

Heavy scalar molecule $B_c^+ B_c^-$

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Mass and full width of the heavy scalar molecule $\mathcal{M}$ composed of the mesons B_c^+ and B_c^- are calculated in the QCD sum rule framework. To find its mass and current coupling, we apply the two-point sum rule method. The width of the hadronic molecule $\mathcal{M} = B_c^+ B_c^-$ is evaluated by taking into account its dissociation to $\eta_c \eta_b$, $J/\psi \Upsilon$, and $B_c^+ B_c^-$ mesons. Decays into D and B meson pairs with appropriate charges and quantum numbers triggered by $\bar{b}b$ and $\bar{c}c$ annihilations to light quark-antiquarks are also included in the analyses. Because the partial widths of $\mathcal{M}$ molecule's decay channels depend on the strong couplings at $\mathcal{M}$ -meson-meson vertices, we estimate them by invoking tools of the three-point sum rule method. Our predictions $m = (12725 \pm 85)$ MeV and $\Gamma[\mathcal{M}] = (155 \pm 23)$ MeV for the parameters of the molecule $\mathcal{M}$ provide useful information for experimental studies of numerous heavy resonances.

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I. INTRODUCTION

Internal organization of multi-quark hadrons, their parameters, production, and decay mechanisms are among important questions of the quark-parton model and quantum chromodynamic (QCD). Two primary motivations drive the investigation of such structures. First, there is a fundamental theoretical interest in exploring particles that could exist within the quark-gluon model framework. Second, and more significantly, these hypothetical structures offer a means to elucidate properties of the particles and reconcile them with experimental data. For instance, this concept was utilized in Ref. [1] to analyze a mass hierarchy within the lowest scalar multiplet. According to this work, the nonet of the light scalars are four-quark states $q^2 \bar{q}^2$ which accounts for the experimentally observed features of these mesons.

Production of numerous $J^{PC} = 1^{--}$ states by e^+e^- annihilation in the energy range 3.9–4.4 GeV triggered attempts to explain this phenomenon by the existence of exotic $c\bar{q}\bar{c}q$ mesons [2]. These four-quark particles are made of color-singlet $c\bar{q}$ and $\bar{c}q$ components. Similar concepts regarding the formation of charmonium molecules were also advanced by other researchers [3,4]. In this

scenario, the four-quark mesons manifest as bound or/and resonant states of the D meson pairs, interacting via conventional light meson exchange.

In subsequent years, the concept of hadronic molecules constructed from ordinary mesons was further elaborated on and advanced in numerous publications [5–28], and it still remains an active area of research. These studies explored the binding mechanisms of such states, calculated their masses using diverse models and approaches, as well as analyzed processes through which these particles might be observed.

Over the past two decades, experimental investigations collected significant and rich information regarding potential candidates to exotic tetraquark mesons. All such resonances were examined within the frameworks of both diquark-antidiquark and hadronic molecule models. Prominent among recent experimental achievements is the discovery of four X resonances in di- J/ψ and $J/\psi\psi'$ mass distributions by the LHCb, ATLAS, and CMS Collaborations [29–31]. These resonances are located in the mass interval 6.2–7.3 GeV and are presumed to be hidden-charm scalar tetraquarks $c\bar{c}\bar{c}c$. These fully charmed four-quark resonances were also investigated in our previous articles [32–35], where we employed the QCD sum rule (SR) method to evaluate their masses and full decay widths. A comparison of our predictions with available experimental data led to the interpretation of the lightest state, $X(6200)$, as an $\eta_c \eta_c$ molecule [33], while $X(6600)$ is likely a diquark-antidiquark state with axial-vector constituents [32]. A superposition of a diquark-antidiquark state and a hadronic $\chi_{c0} \chi_{c0}$ molecule exhibits parameters

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consistent with those of the state $X(6900)$ [33,34]. The heaviest resonance, $X(7300)$, can be considered as an admixture of a $\chi_{c1}\chi_{c1}$ molecule and the first radial excitation of $X(6600)$ [35].

