

Eigenvalue distribution from bootstrap estimates

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The bootstrap method has proven useful for a wide range of matrix models. Here, we show that the computed moments can be used to reconstruct the underlying eigenvalue probability distribution, which, in turn, allows us to compute the free energy of the model—a necessary quantity for identifying the thermodynamically preferred solution. We verify the method on the well-studied quartic potential model and then apply it to a recently analyzed asymmetric multitrace model. We consider an extended class of possible solutions and demonstrate that free-energy analysis reliably selects the correct one, making it an essential tool for studying models with a complex solution structure.

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I. INTRODUCTION

Various physical systems can be expressed in terms of matrices of a certain type, for example, Hermitian matrices. When studying a model, for example, from a statistical or quantum mechanical point of view, integration over an ensemble is performed, which in the case of matrix models is an integration over the space of matrices. There are many approaches to performing this task; sometimes the model is accessible by (semi)analytical approaches, and sometimes one is required to perform numerical simulations; see [1,2], and references therein. Recently, the bootstrap method gained popularity [3]. While it requires some numerical steps, it seems to be able, at least in some cases, to obtain high-precision results with modest computational resources. The basic idea is that one sets up a recurrent equation for the moments of the underlying distribution of the studied field and has a requirement that the solution has to satisfy; this requirement does not have to be interesting from a physical point of view. Then, one needs to scan the space of a small number of initial moments, use the recurrent equation to compute more of them, and then check if they satisfy the set requirement or not. In a series of papers [3–22], this method has been proven to work well for various (not only) matrix quantum mechanical problems.

Matrix models often appear in the context of quantum spaces, and this is the point of view of interest of the authors. In these models—and in many others—one is

often interested in observables that depend only on the matrix eigenvalues and not on the entire matrices. In some cases, referred to as pure-potential matrix models, the entire action is in the form that depends only on the eigenvalues. This, however, does not mean that the other matrix degrees of freedom are not important, as one still needs to perform integration over all of them. The integration over all degrees of freedom can be split into the integration over the eigenvalues and over the rest of the variables, and the other integration can be performed, producing an effective term in the action that can be interpreted as a logarithmic repulsion between the eigenvalues. This new term leads to interesting properties that have been studied before [23]. For the content of this paper, we need just one fact: The behavior of pure-potential matrix models is entirely captured by a single function—the eigenvalue distribution from which everything else can be computed. For some other models, such as fuzzy space models, this function does not capture every relevant aspect, but it is still valuable for analyzing the model, for example, from the perspective of its phase space structure [24–29].

The focus of this paper is on the reconstruction of the eigenvalue distribution from its moments that are accessible by the bootstrapping algorithm. For this, we use the method developed by Tekel and Cohen [30]; other methods are available, but this one is straightforward and works well for our purpose. We show the usefulness of this construction by considering an asymmetric multitrace pure-potential model that has been shown recently to have a coexistence of various solutions, that is, various local minima of free energy, of which only one is the thermodynamically preferred [31,32]. While the bootstrap algorithm finds many (all, in some limit) of these solutions, the reconstructed eigenvalue distribution allows us to compute the free energy and find the preferred one. From this point of view, constructing the eigenvalue distribution is a crucial

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step in the analysis of the models with a complex structure of free-energy minima.

This paper is organized as follows. First, we briefly summarize the bootstrap method and the eigenvalue density reconstruction from the moments. Then, we show the method on a well-studied pure-potential quartic model to show that the method recovers the known results. Finally, we apply the results to the recently studied findings of an asymmetric multitrace matrix model.

II. MATRIX MODELS AND THEIR MOMENTS

We will consider a matrix ensemble [1,28,33] with the partition function of the form

$$Z = \int e^{-N^2 S(\phi)} d[\phi], \quad (1)$$

where $[\phi]$ means the integration goes over all matrix elements—we will consider a set of Hermitian matrices of size N ; other choices also appear in the literature. The degrees of freedom of the model are, therefore, the elements of the matrix ϕ . The observables one is interested in are often expressed as traces of some function of ϕ , and their mean value is taken to be

$$\langle f \rangle = \frac{1}{Z} \int f(\phi) e^{-N^2 S(\phi)} d[\phi]. \quad (2)$$

The simplest version of the action is a trace of some function of ϕ , that is,

$$S(\phi) = \frac{1}{N} \sum_{n=0}^N g_n \text{Tr}(\phi^n); \quad (3)$$

however, later in the paper, we will also consider a multitrace model. In some situations, one can introduce terms with additional (usually) fixed matrices that add a physical context to the model, for example, a kinetic term of the form $\sim \text{Tr}[L_i, \phi]^2$, where L_i are fixed matrices, in fuzzy space models.

For some models, the action does depend only on the eigenvalues of ϕ , denoted λ . However, one cannot just ignore the rest of the degrees of freedom, as integration initially went over all matrix elements, and the change of variables produces a Jacobian that can be added to the action as an additional term in the action:

$$S_J(\lambda) = -\frac{2}{N^2} \sum_{i < j} \log |\lambda_i - \lambda_j|. \quad (4)$$

Together, the total free energy is

$$S_{\text{eff}} = S(\lambda) + S_J(\lambda). \quad (5)$$

An important set of observables is the moments of the eigenvalue probability distribution

$$m'_n = \frac{1}{N} \langle \text{Tr}(\phi^n) \rangle = \int_C \lambda^n \rho(\lambda) d\lambda, \quad (6)$$

where λ denotes the eigenvalues of ϕ , $\rho(\lambda)$ is its probability distribution with respect to (1), and C denotes the support of the distribution.

The recurrent relation between the moments is given by the Dyson-Schwinger equation

$$\int \frac{\partial}{\partial \phi_{ij}} ((\phi^k)_{ij} e^{-N^2 S(\phi)}) d[\phi] = 0, \quad (7)$$

where $k \geq 1$. This yields, for pure-potential single-trace actions $S(\phi) = \frac{1}{N} \text{Tr} V(\phi)$, the following:

$$N \langle \text{Tr}(\phi^k V'(\phi)) \rangle = \sum_{q=0}^{k-1} \langle \text{Tr}(\phi^q) \rangle \langle \text{Tr}(\phi^{k-1-q}) \rangle. \quad (8)$$

Depending on the particular form of the action, some of the moments can be expected to vanish. The first one is a normalization factor, which is set $m'_0 = 1$; the rest is then computed using Eq. (8).

