

Self-gravitating quantum stars with a globally relevant Bohm potential

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The microphysics underlying nonbaryonic dark matter remains unknown. I derive the two-species Schrödinger-Poisson-Yukawa system for spin- $\frac{1}{2}$ dark-sector fermion fields, ψ (mass m_1) and χ (mass m_2), coupled through a scalar mediator of mass m_ϕ via a universal Yukawa coupling, within an orbital-free density-functional framework with the Kirzhnits gradient coefficient $\lambda_B = 1/9$. A central result is that the Bohm potential, far from being negligible in the Thomas-Fermi regime, contributes a species-dependent surface-energy correction analogous to the nuclear liquid-drop model; the heavier fermion species generates an outward quantum-pressure wall whilst the lighter species provides an inward surface tension, with degeneracy pressure furnishing the bulk confinement. In the single-species Schrödinger-Poisson limit the ground state recovers the benchmarked invariants $M_{\text{dim}} \simeq 3.883$ and $x_T \simeq 2.562$, yielding $MR_T \simeq 9.95\lambda_B\hbar^2/(Gm_1^2)$. For polytropic index $\gamma = 5/3$ the mass-radius relation satisfies $R \propto M^{-1/3}$; for $\gamma = 4/3$ a limiting mass emerges above which no stable equilibrium exists. Illustrative configurations span $M = 10^{-8} - 5M_\odot$, $m_1 \sim 10^{-14} - 10^{-6}$ eV, and radii from a few km to $\sim 10^3 R_\odot$, with gravitational-wave contact frequencies in the Einstein Telescope and LISA bands and microlensing signatures accessible to current surveys. The predictive rigidity of the resulting mass-radius relation, in which the single microphysical parameter m_1 determines the equilibrium radius once the total mass is specified, furnishes a reproducible, first-principles reference for constraining the dark-fermion mass in multicomponent dark sectors.

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I. INTRODUCTION

The case for nonbaryonic dark matter rests on independent and mutually reinforcing observational probes; the power spectrum of the cosmic microwave background [1], galaxy rotation curves [2], the universal density profiles of dark-matter halos [3], and strong-lensing observations, amongst others. Taken together, these measurements establish that roughly 27% of the energy density of the Universe resides in a gravitationally active component that neither emits nor absorbs electromagnetic radiation, yet the particle-physics identity of this component remains stubbornly unknown [4]. Were a fraction of that dark component locked in stable, self-gravitating compact objects rather than distributed as a diffuse particle bath, the dark matter problem would be partly recast from particle physics to compact-object astrophysics, with observational signatures accessible to gravitational-wave and microlensing surveys. One of the most instructive proving grounds for specific

dark-matter candidates is the small-scale structure of galactic halos, where collisionless cold dark matter simulations predict steeply cusped density profiles that conflict with the flatter cores inferred from dwarf-galaxy rotation curves [2,5].

Amongst the candidates that naturally suppress small-scale power, ultralight dark matter, whether bosonic (fuzzy/wave dark matter) or fermionic, has attracted considerable attention [6–8]. In the bosonic scenario a particle mass $m \sim 10^{-22}$ eV yields a de Broglie wavelength of order a kiloparsec, so that quantum-pressure effects smooth the central cusps collisionless models produce; in the fermionic case degeneracy pressure furnishes analogous support. The cosmological behavior of ultralight bosonic dark matter is well-described by the Schrödinger-Poisson (SP) system [9], and numerical simulations reveal that virialized halos harbor a dense solitonic core whose profile is the nodeless ground state of the SP eigenvalue problem [10,11].

The theoretical pedigree of self-gravitating quantum equilibria is considerably older; Kaup [12] studied the Klein-Gordon geon, Ruffini, and Bonazzola [13] established the general-relativistic framework for self-gravitating boson systems (see Liebling and Palenzuela [14] for a comprehensive review), and the Newtonian SP limit was placed on a rigorous footing by Lieb [15] and Moroz *et al.* [16]. Self-interacting extensions were

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analyzed by Chavanis [17] and Chavanis and Delfini [18], who obtained mass-radius relations for Newtonian condensates. The same ground-state soliton appears in the theory of self-bound dark-sector objects, where a Yukawa-type scalar interaction generates compact configurations held together by dark forces rather than gravity [19–21]; related nontopological solitons such as Q -balls [22] further illustrate the observational richness of dark-matter microphysics [23,24].

In a parallel line of investigation, dark-matter imprints on ordinary stars have been explored through gravitational capture and energy transport [25,26], with observable consequences for helioseismology [27–35], asteroseismology [36–38], and neutron-star structure [39–41]. Recent work has begun to connect these stellar-scale diagnostics with the broader program of characterizing exotic compact objects in the dark sector [42,43]. The present work extends this program to the Newtonian regime, deriving the equilibrium structure of self-gravitating dark-fermion configurations from first principles within the quantum-hydrodynamic framework.

In the quantum-hydrodynamic (QHD) description, the support provided by wave-mechanical effects is encoded by the Bohm potential, a gradient correction to the classical Euler equation first derived in hydrodynamic form by Madelung [44] and later recast within the hidden-variable interpretation by Bohm [45]. The same gradient term arises naturally in extended Thomas-Fermi theory [46,47] and in quantum-hydrodynamic models of charged fluids [48,49]. For a degenerate fermion fluid the Bohm correction, with the Kirzhnits prefactor $\lambda_B = 1/9$, supplements the Thomas-Fermi degeneracy pressure; in a multicomponent dark sector each species carries its own Bohm potential and degeneracy pressure whilst sharing the gravitational field, so that the equilibrium structure acquires a rich dependence on the particle masses m_i , the mass ratio $q = m_1/m_2$, and the central composition f .

Before proceeding, it is worth remarking on the principal novelties of this study. Whilst previous works have elegantly characterized single-species or purely bosonic configurations, the equilibrium structure of a two-component degenerate dark-fermion fluid has received comparatively little attention, despite being physically well-motivated by multi-species dark-sector models. In this article I restrict attention to the Thomas-Fermi regime, wherein degeneracy pressure furnishes the dominant bulk support, and demonstrate that the Bohm potential, far from vanishing entirely, contributes a species-dependent surface-energy correction whose sign differs between the two fermion species, in close analogy with the nuclear liquid-drop model. The original contributions of the present work are threefold; (i) the derivation of the two-species Schrödinger-Poisson-Yukawa system from a Yukawa Lagrangian with universal coupling, (ii) the identification and quantification of the species-dependent Bohm surface-energy correction in the Thomas-Fermi

regime, and (iii) the computation of benchmarked dimensionless invariants together with the gravitational-wave and microlensing signatures of these exotic configurations.

The treatment is deliberately Newtonian, restricted to equilibrium configurations within the orbital-free density-functional framework where the Bohm term contributes to the bulk force balance rather than merely regularizing a sharp surface [19,21]. Setting $f = \eta_2(0) = 0$ recovers the single-species limit, in which an exact scaling symmetry renders the family tightly constrained; once m_1 and the total mass M are specified, the radius follows uniquely. It follows that any detection of a compact object whose mass and radius are inconsistent with neutron stars or black holes, yet compatible with the sequences derived here, would directly constrain the dark-fermion mass m_1 , thereby furnishing a bridge between compact-object astrophysics and the dark-sector Lagrangian.

The remainder of the paper is organized as follows. Sections II and III derive the Newtonian quantum-hydrodynamic model from a two-species Yukawa Lagrangian, identify the SP-Yukawa reduction at finite m_ϕ , and set out the hydrostatic equilibrium incorporating the Bohm potential. Section IV introduces dimensionless variables and the scaling symmetry governing the mass-radius relation. The numerical implementation is described in Sec. V, whilst Sec. VI presents the SP baseline, the Thomas-Fermi reference model, and parameter-sensitivity tests. Section VII discusses stability, observational prospects, and the bearing of these results on the dark matter problem.

I adopt natural units ($\hbar = c = 1$) throughout, so that all dimensionful quantities are expressed as powers of the electron-volt; the Yukawa coupling g_ϕ is dimensionless. Explicit factors of $\hbar$, c , and G are retained where clarity demands it. Standard conversion factors [4] and the IAU 2015 nominal solar values $M_\odot \simeq 1.9885 \times 10^{30}$ kg, $R_\odot = 6.957 \times 10^8$ m [50] are used for SI and astrophysical scales.

II. MODEL: FROM THE YUKAWA LAGRANGIAN TO THE SCHRÖDINGER-POISSON LIMIT

Before proceeding to the numerical study, it is instructive to situate the SP system within the broader framework of dark-sector Lagrangians, both to clarify the approximations involved and to establish contact with the fermionic soliton models of Lee and Pang [19] and Del Grosso *et al.* [21]. I follow the field-theoretic conventions of Serot and Walecka [51] throughout.

A. Yukawa Lagrangian and field equations

Consider two dark-sector spin- $\frac{1}{2}$ fields, ψ of bare mass m_ψ and χ of bare mass m_χ , both coupled to a common real scalar mediator ϕ of mass m_ϕ through a universal Yukawa coupling of strength g_ϕ . In this notation, $\bar{\psi} = \psi^\dagger \gamma^0$ is the

Dirac adjoint, and the bilinears $\bar{\psi}\psi$, $\bar{\chi}\chi$ reduce, in the nonrelativistic limit, to the respective particle number densities n_1 , n_2 . The baseline renormalizable Lagrangian density, standard in quantum field theory and nuclear mean-field models of the Walecka type [51], reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \bar{\psi}(i\gamma^\mu \partial_\mu - m_\psi)\psi + \bar{\chi}(i\gamma^\mu \partial_\mu - m_\chi)\chi - g_\phi(\bar{\psi}\psi + \bar{\chi}\chi)\phi, \quad (1)$$

with the most general renormalizable scalar potential,

$$V(\phi) = \frac{1}{2} m_\phi^2 \phi^2 + \frac{\kappa}{3!} \phi^3 + \frac{\lambda_\phi}{4!} \phi^4, \quad \lambda_\phi > 0. \quad (2)$$

The Yukawa coupling explicitly breaks $\phi \rightarrow -\phi$, so the cubic interaction is permitted by all symmetries; boundedness from below requires $\lambda_\phi > 0$ [51]. A linear term can be removed by a constant field shift that eliminates the tadpole and redefines the couplings. It is worth noting that the baseline Lagrangian (1) contains no direct four-fermion cross-coupling of the form $(\bar{\psi}\psi)(\bar{\chi}\chi)$; such a vertex has mass dimension six, is therefore nonrenormalizable, and lies outside the minimal model [51]. The two species consequently interact only through the shared mediator ϕ , and in the gravitational limit through the common potential Φ , with no residual contact interaction between m_1 and m_2 . With $\kappa_\phi^2 = m_\phi^2$ in natural units, Eq. (7) below makes this structure explicit.

