

Relativistic quantum kinetic theory: Higher order contributions in assisted Schwinger pair production

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Quantum kinetic theory is an important tool for studying nonequilibrium, nonperturbative and nonlinear interactions within an open quantum system, and as such is able to provide an unprecedented view on particle production in the relativistic, ultrahigh intensity regime of quantum electrodynamics. By reorganizing the relativistic quantum transport equations for Abelian plasmas and integrating them with a perturbative expansion, we significantly expand the scope for kinetic theories to further elucidate the peculiarities of particle production at a spectral level.

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Introduction. Quantum electrodynamics (QED) is widely regarded as the best-tested physical theory conceived [1]. Its success is well illustrated by its consistency in measurements of value of the fine structure constant, $\alpha = e^2/4\pi$ and precision in its prediction of the electron's anomalous magnetic moment, $g - 2$, currently determined perturbatively to 4 loop order [2]. Still, since QED was first developed it has undergone a shift in focus regarding phenomenology, from measuring perturbative single-photon absorption and emission at its inception [3–10], to modern interpretations in cavity [11], circuit [12] and strong-field laser-matter interactions [13–15], testing effects such as vacuum polarization [16], vacuum nonlinearities [17,18] or vacuum decay [19,20]. Nevertheless, despite its success and wide applicability many areas of the theory remain unexplored, cf., recent reviews [21–23]. In particular, research on out-of-equilibrium nonlinear quantum many-body systems remains in its infancy [24–26]. Accordingly, a variety of techniques are used to study one of the theory's most prominent predictions: particle creation [27–31]. These include semiclassical techniques and numerical simulations of various complexity, cf. worldlines methods [32–35], WKB analysis [36–39], Bogoliubov theory [40–43], lattice theory [44,45] and scattering approaches [46–53].

In this article we are interested in the interplay between perturbative and nonperturbative QED at the mean-field level. The physical phenomenon under consideration is the (assisted) Schwinger effect [54], or the creation of electron-positron pairs by exposing the quantum vacuum to strong electromagnetic fields. Here we will consider this process occurring through a combination of nonperturbative quantum tunneling [27] produced by a strong background, combined with photon absorption from a secondary source [55,56].¹

The rationale behind this approach is that instead of a strong field doing the entire work to create a particle pair [57,58], it is known that the absorption of an additional photon from the secondary source serves to reduce the threshold value [59–65], see also the Franz-Keldysh effect in solids as its nonrelativistic counterpart [66,67]. In contrast, this article presents an examination of the distinctive signatures of higher-order effects in a weak-field expansion with respect to the secondary source, see also [68–70], but with particular focus on the particle spectrum [71–74].

We adopt an open quantum system approach, enabling the application of equal-time kinetic theories [75]. In the context of relativistic, adiabatic QED an initial vacuum state is driven out of equilibrium by an external, time-dependent electric background field to produce a particle pair [76–80].

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¹A brief note on terminology. In this paper, the term “Schwinger effect” will be used to refer to particle pair production via nonperturbative tunneling—a form of vacuum decay. High-frequency pair production will be understood as energy absorption by massive particles in well-defined quanta. This underscores that we work within mean-field theory, and therefore do not have quantized “photonic” degrees of freedom.

The formalism is highly general (unifying Vlasov and Bargmann-Michel-Telegdi equations into a single set of coupled differential equations nonlinear in $\hbar$), thereby enabling, *i.a.*, the calculation of the momentum spectrum of the produced particles irrespective of the formation mechanism [81–83]. The challenge thus far has been to distinguish between different contributions to pair creation, such as one- and two-photon effects or mutual assistance, given that this formalism reveals the entire particle spectrum at once. The aim of distinguishing between processes of different photon multiplicity aligns well with recent complementary work in the context of vacuum polarization in [84], where contributions associated to absorption of fixed numbers of photons from an intense laser were identified.

Here, we resolve this issue for pair creation by separating the vector gauge potential into two parts which allows the separation of the underlying transport equations into two sets of coupled equations without losing the nonlinear aspects of the theory. In this context, the procedure to obtain a perturbative expansion with respect to the secondary source is demonstrated.