The second step is to find a constraint that can be used to discard the inconsistent sets of moments m'_i . Usually, in the context of quantum-mechanical matrix models, one can devise a set of positive observables [3]. Here, we use a different method that is closer to what we seek to construct later on, that is, the underlying eigenvalue distribution. The Hamburger problem states that there exists a sequence of moments m'_n such that, for a positive Borel measure μ , the following relation holds:

$$m'_n = \int_{-\infty}^{\infty} x^n d\mu. \quad (9)$$

This condition is satisfied if and only if the matrix constructed from these moments m'_n is positive semi-definite:

$$M_C = \begin{pmatrix} m'_0 & m'_1 & m'_2 & \cdots \\ m'_1 & m'_2 & m'_3 & \cdots \\ m'_2 & m'_3 & m'_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0. \quad (10)$$

This matrix is known as the *Hankel matrix*, and the semidefiniteness condition can equivalently be expressed as

$$\sum_{i,j \geq 0} m'_{i+j} c_i c_j \geq 0, \quad (11)$$

for all sequences $\{c_i\}$ with only finitely many nonzero elements. By computing determinants of various submatrices of Fig. 8, one obtains various constraints against which the obtained set of moments can be tested—and usually discarded.

Having both the recurrence relation (8) and the constraint from (10), we need to specify which of the moments will be considered initial and which are going to be computed; the number depends on the order of the recurrence relation. One then scans over the space of initial values of the moments and verifies which values of initial moments a set of consistent, that is, constraint-satisfying, set.

III. RECONSTRUCTING THE PROBABILITY DISTRIBUTION

The Hamburger problem that we have mentioned already ponders the existence of a probability distribution given a set of moments. The goal is to have a method that takes an initial probability distribution with known moments and deforms it to a different distribution with desired moments. Here, we follow the method to construct it and investigate some details relevant to the application of matrix models under current consideration [30,34]. To begin with, let us define a set of orthonormal functions for $n \geq 0$ as

$$u_n(x) = \frac{1}{\sqrt{N_n}} \sqrt{w(x)} L_n(x), \quad (12)$$

satisfying

$$\int u_n(x) u_m(x) dx = \delta_{nm}, \quad (13)$$

where $L_n(x)$ are polynomials orthogonal under the weight $w(x)$:

$$\int w(x) L_n(x) L_m(x) dx = N_n \delta_{nm}. \quad (14)$$

Here, N_n is a normalization constant. These polynomials serve as a basis for functions on some interval

$$f(x) = \sum_{n=0}^{\infty} c_n u_n(x), \quad c_n = \int f(x) u_n(x) dx. \quad (15)$$

The moments corresponding to the weight $w(x)$ are

$$m_n = \int x^n w(x) dx. \quad (16)$$

Now, given a set of moments $m_0, \dots, m_n$ we can construct a set of orthogonal polynomials:

$$K_n(x) = \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}, \quad (17)$$

that has $K_0 = m_0 = 1$, where $m_0 = 1$ is the normalization condition, and that satisfy

$$\int w(x) K_n^*(x) K_m(x) dx = N_n \delta_{nm}, \quad (18)$$

where the normalization factor N_n is

$$N_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{vmatrix} \times \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ m_2 & m_3 & \cdots & m_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix}. \quad (19)$$

Note the fact that the normalization factor N_n requires first $2n$ moments. Now, any probability distribution can be expanded this way:

$$P(x) = \sqrt{w(x)} \sum_{n=0}^{\infty} c_n u_n(x), \quad (20)$$

where the factor $\sqrt{w(x)}$ makes it so that the coefficients c_n depend only on the moments and one can express that $c_n = \frac{1}{\sqrt{N_n}} \langle L_n(x) \rangle$, where $\langle \cdot \rangle$ is obtained with respect to $P(x)$. Note that since L_n are polynomials, $\langle L_n(x) \rangle$ can be easily expressed in terms of moments of $L_n(x)$. As a result, we have

$$P(x) = w(x) \sum_{n=0}^{\infty} \frac{1}{N_n} \langle L_n(x) \rangle L_n(x). \quad (21)$$

We can now consider $w(x) = P(x)$. Also, we can take another distribution $P'(x)$ with moments m'_n under this weight and expand it in the same way:

$$P'(x) = P(x) \sum_{n=0}^{\infty} \frac{1}{N_n} \langle K_n(x) \rangle' K_n(x), \quad (22)$$

where

$$\langle K_n(x) \rangle' = \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ \int P'(x) dx & \int x P'(x) dx & \int x^2 P'(x) dx & \cdots & \int x^n P'(x) dx \end{vmatrix}. \quad (23)$$

Combining all of this, we obtain an expression for the distribution $P'(x)$:

$$P'(x) = P(x) \sum_{n=0}^M \frac{1}{N_n} \begin{vmatrix} 1 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & m'_1 & m'_2 & \cdots & m'_n \end{vmatrix} \times \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}, \quad (24)$$

where $M = \infty$ would yield an exact expression and finite M corresponds to an approximation. While the expression looks cumbersome, this allows us to construct the probability distribution $P'(x)$ with given moments m'_n starting from an initial distribution $P(x)$ with computable moments m_n . While, given a sufficient number of moments, any initial distribution will converge to the one with desired moments, there are choices that make this procedure more efficient—we will discuss those shortly.

IV. QUARTIC MODEL

Here, we consider perhaps the simplest nontrivial matrix model, that is, the one with a quartic potential. We begin

with reconstructing the work of [5] and extend it by considering asymmetric distributions. The action is

$$S = \frac{1}{N} \text{Tr} \left(\frac{1}{2} r(\phi^2) + g(\phi^4) \right). \quad (25)$$

The equation that allows us to connect different moments, Eq. (7), takes the form

$$\sum_{q=0}^{s-1} \langle \text{Tr}(\phi^q) \rangle \langle \text{Tr}(\phi^{s-1-q}) \rangle - N r \langle \text{Tr}(\phi^{s+1}) \rangle - 4gN \langle \text{Tr}(\phi^{s+3}) \rangle = 0, \quad (26)$$

where $s \geq 1$. This yields the following equation for the moments:

$$m'_{s+3} = \frac{\sum_{q=0}^{s-1} m'_q m'_{s-1-q} - r m'_{s+1}}{4g}. \quad (27)$$