Another significant discovery by LHCb that has profoundly impacted exotic hadron physics is the axial-vector resonance T_{cc}^+ , observed in $D^0 D^0 \pi^+$ mass distribution [36,37]. With a $cc\bar{u}\bar{d}$ content, it represents the first experimentally discovered open-charm tetraquark candidate. This state has garnered considerable interest within the high energy physics community and has been explored using various theoretical approaches (see Refs. [38,39] and references therein). In our previous works [38,39], we evaluated the mass and decay width of this resonance, by modeling it as both a $[cc][\bar{u}\bar{d}]$ diquark-antidiquark and a $D^0 D^{*+}$ molecule state. Our results suggest that T_{cc}^+ preferentially possesses a diquark-antidiquark structure, although the computational accuracy does not preclude its interpretation as a hadronic molecule.

These well-established experimental findings have demonstrated that fully heavy tetraquarks are accessible in ongoing experiments, and thereby encouraged research efforts dedicated to the thorough analysis of tetraquark mesons composed exclusively of c and b quarks. Exotic structures with hidden charm and bottom, such as $bc\bar{b}\bar{c}$ tetraquarks, form an interesting family of such particles. These states were investigated intensively in the literature (see Refs. [40–42] and references therein). In our studies [40–42], we examined the scalar, axial-vector, and tensor $bc\bar{b}\bar{c}$ particles within the diquark-antidiquark framework and calculated their masses and full decay widths.

In this article, we model the scalar $bc\bar{b}\bar{c}$ tetraquark as a hadronic molecule $\mathcal{M} = B_c^+ B_c^-$ and compute its parameters. The mass m and current coupling Λ of $\mathcal{M}$ are found using the two-point sum rule method [43,44]. As it will be demonstrated, the hadronic molecule $\mathcal{M}$ is heavier than two-meson thresholds $\eta_b \eta_c$, $J/\psi \Upsilon$, and $B_c^+ B_c^-$; consequently, dissociations to these meson pairs represent its kinematically allowed decay channels. Other mechanisms also exist for decays of $\mathcal{M}$ into ordinary mesons [45–47]. Indeed, owing to $b\bar{b}$ and $c\bar{c}$ annihilations into $\bar{q}q$ and $\bar{s}s$ pairs, $\mathcal{M}$ readily decays into $D^{(*)+} D^{(*)-}$, $D^{(*)0} \bar{D}^{(*)0}$, and $D_s^{(*)+} D_s^{(*)-}$, as well as $B^{(*)+} B^{(*)-}$, $B^{(*)0} \bar{B}^{(*)0}$, and $B_s^{(*)0} \bar{B}_s^{(*)0}$ meson pairs. Decay widths for all decay channels of the $\mathcal{M}$ molecule are evaluated within the framework of the three-point sum rule method, which is necessary for determining the strong couplings at the corresponding $\mathcal{M}$ -meson-meson vertices.

This work contains the following sections: In Sec. II, we determine the mass and current coupling of the scalar molecule $\mathcal{M}$. The fall-apart processes $\mathcal{M} \rightarrow \eta_b \eta_c$, $J/\psi \Upsilon$, and $B_c^+ B_c^-$ are explored in Sec. III. The decays into mesons $D^{(*)+} D^{(*)-}$, $D^{(*)0} \bar{D}^{(*)0}$, and $D_s^{(*)+} D_s^{(*)-}$ are considered in Sec. IV. Partial widths of the processes with two final-state

B mesons are calculated in Sec. V. Here, we evaluate the full decay width of the molecule $\mathcal{M}$ as well. Our conclusions are collected in Sec. VI.

