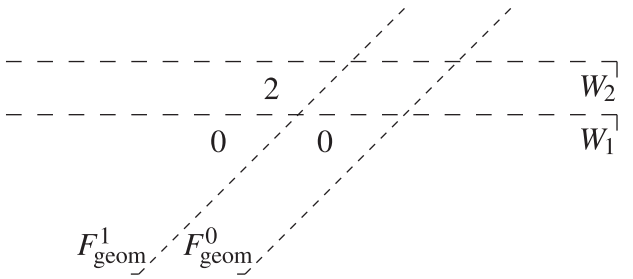
$$\begin{aligned}\Psi_{0110}[1] &= 2\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_1(z_1 - z_0)}, \\ \Psi_{1010}[1] &= -2\varepsilon^4 x^2 C_{\text{Baikov}} U \frac{\eta}{z_0(z_1 - z_0)}.\end{aligned} \tag{270}$$

The exact choice will depend on the unspecified dots in the order relation $(a, w, o, |\mu|, \ldots)$. This choice is of no relevance here. The isomorphism $\iota\colon V^1 \to H^1_\omega$ is given by

$$\begin{aligned}\iota(2\varepsilon^4 x^2 I_{111111100}) &= \Psi_{0110}[1], \\ \iota(-2\varepsilon^4 x^2 I_{1111111(-1)0}) &= \Psi_{1010}[1].\end{aligned} \tag{271}$$

Up to trivial prefactors, this choice corresponds to the example discussed in Ref. [108]. The differential equation for the master integrands defined in Eq. (270) is already in ε-factorized form; therefore, there is nothing to be done in step 2 for the top sector. In the final step, we consider the full system, including all subsectors. For the choice of the master integrands as in Eq. (270), one finds that $(-2\varepsilon^4 x^2 I_{1111111(-1)0})$ will receive corrections from subsectors. These are obtained in a straightforward way with an ansatz as in step 2, and the final result corresponds to the one given in Ref. [108].

In summary, the Hodge-like diagram for the decomposition of V^1 with respect to the $W_\bullet$ filtration and the $F^\bullet_{\text{geom}}$ filtration is



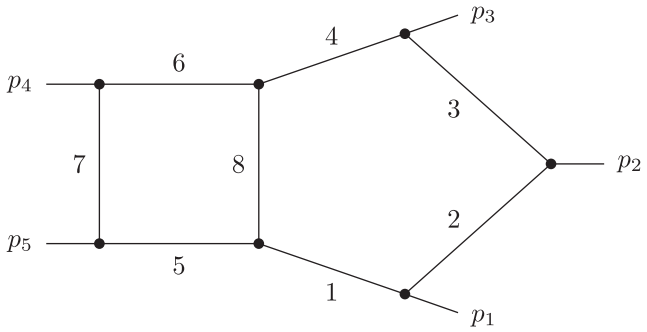


FIG. 12. The two-loop pentagon box graph.

C. The two-loop pentabox integral

The next example we are considering is the two-loop five-point pentagon box integral; see Fig. 12. An ε-factorized form for this family of Feynman integrals has been given in Ref. [109]. This is an example with three master integrals in the top sector. The inverse propagators are defined as

$$\begin{aligned}&\sigma_1 = -k_1^2, && \sigma_2 = -(k_1 + p_1)^2, && \sigma_3 = -(k_1 + p_{12})^2, \\ &\sigma_4 = -(k_1 + p_{123})^2, && \sigma_5 = -k_2^2, && \sigma_6 = -(k_2 + p_{123})^2, \\ &\sigma_7 = -(k_2 + p_{1234})^2, && \sigma_8 = -(k_1 - k_2)^2, && \sigma_9 = -(k_1 + p_{1234})^2, \\ &\sigma_{10} = -(k_2 + p_1)^2, && \sigma_{11} = -(k_2 + p_{12})^2.\end{aligned} \tag{272}$$

The integral depends on four independent (dimensionless) kinematic variables

$$x_1 = \frac{s_{12}}{s_{45}}, \qquad x_2 = \frac{s_{23}}{s_{45}}, \qquad x_3 = \frac{s_{34}}{s_{45}}, \qquad x_4 = \frac{s_{15}}{s_{45}}, \tag{273}$$

where $s_{ij} = 2p_i \cdot p_j$. We set $M^2 = s_{45}$. A loop-by-loop Baikov representation on the maximal cut is

$$e^{2\varepsilon\gamma_E}(s_{45})^{4+2\varepsilon}\int_{\mathcal{C}_{\text{maxcut}}}\frac{d^Dk_1}{i\pi^{2-\varepsilon}}\frac{d^Dk_2}{i\pi^{2-\varepsilon}}\prod_{i=1}^{8}\frac{1}{\sigma_i}$$
$$= C_{\text{Baikov}}\int\frac{dz_1}{(2\pi i)}z_1^{-1-\varepsilon}(1-z_1)^{\varepsilon}((1-x_1-x_2)z_1^2+(x_3-x_2x_3+x_1(x_2-x_4)+x_4)z_1+x_3x_4)^{-1-\varepsilon}, \tag{274}$$

where $z_1=\sigma_9/s_{45}$ and

$$C_{\text{Baikov}}=\frac{32ie^{2\varepsilon\gamma_E}\pi^5}{\Gamma(-2\varepsilon)}x_1^{-1-\varepsilon}x_2^{-1-\varepsilon}\Delta^{\frac{1}{2}+\varepsilon},$$
$$\Delta=(x_1x_2+x_2x_3-x_3+x_4-x_1x_4)^2-4x_1x_2x_3(x_2-x_4-1). \tag{275}$$

By setting

$$C_{\text{abs}}=\frac{\varepsilon^4x_1x_2}{\sqrt{\Delta}}, \tag{276}$$

the product $C_{\text{Baikov}}\cdot C_{\text{abs}}$ is pure of transcendental weight zero. The minimal twist function is

$$U=z_0^{2\varepsilon}z_1^{-\varepsilon}(z_1-z_0)^{\varepsilon}((1-x_1-x_2)z_1^2+(x_3-x_2x_3+x_1(x_2-x_4)+x_4)z_0z_1+x_3x_4z_0^2)^{-\varepsilon} \tag{277}$$

in projective space. We set

$$P_0=z_0,\qquad P_1=z_1,\qquad P_2=z_1-z_0,$$
$$P_3=(1-x_1-x_2)z_1^2+(x_3-x_2x_3+x_1(x_2-x_4)+x_4)z_0z_1+x_3x_4z_0^2. \tag{278}$$

Therefore, the sets I^0_{even} and I^0_{odd} are

$$I^0_{\text{even}}=\{0,1,2,3\},\qquad I^0_{\text{odd}}=\varnothing. \tag{279}$$

We find that

$$\dim V^1=\dim H^1_\omega=3. \tag{280}$$

Consequently, there is an isomorphism between V^1 and H^1_ω, and we do not have to worry about symmetries and supersectors, as in the previous example.

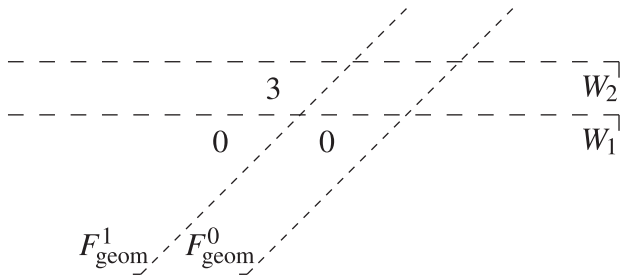
Since $\eta=z_0dz_1-z_1dz_0$ is homogeneous of degree 2 and $d_U=0$, the rational functions $\hat{\Phi}_{\mu_0\mu_1\mu_2\mu_3}[Q]$ have to be homogeneous of degree (-2). We start by looking at all possible localizations. There are five possibilities to localize on a point, and we will get up to five differential forms with $a=-2$. Running the Laporta algorithm on the full sector will reduce them to three master integrands. A possible choice for the master integrands is

$$\Psi_{0001}[1]=-\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}C_{\text{Baikov}}U\frac{\eta}{P_3},$$
$$\Psi_{0101}[z_0]=\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}C_{\text{Baikov}}U\frac{z_0\eta}{P_1P_3},$$
$$\Psi_{1001}[z_1]=-2\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}C_{\text{Baikov}}U\frac{z_1\eta}{P_0P_3}. \tag{281}$$

The exact choice will depend on the unspecified dots in the order relation $(a,w,o,|\mu|,\ldots)$. This choice is of no relevance here. The isomorphism $\iota\colon V^1\to H^1_\omega$ is given by

$$\iota\left(-\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}I_{11111111(-1)00}\right)=\Psi_{0001}[1],$$
$$\iota\left(\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}I_{11111111000}\right)=\Psi_{0101}[z_0],$$
$$\iota\left(-2\varepsilon^4\frac{x_1x_2}{\sqrt{\Delta}}I_{11111111(-2)00}\right)=\Psi_{1001}[z_1]. \tag{282}$$

The decomposition of V^1 with respect to $(W_\bullet,F^\bullet_{\text{geom}})$ is



The differential equation for $(\Psi_{0001}[1],\Psi_{0101}[z_0],\Psi_{1001}[z_1])$ is in Laurent-polynomial form and compatible with the $F^\bullet_{\text{comb}}$ filtration. As there is only one nontrivial part in the

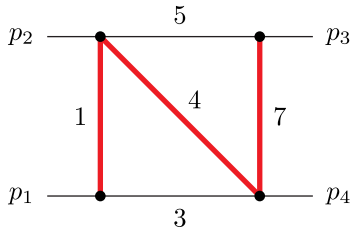


FIG. 13. A two-loop contribution to Møller scattering. Massive propagators are indicated by red lines.