We keep the prime symbol here to denote the known moments of an unknown distribution. The model has a known solution, and the eigenvalue distribution is either one-cut or two-cut, in both cases symmetric. The single-cut is

$$\rho_1(\lambda) := \frac{1}{2\pi} (r + 2gk + 4g\lambda^2) \sqrt{k - \lambda^2}, \quad (28)$$

for the interval $\lambda^2 < k$, where $k = \frac{1}{6g} (\sqrt{r^2 + 48g} - r)$, and the two-cut solution is

$$\rho_2(\lambda) := \frac{2g|\lambda|}{\pi} \sqrt{a^2 - (b - \lambda^2)^2}, \quad (29)$$

for the interval $(-\sqrt{b+a}, -\sqrt{b-a}) \cup (\sqrt{b-a}, \sqrt{b+a})$, where $a = \sqrt{\frac{1}{g}}$ and $b = -\frac{r}{4g}$. The model switches between those at the critical coupling $r = -4\sqrt{g}$. We will now reproduce this result using the bootstrap method. First, from Eq. (26) we obtain for this case the matrix (10) is the following:

$$M_C = \begin{pmatrix} 1 & 0 & m'_2 & \dots \\ 0 & m'_2 & 0 & \dots \\ m'_2 & 0 & \frac{1-rm'_2}{4g} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0. \quad (30)$$

Note that we put $m'_1 = 0$ for obvious reasons, and then the entire matrix depends only on m'_2 . The task is then to test different plausible values of m'_2 and see for which this matrix remains semidefinite.

Figure 1 further suggests that the choice of r significantly influences the width of these intervals, and this dependency must be considered: For smaller values of r , larger matrices were required, whereas for larger r , even 5×5 matrices were sufficient. Table I shows the comparison of the bootstrap estimates and exact values of the eigenvalue distribution for different values of parameters.

Once a suitable set of moments is obtained, we proceed with the method described in the previous section to construct the eigenvalue distribution; the examples where it is one-cut and two-cut are shown in Fig. 2. Construction of the eigenvalue distribution is the novel result of this paper, and, as will be later shown, it is necessary for choosing the thermodynamically preferred solution.

Importantly, the pace of convergence depends on the particular form of the initial distribution and its support. We have tested various options—specifically, constant, Gaussian, and parabolic—and we found them to be working comparably well, showing the robustness of the method. By increasing the number of moments used, for

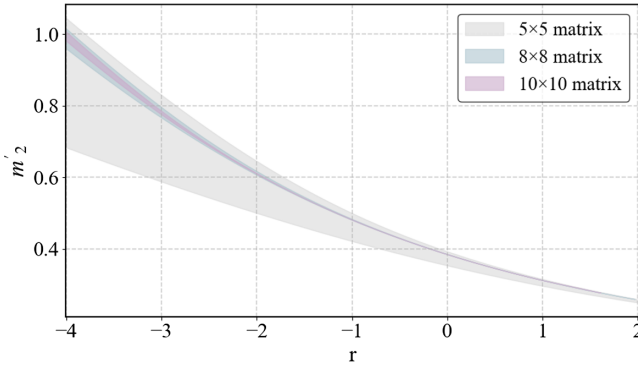


FIG. 1. Dependence of the second moment on the constant r . To determine specific values of m'_2 , it is necessary to consider larger matrices.

TABLE I. Comparison of exact calculations and the bootstrap method for different values of r and $g = 1$. The individual moments were obtained using a bootstrap matrix of size 25. At this size, the intervals of the moments were negligibly narrow, so the set of moments with the average m'_2 was selected.

Exact/BM, $g = 1$			
m'_n	$r = 1$	$r = -4$	$r = -7$
m'_0	1.0000/1.0000	1.0000/1.0000	1.0000/1.0000
m'_2	0.3125/0.3125	1.0000/0.9998	1.7500/1.7499
m'_4	0.1719/0.1719	1.2500/1.2498	3.3125/3.3123
m'_6	0.1133/0.1133	1.7500/1.7497	6.6719/6.6715
m'_8	0.0820/0.0820	2.6250/2.6245	14.0977/14.0969
m'_{10}	0.0630/0.0630	4.1250/4.1241	30.9050/30.9034
m'_{12}	0.0504/0.0504	6.7031/6.7016	69.7141/69.7095
m'_{14}	0.0415/0.0415	11.1719/11.1690	160.838/160.8260

each value of the parameters r and g , the interval of allowed m'_2 quickly shrinks, and the solution quickly converges. By repeating the same process for various values of the model parameters, we have identified either a one-cut or a two-cut solution (details are commented on in Appendixes A and B) and produced the phase diagram, which is shown in Fig. 3.

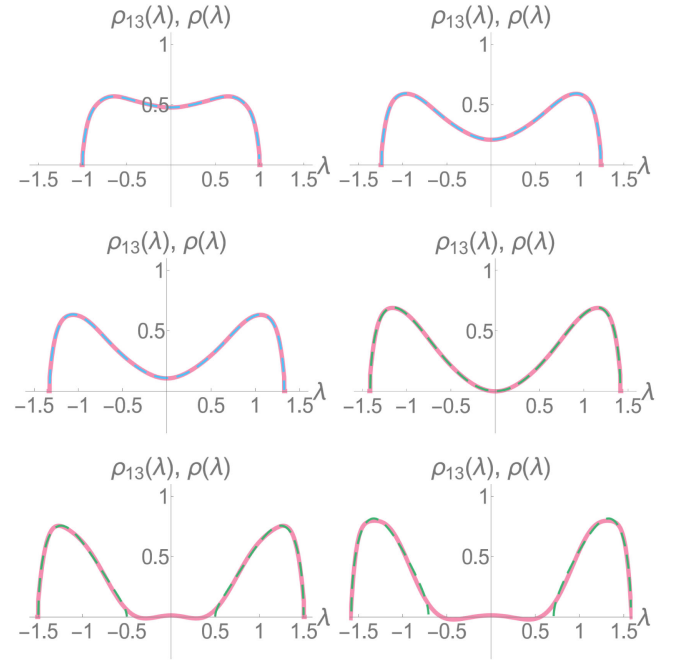


FIG. 2. Bootstrap-generated probability densities for $g = 1$. The approximate distribution $\rho_M(\lambda)$ (red solid line) is shown for $M = 13$ with bootstrap matrix size of 25. The one-cut symmetric solution (28) is plotted as a blue dashed line, and the two-cut symmetric solution (29) as a green dashed line. From upper left to bottom right, the plots correspond to $r = \{1, -2, -3, -4, -5, -6\}$. The approximate distribution was obtained using the expression (24) as $\rho_M(\lambda, t) = P(\lambda)(1 + 0 + (m'_2 - m_2)(m_2 - \lambda^2)(m_2^2 - m_4)^{-1} + \dots)$, where $P(\lambda)$ is the chosen initial distribution.