II. SPECTROSCOPIC PARAMETERS OF $\mathcal{M} = B_c^+ B_c^-$

To derive the SRs for the mass m and current coupling Λ of the scalar molecule $\mathcal{M}$, we consider the correlation function

$$\Pi(p) = i \int d^4 x e^{ipx} \langle 0 | \mathcal{T} \{ J(x) J^\dagger(0) \} | 0 \rangle, \quad (1)$$

where $J(x)$ is the interpolating current for the molecule $\mathcal{M}$ and $\mathcal{T}$ is the time-ordered product of currents.

In the hadronic molecule picture the interpolating current for $\mathcal{M}$ is

$$J(x) = [\bar{b}_a(x) i \gamma_5 c_a(x)] [\bar{c}_b(x) i \gamma_5 b_b(x)], \quad (2)$$

where a and b are the color indices. This current corresponds to the scalar particle with the quantum numbers $J^{PC} = 0^{++}$.

The sum rules for m and Λ can be found by calculating $\Pi(p)$ using the physical parameters of $\mathcal{M}$, which establishes the phenomenological component of SRs $\Pi^{\text{Phys}}(p)$. The correlator $\Pi^{\text{Phys}}(p)$ is given by the expression

$$\Pi^{\text{Phys}}(p) = \frac{\langle 0 | J | \mathcal{M} \rangle \langle \mathcal{M} | J^\dagger | 0 \rangle}{m^2 - p^2} + \dots \quad (3)$$

In Eq. (3) the term presented explicitly is a contribution of the ground-state particle, whereas effects of higher resonances and continuum states are shown by ellipses.

To calculate $\Pi^{\text{Phys}}(p)$, it is useful to write it using the parameters m and Λ . For these purposes, we introduce the matrix element

$$\langle 0 | J | \mathcal{M} \rangle = \Lambda. \quad (4)$$

After simple manipulations, one gets

$$\Pi^{\text{Phys}}(p) = \frac{\Lambda^2}{m^2 - p^2} + \dots \quad (5)$$

A term in $\Pi^{\text{Phys}}(p)$ with the simple Lorentz structure $\Lambda^2/(m^2 - p^2)$ is the amplitude $\Pi^{\text{Phys}}(p^2)$ necessary for our analysis.

We calculate $\Pi(p)$ with certain accuracy also by utilizing the heavy quark propagators and operator product expansion (OPE). The function $\Pi^{\text{OPE}}(p)$ obtained after this computation establishes the QCD side of the SRs. It contains the perturbative and nonperturbative terms: the latter being proportional to $\langle \alpha_s G^2/\pi \rangle$. The reason is that

heavy quark propagators do not contain light quark and mixed quark-gluon condensates. As a result, the next nonperturbative contributions to $\Pi^{\text{OPE}}(p)$ are generated by gluon condensates $\langle g_s^3 G^3 \rangle$ and $\langle \alpha_s G^2/\pi \rangle^2$, which are small and neglected in what follows.

The $\Pi^{\text{OPE}}(p)$ obtained using propagators of the heavy quarks is

$$\Pi^{\text{OPE}}(p) = i \int d^4x e^{ipx} \text{Tr} \left[\gamma_5 S_c^{aa'}(x) \gamma_5 S_b^{a'a}(-x) \right] \times \text{Tr} \left[\gamma_5 S_b^{bb'}(x) \gamma_5 S_c^{b'b}(-x) \right], \quad (6)$$

where propagators of the b and c quarks are denoted by $S_{b(c)}(x)$ [48], respectively.