The Euler-Lagrange equations yield the Dirac equations in the scalar background,

$$(i\gamma^\mu \partial_\mu - m_\psi)\psi = g_\phi \phi \psi, \quad (3)$$

$$(i\gamma^\mu \partial_\mu - m_\chi)\chi = g_\phi \phi \chi, \quad (4)$$

showing that the scalar field generates effective mass shifts $\tilde{m}_\psi = m_\psi + g_\phi \phi$ and $\tilde{m}_\chi = m_\chi + g_\phi \phi$. For compactness I write $m_1 \equiv m_\psi$ and $m_2 \equiv m_\chi$ henceforth, and define the mass ratio $q \equiv m_1/m_2$. The mediator obeys the Klein-Gordon equation,

$$(\partial^2 + m_\phi^2)\phi + \frac{\kappa}{2} \phi^2 + \frac{\lambda_\phi}{6} \phi^3 = -g_\phi(\bar{\psi}\psi + \bar{\chi}\chi). \quad (5)$$

In the static regime, introducing the mean-field potential $\Phi \equiv g_\phi \phi$ and replacing $\langle \bar{\psi}\psi \rangle$, $\langle \bar{\chi}\chi \rangle$ by the number densities n_1 , n_2 in the nonrelativistic limit, Eq. (5) becomes

$$(\nabla^2 - m_\phi^2)\Phi = g_\phi^2(n_1 + n_2) + \frac{\kappa}{2g_\phi} \Phi^2 + \frac{\lambda_\phi}{6g_\phi^2} \Phi^3. \quad (6)$$

The last two terms encode the cubic and quartic self-couplings of the mediator. Their fractional importance relative to the linear source scales as $J_\kappa \sim \kappa g_\phi n_{\text{tot}}/(2m_\phi^4)$

and $J_\lambda \sim \lambda_\phi g_\phi^2 n_{\text{tot}}^2/(6m_\phi^6)$, respectively, where $n_{\text{tot}} = n_1 + n_2$. In the weak-field regime ($\Phi \ll m_\phi/g_\phi$) both ratios are negligible, and Eq. (6) reduces to the linear Yukawa form [51],

$$(\nabla^2 - m_\phi^2)\Phi = g_\phi^2(n_1 + n_2). \quad (7)$$

The inverse Compton wave number of the mediator,

$$\kappa_\phi \equiv \frac{m_\phi c}{\hbar}, \quad (8)$$

sets the Yukawa screening length $\ell_\phi = \kappa_\phi^{-1}$; in natural units ($\hbar = c = 1$) one has simply $\kappa_\phi = m_\phi$ and $\kappa_\phi^2 = m_\phi^2$, consistent with Eq. (7). For the gravitational interpretation, the universal Yukawa coupling is identified with the Newtonian strength,

$$g_\phi^2 = 4\pi G m_1^2, \quad (9)$$

so that the source in Eq. (7) becomes $4\pi G m_1^2(n_1 + n_2)$ for the mean-field potential $\Phi = g_\phi \phi$. Newtonian gravity, however, couples to mass density rather than to number density; the equivalence principle demands that each species contribute to the gravitational source proportionally to $m_i n_i$ [52,53]. This distinction is immaterial for the single-species case but has physical consequences when $m_1 \neq m_2$. I therefore impose the equivalence principle at the level of the source whilst retaining the Yukawa screening. Since the gravitational potential $\Phi_g \equiv \Phi/m_1$ must satisfy $\nabla^2 \Phi_g = 4\pi G(m_1 n_1 + m_2 n_2)$ in the Newtonian limit, the mean-field potential $\Phi = g_\phi \phi$ obeys the two-species Yukawa-Poisson equation,

$$(\nabla^2 - \kappa_\phi^2)\Phi = g_\phi^2 \left(n_1 + \frac{n_2}{q} \right), \quad (10)$$

where $1/q = m_2/m_1$ weights species 2 by the mass ratio, as the equivalence principle demands. For a single species ($n_2 = 0$) the identification $g_\phi^2 = 4\pi G m_1^2$ recovers Eq. (7) identically. In the massless-mediator limit $\kappa_\phi \rightarrow 0$, dividing Eq. (10) by m_1 gives the standard Poisson equation for Φ_g ,

$$\nabla^2 \Phi_g = 4\pi G(m_1 n_1 + m_2 n_2), \quad (11)$$

which is the Newtonian form used in the numerical calculations of Sec. V.

III. HYDROSTATIC EQUILIBRIUM IN QHD FORM

In this section I recast the dynamics of each species as a pair of fluid equations, a continuity equation and an Euler equation, within the quantum-hydrodynamic formulation. Within the orbital-free density-functional framework for a

degenerate fermion fluid, one introduces for each species $i \in \{1, 2\}$ an effective Madelung field $\psi_{\text{eff},i} = \sqrt{n_i} e^{iS_i/\hbar}$, a hydrodynamic auxiliary rather than a true single-particle state [44–46]. The densities $n_i = |\psi_{\text{eff},i}|^2$ and the velocity fields $\mathbf{v}_i = \nabla S_i / m_i$ satisfy, for each species,

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0, \quad (12)$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i = -\frac{1}{m_i n_i} \nabla P_i - \frac{1}{m_i} \nabla \Phi - \frac{1}{m_i} \nabla Q_i, \quad (13)$$

where Φ is the shared gravitational potential satisfying the two-species Yukawa-Poisson equation (10).

A. The Bohm potential

The dispersive stress in Eq. (13) is encoded by the Bohm potential for each species,

$$Q_i = -\frac{\lambda_B \hbar^2}{2m_i} \frac{\nabla^2 \sqrt{n_i}}{\sqrt{n_i}}, \quad i \in \{1, 2\}. \quad (14)$$

The dimensionless prefactor λ_B is set to 1/9 throughout this work, the value mandated by the second-order Kirzhnits gradient expansion of the kinetic-energy density functional for a degenerate Fermi gas [47,49,54].

B. Equation of state and pseudoenthalpy

For each species I adopt a polytropic equation of state [55,56],

$$P_i(n_i) = K_{p,i} n_i^{\gamma_{p,i}}, \quad i \in \{1, 2\}, \quad (15)$$

where $K_{p,i}$ is the polytropic constant and $\gamma_{p,i}$ the polytropic index of species i ; $\gamma_{p,i} = 5/3$ reproduces a nonrelativistic degenerate Fermi gas, whilst $\gamma_{p,i} = 4/3$ corresponds to the ultrarelativistic limit. It is convenient to work with the pseudoenthalpy per particle,

$$U_{p,i}(n_i) = \int_0^{n_i} \frac{dP_i(n'_i)}{n'_i} = \frac{K_{p,i} \gamma_{p,i}}{\gamma_{p,i} - 1} n_i^{\gamma_{p,i}-1}. \quad (16)$$

C. Closure relation from the Euler equation

Setting $\partial_t \mathbf{v}_i = 0$ and $\mathbf{v}_i = 0$ in Eq. (13) for a static equilibrium, the left-hand side vanishes identically. Noting that $\nabla U_{p,i} = (1/n_i) \nabla P_i$ from Eq. (16), one obtains upon integration,

$$\mu_i = U_{p,i} + \Phi + Q_i, \quad i \in \{1, 2\}, \quad (17)$$

where μ_i is the chemical potential (integration constant) for each species. The gradient of Eq. (17) gives the global balance condition for species i , expressing the vanishing of

the net force per unit mass at every radius. For the SP ground state ψ_i attains its maximum at $r = 0$, whence $\nabla^2 \sqrt{n_i} / \sqrt{n_i} < 0$ near the center and $Q_i > 0$ there; Q_i decreases outward and becomes negative in the exponential tail. The outward-supporting force is $-m_i^{-1} \nabla Q_i$, which is positive wherever Q_i decreases with radius, so the Bohm potential provides global support for each species independently. By contrast, in a fermionic Thomas-Fermi configuration the nearly uniform density core gives $\nabla^2 \sqrt{n_i} / \sqrt{n_i} \approx 0$ in the bulk, rendering Q_i negligible in the interior; the Bohm term then contributes only as a surface-energy correction [19,21], a distinction explored quantitatively in Sec. VIB.

IV. DIMENSIONLESS FORMULATION, SCALING SYMMETRY AND MASS-RADIUS RELATION

A. Dimensionless Schrödinger-Poisson system

For spherical symmetry, combining the closure relation (17), the Bohm potential (14), and the Yukawa-Poisson equation (10) (with $n_i = \psi_{\text{eff},i}^2$) gives, for species 1,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi_1}{dr} \right) = \frac{2m_1}{\lambda_B \hbar^2} (U_{p,1} + \Phi - \mu_1) \psi_1, \quad (18)$$

and, for species 2 with $m_2 = m_1/q$,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi_2}{dr} \right) = \frac{2m_1}{q \lambda_B \hbar^2} (U_{p,2} + \Phi - \mu_2) \psi_2, \quad (19)$$

together with the Yukawa-Poisson equation,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = g_\phi^2 \left(\psi_1^2 + \frac{\psi_2^2}{q} \right) + \kappa_\phi^2 \Phi. \quad (20)$$