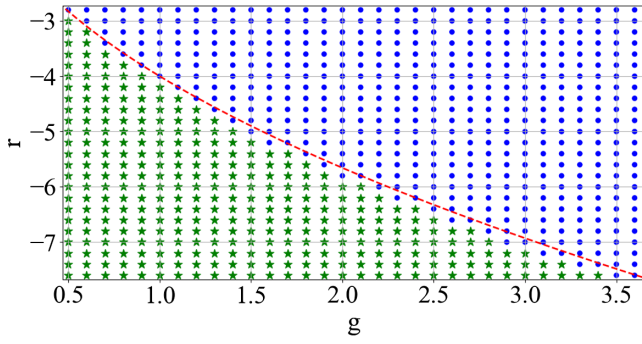


FIG. 3. Phase diagram illustrating the transition between the symmetric one-cut (blue dots) and two-cut (green stars) solutions. The red curve $r = -4\sqrt{g}$ marks the boundary separating the two phases.

This has all been done assuming the resulting solution is symmetric, as has already been done in the aforementioned Ref. [5]. Omitted was the option of asymmetric distributions. We consider it by not fixing $m'_1 = 0$ and making it the parameter from which we start the recurrent relation and whose values we scan over to find possible solutions. Contrary to the case of a symmetric solution, for some choices of parameters r and g , the set of allowed values of m'_1 does not converge to a single value; instead, we obtain an interval—see Fig. 4. For $r < -2\sqrt{15g}$, taking the end points of the interval, we obtain a single-cut asymmetric solution matching the analytic solution—see Fig. 5

$$\begin{aligned} \rho_3(\lambda) &= \frac{1}{2\pi} (4D^2g + 4Dg\lambda + 2\delta g + r + 4g\lambda^2) \\ &\quad \times \sqrt{(D + \sqrt{\delta} - \lambda)(\lambda - D + \sqrt{\delta})} \\ &= \frac{1}{2\pi} (4D^2g + 4Dg\lambda + 2\delta g + r + 4g\lambda^2) \\ &\quad \times \sqrt{\delta - (\lambda - D)^2} \end{aligned} \quad (31)$$

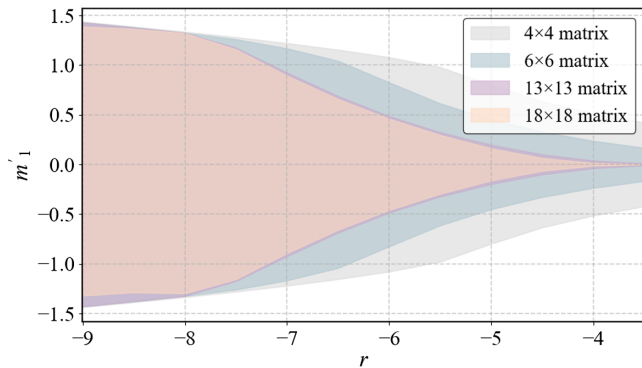


FIG. 4. Dependence of the first moment m'_1 on the constant r for $g = 1$. For $r > -4$, the values converge only to the symmetric solution ($m'_1 = 0$). For smaller r , however, progressively wider symmetric intervals $(-m'_1, m'_1)$ appear, indicating a richer structure of possible first moments than in the symmetric case.

TABLE II. Free energy for selected values of r at $g = 1$ for symmetric two-cut, asymmetric two-cut, and asymmetric one-cut solutions. For asymmetric two-cut solutions, multiple types of solutions exist; here, we show a few randomly selected examples.

Free-energy values for $g = 1$			
r	Sym. two-cut	Asym. two-cut	Asym. one-cut
-8	-3.233	-1.932	-1.895
-9	-4.353	-3.065	-2.885
-12	-8.218	-7.907	-6.657
-15	-13.36	-12.277	-11.598
-20	-24.289	-24.019	-22.382

for the interval $(D - \sqrt{\delta}, D + \sqrt{\delta})$, where

$$\begin{aligned} \delta &= \frac{-r - \sqrt{-60g + r^2}}{15g}, \\ D &= \pm \sqrt{\frac{-3r + 2\sqrt{-60g + r^2}}{20g}}. \end{aligned} \quad (32)$$

For values between those, when reconstructing the probability distribution for values in between, we obtain

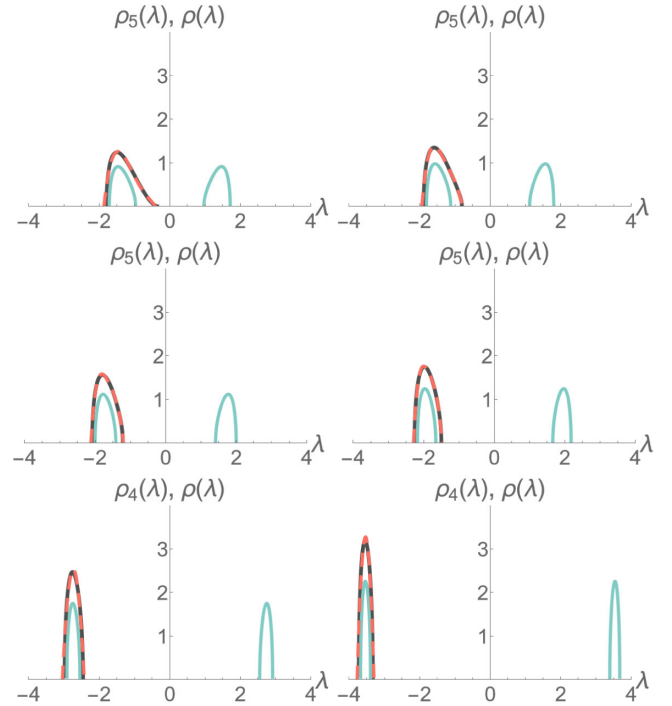


FIG. 5. Plot of the asymmetric one-cut solution obtained using the approximation method and moments generated via the bootstrap method (red dashed line), compared with the exact solution (black line), with a bootstrap matrix of size 15. The green line represents the symmetric two-cut solution. The plots are shown for $g = 1$ and, from left to right, for $r = \{-2\sqrt{15}, -9, -12, -15, -30, -50\}$.