$F^{\bullet}_{\text{comb}}$ filtration, we have that the matrix A is linear in ε. We may then proceed to step 2 of the algorithm and arrive at an ε-factorized basis equivalent to the one given in Ref. [109].

D. A two-loop contribution to Møller scattering

As the next example, we consider a two-loop integral contributing to Møller scattering. This is now a nontrivial example with five master integrals in the top sector and an elliptic curve associated to the top sector. We use this example to illustrate details of the construction of the intermediate basis J. The inverse propagators are defined by

$$\begin{aligned}
&\sigma_1 = -(k_1 - p_1)^2 + m^2, \qquad \sigma_2 = -(k_1 - p_{12})^2, \qquad \sigma_3 = -k_1^2,\\
&\sigma_4 = -(k_1 + k_2)^2 + m^2, \qquad \sigma_5 = -(k_2 + p_{12})^2, \qquad \sigma_6 = -k_2^2,\\
&\sigma_7 = -(k_2 + p_{123})^2 + m^2, \qquad \sigma_8 = -(k_1 - p_{13})^2, \qquad \sigma_9 = -(k_2 + p_{13})^2.
\end{aligned} \tag{283}$$

We are interested in sector 93 with inverse propagators $\{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_7\}$. The graph is shown in Fig. 13. We consider the loop-by-loop Baikov representation, where the inner loop is formed by the inverse propagators $\{\sigma_4, \sigma_5, \sigma_7\}$. We set $x_1 = m^2/s$, $x_2 = t/s$. With

$$r = \frac{s^4(\sigma_8 + m^2 - t)}{\sigma_1\sigma_3\sigma_4\sigma_5\sigma_7}, \tag{284}$$

$z_1 = (\sigma_8 + m^2 - t)/s$, and $z_2 = (\sigma_2 + \sigma_8 + m^2 - t)/s$, we obtain a Baikov representation with a minimal twist function:

$$\begin{aligned}
&e^{2\varepsilon\gamma_{\text{E}}} s^{-4+2\varepsilon} \int_{\mathcal{C}_{\text{maxcut}}} \frac{d^D k_1}{i\pi^{2-\varepsilon}} \frac{d^D k_2}{i\pi^{2-\varepsilon}} r(\sigma)\\
&= C_{\text{Baikov}} \int \frac{dz_1}{(2\pi i)} \frac{dz_2}{(2\pi i)} U(z)|_{z_0=1},
\end{aligned} \tag{285}$$

where

$$C_{\text{Baikov}} = -\frac{32\pi^4 e^{2\varepsilon\gamma_{\text{E}}}}{\Gamma(1-2\varepsilon)} x_1^{-\varepsilon} x_2^{\varepsilon} (1 + x_2)^{\varepsilon}. \tag{286}$$

Setting

$$C_{\text{abs}} = \varepsilon^4 \tag{287}$$

ensures that $C_{\text{Baikov}} \cdot C_{\text{abs}}$ is pure of transcendental weight zero. In projective space, the minimal twist function reads

$$U = P_0^{2\varepsilon} P_1^{2\varepsilon} P_2^{-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon}. \tag{288}$$

The polynomials are given by

$$\begin{aligned}
P_0 &= z_0,\\
P_1 &= z_1,\\
P_2 &= z_1 z_2 + x_1 z_0 z_2 - z_1^2,\\
P_3 &= [x_2 z_1 - (1 + x_2) z_2 + (x_1 - x_2) z_0]^2 + 4 x_1 (1 + x_2) z_0 z_2.
\end{aligned} \tag{289}$$

We have three even polynomials $I^0_{\text{even}} = \{0, 1, 2\}$ and one odd polynomial $I^0_{\text{odd}} = \{3\}$. In this example, the dimensions of V^2 and H^2_ω do not match. We find

$$\dim V^2 = 5, \qquad \dim H^2_\omega = 6. \tag{290}$$

The mismatch in dimensions is related to symmetries. An example of a symmetry relation is

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7 00} = I_{\nu_7\nu_6\nu_5\nu_4\nu_3\nu_2\nu_1 00}. \tag{291}$$

We have $d_U = -1$; hence, $\hat{\Phi}$ must be homogeneous of degree -2. In order to keep the notation compact, we will use whenever there is no ambiguity the preimages $\Psi_{\mu_0\mu_1\mu_2\mu_3}[Q]$ instead of the more lengthy notation $\text{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r} \Psi_{\mu_0\mu_1\mu_2\mu_3}[Q, I, P_s]$.

In the construction of the intermediate basis J, we first consider recursively all possible localizations. We have three even polynomials, and we consider, in turn, the localization on each of them.

We start with the localization at $P_0 = 0$. In this limit, P_2 factorizes:

$$P_2|_{z_0=0} = z_1(z_2 - z_1), \tag{292}$$

and P_3 becomes a perfect square:

$$P_3|_{z_0=0} = [x_2 z_1 - (1+x_2) z_2]^2. \tag{293}$$

We may further localize on P_1, P_2, or P_3, resulting in the localizations on the points

$$[z_0 : z_1 : z_2] \in \{[0:0:1], [0:1:1], [0:1+x_2:x_2]\}. \tag{294}$$

The differential form $\Psi_{1100}[1]$ has nonzero residues at the points $[0:0:1]$ and $[0:1+x_2:x_2]$; the differential form $\Psi_{1010}[z_1]$ has nonzero residues at the points $[0:1:1]$ and $[0:1+x_2:x_2]$. Hence, we obtain at weight $w=4$ the master integrands

$$\Psi_{\mu_0\mu_1\mu_2\mu_3} \in \{\Psi_{1100}[1], \Psi_{1010}[z_1]\}. \tag{295}$$

We assign $a=-4$, $r=2$, and $o=2$ to these three differential forms.

There is no merge procedure in going from zero-dimensional varieties to one-dimensional varieties, as there are no intersections between distinct points.

We then run the Laporta algorithm for the localization at $P_0=0$. We choose P_1 as a scale polynomial. This gives us

$$U^{\text{loc}}_{\langle P_0\rangle} = P_1^{4\varepsilon} P_2^{-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon} \tag{296}$$

as the modified twist function. From an analysis of the number of critical points on the localization $P_0=0$, we expect to find one master integrand, and, indeed, there is one linear relation among the two candidates in Eq. (295). The unspecified dots in the order relation will determine the candidate to be eliminated. We assume that $\Psi_{1010}[z_1]$ is eliminated. Thus, we obtain

$$\Psi_{1100}[1] \tag{297}$$

as master integrand from the localization at $P_0=0$.

Next, we consider the localization at $P_1=0$. In this limit, we have

$$\begin{aligned} P_2|_{z_1=0} &= x_1 z_0 z_2, \\ P_3|_{z_1=0} &= [-(1+x_2) z_2 + (x_1 - x_2) z_0]^2 \\ &\quad + 4 x_1 (1+x_2) z_0 z_2. \end{aligned} \tag{298}$$

In principle, the square root associated with $P_3|_{z_1=0}$ can be rationalized, but this is not needed. After localization on $z_1=0$, we may further localize on $z_0=0$ or on $z_2=0$, i.e., on the points

$$[z_0 : z_1 : z_2] \in \{[0:0:1], [1:0:0]\}. \tag{299}$$

The first possibility corresponds to $\Psi_{1100}[1]$, which we already obtained from the localization on $P_0=0$. The second possibility gives $\Psi_{0110}[z_0]$. We assign $a=-4$ to these two candidates and run the Laporta algorithm on the localization $P_1=0$. As a scale polynomial, we choose P_0, giving us the modified twist function

$$U^{\text{loc}}_{\langle P_1\rangle} = P_0^{4\varepsilon} P_2^{-\varepsilon} P_3^{-\frac{1}{2}-\varepsilon}. \tag{300}$$

There are no further linear relations among these two candidates, and $\{\Psi_{1100}[1], \Psi_{0110}[z_0]\}$ forms a basis for the localization on $P_1=0$.