To expose the parameter dependence and facilitate numerical integration I introduce a characteristic radial scale r_0 , initially free, and define the dimensionless coordinate $x = r/r_0$. All characteristic scales are referred to m_1 ; the mass ratio $q \equiv m_1/m_2$ then enters the equations for species 2 as a dimensionless parameter. The fields are normalized to their central values,

$$\eta_i(x) \equiv \frac{\psi_i(r)}{\psi_0} = \sqrt{\frac{n_i(r)}{n_0}}, \quad \varphi(x) \equiv \frac{\Phi(r)}{\Phi_c}, \quad (21)$$

where $n_0 = \psi_0^2$ is the reference central number density and $\Phi_c \equiv \Phi(0)$ the central gravitational potential. The characteristic scales, defined via m_1 , are

$$\Phi_0 = \frac{\lambda_B \hbar^2}{m_1 r_0^2} = \mu_0, \quad \psi_0^2 = \frac{4\pi \lambda_B \hbar^2}{g_\phi^2 m_1 r_0^4}, \quad (22)$$

and I adopt the normalization convention $\eta_1(0) = 1$. The central value of the second species, $f \equiv \eta_2(0)$, is a free

input parameter that controls the relative abundance of the two components. With these definitions, the radial system (18)–(20) becomes the coupled dimensionless system of three second-order ODEs (primes denoting d/dx),

$$\eta_1'' + \frac{2}{x}\eta_1' = 2v_c\eta_1\varphi + 2\sigma_1\eta_1^{2\gamma_{p,1}-1} - 2\varepsilon_1\eta_1, \quad (23)$$

$$\eta_2'' + \frac{2}{x}\eta_2' = \frac{2}{q}(v_c\eta_2\varphi + \sigma_2\eta_2^{2\gamma_{p,2}-1} - \varepsilon_2\eta_2), \quad (24)$$

$$\varphi'' + \frac{2}{x}\varphi' = \frac{4\pi}{v_c}\left(\eta_1^2 + \frac{\eta_2^2}{q}\right) + \xi^2\varphi. \quad (25)$$

The factor $2/q$ in Eq. (24) is inherited directly from the $1/q$ already visible in the dimensional Eq. (19), where $m_2 = m_1/q$ enters the kinetic prefactor. The corresponding factor in the Poisson source (25) follows from the equivalence principle imposed in Eq. (10); because the gravitational source is $4\pi G(m_1n_1 + m_2n_2)$, species 2 enters weighted by $m_2/m_1 = 1/q$ relative to species 1 once all quantities are normalized to the reference mass. The dimensionless central potential $v_c \equiv \Phi_c/\Phi_0$ is determined self-consistently by the boundary-value problem. The dimensionless pressure parameters for the two species are

$$\sigma_i \equiv \frac{m_1 r_0^2}{\lambda_B \hbar^2} \frac{K_{p,i} \gamma_{p,i}}{\gamma_{p,i} - 1} \psi_0^{2(\gamma_{p,i}-1)}, \quad i \in \{1, 2\}, \quad (26)$$

where the pure SP limit is recovered for $\sigma_1 = \sigma_2 = 0$. In this regime the sole mechanism opposing gravitational collapse is the quantum kinetic (Bohm) pressure encoded in the gradient term $\nabla^2 \sqrt{n_i}/\sqrt{n_i}$, precisely the framework employed in fuzzy dark matter and ultralight boson models. Each species has its own dimensionless eigenvalue,

$$\varepsilon_i \equiv \mu_i/\mu_0, \quad i \in \{1, 2\}, \quad (27)$$

both measured in units of the same reference scale $\mu_0 = \Phi_0$. The dimensionless mediator-mass parameter is

$$\xi \equiv \kappa_\phi r_0 = \frac{m_\phi c r_0}{\hbar}, \quad (28)$$

equal to the ratio of the radial scale r_0 to the screening length $\ell_\phi = \kappa_\phi^{-1}$; setting $\xi = 0$ recovers the massless-mediator limit. Regularity at the origin and the central normalization impose six boundary conditions,

$$\begin{aligned} \eta_1(0) = 1, \quad \eta_1'(0) = 0, \quad \eta_2(0) = f, \quad \eta_2'(0) = 0, \\ \varphi(0) = 1, \quad \varphi'(0) = 0, \end{aligned} \quad (29)$$

and bound states satisfy $\eta_1(x) \rightarrow 0$, $\eta_2(x) \rightarrow 0$ and $\varphi(x) \rightarrow 0$ as $x \rightarrow \infty$.

B. Scaling symmetry and its breaking by m_ϕ

For $\xi = 0$ and $\sigma_1 = \sigma_2 = 0$, the Yukawa screening term vanishes and Eqs. (23)–(25) admit a scaling symmetry; the normalized profiles $\eta_1(x)$, $\eta_2(x)$, and $\varphi(x)$ are shape-invariant, while the physical central values and the radial scale transform as

$$\begin{aligned} \psi_c &\rightarrow \alpha^2 \psi_c, & \Phi_c &\rightarrow \alpha^2 \Phi_c, \\ \mu_i &\rightarrow \alpha^2 \mu_i, & r_0 &\rightarrow r_0/\alpha, \end{aligned} \quad (30)$$

for any $\alpha > 0$. In terms of the unnormalized potential $v \equiv v_c \varphi = \Phi/\Phi_0$, the scaling reads $v_\alpha(x) = \alpha^2 v(\alpha x)$ and $\varepsilon_{i,\alpha} = \alpha^2 \varepsilon_i$. Under this rescaling $\xi \rightarrow \xi/\alpha$; for $\xi \neq 0$ the Yukawa term breaks the symmetry, and each value of ξ defines a distinct family. Since q is itself invariant, the dimensionless profiles remain shape-invariant at fixed q and f . The normalization convention $\eta_1(0) = 1$ is not unique (one could equally normalize to $\int \eta_1^2 x^2 dx = 1$), but it yields particularly transparent expressions for the physical scales.

Because ξ itself depends on r_0 via Eq. (28), and r_0 depends on $M_{\text{dim}}(\xi)$ via Eq. (34), the dimensionless mediator parameter satisfies the self-consistency condition

$$\xi = \Xi \cdot M_{\text{dim}}(\xi), \quad \Xi \equiv \frac{\kappa_\phi \lambda_B \hbar^2}{G m_1^2 M}, \quad (31)$$

which must be solved iteratively for given (m_ϕ, m_1, q, f, M) . In the limit $\Xi \rightarrow 0$ ($m_\phi \rightarrow 0$), $\xi \rightarrow 0$ and the SP invariants of Eq. (37) are recovered.

C. Mass-radius relation

The total dimensionless mass, accounting for both species with their respective gravitational couplings, is

$$M_{\text{dim}} = 4\pi \int_0^\infty \left(\eta_1^2 + \frac{\eta_2^2}{q} \right) x^2 dx, \quad (32)$$

where the weight $1/q = m_2/m_1$ mirrors the Poisson source weighting of Eq. (25). The physical mass is

$$M = \frac{\lambda_B \hbar^2}{G m_1^2 r_0} M_{\text{dim}}(\xi, q, f). \quad (33)$$

For a chosen total mass M this fixes the radial scale,

$$r_0(\xi) = \frac{\lambda_B \hbar^2}{G m_1^2} \frac{M_{\text{dim}}(\xi, q, f)}{M}, \quad (34)$$

where M_{dim} is obtained by solving the coupled system (23)–(25) at given (ξ, q, f) . Because the Yukawa screening term $\xi^2 \varphi$ weakens the gravitational well, M_{dim} increases monotonically with ξ and the radial scale inherits this

dependence. Substituting numerical values and normalizing to the single-species massless-mediator invariant $M_{\text{dim}}(0) \simeq 3.883$ (at $f = 0$),

$$r_0(\xi) \simeq 4.5R_\odot \left(\frac{M_{\text{dim}}(\xi, q, f)}{3.883} \right) \times \left(\frac{m_1}{6 \times 10^{-14} \text{ eV}} \right)^{-2} \left(\frac{M}{1M_\odot} \right)^{-1}, \quad (35)$$

The physical radius $R_T(\xi) = x_T(\xi)r_0(\xi)$ depends on both the Yukawa screening and the two-species parameters (q, f) .

A physically meaningful radius is R_T , defined as the sphere enclosing 99% of the total mass. Writing x_T for the corresponding dimensionless radius, one obtains the explicit mass-radius relation,

$$MR_T = \frac{\lambda_B \hbar^2}{Gm_1^2} (M_{\text{dim}}(\xi, q, f)x_T(\xi, q, f)), \quad (36)$$

where M_{dim} and x_T are determined by the boundary-value problem at the self-consistent ξ satisfying Eq. (31). For the single-species massless-mediator ground state [$\xi = 0$, $f = 0$, $\eta_1(0) = 1$] I find numerically (Sec. V)

$$M_{\text{dim}}(0) \simeq 3.883, \quad x_T(0) \simeq 2.562, \\ MR_T(0)x_T(0) \simeq 9.95, \quad (37)$$

so that the bracketed factor in Eq. (36) at $\xi = 0$ is fixed to better than 0.1% by a single boundary-value solve. For $\xi > 0$ the Yukawa screening shifts $M_{\text{dim}}(\xi)$ and $x_T(\xi)$ relative to these reference values, and the $R \propto M^{-1}$ scaling, which holds exactly only at $\xi = 0$, acquires an additional m_ϕ -dependence through the self-consistency condition (31). In the two-species case the invariants additionally depend on q and f , so the single mass-radius curve is replaced by a family of curves parametrized by (q, f) .

The $R \propto M^{-1}$ scaling at $\xi = 0$, $\sigma_i = 0$ coincides with the standard Newtonian boson-star relation [12,13,17,18,57] and the soliton scaling in wave-dark-matter phenomenology [6,9,10]; in the fermionic case this limit serves as the purely Bohm-supported reference, whilst the physical regime requires $\sigma_i \neq 0$.

Table I collects the dimensionless mass and the mass-radius product from several independent calculations. All three comparison references solve the *bosonic* SP system; however, at $\sigma_i = 0$ the coefficient λ_B cancels from the dimensionless equations, rendering the invariants M_{dim} and $M_{\text{dim}}x_T$ independent of particle statistics, so the comparison is entirely appropriate.

D. Dimensionless energy and escape velocity

It is useful to define the dimensionless kinetic (quantum-gradient) energy and gravitational potential energy for each

TABLE I. Comparison of dimensionless invariants for the SP ground state with $\eta(0) = 1$. The eigenvalue ε depends on conventions, but M_{dim} and $M_{\text{dim}}x_T$ are convention-independent.

Source	M_{dim}	x_T	$M_{\text{dim}}x_T$
This work	3.883	2.562	9.95
Moroz <i>et al.</i> [16]	3.88 ^a	...	...
Chavanis [17]	3.88	...	9.9 ^b
Guzmán and Ureña-López [58]	3.88	...	...

^aConverted from their normalization using $\alpha = 1$.