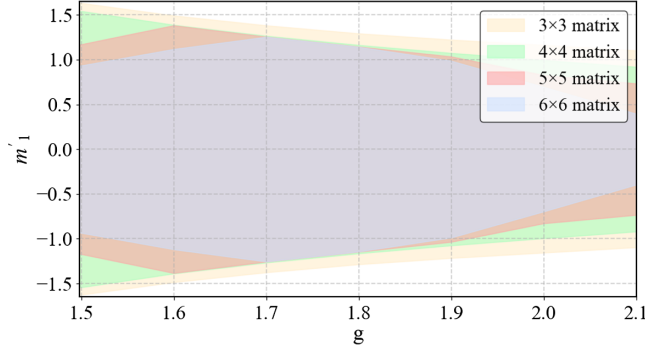


FIG. 6. Interval of suitable values for the first moment at $r = -5.265$ for various g and bootstrap matrix sizes M_C . As in the quartic case, a boundary emerges as the matrix size increases, toward which m'_1 converges. The iteration step used was 0.0005; for $g < 1.7$ or $g > 1.9$, smaller steps would be needed for larger matrices to approach the limiting line.

an asymmetric two-cut solution—see Fig. 5. To choose the correct solution among the symmetric and asymmetric solutions, we compute the free energy (5). The results are shown in Table II.

From this, we can conclude that the correct preferred solution with the minimal free energy is that of the symmetric solutions, namely, the one-cut and two-cut solutions, as shown in the phase diagram in Fig. 3.

To summarize the bootstrap study of the quartic potential model, the method is capable of reproducing the known results, and this effort is greatly enhanced by some choices of the initial probability distribution, and computation of free energy is needed to specify the correct solution. We now move to a far less-known model with an additional asymmetric potential.

V. ASYMMETRIC MULTITRACE MODEL

Let us consider the following model investigated in [31,32]:

$$S = \frac{1}{N} \text{Tr} \left(\frac{1}{2} r \phi^2 + g \phi^4 \right) + \frac{1}{N^2} t \text{Tr}(\phi) \text{Tr}(\phi^3). \quad (33)$$

The Dyson-Schwinger equations (7) give the following relation between the moments:

$$\begin{aligned} \sum_{q=0}^{k-1} \langle \text{Tr}(\phi^q) \rangle \langle \text{Tr}(\phi^{k-1-q}) \rangle - N r \langle \text{Tr}(\phi^{k+1}) \rangle \\ - N 4g \langle \text{Tr}(\phi^{k+3}) \rangle - t \langle \text{Tr}(\phi^3) \rangle \langle \text{Tr}(\phi^k) \rangle \\ - 3 \langle \text{Tr}(\phi) \rangle \langle \text{Tr}(\phi^{k+2}) \rangle = 0. \end{aligned} \quad (34)$$

As we are dealing with an asymmetric model, we cannot set $m'_1 = 0$ and we are left with two unspecified moments, m'_1 and m'_2 . We keep both of them unfixed and scan over their possible values—a two-dimensional parameter

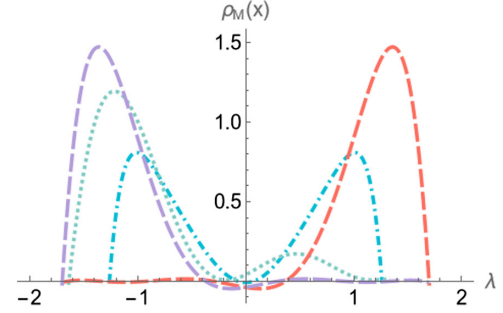


FIG. 7. Plotted approximate distributions $\rho_M(\lambda)$ for $g = 1.733$ and $r = -5.265$. The purple and red dashed lines represent the extreme values of the first moment, corresponding to one-cut asymmetric solutions on the right and left sides, respectively. We also observe the persistent presence of a symmetric two-cut solution (blue dash-dotted line), as well as several asymmetric two-cut solutions, one of which is shown as the green dotted line.

space (m'_1, m'_2) —and for each such parameter space we test given the condition:

$$M_C = \begin{pmatrix} 1 & m'_1 & m'_2 & \cdots \\ m'_1 & m'_2 & -\frac{3tm'_1 m'_2 + rm'_1}{4g+t} & \cdots \\ m'_2 & -\frac{3tm'_1 m'_2 + rm'_1}{4g+t} & \frac{1-rm'_2 - 4tm'_1 m'_3}{4g} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0. \quad (35)$$

Now, we can proceed as before—scan over a two-dimensional lattice of parameters r and g and find which of the values are permitted by Eq. (35). As in the case of asymmetric solutions of the quartic potential model, we find for different values of g an interval of allowed solutions that does not shrink with additional moments included; see Fig. 6. Again, this is caused by various allowed asymmetric and symmetric solutions; see Fig. 7. In Figs. 7–10 and Table III, we used bootstrap matrices of size 26 for the symmetric solutions and, as shown in the previous figure, smaller matrices of size 6 for the asymmetric solutions.

TABLE III. Free energy of the model (33) as a function of r and g .

M	$r = -5.265, g = 1.732$		$r = -5, g = 1.5$	
	Asym	Sym	Asym	Sym
2	-0.7461	0.0581	-2.3571	0.0078
4	-0.2644	-0.1163	-0.6141	-0.1727
6	-0.1424	-0.1483	-0.8541	-0.2033
8	-0.1442	-0.1492	-0.8202	-0.2042
10	-0.1441	-0.1494	-0.8170	-0.2043
12	-0.1439	-0.1494	-0.8156	-0.2043
14	-0.1431	-0.1494	...	...