We then consider the localization at $P_2=0$. The polynomial P_2 is of degree 2; however, it is linear in z_0 and z_2. From an analysis of the number of critical points on the localization $P_2=0$, we expect to find five master integrands. We first consider the possible localizations on points. We find three points

$$\{[0:0:1], [0:1:1], [1:0:0]\}, \tag{301}$$

giving us the master integrands

$$\{\Psi_{1010}[z_2], \Psi_{1010}[z_1], \Psi_{0110}[z_0]\}, \tag{302}$$

to which we assign $a=-4$. As a scale polynomial for the localization $\langle P_2\rangle$, we choose P_3, giving us the modified twist function

$$U^{\text{loc}}_{\langle P_2\rangle} = P_0^{2\varepsilon} P_1^{2\varepsilon} P_3^{-\frac{1}{2}-2\varepsilon}. \tag{303}$$

Note that the choice of P_0 or P_1 as scale polynomial would give us a nongeneric modified twist function, where either the divisor P_0 or P_1 is absent. Running the Laporta algorithm on the localization $\langle P_2\rangle$, one obtains two additional master integrands

$$\{\Psi_{0010}[1], \Psi_{0011}[z_0^2]\}, \tag{304}$$

to which we assign $a=-3$.

We have now done all localizations. It is instructive to discuss the merge step on the 1-skeleton in detail. In order to do so, we switch to the longer notation $\text{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r} \Psi_{\mu_0\mu_1\mu_2\mu_3}[Q, I, P_s]$. The 1-skeleton is given by

$$V(\langle P_0\rangle) \cup V(\langle P_1\rangle) \cup V(\langle P_2\rangle). \tag{305}$$

The 0-skeleton is given by the four points

$$[0:0:1] \cup [0:1:1] \cup [1:0:0] \cup [0:1+x_2:x_2]. \tag{306}$$

The intersections of the components of the 1-skeleton are given by the three points

$$\bigcup_{(ij)} V(\mathcal{I}_{ij}) = [0:0:1] \cup [0:1:1] \cup [1:0:0]. \tag{307}$$

In the notation of Sec. IV A, we write

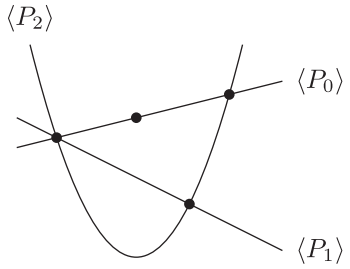


FIG. 14. A sketch of the 1-skeleton and the 0-skeleton.

$$\bigcup_{(ij)} V(\mathcal{I}_{ij}) = C_1 \cup C_2 \cup C_3 \tag{308}$$

with

$$C_1 = \{[0:0:1]\}, \quad C_2 = \{[0:1:1]\}, \quad C_3 = \{[1:0:0]\}. \tag{309}$$

The skeleton is sketched in Fig. 14. Note that

$$C_1 = V(\langle P_0\rangle) \cap V(\langle P_1\rangle), \qquad C_1 \subset V(\langle P_0\rangle) \cap V(\langle P_2\rangle),$$
$$C_1 \subset V(\langle P_1\rangle) \cap V(\langle P_2\rangle). \tag{310}$$

In the notation of Sec. IV A, we have

$$S_1 = \{0, 1, 2\}. \tag{311}$$

The bases on the localizations of dimension 1 are

$$\begin{aligned}
&V(\langle P_0\rangle)\colon \mathrm{Res}_{P_0}\Psi_{1100}[1, \langle P_0\rangle, P_1],\\
&V(\langle P_1\rangle)\colon \mathrm{Res}_{P_1}\Psi_{1100}[1, \langle P_1\rangle, P_0], \quad \mathrm{Res}_{P_1}\Psi_{0110}[z_0, \langle P_1\rangle, P_0],\\
&V(\langle P_2\rangle)\colon \mathrm{Res}_{P_2}\Psi_{1010}[z_2, \langle P_2\rangle, P_3], \quad \mathrm{Res}_{P_2}\Psi_{1010}[z_1, \langle P_2\rangle, P_3], \quad \mathrm{Res}_{P_2}\Psi_{0110}[z_0, \langle P_2\rangle, P_3],\\
&\qquad \mathrm{Res}_{P_2}\Psi_{0010}[1, \langle P_2\rangle, P_3], \quad \mathrm{Res}_{P_2}\Psi_{0011}[z_0^2, \langle P_2\rangle, P_3].
\end{aligned} \tag{312}$$

Let us assume that we have chosen at the point $[0:0:1]$ the basis

$$[0:0:1]\colon \mathrm{Res}_{P_0,P_1}\Psi_{1100}[1, \langle P_0, P_1\rangle, P_3]. \tag{313}$$

There are eight different objects appearing in Eq. (312). From the intersections of the components of the 1-skeleton, we obtain three equations:

$$\begin{aligned}
[0:0:1]\colon\ & \mathrm{Res}_{P_0}\Psi_{1100}[1, \langle P_0\rangle, P_1] = \mathrm{Res}_{P_1}\Psi_{1100}[1, \langle P_1\rangle, P_0],\\
[0:1:1]\colon\ & \mathrm{Res}_{P_0}\Psi_{1010}[z_1, \langle P_0\rangle, P_1] = \mathrm{Res}_{P_2}\Psi_{1010}[z_1, \langle P_2\rangle, P_3],\\
[1:0:0]\colon\ & \mathrm{Res}_{P_1}\Psi_{0110}[z_0, \langle P_1\rangle, P_0] = \mathrm{Res}_{P_2}\Psi_{0110}[z_0, \langle P_2\rangle, P_3].
\end{aligned} \tag{314}$$

Note that there is only one equation for the point $[0:0:1]$, as $\Psi_{1100}[1]$ has a residue on $\langle P_0\rangle$ and $\langle P_1\rangle$ but not on $\langle P_2\rangle$. Note further that $\mathrm{Res}_{P_0}\Psi_{1010}[z_1, \langle P_0\rangle, P_1]$ is not a master integrand on $\langle P_0\rangle$. We, therefore, add one equation, which relates this integrand to the master integrand on $\langle P_0\rangle$:

$$c_1\mathrm{Res}_{P_0}\Psi_{1100}[1, \langle P_0\rangle, P_1] + c_2\mathrm{Res}_{P_0}\Psi_{1010}[z_1, \langle P_0\rangle, P_1] = 0. \tag{315}$$

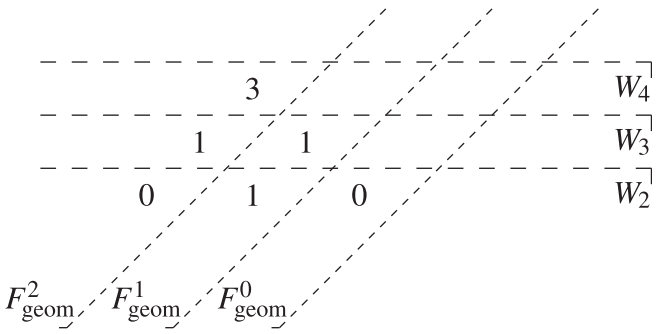
The coefficients are obtained from the reduction on $\langle P_0\rangle$. The actual values do not matter; it is relevant for us only that $c_1 \neq 0$ and $c_2 \neq 0$. We now have nine objects and four equations, leaving us with five independent integrands. After renaming, we have

$$\Psi_{1100}[1], \quad \Psi_{1010}[z_2], \quad \Psi_{0110}[z_0], \quad \Psi_{0010}[1], \quad \Psi_{0011}[z_0^2]. \tag{316}$$

Finally, we run the reduction on the full system without any localization. This will give one additional master integrand

$$\Psi_{0001}[1]. \tag{317}$$

The decomposition of H^2_ω with respect to $(W_\bullet, F^\bullet_{\mathrm{geom}})$ is