This approach shares some similarities with the Furry picture of strong field QED [85], where the photon field is split into a fixed semiclassical background plus a quantum fluctuation. The former is treated nonperturbatively to define dressed fermion states, while the latter is treated in the usual perturbation theory. Here, we treat the primary part of the background field nonperturbatively and expand in the secondary field. This leads naturally to an expansion in a suitable nonlinearity parameter, $\xi = eE_{\text{cr}}/m\Omega$, where eE_{cr} is the magnitude of the weak electric field in terms of the critical Schwinger field strength $E_{\text{cr}} = m^2/e \approx 1.3 \times 10^{18}$ V/m and Ω its characteristic frequency. This “classical” nonlinearity parameter characterizes the likelihood of multiple interactions with photons from the weak field, well-illustrated by the exponential suppression of the pair creation amplitude for electric fields below the Schwinger field strength.

The structure of this paper is as follows. In Sec. II we present the relativistic quantum kinetic equations for linearly polarized, purely time-dependent electromagnetic fields in the mean-field approximation. Then we present their decomposition into strong field contributions with weak field perturbations, which allows them to be separated into two sets, and a perturbative expansion to be developed. In Sec. III, we then solve the corresponding differential equations numerically and discuss their significance for specific examples. A summary of our findings is provided in Sec. IV. Nonessential technical details and background information are included in the Supplemental Material [86]. Throughout the manuscript we use natural units, $\hbar = c = 1$ and express results in terms of the electron mass, $m = 511$ keV.

Theoretical formalism. Kinetic theories are widely employed to describe transport phenomena for systems in and out-of equilibrium. Their quantum mechanical counterparts [91], quantum kinetic field theories, extend their applicability into the high-energy regime [92–95].

Nevertheless, at the lowest order and in particular for Abelian plasmas, the Boltzmann transport equation, cf., [96], can be replaced by simpler equations of Vlasov type [97–99]. Furthermore, we assume that backreaction can be ignored [100], meaning that the dynamical field created by the charge carriers is negligible compared to the (essentially fixed) background field driving the time evolution. Finally, in homogeneous quantum systems, spatial diffusion is absent.

For the system under consideration energy is continually deposited by the environment, in this case, a time-dependent electric field, cf., [101,102]. We take the energy density of the field to be sufficiently high such that the particle number of the plasma is not a conserved quantity, as particles can be created dynamically over time (cf., Schwinger pair creation).

We begin by defining the quantities under consideration (see the Supplemental Material [86] for more details). We introduce a scalar function representing particle density (per volume of phase space) at fixed momentum, $f(t, \mathbf{p})$. This particle distribution function satisfies the quantum kinetic equation [103–107]

$$\dot{f}(t, \mathbf{p}) + e\mathbf{E}(t) \cdot \nabla_{\mathbf{p}} f(t, \mathbf{p}) = C[f](t, \mathbf{p}). \quad (1)$$

The operator C acts, *i.a.*, as a source term for nonperturbative particle pair creation. Specifically, for linearly polarized external fields $\mathbf{E}(t) = (E(t), 0, 0)$ it is found that (see Supplemental Material [86])

$$\begin{aligned} C[f](t, \mathbf{p}) &= Q(t, \mathbf{p}) \mathcal{W}[\cos](t, t_i, \mathbf{p}) \\ &= Q(t, \mathbf{p}) \int_{t_i}^t dt' Q(t', \mathbf{p}) (1 - f(t', \mathbf{p})) \\ &\quad \times \cos(2\Theta(t, t', \mathbf{p})), \end{aligned} \quad (2)$$

with field-dependent quantities Q and Θ , and the quantum statistical factor $(1 - f(t', \mathbf{p}))$ which facilitates, for example, Pauli blocking. Note that in this formalism the fermion dynamics are nonlocal in time, relying on the entirety of the quantum system’s history. Therefore, Eq. (2) represents a non-Markovian problem, which are notoriously challenging [108,109]. In this regard, a physical interpretation of the particle distribution’s nontrivial time evolution has been proposed only recently [110,111].