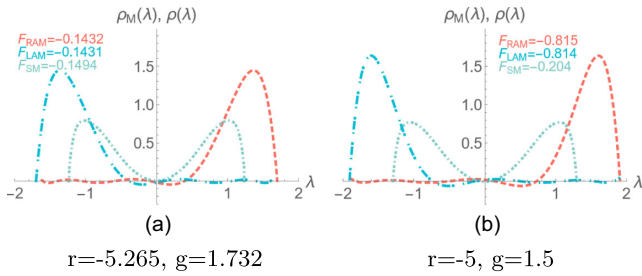


FIG. 8. Comparison of the free energies of the model (33): F_{RAM} (right asymmetric, dashed red line), F_{LAM} (left asymmetric, blue dash-dotted line), and F_{SM} (symmetric, green dotted line) for given r and g .

The parameter M used in the approximation method was set to 15.

Therefore, to find the correct solution, we again compute the free energy and find which solution minimizes it. Table III shows a quick convergence of the free energy as a function of M . In Fig. 8, we can see a comparison of the free energies for symmetric solutions and one-cut asymmetric solutions for specific values of r and g .

Doing this for every chosen value of r and g , we construct the phase diagram of the model (33), and we find it to be in good agreement with the one obtained in the previous study [32]. Note how easily the used method can be modified for a rather different form of matrix action.

The following graph focuses on the region where, according to the referenced article, a triple point, where all three phases coexist (symmetric one-cut, two-cut, and asymmetric one-cut), is expected to exist.

Looking at the triple point in Fig. 10, we can see that, in our approximate approach, it appears around the point $g = 1.75$, $r = -5.29$. In the article [32], the triple point was found at $g = 1.7325$, $r = -5.265$ and through Monte Carlo simulations at $g = 1.7706$, $r = -5.5432$. The value

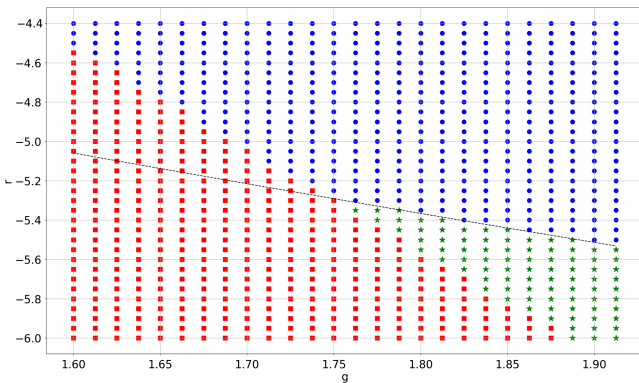


FIG. 9. Phase diagram for the cubic asymmetric multitrace matrix model. Symmetric solutions are represented by blue dots (one-cut) and green stars (two-cut), while the red squares correspond to asymmetric one-cut solutions. The black line indicates the transition between symmetric solutions, given by $r = -4\sqrt{g}$.

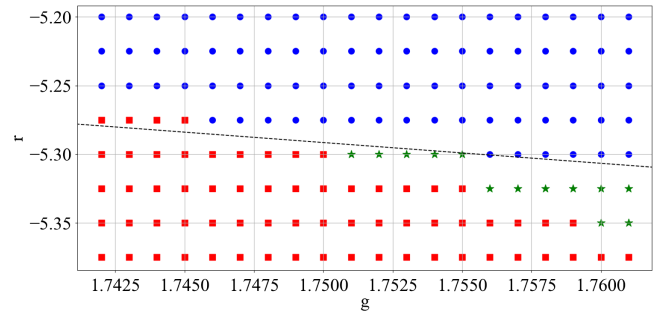


FIG. 10. Triple point for the cubic asymmetric multitrace matrix model, color coding as in Fig. 9.

obtained here is theoretically closer because the method naturally works in the large- N limit.

VI. CONCLUSION

We have extended the bootstrap method for matrix models by computing the eigenvalue distribution from the moments already being used in the procedure. This allows us not only to track phase transitions, but also to compute the free energy, which is necessary for selecting the correct, that is, the thermodynamically preferred, solution among the plethora found by the bootstrap method.

In more detail, we have focused on the quartic single-trace potential with a single (scalar) matrix model. This is one of the simplest and most studied examples, and we verified the method on it. Usually, the asymmetric solutions are discarded by hand by considering odd moments to vanish. A similar insight about the structure of solutions might be less accessible in complex models, and a method not relying on it is warranted. We have shown that, without these assumptions, one also recovers asymmetric solutions which are known to exist but also to have larger free energy than the symmetric ones.

This simple model allowed us to test the sensitivity of the result on the choice of initial probability distribution, which is being updated and transformed into the correct distribution by the method of [30]. We found all tested initial distributions to be working well. Also, the convergence of the free energy was quick, which explains why we were able to reproduce the phase diagram well and efficiently; see Table VII. In practice, a good approximation was obtained using 10–15 moments. Note that computing the moments is simple; checking the positivity is less so. However, the total computation time relies heavily on the size of the scanned parameter space, which can be reduced by various assumptions, such as the discussed parity of the solution. Contrary to this, the usual numerical approach of Hamiltonian Monte Carlo allows for fewer such simplifications [2].

We have utilized a particular form of the recurrent equation, the constraint, and the set of orthogonal

polynomials—each of those could have been chosen differently and should be, if it is proffered in the current study. We were searching for a distribution on a real line; for a bounded interval, one has the Hausdorff moment problem, for a semibounded interval, the Stieltjes moment problem, or the trigonometric moment problem for a unit circle [35].

We have then used our findings to study an asymmetric multitrace potential model recently analyzed in [32]. We were able to produce the phase diagram of the model by computing the free energy using the reconstructed eigenvalue distribution.

In summary, obtaining the eigenvalue distribution is a vital part of the bootstrap procedure, and we have shown how it is necessary to pick the correct solution among those found. We have also shown that constructing the distribution using a limited number of moments is of limited precision, and the result has to be understood as an approximation; for example, one can observe $\rho < 0$ due to this inaccuracy. Careful handling, however, still allows it to be used for obtaining valuable insight into the model.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [36].

APPENDIX A: IMPORTANCE OF THE INITIAL DISTRIBUTION

In applying the approximation method (22), we found that the choice of the initial distribution $P(x)$ influences the pace of convergence and the overall accuracy. Two aspects were tested: the normalization interval and the functional form of the initial distribution.

The support of $P(x)$ should approximately match that of the target distribution. If the interval is too narrow, the method diverges quickly and fails to approximate the desired density even for small values of M . If it is too wide, the approximation still converges but requires larger M , which can lead to numerical complications.