Ordered by $|\mu|$ we have the following basis for H^2_ω:

$$\Psi \in \{\Psi_{0001}[1], \Psi_{0010}[1], \Psi_{0011}[z_0^2], \Psi_{1100}[1], \Psi_{1010}[z_2], \Psi_{0110}[z_0]\}. \tag{318}$$

For this basis, one finds a differential equation of the form as in Eq. (202). For the map ι: $V^2 \to H^2_\omega$, we have at weight four

$$\begin{aligned} \iota(4\varepsilon^4 I_{101110100}) &= \Psi_{1100}[1], \\ \iota(-2\varepsilon^3 I_{101210100}[z_2]) &= \Psi_{1010}[z_2], \\ \iota\left(\varepsilon^3\left(I_{101120100} + \frac{1}{x_1} I_{101210100}[z_2]\right)\right) &= \Psi_{0110}[z_2], \end{aligned} \tag{319}$$

where we used the notation

$$\begin{aligned} I_{101210100}[z_2] = I_{1(-1)1210100} + I_{1012101(-1)0} \\ + (x_1 - x_2) I_{101210100}. \end{aligned} \tag{320}$$

The preimage of $\Psi_{0010}[1]$ is given by

$$\iota(-\varepsilon^3 I_{101210100}) = \Psi_{0010}[1]. \tag{321}$$

Note that instead of $\Psi_{0011}[z_0^2]$ we could have chosen $\Psi_{0011}[\partial_{x_2} P_3]$; in the order relation, they differ only in the unspecified dots. The latter integrand has the preimage

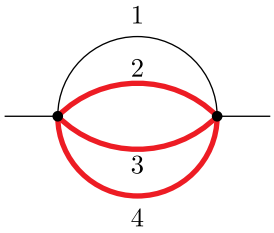


FIG. 15. A three-loop contribution to the electron self-energy. Massive propagators are indicated by red lines.

$$\begin{aligned} &\iota\left(-\varepsilon^2 \partial_{x_2} I_{101210100} + \varepsilon^3 \left(\frac{1}{x_2} + \frac{1}{1+x_2}\right) I_{101210100}\right) \\ &= \Psi_{0011}[\partial_{x_2} P_3]. \end{aligned} \tag{322}$$

We may either replace $\Psi_{0011}[z_0^2]$ by $\Psi_{0011}[\partial_{x_2} P_3]$ or express $\Psi_{0011}[\partial_{x_2} P_3]$ in terms of $\Psi_{0011}[z_0^2]$ and the other basis elements of H^2_ω.

The sector discussed in this example is a subsector of the planar double-box integral with three Z-boson exchanges contributing to Møller scattering. The planar and the nonplanar double-box integrals with all subsectors included will be discussed in detail in a forthcoming publication.

E. A three-loop contribution to the electron self-energy

As the next example, we consider a specific three-loop contribution to the electron self-energy [48]. We discuss the three-loop banana integral with one massless propagator. This is an example with non-normal crossing singularities. The example has three master integrals in the top sector and an elliptic curve associated with the top sector. The inverse propagators are defined by

$$\begin{aligned} &\sigma_1 = -k_1^2, && \sigma_2 = -k_2^2 + m^2, && \sigma_3 = -k_3^2 + m^2, \\ &\sigma_4 = -(k_1 + k_2 + k_3 - p)^2 + m^2, && \sigma_5 = -(k_1 + k_2 - p)^2, && \sigma_6 = -(k_1 - p)^2, \\ &\sigma_7 = -(k_1 + k_2)^2, && \sigma_8 = -(k_1 + k_3)^2, && \sigma_9 = -(k_2 + k_3)^2. \end{aligned} \tag{323}$$

We are interested in sector 15 with inverse propagators $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. The graph is shown in Fig. 15. It will be convenient to discuss this example in $D = 2 - 2\varepsilon$ space-time dimensions. We consider the loop-by-loop Baikov representation, where the innermost loop is formed by the inverse propagators $\{\sigma_3, \sigma_4\}$, followed by the loop where we add the inverse propagator σ_2. For the third and final loop, we add the inverse propagator σ_1. This yields a two-dimensional Baikov representation with Baikov variables σ_5 and σ_6. We set $x = m^2/p^2$, $z_1 = \sigma_5/p^2$, and $z_2 = \sigma_6/p^2$. We have three master integrals in sector 15 (this is the sector of interest) and one master integral in sector 47 (this is a supersector). In addition, there are two independent symmetry relations in V^2, which can be taken as

$$\begin{aligned} 2I_{121100000} - I_{112100000} - I_{111200000} &= 0, \\ \frac{d}{dx}(2I_{121100000} - I_{112100000} - I_{111200000}) &= 0. \end{aligned} \tag{324}$$

Thus, we have

$$\dim V^2 = 3, \qquad \dim H^2_\omega = 6. \tag{325}$$

With

$$r = \frac{(p^2)^3(\sigma_6 + p^2)}{\sigma_1\sigma_2\sigma_3\sigma_4}, \tag{326}$$

we obtain a Baikov representation with a minimal twist function:

$$e^{3\varepsilon\gamma_{\mathrm{E}}}(p^2)^{-3+3\varepsilon}\int_{\mathcal{C}_{\mathrm{maxcut}}}\frac{d^Dk_1}{i\pi^{1-\varepsilon}}\frac{d^Dk_2}{i\pi^{1-\varepsilon}}\frac{d^Dk_3}{i\pi^{1-\varepsilon}}r(\sigma) = C_{\mathrm{Baikov}}\int\frac{dz_1}{(2\pi i)}\frac{dz_2}{(2\pi i)}U(z)|_{z_0=1}, \tag{327}$$

where

$$C_{\mathrm{Baikov}} = -\frac{2^{6+6\varepsilon}i\pi^{\frac{9}{2}}e^{3\varepsilon\gamma_{\mathrm{E}}}}{\Gamma(\frac{1}{2}-\varepsilon)^3}. \tag{328}$$

Setting

$$C_{\mathrm{abs}} = \varepsilon^3 \tag{329}$$

ensures that $C_{\mathrm{Baikov}}\cdot C_{\mathrm{abs}}$ is pure of transcendental weight zero. The minimal twist function reads

$$U(z_0,z_1,z_2) = P_0^{4\varepsilon}P_1^{\varepsilon}P_2^{-2\varepsilon}P_3^{-\frac{1}{2}}P_4^{-\frac{1}{2}-\varepsilon}P_5^{-\frac{1}{2}-\varepsilon}, \tag{330}$$

with

$$\begin{aligned} &P_0 = z_0, \quad P_3 = z_1,\\ &P_1 = z_2, \quad P_4 = z_1 + 4xz_0,\\ &P_2 = z_2 + z_0, \quad P_5 = (z_2-z_1)^2 + 2xz_0(z_1+z_2) + x^2z_0^2. \end{aligned} \tag{331}$$

We have $d_U = -2$; hence, $\hat{\Phi}$ must be homogeneous of degree -1. We have three even polynomials ($I^0_{\mathrm{even}} = \{0,1,2\}$), and we consider, in turn, the localizations on each of the even polynomials. As we have two Baikov variables, we have to consider for each of those possibilities the localization on a further polynomial. As in the previous example, we keep the notation compact and use whenever there is no ambiguity the preimages $\Psi_{\mu_0\mu_1\mu_2\mu_3}[Q]$ instead of the more lengthy notation $\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r}\Psi_{\mu_0\mu_1\mu_2\mu_3}[Q,I,P_s]$.

We start with the localization at $P_0 = 0$. From the analysis of the critical points, we expect one master integrand from the localization on $P_0 = 0$. With the choice of P_1 as a scale polynomial, we obtain the (nonminimal) modified twist function

$$U^{\mathrm{loc}}_{\langle P_0\rangle} = P_1^{5\varepsilon}P_2^{-2\varepsilon}P_3^{-\frac{1}{2}}P_4^{-\frac{1}{2}-\varepsilon}P_5^{-\frac{1}{2}-\varepsilon}. \tag{332}$$

We have

$$U^{\mathrm{loc}}_{\langle P_0\rangle}|_{z_0=0} = z_2^{3\varepsilon}z_1^{-1-\varepsilon}(z_2-z_1)^{-1-2\varepsilon}. \tag{333}$$

We may take a second residue at the points

$$[0:0:1], \qquad [0:1:1], \qquad [0:1:0]. \tag{334}$$

The differential form $\Psi_{100000}[1]$ has a nonzero residue on the first two points; the differential form $\Psi_{110000}[z_1]$ has a nonzero residue at the third point (and at the second point). Therefore, we obtain from the three points two candidates:

$$\Psi_{100000}[1], \qquad \Psi_{110000}[z_1], \tag{335}$$

to which we assign $a = -4$.