For the purposes of this article we first bring the quantum kinetic equation (1) into a more appropriate form. On the basis of the auxiliary quantities $\mathcal{G} = \mathcal{W}[\cos](t, t_i, \mathbf{p})$ and $\mathcal{H} = \mathcal{W}[\sin](t, t_i, \mathbf{p})$, cf. Eq. (2), Eq. (1) is transformed into a closed set of equations

$$\begin{aligned}
\dot{f} + e\mathbf{E} \cdot \nabla_p f &= Q\mathcal{G}, \\
\dot{\mathcal{G}} + e\mathbf{E} \cdot \nabla_p \mathcal{G} &= Q(1-f) - 2\omega\mathcal{H}, \\
\dot{\mathcal{H}} + e\mathbf{E} \cdot \nabla_p \mathcal{H} &= +2\omega\mathcal{G},
\end{aligned} \tag{3}$$

with the one-particle on-shell energy $\omega^2 = m^2 + \mathbf{p}^2$. In anticipation of a perturbative expansion, we introduce new functions, $\mathbb{s}$, $\mathbb{v}_{\parallel}$ and $\mathbb{v}_{\perp}$. In this way, for example the particle distribution is decomposed into

$$\omega f(t, \mathbf{p}) = m\mathbb{s} + p_{\parallel}\mathbb{v}_{\parallel} + p_{\perp}\mathbb{v}_{\perp}. \tag{4}$$

Similar expressions hold for $\mathcal{G}$ and $\mathcal{H}$. If instead formulated in terms of canonical momenta, $\mathbf{q} = \mathbf{p} + e\mathbf{A}$, we ultimately find the coupled equations

$$\begin{aligned}
\dot{\mathbb{s}} - 2q_{\parallel}\mathbb{v}_{\perp} + 2q_{\perp}\mathbb{v}_{\parallel} + 2e\mathcal{A}\mathbb{v}_{\perp} &= 0, \\
\dot{\mathbb{v}}_{\parallel} - 2q_{\perp}\mathbb{s} &= -2m\mathbb{v}_{\perp}, \\
\dot{\mathbb{v}}_{\perp} + 2q_{\parallel}\mathbb{s} - 2e\mathcal{A}\mathbb{s} &= +2m\mathbb{v}_{\parallel}.
\end{aligned} \tag{5}$$

Developing an expansion scheme: We start by decomposing the so-far arbitrary gauge potential $\mathbf{A} = \mathcal{A} + \mathbf{A}^{\gamma}$ into two vector potentials, corresponding to a spatially homogeneous pulse, with field strength $\mathcal{E}$, pulse duration $\mathcal{T}$, and temporal profile

$$\mathcal{A}(t) = \mathcal{E}E_{\text{cr}}t \exp(-t^2/\mathcal{T}^2), \tag{6}$$

and a secondary, oscillating field, also homogeneous with field strength ε , pulse envelope τ , frequency Ω , and profile

$$A^{\gamma}(t) = \varepsilon E_{\text{cr}}/\Omega \exp(-t^2/\tau^2) \sin(\Omega t), \tag{7}$$

such that $\Omega\tau \gg 1$. (In our gauge the temporal components of the gauge potentials vanish).

The Ansatz that allows us to split the complete phase-space information into components is as follows:

$$f = f^{(0)} + f^{(1)} + \sum_{i=1}^{\infty} f^{(2,i)}. \tag{8}$$

This expansion at the probability level holds for all constituents in the decomposition (4). For clarity, $f^{(0)}$ is associated with the phase-space distribution for free fermions, $f^{(1)}$ with a nonperturbative “dressing” by the first field, $\mathcal{A}$, and the remaining sum over the $f^{(2,i)}$ provides an order-by-order decomposition of the additional contributions from the presence of the oscillating field, A^{γ} [that is, $f^{(2,i)}$ and its components in the decomposition (4) are $\mathcal{O}(\xi^i)$]. This proposed perturbation scheme is, in principle, applicable as long as the Keldysh parameter $\gamma = 1/\xi$ of the oscillating field is larger than one [112].