Figure 11 compare three initial functions—parabolic, constant, and Gaussian—normalized on intervals $[-1, 1]$, $[-2, 2]$, and $[-3, 3]$. The target density $P'(x)$ corresponds to

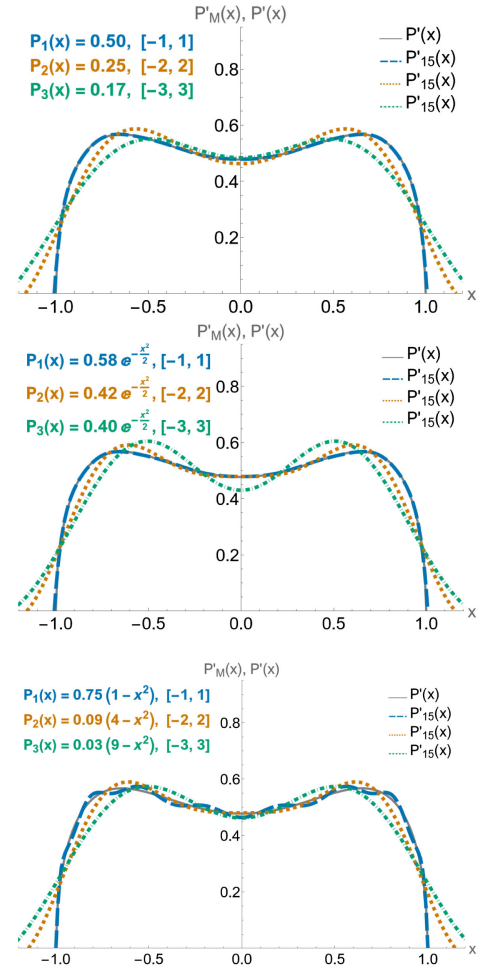


FIG. 11. Comparison of the approximation $P'_{M=15}(x)$ distribution with the target density $P'(x)$ for different initial functions and normalization intervals, obtained using Eq. (24). Each panel shows one example of an initial function or its corresponding approximation.

TABLE IV. Quadratic error of the approximation for different intervals and initial distributions.

Interval	Parabolic	Constant	Gaussian
$[-1, 1]$	0.00026	0.00001	0.00001
$[-2, 2]$	0.00189	0.00257	0.00209
$[-3, 3]$	0.00447	0.00508	0.00678

TABLE V. Quadratic error of the approximation for different intervals and initial distributions.

Interval	Constant	Gaussian	Parabolic
$[-1.9, 1.9]$	0.001457	0.001775	0.001110
$[-3, 3]$	0.014865	0.020695	0.014590
$[-4, 4]$	0.036175	0.043406	0.027935

TABLE VI. Quadratic error of the approximation for different values of M for symmetric and asymmetric cases.

Symmetric			
M	Constant	Gaussian	Parabolic
1	0.004849	0.002656	0.013637
3	0.002659	0.001934	0.001461
5	0.000034	0.000533	0.000624
15	0.000332	0.000041	0.000152

Asymmetric			
M	Constant	Gaussian	Parabolic
1	0.616616	0.796089	0.791123
3	0.093591	0.122212	0.188274
7	0.002506	0.009583	0.010113
15	0.000852	0.000737	0.000844

the one-cut symmetric solution (28) for $r = 1$ and $g = 1$, with the support over the interval $[-1, 1]$, shown as the solid gray line.

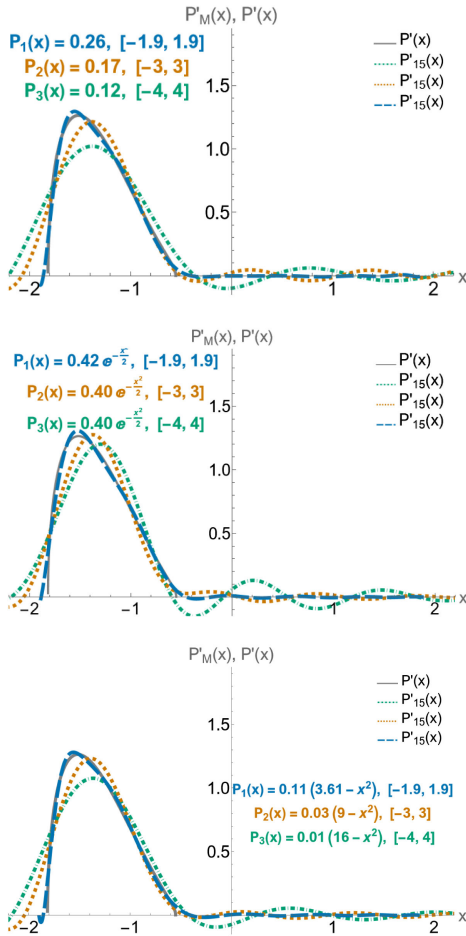


FIG. 12. Comparison of different initial functions and their corresponding approximations.

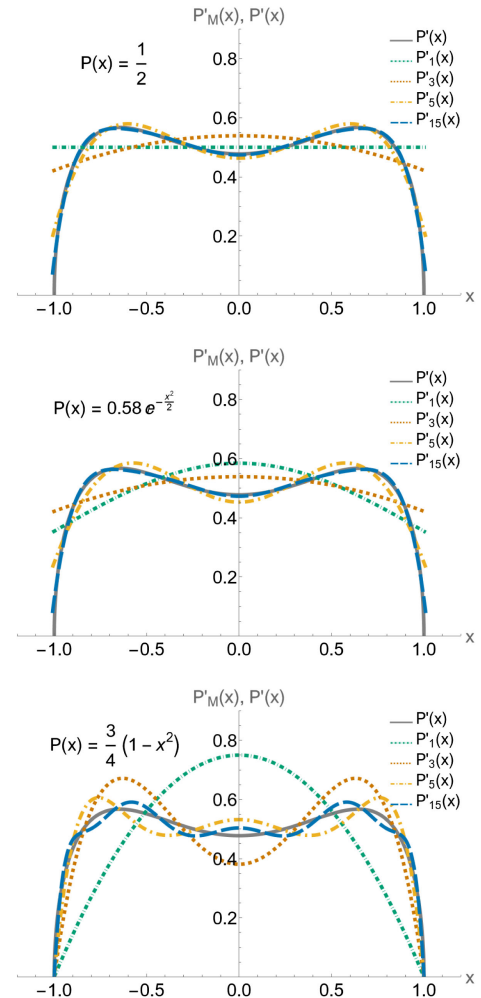
The quadratic error values in Tables IV–VI quantify this effect.