Running then in dimension one (i.e., at weight 3) will find a relation among those and eliminate one candidate. In summary, we get one master from this localization, which we may take as

$$\Psi_{100000}[1] \tag{336}$$

with $|\mu| = 1$, $r = 2$, and $o = 2$.

Next, we consider the localization at $\langle P_1\rangle$. The analysis of the critical points indicates two master integrands. We choose P_0 as a scale polynomial and obtain the (nonminimal) modified twist function

$$U^{\mathrm{loc}}_{\langle P_1\rangle} = P_0^{5\varepsilon}P_2^{-2\varepsilon}P_3^{-\frac{1}{2}}P_4^{-\frac{1}{2}-\varepsilon}P_5^{-\frac{1}{2}-\varepsilon}. \tag{337}$$

We have

$$U^{\mathrm{loc}}_{\langle P_1\rangle}|_{z_2=0} = z_0^{3\varepsilon}z_1^{-\frac{1}{2}}(z_1+4xz_0)^{-\frac{1}{2}-\varepsilon}(z_1+xz_0)^{-1-2\varepsilon}. \tag{338}$$

We may take a second residue at the points

$$[0:1:0], \qquad [1:-x:0]. \tag{339}$$

The differential form $\Psi_{110000}[z_1]$ has a twofold nonzero residue at $[0:1:0]$. We choose this differential form as a master integrand on the point $[0:1:0]$. This differential form also has a twofold nonzero residue at the other point $[1:-x:0]$; however, at this point, there is a simpler choice, given by the differential form $\Psi_{010000}[1]$. The algorithm will pick the latter. From dimension zero, we obtain these two candidates. Running then in dimension one, we find that there are no further linear relations and these two differential forms constitute a basis for the localization on $\langle P_1\rangle$. In summary, we get two masters from the localization $P_1 = 0$, given by

$$\Psi_{010000}[1], \qquad \Psi_{110000}[z_1]. \tag{340}$$

The former differential form has the values $(|\mu|, r, o) = (1,2,2)$; the latter has the values $(|\mu|, r, o) = (2,2,2)$.

We then consider the localization at $\langle P_2\rangle$. An analysis of the critical points indicates three master integrands at $\langle P_2\rangle$. In order to reduce polynomials modulo the ideal $\langle P_2\rangle = \langle z_0 + z_2\rangle$, we need to choose a monomial order for (z_0, z_1, z_2). Let us assume that we eliminate z_2 in favor of z_0, e.g., $z_2 = -z_0$ on $\langle P_2\rangle$. We choose P_0 as a scale polynomial and obtain, up to a prefactor, the modified twist function

$$U^{\mathrm{loc}}_{\langle P_2\rangle} = P_0^{2\varepsilon} P_1^{\varepsilon} P_3^{-\frac{1}{2}} P_4^{-\frac{1}{2}-\varepsilon} P_5^{-\frac{1}{2}-\varepsilon}. \tag{341}$$

We have

$$U^{\mathrm{loc}}_{\langle P_2\rangle}\big|_{z_2=-z_0} = z_0^{3\varepsilon} z_1^{-\frac{1}{2}} (z_1 + 4xz_0)^{-\frac{1}{2}-\varepsilon} \Big[(z_0+z_1)^2 + 2xz_0(z_1 - z_0) + x^2 z_0^2\Big]^{-\frac{1}{2}-\varepsilon}. \tag{342}$$

At the point $[0:1:0]$, we may take a second residue, and we obtain for this point the master integrand

$$\Psi_{101000}[z_1], \tag{343}$$

with $|\mu| = 2$, $r = 2$, and $o = 2$. We assign $a = -4$ to this integrand. The remaining two master integrands on the localization $P_2 = 0$ are related to an elliptic curve and are obtained by running the Laporta algorithm on the localization $P_2 = 0$. It is instructive to discuss this in detail. The first "elliptic" integrand is straightforward:

$$\Psi_{001000}[1] \tag{344}$$

and has $|\mu| = 1$, $r = 1$, and $o = 1$. For the second one, there is a choice. The candidates should have $|\mu| = 2$, $r = 1$, and $o = 2$. If we further require that the numerator polynomial has minimal degree, we need to consider

$$\Psi_{001100}[Q], \qquad \Psi_{001010}[Q] \tag{345}$$

with $\deg Q = 1$. We require that the candidates have poles of the order of at most two on $\mathbb{CP}^2$, not just on the localization. This excludes, for example,

$$\Psi_{001100}[z_0], \tag{346}$$

which has a pole of the order of three (after blowup) at $z_1 = 0$ and $z_2 = -xz_0$. We take

$$\Psi_{001010}[z_0] \tag{347}$$

as the second master integrand related to the elliptic curve. This integrand has $|\mu| = 2$, $r = 1$, and $o = 2$.

We have now done all localization. In the next step, we merge the preferred candidates from the localizations of dimension 1. We switch back to the longer notation

$$\mathrm{Res}_{\mathcal{P}_1,\ldots,\mathcal{P}_r} \Psi_{\mu_0\mu_1\mu_2\mu_3\mu_4\mu_5}[Q, I, P_s]. \tag{348}$$

The 1-skeleton is given by

$$V(\langle P_0\rangle) \cup V(\langle P_1\rangle) \cup V(\langle P_2\rangle). \tag{349}$$

The 0-skeleton is given by the four points

$$[0:0:1] \cup [0:1:1] \cup [0:1:0] \cup [1:-x:0]. \tag{350}$$

The intersection of the components of the 1-skeleton is given by a single point

$$\bigcup_{(ij)} V(\mathcal{I}_{ij}) = C_1 = [0:1:0]. \tag{351}$$

We have $S_1 = \{0, 1, 2\}$. The skeleton is sketched in Fig. 16. The bases on the localizations of dimension 1 are

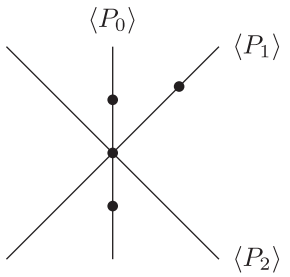


FIG. 16. A sketch of the 1-skeleton and the 0-skeleton.

$$\begin{aligned}
&V(\langle P_0\rangle)\colon \mathrm{Res}_{P_0}\Psi_{100000}[1, \langle P_0\rangle, P_1],\\
&V(\langle P_1\rangle)\colon \mathrm{Res}_{P_1}\Psi_{010000}[1, \langle P_1\rangle, P_0], \quad \mathrm{Res}_{P_1}\Psi_{110000}[z_1, \langle P_1\rangle, P_0],\\
&V(\langle P_2\rangle)\colon \mathrm{Res}_{P_2}\Psi_{101000}[z_1, \langle P_2\rangle, P_0], \quad \mathrm{Res}_{P_2}\Psi_{001000}[1, \langle P_2\rangle, P_0], \quad \mathrm{Res}_{P_2}\Psi_{001010}[z_0, \langle P_2\rangle, P_0].
\end{aligned} \tag{352}$$

These are six different objects. Let us assume that we have chosen $\Psi_{110000}[z_1]$ as a master integrand on the point $[0:1:0]$. This differential form has a residue on $\langle P_0\rangle$ and $\langle P_1\rangle$ but not on $\langle P_2\rangle$. From the intersections of the components of the 1-skeleton, we get, therefore, only one equation:

$$[0:1:0]\colon \mathrm{Res}_{P_0}\Psi_{110000}[z_1,\langle P_0\rangle,P_1] = \mathrm{Res}_{P_1}\Psi_{110000}[z_1,\langle P_1\rangle,P_0]. \tag{353}$$

$\mathrm{Res}_{P_0}\Psi_{110000}[z_1,\langle P_0\rangle,P_1]$ is not a master integrand on $\langle P_0\rangle$. We add one equation, which relates this integrand to the master integrand on $\langle P_0\rangle$:

$$c_1\mathrm{Res}_{P_0}\Psi_{110000}[z_1,\langle P_0\rangle,P_1] + c_2\mathrm{Res}_{P_0}\Psi_{100000}[1,\langle P_0\rangle,P_1] = 0. \tag{354}$$