We proceed order by order in ξ , recalling that A^{γ} is $\mathcal{O}(\xi)$. At zeroth order we find the algebraic relations

$$q_{\parallel}\mathbb{v}_{\perp}^{(0)} = q_{\perp}\mathbb{v}_{\parallel}^{(0)}, \quad m\mathbb{v}_{\perp}^{(0)} = q_{\perp}\mathbb{s}^{(0)}, \quad q_{\parallel}\mathbb{s}^{(0)} = m\mathbb{v}_{\parallel}^{(0)}. \tag{9}$$

We require that at the vacuum level we uniformly have no particle excitations. We find initial values, cf., Eq. (4),

$$\mathbb{s}^{(0)} = -m/\omega, \quad \mathbb{v}_{\parallel}^{(0)} = -q_{\parallel}/\omega, \quad \mathbb{v}_{\perp}^{(0)} = -q_{\perp}/\omega. \tag{10}$$

These solutions act as source terms for the components of $f^{(1)}$, which must satisfy

$$\begin{aligned}
\dot{\mathbb{s}}^{(1)} - 2q_{\parallel}\mathbb{v}_{\perp}^{(1)} + 2q_{\perp}\mathbb{v}_{\parallel}^{(1)} + 2e\mathcal{A}\mathbb{v}_{\perp}^{(0)} + 2e\mathcal{A}\mathbb{v}_{\perp}^{(1)} &= 0, \\
\dot{\mathbb{v}}_{\parallel}^{(1)} - 2q_{\perp}\mathbb{s}^{(1)} &= -2m\mathbb{v}_{\perp}^{(1)}, \\
\dot{\mathbb{v}}_{\perp}^{(1)} + 2q_{\parallel}\mathbb{s}^{(1)} - 2e\mathcal{A}\mathbb{s}^{(0)} - 2e\mathcal{A}\mathbb{s}^{(1)} &= +2m\mathbb{v}_{\parallel}^{(1)}.
\end{aligned} \tag{11}$$

Correspondingly, the equations at $\mathcal{O}(\xi)$ receive contributions from both lower-order results acting as sources

$$\begin{aligned}
\dot{\mathbb{s}}^{(2,1)} - 2q_{\parallel}\mathbb{v}_{\perp}^{(2,1)} + 2q_{\perp}\mathbb{v}_{\parallel}^{(2,1)} \\
+ 2eA^{\gamma}\mathbb{v}_{\perp}^{(0)} + 2e\mathcal{A}\mathbb{v}_{\perp}^{(2,1)} + 2eA^{\gamma}\mathbb{v}_{\perp}^{(1)} &= 0, \\
\dot{\mathbb{v}}_{\parallel}^{(2,1)} - 2q_{\perp}\mathbb{s}^{(2,1)} &= -2m\mathbb{v}_{\perp}^{(2,1)}, \\
\dot{\mathbb{v}}_{\perp}^{(2,1)} + 2q_{\parallel}\mathbb{s}^{(2,1)} - 2eA^{\gamma}\mathbb{s}^{(0)} - 2e\mathcal{A}\mathbb{s}^{(2,1)} \\
- 2eA^{\gamma}\mathbb{s}^{(1)} &= +2m\mathbb{v}_{\parallel}^{(2,1)}.
\end{aligned} \tag{12}$$

All higher orders in the expansion follow the same routine, thus we use index notation where $i \geq 2$,

$$\begin{aligned}
\dot{\mathbb{s}}^{(2,i)} - 2q_{\parallel}\mathbb{v}_{\perp}^{(2,i)} + 2q_{\perp}\mathbb{v}_{\parallel}^{(2,i)} + 2e\mathcal{A}\mathbb{v}_{\perp}^{(2,i)} + 2eA^{\gamma}\mathbb{v}_{\perp}^{(2,i-1)} &= 0, \\
\dot{\mathbb{v}}_{\parallel}^{(2,i)} - 2q_{\perp}\mathbb{s}^{(2,i)} &= -2m\mathbb{v}_{\perp}^{(2,i)}, \\
\dot{\mathbb{v}}_{\perp}^{(2,i)} + 2q_{\parallel}\mathbb{s}^{(2,i)} - 2e\mathcal{A}\mathbb{s}^{(2,i)} - 2eA^{\gamma}\mathbb{s}^{(2,i-1)} &= +2m\mathbb{v}_{\parallel}^{(2,i)}.
\end{aligned} \tag{13}$$

By truncating the last, infinite set of equations at a suitable order, we can thus calculate both nonperturbative and perturbative contributions to the particle density (8), in particular for asymptotic times (see Supplemental Material [86]).²