^bInferred from their mass-radius formula.

species. The dimensionless kinetic energy density for species i is

$$\mathcal{T}_i(x) = \left(\frac{d\eta_i}{dx} \right)^2, \quad (38)$$

and the dimensionless gravitational energy density is

$$\mathcal{W}_i(x) = v_c \eta_i^2(x) \varphi(x), \quad (39)$$

so that the integrated energies are $T_i = 4\pi \int_0^\infty \mathcal{T}_i x^2 dx$ and $W_i = 4\pi \int_0^\infty \mathcal{W}_i x^2 dx$. These satisfy the virial relation $2T + W = 0$ for bound states, up to corrections of order σ_i when the polytropic pressure is non-negligible.

The escape speed from the surface of a configuration of mass M and radius R_T is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R_T}}. \quad (40)$$

The escape speed characterizes the binding of the configuration; illustrative values are reported in Sec. VIC.

V. NUMERICAL METHOD AND DIAGNOSTIC CHECKS

The ground state of Eqs. (23)–(25) is obtained by treating ε_1 and ε_2 as eigenvalues and solving a two-point boundary-value problem on a truncated domain $x \in [0, x_{\text{max}}]$ with $x_{\text{max}} \gg x_T$. For each target value of (ξ, q, f) , including the single-species reference $f = 0$, $\xi = 0$, a separate eigenvalue solve is performed; the self-consistent ξ for a given physical configuration is then obtained by iterating Eq. (31). A broadly similar numerical approach was employed by Guzmán and Ureña-López [58] for the single-species $\xi = 0$ system.

A. Implementation

For the purposes of numerical integration, the second-order system (23)–(25) is rewritten in first-order form by introducing the enclosed radial fluxes

$$\zeta_i(x) \equiv x^2 \frac{d\eta_i}{dx}, \quad \varpi(x) \equiv x^2 \frac{d\varphi}{dx}. \quad (41)$$

The six-component first-order system becomes

$$\frac{d\eta_1}{dx} = \frac{\zeta_1}{x^2}, \quad \frac{d\zeta_1}{dx} = x^2(2v_c\eta_1\varphi + 2\sigma_1\eta_1^{2\gamma_{p,1}-1} - 2\varepsilon_1\eta_1), \quad (42)$$

$$\frac{d\eta_2}{dx} = \frac{\zeta_2}{x^2}, \quad \frac{d\zeta_2}{dx} = \frac{x^2}{q}(2v_c\eta_2\varphi + 2\sigma_2\eta_2^{2\gamma_{p,2}-1} - 2\varepsilon_2\eta_2), \quad (43)$$

$$\frac{d\varphi}{dx} = \frac{\varpi}{x^2}, \quad \frac{d\varpi}{dx} = x^2 \left(\frac{4\pi}{v_c}(\eta_1^2 + \eta_2^2/q) + \xi^2\varphi \right), \quad (44)$$

with coefficients as defined in the preceding section. The boundary-value problem is solved using a collocation method with $x_{\max} = 30$, 900 initial mesh points and a relative residual tolerance $\text{tol} = 5 \times 10^{-7}$. Setting $f = 0$ and $\xi = 0$ reproduces the single-species SP solver. Convergence is verified by increasing $x_{\max}$ and the mesh until M_{dim} and x_T stabilize to the quoted precision. The outer boundary condition $\varphi(x_{\max}) = 0$ approximates $\varphi(\infty) = 0$ and fixes the additive constant of the potential; changing $x_{\max}$ shifts ε_i slightly but leaves $v_c\varphi - \varepsilon_i$ and derived force-balance diagnostics invariant to numerical accuracy.

The nine boundary conditions that close the system are $\eta_1(0) = 1$ (central normalization), $\eta'_1(0) = 0$ (regularity), $\eta_2(0) = f$ (second-species central value), $\eta'_2(0) = 0$ (regularity), $\varphi(0) = 1$ (central normalization), $\varphi'(0) = 0$ (regularity), $\eta_1(x_{\max}) = 0$ (bound-state decay), $\eta_2(x_{\max}) = 0$ (bound-state decay), and $\varphi(x_{\max}) = 0$ (potential zero at

infinity). The eigenvalues ε_1 , ε_2 and the dimensionless central potential v_c are treated as free parameters adjusted by the solver to satisfy the outer boundary conditions.

B. Internal diagnostics

Two internal checks serve as stringent consistency tests for each species. First, the ground state is nodeless; $\eta_i(x) > 0$ for all $x \in (0, x_{\max})$ and $i \in \{1, 2\}$. Second, the force balance implied by Eq. (17) is verified by computing the dimensionless Bohm diagnostic,

$$Q_{\text{dim},i}(x) \equiv \varepsilon_i - v_c\varphi(x) - \sigma_i\eta_i^{2(\gamma_{p,i}-1)}(x), \quad i \in \{1, 2\}, \quad (45)$$

which is the dimensionless form of the closure relation $Q_i = \mu_i - \Phi - U_{P,i}$, Eq. (17), divided by Φ_0 . For the pure SP case ($\sigma_i = 0$) this reduces to $Q_{\text{dim},i} = \varepsilon_i - v_c\varphi$. The identity $Q_{\text{dim},i}(x) + v_c\varphi(x) + \sigma_i\eta_i^{2(\gamma_{p,i}-1)}(x) = \varepsilon_i$ holds at all radii up to a numerical error of order 10^{-15} , confirming that the solver has converged to a self-consistent solution for each species independently. A third check, the Bohm potential cross-check, reconstructs Q_i directly from the numerical profile of $\sqrt{n_i} \propto \eta_i$ and compares it with $Q_{\text{dim},i}$; the maximum discrepancy is below 10^{-6} on the mesh used.

VI. RESULTS

The results are organized in order of increasing physical realism, each layer of complexity being introduced against a well-understood baseline. Section VIA establishes the SP ($\sigma_i = 0$) reference. Section VIB presents the Thomas-Fermi model, in which degeneracy pressure provides bulk support and the Bohm potential contributes a quantum surface correction. Sections VIC and VID translate to

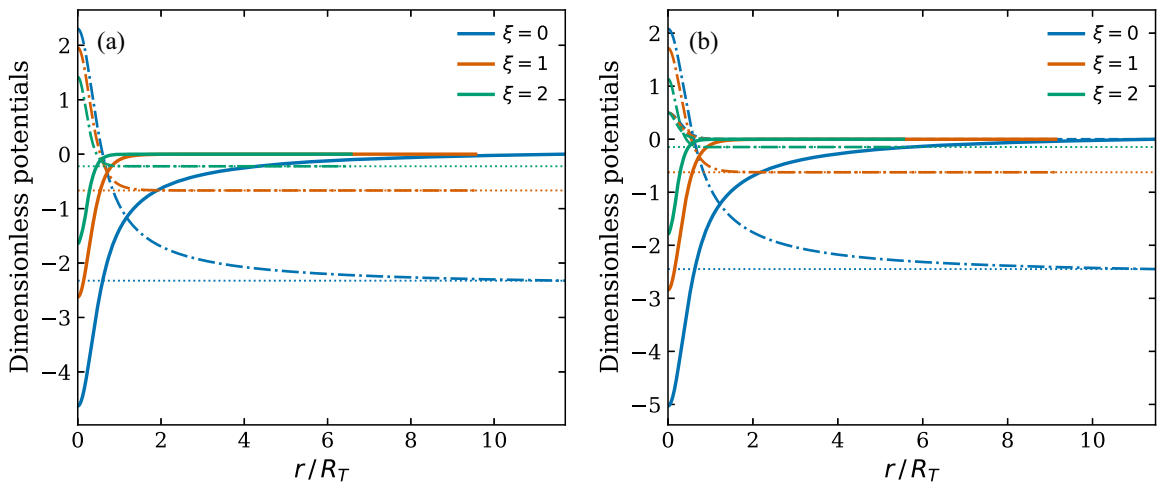


FIG. 1. Dimensionless potential balance for the nodeless ground state at $\xi = 0$ (blue), $\xi = 1$ (vermillion) and $\xi = 2$ (green), plotted against the normalized radius r/R_T . Solid curves show $v_c\varphi$, dot-dashed curves the Bohm diagnostic Q_{dim} [Eq. (45)], and dotted horizontal lines mark ε . (a) Pure SP case ($\sigma = 0$). (b) With polytropic pressure ($\sigma = 0.5$, $\gamma_p = 5/3$); dashed curves show the pseudoenthalpy contribution $\sigma\eta^{2(\gamma_p-1)}$.

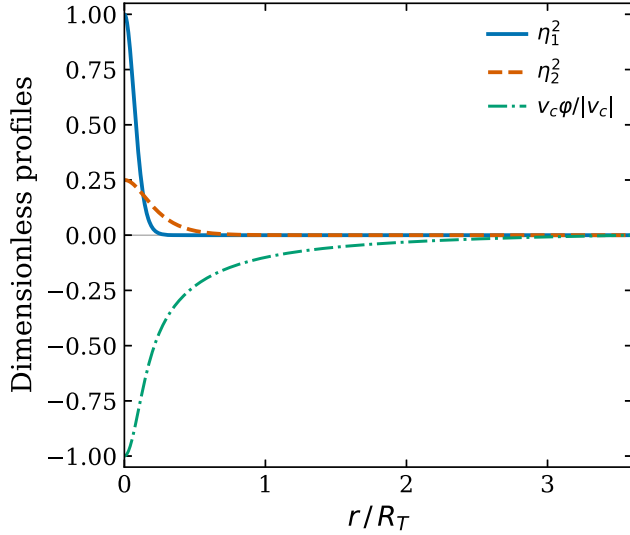


FIG. 2. Two-species dimensionless profiles for $q = 10$, $f = 0.5$ and $\xi = 0$ (SP baseline); density profiles $\eta_1^2(x)$ (blue, solid) and $\eta_2^2(x)$ (vermillion, dashed), and the shared gravitational potential $v_c \phi(x)$ (green, dot-dashed). The lighter species ($m_2 = m_1/10$) is more spatially extended.

physical units and summarize the polytropic mass-radius relation together with parameter-sensitivity tests.