The coefficients are obtained from the reduction on $\langle P_0\rangle$. The actual values do not matter; for us it is relevant only that $c_1 \neq 0$ and $c_2 \neq 0$. We now have seven objects and two equations, leaving us with five independent integrands. After renaming, we have

$$\Psi_{100000}[1],\quad \Psi_{010000}[1],\quad \Psi_{101000}[z_1],\quad \Psi_{001000}[1],\quad \Psi_{001010}[z_0]. \tag{355}$$

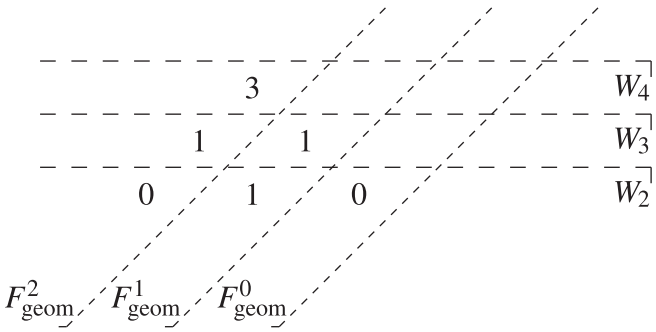
We then run the full Laporta algorithm and find the last master integrand:

$$\Psi_{000010}[1]. \tag{356}$$

Note that the algorithm excludes the candidate $\Psi_{000100}[1]$, as this integrand has a pole of the order of three (after blowup) at $z_1 = 0$ and $z_2 = -xz_0$. In summary, we find the basis

$$\begin{aligned}&\text{weight4}\colon \Psi_{100000}[1],\quad \Psi_{010000}[1],\quad \Psi_{101000}[z_1],\\ &\text{weight3}\colon \Psi_{001000}[1],\quad \Psi_{001010}[z_0],\\ &\text{weight2}\colon \Psi_{000010}[1].\end{aligned} \tag{357}$$

The decomposition of H^2_ω with respect to $(W_\bullet, F^\bullet_{\mathrm{geom}})$ is



We order the basis by $|\mu|$

$$\{\Psi_{000010}[1],\ \Psi_{001000}[1],\ \Psi_{100000}[1],\ \Psi_{010000}[1],\ \Psi_{001010}[z_0],\ \Psi_{101000}[z_1]\} \tag{358}$$

and verify that it leads to a $F^\bullet_{\mathrm{comb}}$-compatible differential equation.

Before proceeding to step 2, it is advantageous to map back to the space V^2 of Feynman integrals. The integrand on the left-hand side of the first symmetry relation in Eq. (324) corresponds in Ω^2_ω to

$$\Psi_{000001}[z_1+z_2+xz_0] - \Psi_{000010}[1]. \tag{359}$$

The symmetry relation tells us that this integrand integrates to zero. In H^2_ω we may reduce the integrand by integration-by-parts identities, and we obtain the relation

$$6x\Psi_{000010}[1] + \Psi_{001000}[1] + \Psi_{100000}[1] = 0. \tag{360}$$

We may proceed in a similar way with the second symmetry relation in Eq. (324). We obtain a second symmetry relation in H^2_ω, which we may write (by adding a suitable multiple of the first one) as

$$4(1+3x)\Psi_{001000}[1] + 2\Psi_{100000}[1] + 3\Psi_{101000}[z_1] = 0. \tag{361}$$

Note that these two equations are independent of ε. We may use these equations to eliminate in each equation the integrand with the highest $|\mu|$ value; this will preserve the compatibility with the $F^\bullet_{\mathrm{comb}}$ filtration. Thus, we eliminate $\Psi_{101000}[z_1]$ from the second equation. For the first equation, there is a choice, as all differential forms have $|\mu| = 1$. Let us assume that we eliminate $\Psi_{000010}[1]$.

We may take $\Psi_{010000}[1]$ as the master integrand for the supersector 47. This leaves us with the integrands

$$\{\Psi_{001000}[1], \Psi_{100000}[1], \Psi_{001010}[z_0]\} \tag{362}$$

for sector 15. For the map $\iota\colon V^2 \to H^2_\omega$, we have

$$\begin{aligned}\iota(-2\varepsilon^3 I_{111100000}) &= \Psi_{001000}[1],\\ \iota(4\varepsilon^3(I_{111100000} + I_{11110(-1)000})) &= \Psi_{100000}[1],\\ \iota(\varepsilon^2 I_{111200000}) &= \Psi_{001010}[z_0].\end{aligned} \tag{363}$$

We, therefore, set

$$\begin{aligned}J_1 &= -2\varepsilon^3 I_{111100000},\\ J_2 &= 4\varepsilon^3(I_{111100000} + I_{11110(-1)000}),\\ J_3 &= \varepsilon^2 I_{111200000}.\end{aligned} \tag{364}$$

For $J = (J_1, J_2, J_3)^T$, one obtains the differential equation in the form of Eq. (206):

$$d_B J = AJ, \qquad A = B^{(-1)} + B^{(0)} + B^{(1)}. \tag{365}$$

The matrices $B^{(-1)}$, $B^{(0)}$, and $B^{(1)}$ are explicitly given by

$$B^{(-1)} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -\frac{3x-1}{2\varepsilon x(x-1)(9x-1)} & 0 & -\frac{27x^2-20x+1}{x(x-1)(9x-1)} \end{array}\right),$$

$$B^{(0)} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -\frac{21x-5}{2x(x-1)(9x-1)} & 0 & 0 \end{array}\right),$$

$$B^{(1)} = \left(\begin{array}{cc|c} 0 & 0 & 6\varepsilon \\ -\frac{4\varepsilon}{x} & -\frac{4\varepsilon}{x} & 0 \\ \hline -\frac{(18x^2-3x+1)\varepsilon}{x^2(x-1)(9x-1)} & -\frac{\varepsilon}{x^2(x-1)(9x-1)} & -\frac{(63x^2-30x-1)\varepsilon}{x(x-1)(9x-1)} \end{array}\right). \tag{366}$$

In these matrices, we indicated the block structure due to the $F^{\bullet}_{\text{comb}}$ filtration. We then rotate the system to an ε-form with the rotation matrix

$$R_2 = R_2^{(-1)} R_2^{(0)}. \tag{367}$$

The ansatz for $R_2^{(-1)}$ and $R_2^{(0)}$ is

$$R_2^{(-1)} = \left(\begin{array}{cc|c} R_{11}^{(-1)} & R_{12}^{(-1)} & 0 \\ R_{21}^{(-1)} & R_{22}^{(-1)} & 0 \\ \hline \frac{1}{\varepsilon} R_{31}^{(-1)} & \frac{1}{\varepsilon} R_{32}^{(-1)} & R_{33}^{(-1)} \end{array}\right), \qquad R_2^{(0)} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline R_{31}^{(0)} & R_{32}^{(0)} & 1 \end{array}\right). \tag{368}$$

Note that the matrix $R_2^{(-1)}$ is required to be invertible. We first consider the differential equation for $R_2^{(-1)}$. For this matrix, we start with the first block column, involving the six unknown functions $R_{11}^{(-1)}$, $R_{12}^{(-1)}$, $R_{21}^{(-1)}$, $R_{22}^{(-1)}$, $R_{31}^{(-1)}$, and $R_{32}^{(-1)}$. We immediately obtain

$$\frac{d}{dx} R_{21}^{(-1)} = 0, \qquad \frac{d}{dx} R_{22}^{(-1)} = 0. \tag{369}$$

Thus, these two functions are constant, and we can choose $R_{21}^{(-1)} = 0$ and $R_{22}^{(-1)} = 1$. Note that we cannot choose $R_{21}^{(-1)} = 0$ and $R_{22}^{(-1)} = 0$, as the matrix $R_2^{(-1)}$ would then not be invertible. For the four remaining functions of the first block column, we obtain

$$\frac{d}{dx} R_{1j}^{(-1)} = 6 R_{3j}^{(-1)}, \qquad \frac{d}{dx} R_{3j}^{(-1)} = -\frac{3x-1}{2x(x-1)(9x-1)} R_{1j}^{(-1)} - \frac{27x^2-20x+1}{x(x-1)(9x-1)} R_{3j}^{(-1)}, \qquad j \in \{1,2\}. \tag{370}$$

For $R_{12}^{(-1)}$ and $R_{32}^{(-1)}$ (or $R_{11}^{(-1)}$ and $R_{31}^{(-1)}$), we are free to pick the trivial solution $R_{12}^{(-1)} = 0$ and $R_{32}^{(-1)} = 0$, but we cannot do this simultaneously for the second pair; otherwise, the matrix $R_2^{(-1)}$ would not be invertible. Eliminating $R_{31}^{(-1)}$, we obtain a second-order differential equation for $R_{11}^{(-1)}$:

$$\left[\frac{d^2}{dx^2} + \frac{27x^2-20x+1}{x(x-1)(9x-1)} \frac{d}{dx} + \frac{3(3x-1)}{x(x-1)(9x-1)}\right] R_{11}^{(-1)} = 0. \tag{371}$$

This is the Picard-Fuchs equation of a family of elliptic curves, parametrized by x. The function $R_{11}^{(-1)}$ is, therefore, a period of the elliptic curve.