In the case under consideration $\Pi^{\text{OPE}}(p)$ is equal to the invariant amplitude $\Pi^{\text{OPE}}(p^2)$. To find SRs for the m and Λ one equates $\Pi^{\text{OPE}}(p^2)$ and $\Pi^{\text{Phys}}(p^2)$, uses the Borel transformation, and, benefiting from the quark-hadron duality assumption, carries out continuum subtraction [43,44]. As a result, one obtains

$$m^2 = \frac{\Pi'(M^2, s_0)}{\Pi(M^2, s_0)}, \quad (7)$$

with $\Pi'(M^2, s_0) = \frac{d\Pi(M^2, s_0)}{d(-1/M^2)}$ and

$$\Lambda^2 = e^{m^2/M^2} \Pi(M^2, s_0), \quad (8)$$

which are the SRs for the parameters of $\mathcal{M}$. Above $\Pi(M^2, s_0)$ is the amplitude $\Pi^{\text{OPE}}(p^2)$ after the Borel transformation and continuum subtraction, and it depends on Borel and continuum subtraction parameters M^2 and s_0 .

The amplitude $\Pi(M^2, s_0)$ is a sum of the integral of the spectral density $\rho(s)$ and the term $\Pi(M^2)$:

$$\Pi(M^2, s_0) = \int_{4(m_b+m_c)^2}^{s_0} ds \rho(s) e^{-s/M^2} + \Pi(M^2). \quad (9)$$

The function $\rho(s)$ is equal to the imaginary part of $\Pi^{\text{OPE}}(p^2)$, whereas $\Pi(M^2)$ is evaluated directly from $\Pi^{\text{OPE}}(p^2)$ and does not contain terms included in $\rho(s)$. We do not write down explicitly $\rho(s)$ and $\Pi(M^2)$ because they are given by lengthy formulas.

To perform numerical computations, one needs to specify parameters which enter Eqs. (7) and (8). The masses of the quarks m_b and m_c as well as gluon condensate $\langle \alpha_s G^2/\pi \rangle$ are universal quantities that do not depend on a considering problem:

$$\begin{aligned} m_b &= (4.183 \pm 0.007) \text{ GeV}, \\ m_c &= (1.2730 \pm 0.0046) \text{ GeV}, \\ \langle \alpha_s G^2/\pi \rangle &= (0.012 \pm 0.004) \text{ GeV}^4. \end{aligned} \quad (10)$$

In Eq. (10) m_b and m_c are the running (scale-dependent) b and c quark masses $\bar{m}_b$ and $\bar{m}_c$ in the $\overline{\text{MS}}$ scheme at the scales $\mu = \bar{m}_b$ and $\mu = \bar{m}_c$ [49], respectively. Note that there are various mass definitions, i.e., schemes for the heavy quarks (see the review “Quark Masses” in Ref. [49]). The masses of quarks expressed in some of such schemes can be converted to another one by means of known formulas. The $\overline{\text{MS}}$ scheme is appropriate for our purposes, because it effectively suppresses nonperturbative effects ensuring convergence of OPE and improves precision of calculations. The gluon condensate $\langle \alpha_s G^2/\pi \rangle$ is a μ -scale independent parameter: Its numerical value was extracted from the analysis of hadronic processes [43,44].

In contrast, parameters M^2 and s_0 depend on a task under analysis, and their choice should meet the requirements of the SR calculations. Namely, they have to be fixed in such a way to ensure $\text{PC} \geq 0.5$, where PC is the pole contribution defined as

$$\text{PC} = \frac{\Pi(M^2, s_0)}{\Pi(M^2, \infty)}. \quad (11)$$

The convergence of the operator product expansion and stability of extracted quantities m and Λ on these parameters are also among necessary conditions. In the present computations we take into account the dimension-four nonperturbative term. Therefore, to guarantee the convergence of OPE it is enough to impose a constraint on this term. We restrict this term by the requirement

$$|\Pi^{\text{Dim4}}(M^2, s_0)| \leq 0.05 |\Pi(M^2, s_0)|. \quad (12)$$

Note that the upper border of M^2 is fixed using Eq. (11). The constraint Eq. (12) allows one to fix the low limit of the Borel parameter.