This procedure relies on numerically solving a comparatively simple system of differential equations, rather

²Although this procedure is presented as a perturbative expansion with respect to the oscillating field we note that due to the source terms in the differential equations, each level of the expansion can still include nonperturbative physics. And so are, for instance, quantum interference contributions an inherent aspect of the perturbative expansion at all levels. It is therefore important to interpret the results with caution, as only the total probability, summing over all contributions in f , represents a physical quantity.

than evaluating more complex analytical structures. We will show below that it can provide greater physical insight on contributions to the particle distribution function from a fixed number of interactions with the secondary field. Moreover, the hierarchical quantum kinetic theory (HQKT) developed here is sufficiently general to function in a wide variety of different regimes of pair production, thus complementing [65,113] and connecting with [59,60], while retaining the flexibility to go beyond these works.

Results and discussion. In past literature on the assisted Schwinger mechanism the primary focus has been on optimizing the secondary pulse profile to maximize the total particle yield, looking toward future experimental detection. In contrast, working at the spectral level permits differentiation of pair production rates as a function of particle momenta and can identify effects that may have already been integrated out in the total yield. As an example, one can consider the alternative perspective of identifying momenta at which *no* pairs are created [107].

To illustrate the advantages of the proposed formalism we consider a strong electric pulse of field strength $0.3E_{\text{cr}}$, cf. $\mathcal{A}(t)$ in Eq. (6), superposed by a fast oscillating, but weak linearly polarized light field, cf. $A^\gamma(t)$ in Eq. (7) with a field frequency of $\Omega = m$ and field strength $0.05E_{\text{cr}}$. Due to its high field strength, the first field induces non-perturbative particle creation. The second field with a Keldysh parameter of $\gamma = 20$ is demonstrably within the perturbative regime.

Calculation of the particle spectra for the combined field $\mathcal{A}(t) + A^\gamma(t)$, cf. Fig. 1, reveals the nonlinear nature of strong-field pair production. As expected, the number of created particle pairs $f_{\mathcal{A}+A^\gamma}$ is greater than the sum of the individual contributions $f_{\mathcal{A}} + f_{A^\gamma}$. However, the formalism developed here permits signals in the combined spectrum to be associated with the two fields separately, particularly multiple interactions with the oscillating field. We argue this can enhance understanding of the mechanisms governing the overall particle production. For example, Fig. 1 shows that for this scenario the most significant contributions are the first, $\mathcal{O}(\xi^1)$, and second order terms, $\mathcal{O}(\xi^2)$, from the perturbative expansion.

To give a more accessible interpretation of the HQKT results, a semiclassical picture may be illustrative. From this perspective, a high-intensity background field enables the particle pairs to come into existence via quantum tunneling through the rest-mass threshold energy barrier, while perturbative interactions with the oscillating field can cause the particles to gain energy and cross over the mass threshold of $2m$. When combined, the net effect is a conversion of energy from the electromagnetic field into fermion rest-mass and kinetic energy, with the assisting field lowering the effective barrier for tunneling to occur.

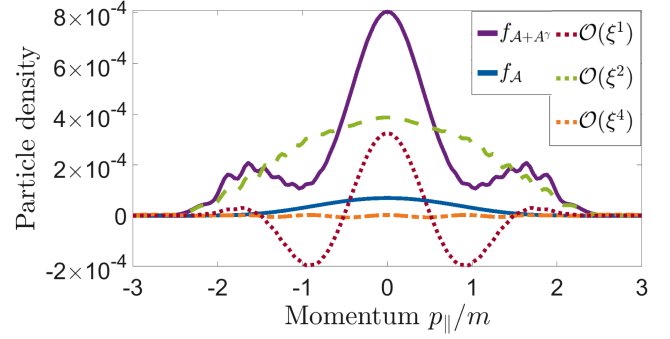


FIG. 1. Spectral plot of the asymptotic particle density against the momentum $p_{\parallel}$, with both vector potentials (purple) compared to the strong-field (blue) contribution, with $p_{\perp} = 0$. The dominant contributions to the full spectrum come from absorbing one energy quantum (green) and its interference with the strong-field (red) (note this lowers the particle density for small $p_{\parallel}$). The orange curve corresponds to the perturbative contribution of order $\mathcal{O}(\xi^4)$, i.e., absorbing two quanta of energy. Field parameters: $\mathcal{E} = 0.3$, $\varepsilon = 0.05$, $\Omega = m$, $T = \tau = 20/m$.