A. Schrödinger-Poisson baseline ($\sigma_i = 0$)

In the formal limit $\sigma_i = 0$ (vanishing degeneracy pressure), the Bohm term is the sole source of repulsive stress, a regime that is not physically representative of degenerate fermions but that furnishes an indispensable numerical reference. The dimensionless potential balance for the single-species ground state (Fig. 1) confirms that the closure relation (17) holds at every radius; increasing ξ weakens the gravitational confinement, as one would expect. At $\xi = 0$ the dimensionless invariants are $M_{\text{dim}}(0) \simeq 3.883$, $x_T(0) \simeq 2.562$ and $M_{\text{dim}} x_T \simeq 9.95$, in excellent agreement with independent determinations (Table I), a concordance that is rather pleasing and confirms that the numerical procedure is well-controlled. The enclosed-mass profile is universal at $\xi = 0$ owing to the scaling symmetry (30), with half the mass enclosed within $r \simeq 0.45 R_T$; for $\xi > 0$ the profile broadens as the confining potential is softened.

In the two-species SP baseline ($q = 10$, $f = 0.5$, $\sigma_i = 0$, $\xi = 0$) the heavier species (η_1) is centrally concentrated whilst the lighter species (η_2) extends to larger radii, a direct consequence of the $1/q$ factor in Eq. (24) (Fig. 2). The solver yields $\varepsilon_1 \simeq -3.035$, $\varepsilon_2 \simeq -1.008$, $v_c \simeq -5.383$, $M_{\text{dim}} \simeq 6.232$, and $x_T \simeq 8.330$, with species 2 contributing $M_{2,\text{grav}}/M \simeq 40.4\%$ of the gravitational mass (after $1/q$ weighting). The Bohm potential provides global support for both species, $|Q_1|/|\Phi| \sim \mathcal{O}(1)$ and $|Q_2|/|\Phi| \sim q$ throughout the interior (Fig. 3).

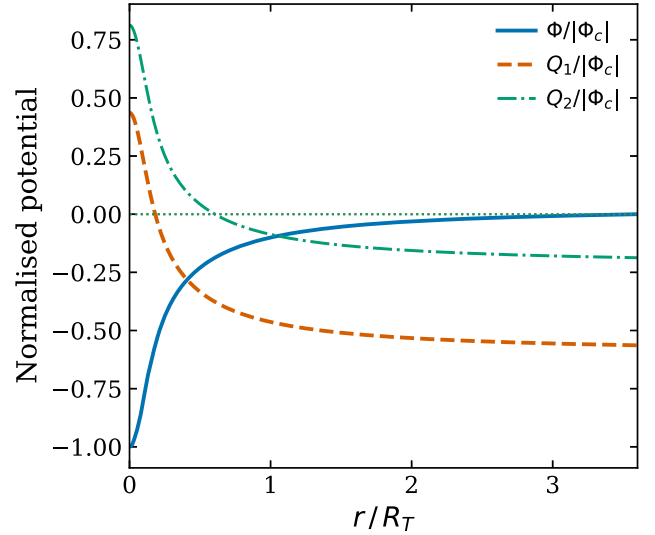


FIG. 3. Combined integrated balance for $q = 10$, $f = 0.5$, $\xi = 0$ (SP baseline). Blue (solid): Shared gravitational potential $\Phi/|\Phi_c|$; vermillion (dashed): Bohm potential $Q_1/|\Phi_c|$ for species 1; green (dash-dotted): Bohm potential $Q_2/|\Phi_c|$ for species 2; dotted curves: Closure residuals. The Bohm potential of the lighter species exceeds $|\Phi|$ by roughly an order of magnitude.

B. Reference model: Thomas-Fermi regime with quantum surface correction

I now turn to the physically more representative Thomas-Fermi regime, where degeneracy pressure furnishes the dominant bulk support. The reference model adopts $q = 10$, $f = 0.35$, $\sigma_1 = 12$, $\sigma_2 = q\sigma_1 = 120$, $\gamma_{p,i} = 5/3$ and $\xi = 0$.

1. Bohm-Thomas-Fermi crossover

It is worth remarking that σ_1 and σ_2 are not independent parameters; for a nonrelativistic degenerate Fermi gas ($\gamma_{p,i} = 5/3$) the polytropic constant scales as $K_{p,i} \propto \hbar^2/m_i$, whence

$$\frac{\sigma_2}{\sigma_1} = \frac{K_{p,2}}{K_{p,1}} = \frac{m_1}{m_2} = q. \quad (46)$$

Figure 4 quantifies this crossover. As σ_1 increases from zero, $|Q_1(0)|/|\Phi_c|$ drops from $\simeq 0.46$ to $\simeq 0.016$ at $\sigma_1 = 12$, whilst $|U_{p,1}(0)|/|\Phi_c|$ rises from zero to $\simeq 0.47$; species 1 crosses over near $\sigma_1 \simeq 2$. For species 2 the physical scaling $\sigma_2 = q\sigma_1$ implies that its degeneracy pressure is q times stronger at the same σ_1 ; the competition between the enhanced kinetic coefficient $\hbar^2/(2m_2) = q\hbar^2/(2m_1)$ and the stronger degeneracy pressure determines the crossover locus for each species independently.

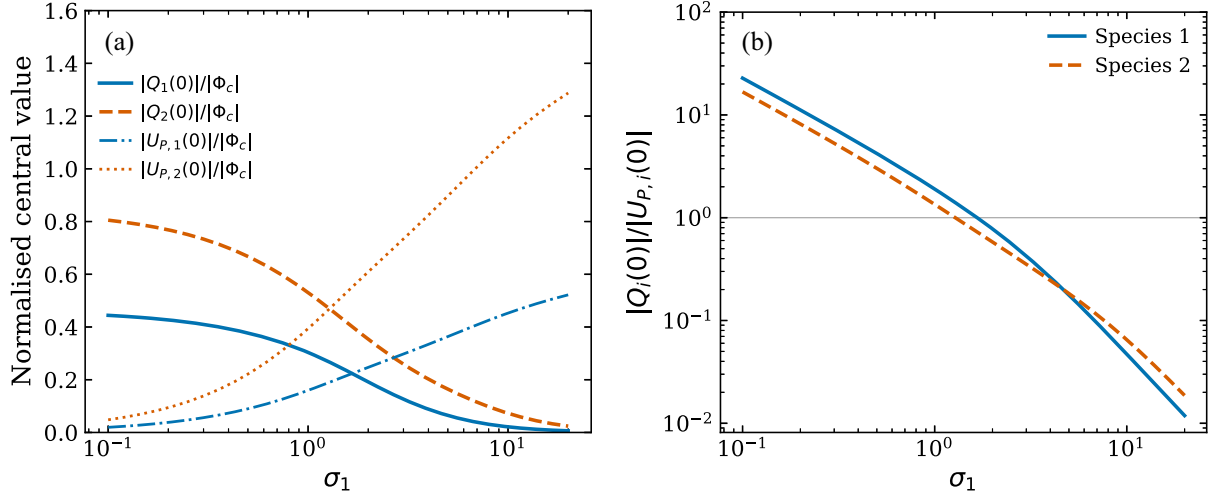


FIG. 4. Bohm-Thomas-Fermi crossover for $q = 10$, $f = 0.35$, $\xi = 0$. (a) Normalized central Bohm potential $|Q_i(0)|/|\Phi_c|$ as a function of σ_1 ; thick curves: physical scaling $\sigma_2 = q\sigma_1$ [Eq. (46)], thin faded curves: equal- σ reference. (b) Ratio $|Q_i(0)|/|U_{P,i}(0)|$ for the physical scaling. The horizontal gray line marks $|Q_i| = |U_{P,i}|$; above this line, the Bohm potential dominates over degeneracy pressure, below it, degeneracy pressure prevails. Species 1 (blue solid line) crosses over near $\sigma_1 \simeq 2$, identifying the transition from the Bohm-supported to the Thomas-Fermi regime.

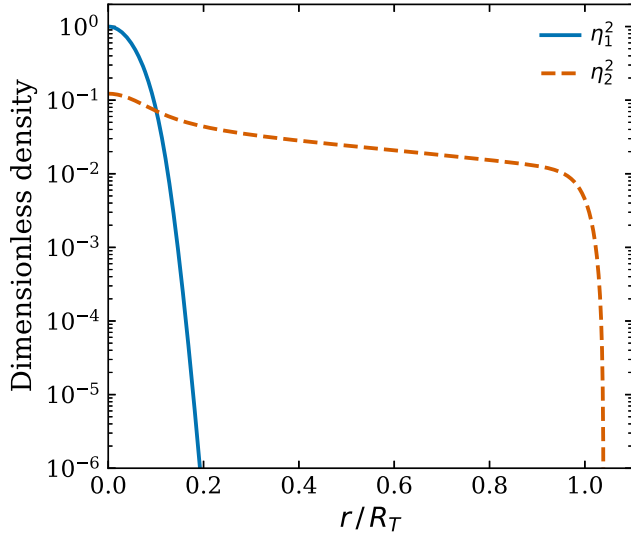


FIG. 5. Density profiles $\eta_i^2(x)$ for the reference Thomas-Fermi model ($q = 10$, $f = 0.35$, $\sigma_1 = 12$, $\sigma_2 = 120$, $\xi = 0$). Species 1 (blue, solid) and species 2 (vermillion, dashed).

2. The Bohm force and the liquid-drop analogy

The Bohm potential Q_i [Eq. (14)] follows the standard Madelung-Bohm convention [44,45] with the Kirzhnits coefficient $\lambda_B = 1/9$ [47,49,59,60]. The physically observable quantity is the Bohm force per unit mass,

$$\mathbf{F}_{B,i} = -\frac{1}{m_i} \nabla Q_i, \quad (47)$$

which encodes the dispersive stress arising from spatial inhomogeneity in the density field [48,61]. Within the

quantum-hydrodynamic framework the Bohm force plays a role analogous to that of the quantum-pressure tensor in the self-consistent fluid model of Manfredi and Haas [48], and its regularizing properties at steep density gradients have been analyzed in the smooth quantum-potential formulation of Gardner and Ringhofer [62]. For spherical equilibria one has $F_{B,i} = -m_i^{-1} dQ_i/dr$; outward support requires $dQ_i/dr < 0$, whilst $dQ_i/dr > 0$ produces an inward restoring force.

In the asymptotic region the closure relation reduces to $Q_i(r \rightarrow \infty) \rightarrow \varepsilon_i$ [cf. Eq. (17)], so the eigenvalue sign determines the direction of the Bohm force near the stellar surface. For species 1, whose eigenvalue remains negative at all σ_1 , the Bohm force $F_{B,1} > 0$ is directed outward, establishing a quantum-pressure wall that defines the stellar boundary; for species 2, whose eigenvalue crosses zero near $\sigma_1 \simeq 3$ (Fig. 8), the force $F_{B,2} < 0$ is directed inward when $\varepsilon_2 > 0$, acting as quantum surface tension whilst degeneracy pressure $U_{P,2}$ furnishes the bulk confinement.