We then turn to the second block column. In this block column, there is only one unknown function $R_{33}^{(-1)}$. The differential equation for $R_{33}^{(-1)}$ reads

$$\frac{d}{dx}R_{33}^{(-1)} = \left[-\frac{27x^2-20x+1}{x(x-1)(9x-1)} - \frac{d}{dx}\ln R_{11}^{(-1)}\right]R_{33}^{(-1)}. \quad (372)$$

A possible solution is

$$R_{33}^{(-1)} = \frac{1}{x(x-1)(9x-1)R_{11}^{(-1)}}. \quad (373)$$

Thus, the matrix $R_2^{(-1)}$ reads

$$R_2^{(-1)} = \left(\begin{array}{cc|c} R_{11}^{(-1)} & 0 & 0 \\ 0 & 1 & 0 \\ \hline \frac{1}{6\varepsilon}\frac{d}{dx}R_{11}^{(-1)} & 0 & \frac{1}{x(x-1)(9x-1)R_{11}^{(-1)}} \end{array}\right), \quad (374)$$

where $R_{11}^{(-1)}$ is determined by Eq. (371).

Rotating by $R_2^{(-1)}$, we obtain a matrix $\tilde{A}^{(0)}$, which contains terms of B-order 0 and 1. The terms of B-order 0 are then removed by the rotation with the matrix $R_2^{(0)}$. The matrix $R_2^{(0)}$ contains two unknown functions $R_{31}^{(0)}$ and $R_{32}^{(0)}$. The differential equation for the latter is trivial:

$$\frac{d}{dx}R_{32}^{(0)} = 0, \quad (375)$$

and we may set $R_{32}^{(0)} = 0$. The former satisfies the differential equation

$$\frac{d}{dx}R_{31}^{(0)} = -\frac{1}{6}R_{11}^{(-1)}\left[(63x-15)R_{11}^{(-1)} + (63x^2-30x-1)\frac{d}{dx}R_{11}^{(-1)}\right], \quad (376)$$

which can be solved by direct integration. One obtains

$$R_{31}^{(0)} = -\frac{1}{12}(63x^2-30x-1)(R_{11}^{(-1)})^2. \quad (377)$$

Note that we may freely choose the integration constant. Thus, $R_2^{(0)}$ reads

$$R_2^{(0)} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline R_{31}^{(0)} & 0 & 1 \end{array}\right), \quad (378)$$

with $R_{31}^{(0)}$ given by Eq. (377). After rotation by $R_2^{(0)}$, we then obtain the ε-factorized form. In this example, there is nothing to be done for the tadpole subsector.

The tadpole integral is rather simple. We set

$$K_0 = \varepsilon^3 I_{011100000}. \quad (379)$$

The master integrals integrals K_1, K_2, and K_3 for the sector 15 are obtained by the rotating $R_2 = R_2^{(-1)}R_2^{(0)}$ from the basis J. This yields

$$\begin{aligned} K_1 &= \frac{J_1}{R_{11}^{(-1)}}, \\ K_2 &= J_2, \\ K_3 &= \left[\frac{1}{12}(63x^2-30x-1)R_{11}^{(-1)} - \frac{1}{6\varepsilon}x(x-1)(9x-1)\frac{d}{dx}R_{11}^{(-1)}\right]J_1 + x(x-1)(9x-1)R_{11}^{(-1)}J_3. \end{aligned} \quad (380)$$

$K = (K_0, K_1, K_2, K_3)^T$ is a basis of the full system with an ε-factorized differential equation

$$d_B K = \varepsilon A K, \quad (381)$$

where

$$A = \begin{pmatrix} -\frac{3}{x} & 0 & 0 & 0 \\ 0 & -\frac{(63x^2-30x-1)}{2x(x-1)(9x-1)} & 0 & \frac{6}{x(x-1)(9x-1)(R_{11}^{(-1)})^2} \\ \frac{4}{x} & -\frac{4R_{11}^{(-1)}}{x} & -\frac{4}{x} & 0 \\ \frac{2R_{11}^{(-1)}}{x} & \frac{(81x^4+1188x^3-594x^2+372x-23)(R_{11}^{(-1)})^2}{24x(x-1)(9x-1)} & -\frac{R_{11}^{(-1)}}{x} & -\frac{(63x^2-30x-1)}{2x(x-1)(9x-1)} \end{pmatrix}. \quad (382)$$

F. The four-loop equal-mass banana integral

As our final example, we consider the four-loop equal mass banana integral [57]. This is an example with a three-dimensional Baikov representation. The associated geometry is a Calabi-Yau threefold. The inverse propagators are defined by

$$\begin{aligned}
&\sigma_1 = -k_1^2 + m^2, && \sigma_2 = -k_2^2 + m^2, && \sigma_3 = -k_3^2 + m^2,\\
&\sigma_4 = -k_4^2 + m^2, && \sigma_5 = -(k_{1234} - p)^2 + m^2, && \sigma_6 = -(k_{123} - p)^2,\\
&\sigma_7 = -(k_{12} - p)^2, && \sigma_8 = -(k_1 - p)^2, && \sigma_9 = -k_{12}^2,\\
&\sigma_{10} = -k_{13}^2, && \sigma_{11} = -k_{23}^2, && \sigma_{12} = -k_{14}^2,\\
&\sigma_{13} = -k_{24}^2, && \sigma_{14} = -k_{34}^2,
\end{aligned} \tag{383}$$

with $k_{ij} = k_i + k_j$, $k_{ijk} = k_{ij} + k_k$, and $k_{ijkl} = k_{ijk} + k_l$. The graph is shown in Fig. 17. It will be convenient to discuss this example in $D = 2 - 2\varepsilon$ space-time dimensions.

Following the notation of sunrise and three-loop banana integrals, we define $x = m^2/(-p^2)$, $\sigma_6 = -p^2 z_1$, $\sigma_7 = -p^2 z_2$, and $\sigma_8 = -p^2 z_3$. The loop-by-loop Baikov representation under the maximal cut of sector 31 (with inverse propagator $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$) reads

$$e^{4\varepsilon\gamma_E}(-s)^{1+4\varepsilon}\int_{C_{\text{maxcut}}}\left(\prod_{i=1}^{4}\frac{d^D k_i}{i\pi^{1-\varepsilon}}\right)\frac{1}{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} = C_{\text{Baikov}}\int\left(\prod_{i=1}^{3}\frac{dz_i}{2\pi i}\right)U(z)|_{z_0=1}, \tag{384}$$

with

$$C_{\text{Baikov}} = e^{4\varepsilon\gamma_E}\frac{2^{8+8\varepsilon}\pi^6}{\Gamma(\frac{1}{2}-\varepsilon)^4}. \tag{385}$$

Setting $C_{\text{abs}} = \varepsilon^4$ ensures that $C_{\text{Baikov}} \cdot C_{\text{abs}}$ is of uniform transcendental weight zero.

The minimal twist function U for four loops reads

$$U(z_0, z_1, z_2, z_3) = P_0^{5\varepsilon} P_2^{\varepsilon} P_3^{\varepsilon} P_1^{-\frac{1}{2}} P_4^{-\frac{1}{2}-\varepsilon} P_5^{-\frac{1}{2}-\varepsilon} P_6^{-\frac{1}{2}-\varepsilon} P_7^{-\frac{1}{2}-\varepsilon}, \tag{386}$$

where

$$\begin{aligned}
&P_0 = z_0, \quad P_1 = z_1, \quad P_2 = z_2, \quad P_3 = z_3, \quad P_4 = z_1 + 4xz_0,\\
&P_5 = (z_2 - z_1)^2 + 2x(z_2 + z_1)z_0 + x^2 z_0^2,\\
&P_6 = (z_3 - z_2)^2 + 2x(z_3 + z_2)z_0 + x^2 z_0^2,\\
&P_7 = z_3^2 - 2(1-x)z_3 z_0 + (1+x)^2 z_0^2.
\end{aligned} \tag{387}$$

One can generalize the minimal twist function to l loops as

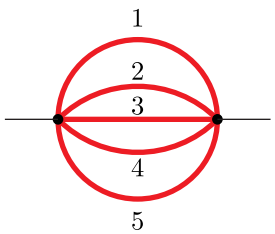


FIG. 17. The four-loop equal mass banana integral. Red lines are massive propagators.