In the specific case of Fig. 1, we interpret the contribution $\mathcal{O}(\xi^2)$ to the result obtained as the electron-positron pair being “activated” through absorbing one quantum of energy from the oscillating field so that its effective tunneling mass is reduced by Ω . Consequently, the effective threshold is only m , making subsequent quantum tunneling much easier. Furthermore, as this effect relies on a strong, slowly varying field as a background for the energy transfer, and that field is still capable of producing particles through quantum tunneling, we also identify interference between nonperturbative Schwinger and dynamically assisted particle production. As such, quantum interference at $\mathcal{O}(\xi)$ contribute significantly toward the spectrum.

The application of HQKT offers further insight when higher-order perturbations are significant. In this case, even a relatively weak, slowly varying field can significantly enhance pair production if the oscillating field “activates” the pair to be just below the threshold energy [60]. In Fig. 2 the frequency of the oscillating field, $A^\gamma(t)$, is $\Omega = 0.9m$. Therefore, only the signal resulting from the absorption of three field quanta, associated with the peaks at $p_{\parallel} \sim \pm m$, is fully developed; pair production through the absorption of two field quanta is suppressed at $p_{\parallel} \sim \pm 0.1m$. However, when an additional pulse, $\mathcal{A}(t)$, is applied the resulting particle spectrum exhibits a significant alteration, particularly in the vicinity of vanishing momentum.

Intuitively, the absorption of two field quanta results in an energy gain of $1.8m$, which in turn leads to an effective mass that is only ten percent of the particle pair mass in the context of quantum tunneling. Consequently, a much lower field strength, exemplified in this scenario by a pulse $E = 0.01E_{\text{cr}}$, can induce tunneling. In contrast, for the specific scenario shown in Fig. 2, the particle distribution at

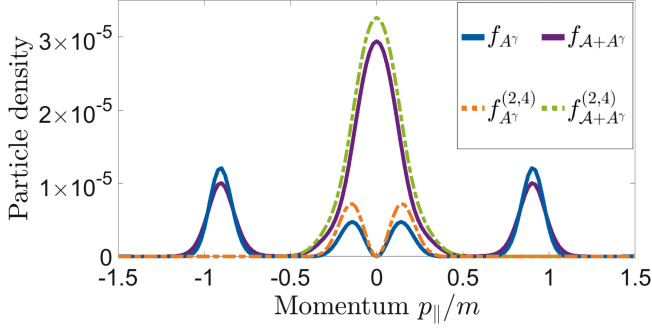


FIG. 2. Asymptotic particle density as a function of momentum $p_{\parallel}$ with $p_{\perp} = 0$ for an oscillating field, $A'(t)$, only (blue and orange lines), and in conjunction with a weak, slowly varying field, $A(t) + A'(t)$, (purple and green lines). In both cases, the total particle yield is shown as a solid line and the particle density at order $\mathcal{O}(\xi^4)$ are displayed as dashed lines. Field parameters: $\mathcal{E} = 0.01$, $\varepsilon = 0.1$, $\Omega = 0.9m$, $T = \tau = 20/m$.

large momenta $p_{\parallel} \approx \pm m$ is hardly affected by the addition of the auxiliary field, since the absorption of, e.g., three field quanta is already pushing the pair above the threshold.

Another crucial detail is that for purely oscillating fields, symmetry constraints in n -photon pair production can prevent the final particle distribution from being nonzero at certain momenta and for certain field frequencies. By simply adding a secondary field, this underlying symmetry is broken and, as a consequence, such restrictions are lifted. As a result, in Fig. 2 the particle spectra at $p_{\parallel} = 0$ are qualitatively different.

Access to the particle spectrum and the ability to distinguish contributions permit a more detailed examination of the role played by higher-order contributions in Schwinger pair production. So far, we have analyzed effects where tunneling particle creation was aided by an n th order perturbative process, or vice versa. In contrast, in Fig. 3, we have displayed results where it is the interplay of one- and two-energy packet absorption that drives particle creation.