Our calculations prove that the working windows

$$M^2 \in [12.5, 14.5] \text{ GeV}^2, \quad s_0 \in [180, 185] \text{ GeV}^2 \quad (13)$$

satisfy these constraints. Indeed, at $M^2 = 14.5 \text{ GeV}^2$ one gets $\text{PC} \approx 0.51$ (on the average in s_0), whereas at $M^2 = 12.5 \text{ GeV}^2$ the pole contribution is equal to $\text{PC} \approx 0.68$. The term $|\Pi^{\text{Dim4}}(M^2, s_0)|$ at $M^2 = 12.5 \text{ GeV}^2$ does not exceed 1% of the whole result. In Fig. 1, we plot the dependence of PC on the parameter M^2 at some values of s_0 . It is clear that PC overshoots 0.5 for almost all M^2 and s_0 from the working windows.

The spectral parameters of the molecule $\mathcal{M}$ are computed as their average values in the domains Eq. (13) and amount to

$$\begin{aligned} m &= (12725 \pm 85) \text{ MeV}, \\ \Lambda &= (1.40 \pm 0.16) \text{ GeV}^5. \end{aligned} \quad (14)$$

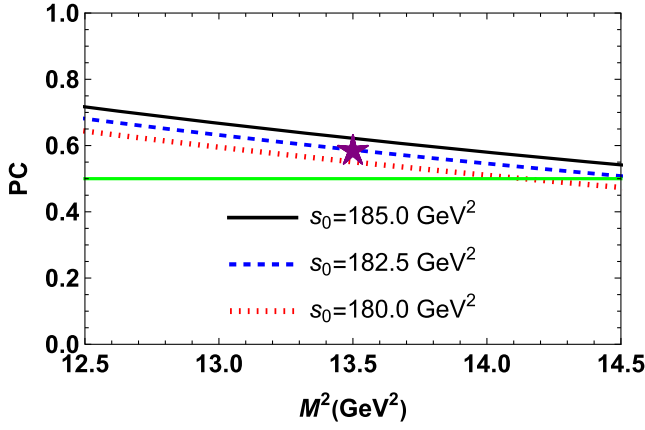


FIG. 1. The pole contribution PC as a function of M^2 at fixed s_0 . The red star marks the point $M^2 = 13.5 \text{ GeV}^2$ and $s_0 = 182.5 \text{ GeV}^2$.

These results coincide with the sum rule predictions at the point $M^2 = 13.5 \text{ GeV}^2$ and $s_0 = 182.5 \text{ GeV}^2$, where the pole term forms $\text{PC} \approx 59\%$ of the whole contribution. This ensures the prevalence of the pole contribution in the obtained results and shows the ground-state character of the molecule $\mathcal{M}$ in its class. As the function of the parameters M^2 and s_0 the mass of the molecule $\mathcal{M}$ is depicted in Fig. 2.

III. DISSOCIATIONS OF $\mathcal{M}$ TO CONVENTIONAL MESONS

In this section, we consider dissociations of the scalar molecule $\mathcal{M}$ into pairs of ordinary mesons and calculate partial widths of these channels. In these processes constituent quarks of $\mathcal{M}$ form directly the final-state particles. The mass m and spin-parities $J^{\text{PC}} = 0^{++}$ of $\mathcal{M}$ are key parameters that limit its possible decay channels. In fact, it is evident that the mass of $\mathcal{M}$ should exceed the corresponding two-meson thresholds. It is easy to see that decays to mesons $\eta_b \eta_c$, $J/\psi \Upsilon$, and $B_c^+ B_c^-$ meet this requirement. The quantum

numbers of the final-state mesons are also compatible with ones of the hadronic molecule $\mathcal{M}$.