This species-dependent directionality of $\mathbf{F}_{B,i}$ is the hallmark of the two-component model, and the physical picture is best illuminated by the nuclear liquid-drop decomposition [19,63]; degeneracy pressure furnishes the volume binding energy, whilst the Bohm force provides a surface-energy correction whose sign differs between the two species. For the heavier species, whose degeneracy pressure is weaker, $\mathbf{F}_{B,1}$ retains its outward-supporting role across the entire profile. For the lighter, deeply degenerate species the Bohm force transitions to an inward surface-tension regime, reflecting the sign change of ε_2 as the Thomas-Fermi parameter increases. The competition between volume and surface terms governs the equilibrium

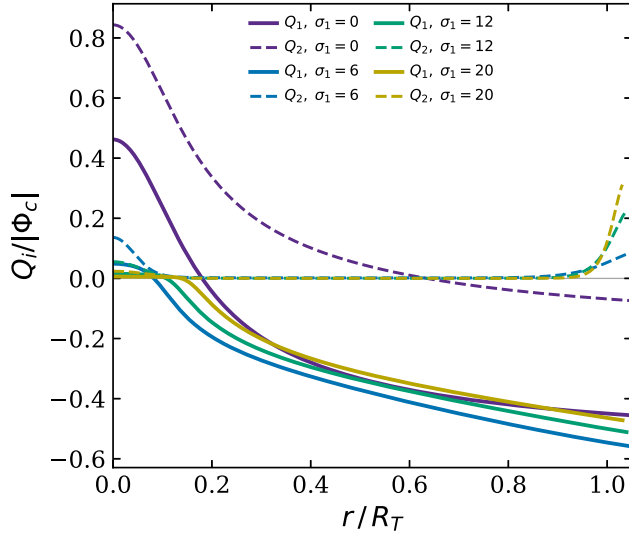


FIG. 6. Signed Bohm potential $Q_i(r)/|\Phi_c|$ for the reference Thomas-Fermi model ($q = 10$, $f = 0.35$, $\sigma_1 = 12$, $\sigma_2 = 120$, $\xi = 0$), showing the surface localization characteristic of the Thomas-Fermi regime.

shape and stability of the configuration [21], much as it does for nuclear matter.

3. Full diagnostic of the reference model

Figures 5–9 present the complete diagnostic for $q = 10$, $f = 0.35$, $\sigma_2 = q\sigma_1$. At $\sigma_1 = 12$ ($\sigma_2 = 120$) the solver yields $\varepsilon_1 \simeq -13.0$, $\varepsilon_2 \simeq 5.7$, $v_c \simeq -25.4$, $M_{\text{dim}} \simeq 205.0$ and $x_T \simeq 28.9$. The central Bohm potentials are small, $|Q_1(0)|/|\Phi_c| \simeq 0.016$ and $|Q_2(0)|/|\Phi_c| \simeq 0.056$, confirming that the configuration resides firmly in the Thomas-Fermi regime. Both density profiles (Fig. 5) decrease monotonically from their central values, as befits the nodeless ground state; species 1 is more centrally concentrated whilst species 2, being lighter, extends to larger radii. The signed $Q_i(r)/|\Phi_c|$ (Fig. 6) exhibits progressive localization to a thin shell around $r \simeq R_T$, with Q_1 becoming negative (outward force) and Q_2 becoming positive (inward surface tension). The Bohm force $F_{B,i} = -dQ_i/dx$ (Fig. 7) confirms that $F_{B,1} > 0$ throughout the interior whilst $F_{B,2} < 0$ near the surface. The eigenvalues (Fig. 8) show ε_1 remaining negative whilst ε_2 crosses zero near $\sigma_1 \simeq 3$, marking the onset of the surface-tension regime. The closure relation $\varepsilon_i = U_{P,i} + v_c \varphi + Q_i$ (Fig. 9) reveals that the interior balance is between degeneracy pressure and gravity, with Q_i negligible; near the surface, $Q_1 \rightarrow \varepsilon_1 < 0$ provides the quantum wall and $Q_2 \rightarrow \varepsilon_2 > 0$ acts as surface tension. This tripartite balance represents the regime most relevant to astrophysical dark-fermion stars.

C. Physical scales and illustrative objects

I now translate the dimensionless solutions to physical units using Eqs. (22)–(35). Table II collects the

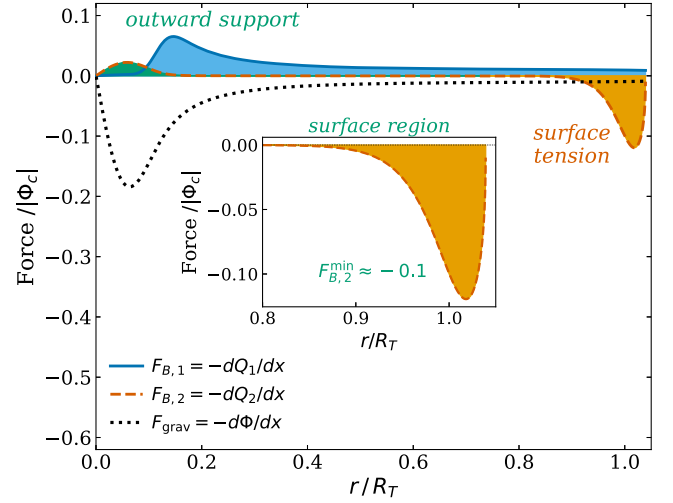


FIG. 7. Bohm force $F_{B,i} = -dQ_i/dx$ normalized to $|\Phi_c|$, for $\sigma_1 = 12$ ($q = 10$, $f = 0.35$, $\sigma_2 = 120$). Species 1 (solid blue) exerts a purely outward force throughout the interior. Species 2 (dashed orange) exhibits two regimes; modest outward support in the bulk (green shading) and a sharp inward surface-tension dip near the stellar surface (orange shading), whose magnitude exceeds the local gravitational force. The inset magnifies the surface region to reveal the full depth of this dip, which the main panel clips. The gravitational force $F_{\text{grav}} = -d\Phi/dx$ (dotted) is included for comparison.

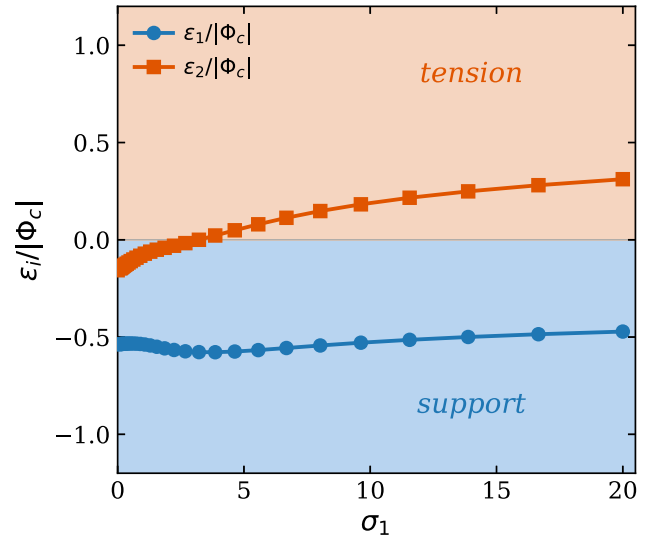


FIG. 8. Eigenvalues $\varepsilon_i/|\Phi_c|$ as a function of σ_1 ($q = 10$, $f = 0.35$, $\sigma_2 = q\sigma_1$). The eigenvalue ε_2 crosses zero near $\sigma_1 \simeq 3$, marking the onset of the surface-tension regime: in the blue-shaded region ($\varepsilon < 0$), the Bohm force acts outward; in the vermilion-shaded region ($\varepsilon > 0$), it acts inward as surface tension.

dimensionless invariants and radial scales for three Yukawa parameters at $m_1 = 6 \times 10^{-14}$ eV and $M = 1M_\odot$ in the single-species SP reference ($\sigma_i = 0$, $f = 0$); at $\xi = 0$ one obtains $r_0 \simeq 4.5R_\odot$ via Eq. (35).

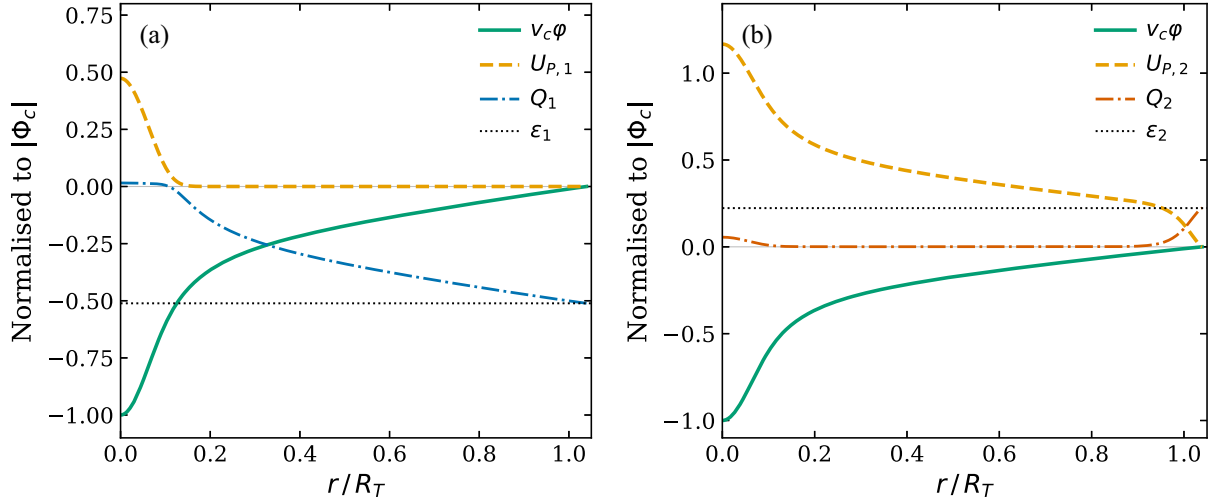


FIG. 9. Closure balance $\varepsilon_i = U_{P,i} + v_c \phi + Q_i$ at $\sigma_1 = 12$ ($q = 10$, $f = 0.35$, $\sigma_2 = 120$): gravity (blue, solid), degeneracy pressure (green, dot-dashed) and Bohm potential (vermillion, dashed). (a) Species 1 (m_1). (b) Species 2 ($m_2 = m_1/q$).