Figure 12 illustrates what happens in asymmetric solutions when the interval is varied for the constant, Gaussian, and parabolic initial distribution. The target density $P'(x)$ corresponds to the one-cut asymmetric solution (31) for $r = -8$ and $g = 1$, with support over the interval (32) indicated by the solid gray line.

These results clearly show that using the correct interval significantly improves convergence. Constant and Gaussian functions perform equally well in this respect, whereas the parabolic function is slightly less stable in some cases yet overall is comparable.

With an appropriate interval, the type of initial distribution also affects convergence. Figures 13 and 14 illustrate how the initial distribution functions behave for $M = 1, 3, 5, 15$.

Constant and Gaussian distributions converge rapidly toward $P'(x)$ even for small M . For Gaussian initial distributions, the parameter σ influences accuracy; see Fig. 15. Values of $\sigma \geq 1$ yield significantly better results

FIG. 13. Approximation $P'_M(x)$ compared with $P'(x)$ for various M and initial functions. Panels show the first three cases.

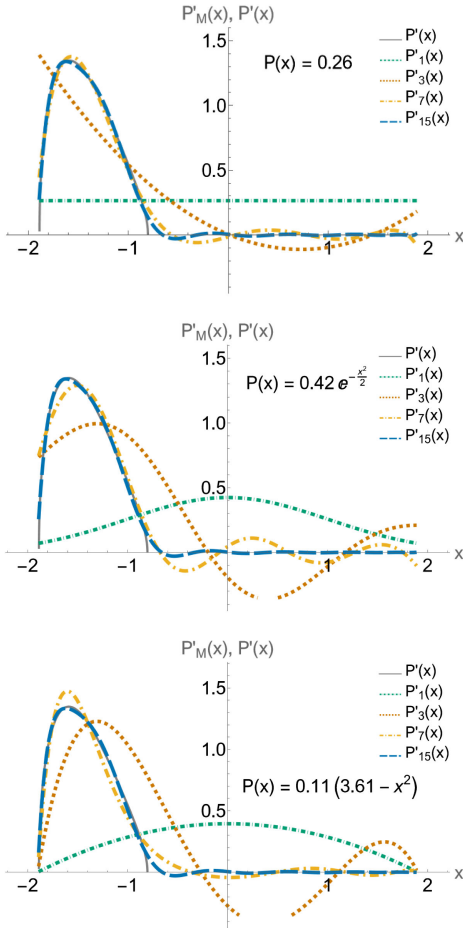


FIG. 14. Approximation $P'_M(x)$ compared with $P'(x)$ for various M and initial functions. Panels show the remaining three cases.

than smaller σ . In our calculations, a constant initial distribution was typically sufficient: Its convergence properties were comparable to the Gaussian case while avoiding the need to tune σ . Nonetheless, these comparisons highlight that both interval and functional form must be chosen carefully to ensure stability and accuracy of the approximation method.

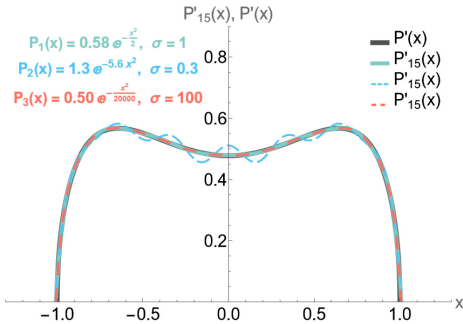


FIG. 15. Influence of the Gaussian parameter σ on approximation accuracy.

APPENDIX B: NUMERICAL DETAILS

Our calculations were performed using Python and Wolfram *Mathematica*. The code is structured into three main parts.

- (1) *Bootstrap method.* This part generates the moments required for the approximation method. The bootstrap procedure is sensitive to several factors, notably the size of the bootstrap matrix. For symmetric solutions, the relevant matrix sizes are shown in Fig. 4. Increasing the matrix size requires a corresponding reduction in the iterative step used for searching for the desired moments. For symmetric solutions, where only the second moment is needed, a smaller precision is sufficient. In contrast, when considering also asymmetric solutions, iteration over both m_1 and m_2 moments is required, demanding higher precision and smaller iterative steps. The matrix sizes needed for good convergence of m_1 in asymmetric cases are shown in Fig. 6.
- (2) *Approximation method.* The second part computes the approximate probability distribution. Here, the choice of the *initial distribution* and its normalized interval plays a crucial role. For symmetric solutions and one-cut asymmetric cases, we chose symmetric intervals. Typical parameters were $M = 10$ – 15 for symmetric solutions and $M \approx 8$ for asymmetric one-cut solutions. It is worth noting that, when a larger interval is chosen, one might expect that M could be increased correspondingly. However, in practice, convergence sometimes fails. The issue arises due to the normalisation in the relevant formula: For large M , the method involves dividing by small numbers, which can render the solution numerically unstable.
- (3) *Free-energy calculation.* Once the probability density is obtained, its free energy is computed. This process is repeated for various values of r and g , separately for symmetric and asymmetric solutions. The preferred solutions, which are then visualized in

TABLE VII. Free energy as a function of M ; notice that reasonably good estimates are obtained already for $M = 6$.

M	Asymmetric		Symmetric	
	$r = -8$	$r = -12$	$r = -4$	$r = -8$
2	-0.8037	-3.3436	-0.0634	-1.9327
4	-1.7259	-6.6704	-0.2326	-3.2395
6	-1.9032	-6.7358	-0.2632	-3.3089
8	-1.9015	-6.6686	-0.2640	-3.2468
10	-1.9013	-6.6665	-0.2642	-3.2481
12	-1.9009	-6.6647	-0.2642	-3.2468
14	-1.9008	-6.6621	-0.2642	-3.2452

a phase diagram, are determined by comparing free energies (see Table VII).

To distinguish between one-cut and two-cut solutions, we applied simple threshold-based criteria. For symmetric

solutions, a distribution was classified as two-cut if its value at $x = 0$ remained below 0.05; otherwise, it was considered one-cut. For asymmetric solutions, a distribution was classified as two-cut if it exceeded 0.01 on both sides; otherwise, it was considered one-cut.

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