A. $\mathcal{M} \rightarrow \eta_b \eta_c$

We first explore the process $\mathcal{M} \rightarrow \eta_b \eta_c$. The width of this decay, besides the masses and decay constants (current couplings) of the initial- and final-state particles, contains the coupling g_1 which characterizes their strong interaction at the vertex $\mathcal{M} \eta_b \eta_c$. To evaluate g_1 , we consider the QCD correlator

$$\Pi_1(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | T \{ J^{\eta_b}(y) \times J^{\eta_c}(0) J^\dagger(x) \} | 0 \rangle, \quad (15)$$

with

$$J^{\eta_b}(x) = \bar{b}_i(x) i \gamma_5 b_i(x), \quad J^{\eta_c}(x) = \bar{c}_j(x) i \gamma_5 c_j(x) \quad (16)$$

being the pseudoscalar η_b and η_c mesons' currents, respectively.

Analysis of this correlation function enables us to obtain the SR for the form factor $g_1(q^2)$, which at the mass shell $q^2 = m_{\eta_c}^2$ gives g_1 . To get the sum rule for $g_1(q^2)$, we recast $\Pi_1(p, p')$ into the form

$$\begin{aligned} \Pi_1^{\text{Phys}}(p, p') &= \frac{\langle 0 | J^{\eta_b} | \eta_b(p') \rangle \langle 0 | J^{\eta_c} | \eta_c(q) \rangle}{p'^2 - m_{\eta_b}^2} \frac{q^2 - m_{\eta_c}^2}{q^2 - m_{\eta_c}^2} \\ &\times \langle \eta_b(p') \eta_c(q) | \mathcal{M}(p) \rangle \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{p^2 - m^2} + \dots, \end{aligned} \quad (17)$$

where $m_{\eta_b} = (9398.7 \pm 2.0) \text{ MeV}$ and $m_{\eta_c} = (2983.9 \pm 0.4) \text{ MeV}$ are the masses of the quarkonia η_b and η_c [49], respectively. The term in the correlation function $\Pi_1^{\text{Phys}}(p, p')$ is a contribution of the ground-state particles, whereas the dots denote effects of higher resonances

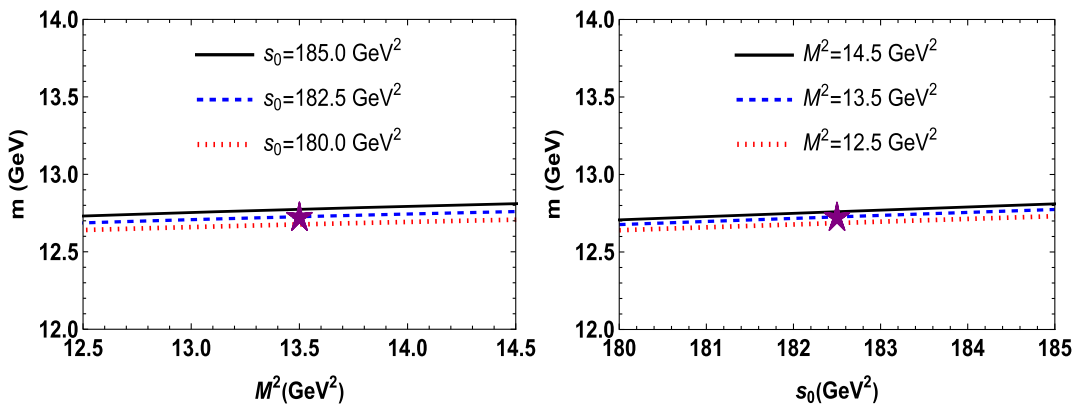


FIG. 2. Mass m of the hadronic molecule $\mathcal{M}$ vs the parameters M^2 (left panel) and s_0 (right panel).

and continuum states. Next, we introduce the matrix elements

$$\langle 0 | J^{\eta_b} | \eta_b(p') \rangle = \frac{f_{\eta_b} m_{\eta_b}^2}{2m_b}, \quad \langle 0 | J^{\eta_c} | \eta_c(q) \rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}, \quad (18)$$

where $f_{\eta_b} = 724$ MeV and $f_{\eta_c} = (421 \pm 35)$ MeV are decay constants of the mesons η_b and η_c .