To illustrate the breadth of the parameter space, Table III presents physical configurations at three representative total masses: $M = 10^{-8} M_\odot$ (subsolar), $M = 1 M_\odot$ (solar) and $M = 5 M_\odot$ (suprasolar). All entries satisfy $\kappa < 10^{-3}$, confirming that the Newtonian approximation is well-justified. The particle mass m_1 is chosen so that R_T spans from ~ 10 km to $\sim 10^2 R_\odot$, corresponding to $m_1 \sim 10^{-14} - 10^{-7}$ eV, an enormous range that reflects the $R_T \propto m_1^{-2}$ scaling at fixed M . The lower panel of Table III additionally presents two-species configurations in which neither σ_i nor ξ vanish, demonstrating the richer parameter space accessible when degeneracy pressure and a finite-mass mediator both contribute.

The central densities span thirteen orders of magnitude and escape speeds range from ~ 16 to $\sim 1.2 \times 10^4$ km s $^{-1}$. The steep central-density scaling, $\rho_c \propto m_1^6$ at fixed M , follows from $r_0 \propto m_1^{-2}$ combined with $\rho_c \propto 1/(m_1 r_0^4)$.

The two-species entries with $\sigma_1 = 1$, $\sigma_2 = 10$ (lower panel) illustrate how degeneracy pressure inflates the equilibrium radius by roughly an order of magnitude relative to the pure Bohm reference at the same (m_1, M) , reflecting the larger M_{dim} of the pressure-supported solution. Entries with a finite-mass mediator ($\xi = 1$)

lie intermediate between the pure Bohm and high-pressure extremes.

D. Polytropic scaling, parameter sensitivity, and limiting cases

In the Thomas-Fermi regime ($\sigma_i \gg 1$), the bulk equilibrium reduces to the Lane-Emden equation; for $\gamma = 5/3$ the radius scales as $R \propto M^{-1/3}$ with no upper mass bound, whereas for $\gamma = 4/3$ a limiting mass emerges above which no stable equilibrium exists. These two limiting cases delimit the nonrelativistic and ultrarelativistic regimes, respectively, and are well-established in the polytropic literature (see, e.g., Chandrasekhar [55]); the corresponding mass-radius curves are therefore not reproduced here.

The two-species mass-radius relation recovers the universal single-species scaling $MR_T = \text{const}/m_1^2$ for $f = 0$; for $f > 0$ the displacement grows with q , owing to the broader spatial envelope of the lighter species. Yukawa screening ($\xi > 0$) shifts the mass-radius curves to larger radii, the shift increasing with ξ through the self-consistency condition (31). These dependences follow directly from the invariants tabulated in Table II; the corresponding mass-radius curves, being single-parameter rescalings of the universal profile, are not reproduced here.

E. Scope and limitations of the present treatment

The Thomas-Fermi reference model ($\sigma_1 = 12$, $\sigma_2 = 120$, $q = 10$, $f = 0.35$) is parametrized in the single-species limit by the triple (m_ϕ, m_1, M) (with λ_B fixed) and in the two-component case by the quintuple (m_ϕ, m_1, m_2, M, f) . Equations (31)–(36) provide a compact mapping from these parameters to a characteristic radius; the massless-mediator limit reproduces the SP invariants of Eq. (37). The existence of such equilibria does not, of course, establish that Nature produces them in abundance; at least three

TABLE II. Dimensionless invariants and physical radial scale for three Yukawa parameters, with $m_1 = 6 \times 10^{-14}$ eV and $M = 1 M_\odot$ in the single-species SP reference ($\sigma_i = 0$, $f = 0$). The radial scale r_0 follows from Eq. (35); the mediator mass is $m_\phi = \xi \hbar / (c r_0)$.

ξ	B_ϕ	m_ϕ [eV]	M_{dim}	x_T	$M_{\text{dim}} x_T$	r_0 [$R_\odot$]	R_T [$R_\odot$]
0	0	0	3.883	2.562	9.95	4.5	11.6
1	1	4.48×10^{-17}	5.414	3.139	17.00	6.3	19.9
2	4	4.69×10^{-17}	10.346	4.572	47.30	12.1	55.3

neglected ingredients appear to matter in realistic settings; self-interactions beyond the polytropic approximation [17,57], finite temperature or incoherent phase-space structure, and external tidal fields capable of stripping weakly bound configurations [9]. The solar-mass configurations have escape speeds of 60–604 km s^{−1}, confirming that they are only modestly bound; tidal-survival estimates would be required for any astrophysical application.

VII. DISCUSSION AND CONCLUSIONS

A. Dynamical stability of the ground state

The nodeless ground state of the SP system is linearly stable under radial perturbations, as demonstrated both by direct numerical evolution [58,64,65] and by the absence of growing modes in the linearized perturbation spectrum. Excited states with one or more nodes are dynamically unstable and decay to the ground state via gravitational cooling on a dynamical timescale [65]. This result is reassuring: once formed, the ground-state configurations persist indefinitely in the absence of external perturbations. For the fermionic case with $\sigma \neq 0$ the perturbation spectrum is modified by degeneracy pressure and stability must be reassessed; the stability of self-interacting generalizations has a richer structure, depending on the sign and magnitude of the coupling [17].

B. Gravitational-wave signatures

Were dark-fermion stars to exist in astrophysical environments, their inspiral and merger in binary systems would generate gravitational-wave emission potentially detectable by next-generation observatories. For an equal-mass binary of total mass M_{tot} at contact separation $a = 2R_T$, the gravitational-wave frequency at contact is [66]

$$f_{\text{GW}} = \frac{1}{\pi} \sqrt{\frac{GM_{\text{tot}}}{(2R_T)^3}}, \quad (48)$$

and the characteristic strain at luminosity distance d_L is

$$h_c = \frac{4G \mathcal{M}_c^{5/3} (\pi f_{\text{GW}})^{2/3}}{c^4 d_L}, \quad (49)$$

where $\mathcal{M}_c = M_{\text{tot}}/2^{6/5}$ is the chirp mass for equal-mass components.

Figure 10 shows the contact gravitational-wave frequency as a function of the total binary mass for several representative compactnesses, together with the design-sensitivity frequency windows of LISA [67], the Einstein Telescope [68], and Cosmic Explorer [69]. The contact frequency sets the highest gravitational-wave frequency emitted before merger; any inspiral signal sweeps upward through the detector band until f_{GW} is reached. Compact configurations with $R_T \sim 10$ km and $M \sim 10^{-8} M_\odot$ produce contact frequencies of order 10^{−1} Hz, placing them near the

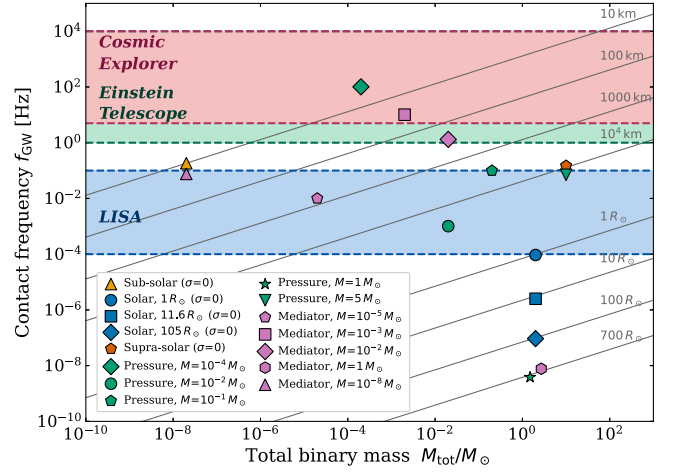


FIG. 10. Gravitational-wave contact frequency f_{GW} [Eq. (48)] as a function of total binary mass for equal-mass dark-fermion star binaries. Diagonal gray lines indicate loci of constant stellar radius (labeled). Shaded horizontal bands, bounded by dashed lines, mark the approximate design-sensitivity windows of three planned or proposed detectors: LISA (10^{−4} – 10^{−1} Hz; blue; Amaro-Seoane *et al.* [67]), the Einstein Telescope (ET, 1–10⁴ Hz; green; Punturo *et al.* [68]), and Cosmic Explorer (CE, 5–10⁴ Hz; pink; Reitze *et al.* [69]). Colored markers show all fifteen illustrative configurations of Table III; the pure Bohm solutions ($\sigma_i = 0$, $\xi = 0$), the two-species models with thermal pressure ($\sigma_1 = 1$, $\sigma_2 = 10$), and the two-species models with mediator coupling ($\xi = 1$). Compact configurations ($R_T \lesssim 10^4$ km) fall within or above the LISA band, whilst the most extended solar-mass objects ($R_T \gtrsim 600R_\odot$) lie far below it.

upper edge of the LISA band, whilst suprasolar objects ($M \sim 5M_\odot$, $R_T \sim 10^4$ km) yield frequencies of order 10^{−1} Hz, likewise accessible to the LISA band. A dedicated signal-analysis study would be required to establish precise detectability thresholds; the present estimates serve to identify the frequency windows in which dark-fermion star mergers ought to be sought.

The illustrative configurations of Table III populate distinct regions of the $f_{\text{GW}}-M_{\text{tot}}$ plane (Fig. 10). The subsolar object ($M = 10^{-8} M_\odot$) yields $f_{\text{GW}} \simeq 0.18$ Hz, in the decihertz regime that proposed missions such as DECIGO [70] and the big bang observer [71] are designed to explore. The suprasolar case ($M = 5M_\odot$, $R_T \simeq 9100$ km) produces $f_{\text{GW}} \simeq 0.15$ Hz, whilst the most extended solar-mass entry ($R_T \simeq 726R_\odot$) yields $f_{\text{GW}} \sim 5 \times 10^{-9}$ Hz, well-below the LISA band. In each case m_1 , through its control of R_T , determines whether a given merger falls within the sensitivity window of space-borne or ground-based interferometers.