In this case, with the frequency of the secondary field of $\Omega = 0.7m$, the field $A(t)$ occupies a “Goldilocks zone,” exhibiting sufficient strength to provoke a sizeable increase in total particle yield compared to the unassisted scenario, while remaining too weak to entirely dominate the pair production process. Accordingly, assistance through the absorption of a single quantum of energy from the oscillating field yields a substantial contribution. However, another fraction originates from pairs that absorb two quanta of energy. Without the field $A(t)$ the latter would be a less probable route to pair production due to its scaling by fourth order in the field strength. Nevertheless, the effective mass of such a pair is $2m_{\text{eff}} = 2m - 1.4m$ and therefore sufficiently small that in the second stage of the formation process it has a distinct advantage over the alternative.

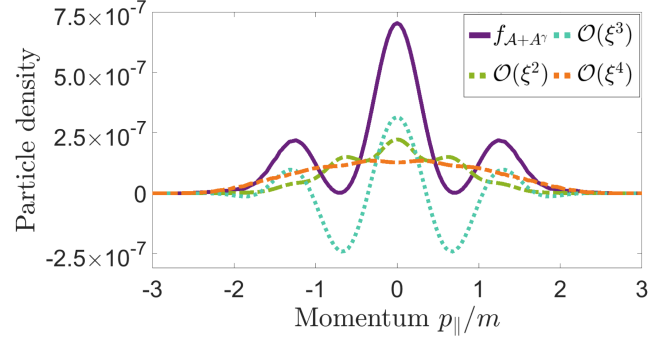


FIG. 3. Asymptotic particle density as a function of momentum $p_{\parallel}$ with $p_{\perp} = 0$ for both fields, $A(t) + A'(t)$. The total particle yield is shown as the solid purple line. In comparison, the contributions from orders $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^4)$ are displayed as green, dashed and orange, dashed lines, respectively. The most important strong-field interference term is of order ξ^3 and displayed in teal. Field parameters: $\mathcal{E} = 0.15$, $\varepsilon = 0.02$, $\Omega = 0.7m$, $T = \tau = 40/m$.

Furthermore, interference has a substantial impact on the particle spectra, e.g., at $p_{\parallel} \approx \pm 0.72m$ in Fig. 3 the particle distribution is reduced by one order of magnitude. The latter aspect represents another significant advantage of HQKT, as this feature is not discernible in the total yield. To illustrate, in this specific scenario, roughly 99 percent of the total yield is concentrated within orders ξ^2 and ξ^4 . Additional figures showing how the relative contributions from the competing pair creation mechanisms varies with field strengths are provided in the Supplemental Material [86] (cf. Fig. S4.).

Conclusion. We have shown how ideas from perturbation theory can be applied at the level of *probabilities* within the mean-field approximation in relativistic quantum kinetic theory, without sacrificing access to nonperturbative physics. This led to a hierarchical approach to quantum kinetic theory characterized by an infinite tower of quantum kinetic equations. These can be solved order by order in a natural expansion related to a secondary (in our case oscillating) field. This approach gives physically intuitive results at each order in that field’s “classical” nonlinearity parameter.

In this regard, we have analyzed the momentum spectra of created particles in terms of both perturbative and nonperturbative contributions. We have shown how this approach can be applied in three examples of dynamical assistance to the particle pair creation process. Not only can the overall particle yield be increased, but we find that the superposition of an oscillating field with a weak, slowly varying pulse can lead to particle production in an otherwise forbidden momentum region. Conversely, quantum interference between different paths of assistance—and indeed in interplay with the strong background field—has the potential to alter the spectrum, creating regions where the occupation probability is drastically modified. In the context

of strong field QED, analogous interference effects that contribute to back reaction onto the background field were studied for the radiated waveform in [114].

The method of splitting the particle distribution function into components and then expanding the transport equations is also applicable to more complex scenarios and field geometries. In this respect, the approach introduced in this article may well become the starting point for further explorations regarding new perturbative approaches to theoretical calculations in strong-field quantum physics.

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Data availability. The data that support the findings of this article are openly available [115].

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