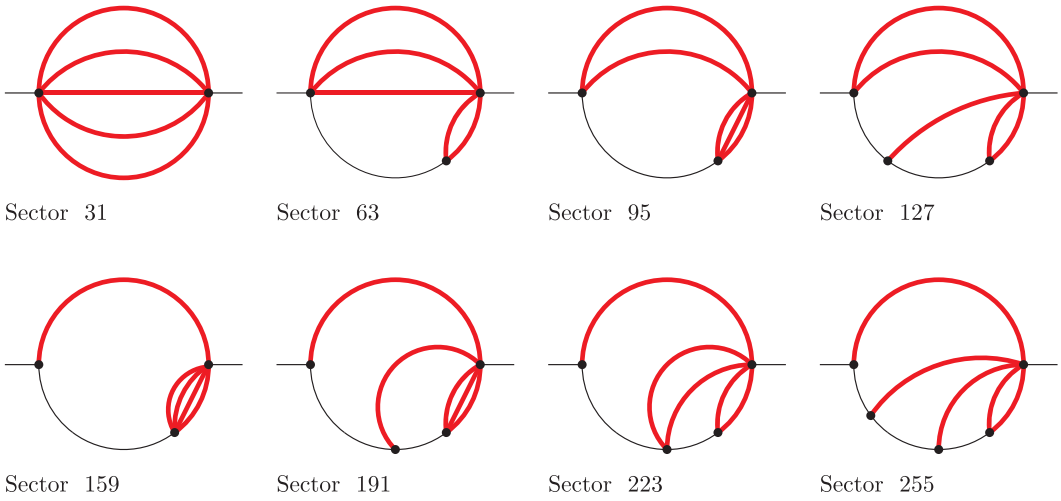


FIG. 18. The sector of the equal-mass banana integral at four loops together with the relevant supersectors.

$$U_l(z_0,z_1,\ldots,z_{l-1}) = P_0^{(l+1)\varepsilon} P_1^{-\frac{1}{2}} P_l^{-\frac{1}{2}-\varepsilon} \left(\prod_{i=2}^{l-1} P_i^{\varepsilon} P_{l-1+i}^{-\frac{1}{2}-\varepsilon}\right) P_{2l-1}^{-\frac{1}{2}-\varepsilon}, \tag{388}$$

where

$$\begin{aligned} &P_0 = z_0, \qquad P_1 = z_1, \qquad P_l = z_1 + 4xz_0, \\ &P_{2l-1} = z_{l-1}^2 - 2(1-x)z_{l-1}z_0 + (1+x)^2 z_0^2, \end{aligned} \tag{389}$$

and

$$\begin{aligned} P_i &= z_i, \\ P_{l-1+i} &= (z_i - z_{i-1})^2 + 2x(z_i + z_{i-1})z_0 \\ &\quad + x^2 z_0^2, \qquad 2 \le i \le l-1. \end{aligned} \tag{390}$$

The subscript in the twist in (388) manifests the loop number. One can find a generalization for the unequal-mass cases in a similar way.

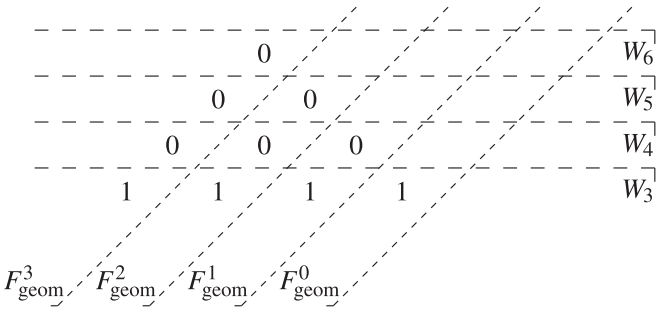
In the equal-mass four-loop case, $d_U = -4$; hence, $\hat{\Phi}$ must be homogeneous of degree 0. We can read off from Eq. (386) that there are three even polynomials $I^0_{\text{even}} = \{0,2,3\}$ and five odd polynomials $I^0_{\text{odd}} = \{1,4,5,6,7\}$. The dimensions of V^3 and H^3_ω do not match, and we find that

$$\dim V^3 = 4, \qquad \dim H^3_\omega = 25. \tag{391}$$

The mismatch is due to supersectors and symmetry relations. Figure 18 shows all relevant supersectors. We find that the supersector 95 has three extra master integrals and that super-sector 159 has one extra master integral. All the other supersectors have no extra master integrals.

The four-loop equal-mass banana integral has a high degree of symmetry, and, as a consequence, there are many symmetry relations (to be concrete, 17 in this example). This explains the high dimension of H^3_ω. As an outlook, let us note that one may incorporate symmetries directly in twisted cohomology following the ideas of [110,111]. This will improve the efficiency in examples which have a high degree of symmetry.

The decomposition of V^3 with respect to $(W_\bullet, F^\bullet_{\text{geom}})$ is



Step 2 of the algorithm proceeds similarly to the calculation of Ref. [57].

VI. CONCLUSIONS

In this work, we provided a detailed description of an algorithm outlined in Ref. [70]. This algorithm identifies a basis of master integrals whose differential equations are ε-factorized. The approach has the advantage of not requiring any previous knowledge of the underlying geometry associated with the maximally cut Feynman integral. The algorithm consists of two steps: first, the selection of master integrands that satisfy differential equations in Laurent polynomial form in ε and compatible with a filtration, and, second, the rotation of these differential equations to ε-factorized form.

In the first step, we perform integration-by-parts reduction within the framework of twisted cohomology,

supplemented with a specific ordering criterion in the Laporta algorithm. We observe that, for all tested cases, the corresponding system of differential equations exhibits a Laurent polynomial form with respect to the regulator ϵ. Even more constraining, the deepest pole in the differential equation is dictated by a certain combinatorial filtration of the basis elements, whereas the maximum exponent in the ϵ expansion is bounded to be one. We say that such a differential equation is $F^{\bullet}$ compatible.

In the second step of the algorithm, we employ a recursive procedure to find a transformation that leads to ϵ-factorized form. We prove that such a transformation can be systematically found if the starting differential equation is $F^{\bullet}$ compatible. At each step of this procedure, we transform the differential equations such that the deepest pole in ϵ is removed until all terms are ϵ-factorized.

We demonstrate the applicability of the method on a range of examples of varying complexity. In particular, we first show that the algorithm correctly produces a basis with desired properties in polylogarithmic cases, such as the massless double-box integral and the massless two-loop pentabox. More importantly, the algorithm is designed for cutting-edge Feynman integrals, characterized by nontrivial geometries with substructures, like an elliptic curve with additional punctures or a K3 surface with punctures. For the elliptic case, we present two nontrivial examples in Secs. V D and V E. The case of a K3 surface appeared in a separate publication [63].

Among possible future avenues, it would be interesting to establish, for step one, mathematical proofs of our findings. In particular, one can attempt to provide a proof of existence for our candidate basis following the dlog filtration of twisted cohomology introduced in [112].

From a practical standpoint, it will be interesting to stress test the algorithm against even more demanding examples with more scales, more loops, and more Baikov variables. Furthermore, the suggested algorithm provides an alternative setup for integration-by-parts reduction, in both the generation of equations and ordering criteria. It would be interesting to combine those with other efficiency improvements, e.g., the ones proposed in Refs. [26,113–115].

As a potential improvement of the algorithm, we can exploit the remaining degeneracy among the differential forms that are equal under the ordering criteria on the maximal cut but differ by lower-sector terms. Presently, we do not take advantage of this. This could yield additional gains in the performance of the algorithm or provide a more compact lower sector coupling coefficients. The latter seems to be hinted at in Ref. [63], where projecting onto the dot basis has resulted in the same expression on the maximal cut while offering a significant simplification of lower sector couplings.

An additional improvement can be the incorporation of symmetries directly in twisted cohomology following the ideas of [110,111]. This will improve the efficiency in examples which have a high degree of symmetry.

Furthermore, in Ref. [8], it has proven useful to find a self-dual form of the ϵ-factorized differential equations. This was demonstrated to reduce the number of implicitly defined functions in the system. It would be interesting to see whether this operation can be performed independently of the underlying geometry, as it would improve the performance of the second step of the algorithm.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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