C. Microlensing and complementary observational signatures

Gravitational microlensing offers an independent and complementary detection channel, since it requires only

TABLE III. Illustrative dark-fermion star configurations spanning three total-mass regimes. The gravitational compactness is $\kappa \equiv GM/(R_T c^2)$; the escape speed is $v_{\text{esc}} = \sqrt{2GM/R_T}$. All entries satisfy $\kappa < 10^{-3}$, confirming the validity of the Newtonian approximation. The upper panel collects the pure Bohm reference ($\sigma_i = 0, \xi = 0$); the lower panel presents two-species configurations with nonzero degeneracy pressure and mediator coupling. The “Fig.” column shows the colored marker used to identify each configuration in Fig. 10.

Fig.	$M [M_\odot]$	$m_1 [\text{eV}]$	R_T	$\rho_c [\text{g cm}^{-3}]$	$\bar{\rho} [\text{g cm}^{-3}]$	$v_{\text{esc}} [\text{km s}^{-1}]$	κ
<i>Pure Bohm reference ($\sigma_i = 0, \xi = 0$)</i>							
<i>Subsolar mass</i>							
▲	10^{-8}	5.4×10^{-7}	10 km	8.6×10^7	4.8×10^6	16	1.5×10^{-9}
<i>Solar mass</i>							
●	1	2×10^{-13}	$1.0 R_\odot$	2.2×10^1	1.2	604	2.1×10^{-6}
■	1	6×10^{-14}	$11.6 R_\odot$	1.6×10^{-2}	9.0×10^{-4}	181	1.8×10^{-7}
◆	1	2×10^{-14}	$105 R_\odot$	2.2×10^{-5}	1.2×10^{-6}	60	2.0×10^{-8}
<i>Supra-solar mass</i>							
◆	5	8×10^{-13}	9108 km	5.7×10^7	3.1×10^6	12071	8.1×10^{-4}
<i>Two-species with pressure ($\sigma_1 = 1, \sigma_2 = 10, q = 10, f = 0.5, \xi = 0$)</i>							
◆	10^{-4}	4.0×10^{-8}	3.2 km	2.4×10^{16}	1.4×10^{12}	2880	4.6×10^{-5}
●	10^{-2}	4.0×10^{-11}	3.2×10^4 km	2.4×10^6	1.4×10^2	288	4.6×10^{-7}
◆	10^{-1}	4.0×10^{-11}	3244 km	2.3×10^{10}	1.4×10^6	2860	4.6×10^{-5}
★	1	3.2×10^{-14}	$726 R_\odot$	6.1×10^{-5}	3.6×10^{-9}	23	2.9×10^{-9}
▼	5	2.6×10^{-12}	1.5×10^4 km	1.2×10^{10}	6.3×10^5	9406	4.9×10^{-4}
<i>Two-species with mediator ($\sigma_1 = 0.5, \sigma_2 = 5, q = 10, f = 0.5, \xi = 1$)</i>							
◆	10^{-5}	1.4×10^{-8}	695 km	8.7×10^7	1.4×10^4	62	2.1×10^{-8}
■	10^{-3}	6.6×10^{-9}	32 km	8.9×10^{13}	1.4×10^{10}	2880	4.6×10^{-5}
◆	10^{-2}	7.2×10^{-10}	269 km	1.5×10^{12}	2.4×10^8	3141	5.5×10^{-5}
●	1	5.4×10^{-14}	$673 R_\odot$	2.9×10^{-5}	4.6×10^{-9}	24	3.2×10^{-9}
▲	10^{-8}	2.8×10^{-6}	18 km	5.0×10^9	8.1×10^5	12	8.2×10^{-10}

that the object possess sufficient mass to deflect background starlight and is entirely agnostic to the existence of a binary companion. For a point-mass lens of mass M at luminosity distance d_L from the observer (with the source at $d_S \gg d_L$), the Einstein radius is

$$R_E = \sqrt{\frac{4GM}{c^2}} d_L, \quad (50)$$

and the corresponding angular Einstein radius $\theta_E = R_E/d_L$. The microlensing timescale, defined as the time for the lens to traverse one Einstein radius at transverse velocity $v_\perp$, is

$$t_E = \frac{R_E}{v_\perp}. \quad (51)$$

At a fiducial distance $d_L = 8$ kpc and transverse velocity $v_\perp = 200$ km s $^{-1}$, the subsolar and suprasolar configurations of Table III are compact relative to R_E and produce standard point-lens events with timescales $t_E \simeq 21$ h and 87 days, respectively, well within the sensitivity of OGLE [72,73], MOA [74], MACHO [75], and EROS-2 [76].

The extended solar-mass configurations ($R_T \sim 1\text{--}10^2 R_\odot$) exceed $R_E \simeq 6.9 R_\odot$, so the point-lens approximation breaks down and finite-size corrections must be included [77]; a detailed finite-lens calculation using the density profiles derived here lies beyond the present scope. For the supra-solar case, distinguishing the dark-fermion star from an ordinary stellar remnant would require the absence of an electromagnetic counterpart or an anomalous astrometric signal detectable by *Gaia* [78]. Microlensing, the gravitational-wave channel (Sec. VII B), astrometric perturbations and pulsar-timing residuals [79] are complementary in scope, together constraining the abundance, mass function and internal structure of dark-fermion stars.

D. Observational constraints on the particle mass

The superradiance constraints frequently invoked for ultralight dark matter apply exclusively to *bosonic* fields. For spin-0 and spin-1 particles the superradiant instability of rotating black holes enables stringent exclusions in the mass window $m \sim 10^{-13} - 10^{-10}$ eV [80,81]. For spin- $\frac{1}{2}$ fermions the Pauli exclusion principle prevents exponential

amplification, and no superradiant instability develops [82,83]. The illustrative masses employed here ($m_1 \sim 10^{-14} - 10^{-7}$ eV) are therefore *not* excluded by black-hole superradiance, a point that merits emphasis given the prevalence of bosonic constraints in the literature.

The relevant fermionic bounds arise from phase-space considerations. The classical Tremaine-Gunn argument [84] requires $m \gtrsim \mathcal{O}(10^2)$ eV for a single species constituting the totality of dark matter; this bound is substantially relaxed, to $m \gtrsim \mathcal{O}(10^{-14})$ eV, when a large number of quasidegenerate fermionic species is invoked [85], or when the dark fermions constitute only a subcomponent of the dark sector. All m_1 values in Table III satisfy the relaxed bound; however, for the two-species entries the lightest secondary species ($m_2 = m_1/q$ with $q = 10$) lies at or below the Tremaine-Gunn threshold for the most extended configurations and would require either a richer dark-sector spectrum or a subdominant mass fraction to remain consistent. Additional constraints from galactic rotation curves [86], phase-space densities in dwarf spheroidals [87] and laboratory searches with atomic clocks [88] primarily target bosonic couplings; the fermionic hypothesis in this mass range is not excluded by current data, though any astrophysical interpretation must be cross-checked against the latest phase-space and small-scale structure bounds.

E. Connection to the dark matter problem

The configurations studied here contribute to our understanding of the dark matter problem in a specific, and deliberately circumscribed, manner. The rigid mass-radius product at $\xi = 0$ [Eq. (37)] admits no adjustable equation of state; once the total mass M is specified, the radius is entirely determined by the single microphysical parameter m_1 . This predictive rigidity, inherited from the scaling symmetry of the SP system and the fixed Kirzhnits coefficient $\lambda_B = 1/9$, is the central virtue of the model. Consequently, any detection of a self-gravitating dark-fermion star with independently measured mass and radius would immediately constrain m_1 , or in the two-species case the combination (m_1, q, f) , thereby furnishing a direct bridge between astrophysical observables and the dark-sector Lagrangian. In the Thomas-Fermi regime, the Bohm term is confined to a thin shell near the stellar surface, yet it imprints a species-dependent signature, outward quantum pressure for the heavier species, inward surface tension for the lighter, that distinguishes these objects both from conventional compact remnants and from purely bosonic solitons. This distinction is by no means trivial; whilst bosonic solitons are governed by a single scalar-field mass, the fermionic equilibria derived here encode the richer structure of a multicomponent dark sector through the mass ratio q and the central composition f . The present results should therefore be understood as establishing a baseline; the irreducible quantum-gradient contribution to the equilibrium of a self-gravitating degenerate dark-fermion fluid,

upon which more elaborate models incorporating finite temperature, self-interactions beyond the polytropic approximation, and phase-space coarse-graining may be constructed as the relevant physics demands [9,10].

F. Formation channels

Self-gravitating dark-sector fields can relax towards solitonic configurations via gravitational cooling [65], and it is reasonable to suppose that analogous mechanisms operate for degenerate fermion clumps in sufficiently high-density environments. In the bosonic case, dense axion miniclusters have been proposed as formation sites [89], with more recent work quantifying kinetic-regime condensation and subsequent growth [90,91]; a comprehensive review is given by Ferreira [8]. The fermionic analog remains largely unexplored, and establishing whether gravitational cooling or alternative dissipative mechanisms can assemble the configurations derived here constitutes an open question whose resolution would ultimately determine the astrophysical relevance of dark-fermion stars.

G. Summary

I have shown that a two-component self-gravitating degenerate dark-fermion fluid, described within an orbital-free density-functional framework with the Kirzhnits gradient coefficient $\lambda_B = 1/9$, admits equilibria in which the Bohm potential contributes a species-dependent surface-energy correction analogous to the nuclear liquid-drop model. The heavier fermion species establishes an outward quantum-pressure wall, the lighter species provides an inward surface tension, and degeneracy pressure furnishes the dominant bulk confinement: a tripartite balance whose structure is governed by the dimensionless pressure parameter σ_i and the mass ratio q . The predictive rigidity of the resulting mass-radius relation, $MR_T \simeq 9.95\lambda_B \hbar^2 / (Gm_1^2)$ in the SP limit, is a noteworthy feature, for it implies that a single microphysical parameter, m_1 , determines the equilibrium once the total mass is specified, distinguishing these configurations from phenomenological models with adjustable equations of state.

The observational channels through which these objects might reveal themselves, gravitational-wave contact frequencies in the LISA and Einstein Telescope bands, microlensing signatures accessible to OGLE [72] and MOA [74], and astrometric perturbations, are complementary in scope and sensitivity. The mass-radius relations and benchmarked dimensionless invariants presented here furnish a reproducible, first-principles reference for constraining the dark-fermion mass in multicomponent dark sectors. In this sense, the quantum-gradient structure of a degenerate Fermi fluid leaves an indelible imprint on the equilibrium of self-gravitating dark matter, one that current and forthcoming

gravitational-wave observatories are, in principle, well-placed to detect.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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