The vertex $\mathcal{M}_{\eta_b \eta_c}$ is given by the formula

$$\langle \eta_b(p') \eta_c(q) | \mathcal{M}(p) \rangle = g_1(q^2) p \cdot p'. \quad (19)$$

Having performed all calculations, for $\Pi_1^{\text{Phys}}(p, p')$ we get

$$\begin{aligned} \Pi_1^{\text{Phys}}(p, p') &= g_1(q^2) \frac{\Lambda f_{\eta_b} m_{\eta_b}^2 f_{\eta_c} m_{\eta_c}^2}{8m_b m_c (p^2 - m^2)} \\ &\times \frac{m^2 + m_{\eta_b}^2 - q^2}{(p'^2 - m_{\eta_b}^2)(q^2 - m_{\eta_c}^2)} + \dots \end{aligned} \quad (20)$$

The correlator $\Pi_1^{\text{Phys}}(p, p')$ has a simple Lorentz structure; therefore, it is equal to the invariant amplitude $\Pi_1^{\text{Phys}}(p^2, p'^2, q^2)$.

The same correlation function $\Pi_1(p, p')$ computed in terms of the heavy quark propagators reads

$$\begin{aligned} \Pi_1^{\text{OPE}}(p, p') &= i^4 \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr}[\gamma_5 S_b^{ib}(y-x) \\ &\times \gamma_5 S_c^{bj}(x) \gamma_5 S_c^{ja}(-x) \gamma_5 S_b^{ai}(x-y)]. \end{aligned} \quad (21)$$

Due to a triviality of its structure, $\Pi_1^{\text{OPE}}(p, p')$ is equal to the amplitude $\Pi_1^{\text{OPE}}(p^2, p'^2, q^2)$. Having equated the amplitudes $\Pi_1^{\text{Phys}}(p^2, p'^2, q^2)$ and $\Pi_1^{\text{OPE}}(p^2, p'^2, q^2)$, performed the double Borel transformations over $-p^2$, $-p'^2$, and continuum subtractions, one gets the sum rule for $g_1(q^2)$:

$$\begin{aligned} g_1(q^2) &= \frac{8m_b m_c}{\Lambda f_{\eta_b} m_{\eta_b}^2 f_{\eta_c} m_{\eta_c}^2} \frac{q^2 - m_{\eta_c}^2}{m^2 + m_{\eta_b}^2 - q^2} \\ &\times e^{m^2/M_1^2} e^{m_{\eta_b}^2/M_2^2} \Pi_1(\mathbf{M}^2, \mathbf{s}_0, q^2). \end{aligned} \quad (22)$$

In Eq. (22) the correlator $\Pi_1(\mathbf{M}^2, \mathbf{s}_0, q^2)$ has the following form:

$$\begin{aligned} \Pi_1(\mathbf{M}^2, \mathbf{s}_0, q^2) &= \int_{4(m_b+m_c)^2}^{s_0} ds \int_{4m_b^2}^{s'_0} ds' \rho_1(s, s', q^2) \\ &\times e^{-s/M_1^2} e^{-s'/M_2^2}. \end{aligned} \quad (23)$$

It is written using the spectral density $\rho_1(s, s', q^2)$ which contains the perturbative and dimension-four components. The function $\Pi_1(\mathbf{M}^2, \mathbf{s}_0, q^2)$ depends on the Borel $\mathbf{M}^2 = (M_1^2, M_2^2)$ and continuum threshold parameters $\mathbf{s}_0 = (s_0, s'_0)$. The pair of parameters (M_1^2, s_0) corresponds

to the hadronic $\mathcal{M}$ molecule channel, whereas (M_2^2, s'_0) describes the channel of the η_b meson.

To perform numerical analysis, in the $\mathcal{M}$ channel we employ M_1^2 and s_0 from Eq. (13). In the η_b channel as the parameters (M_2^2, s'_0) we use

$$M_2^2 \in [9, 11] \text{ GeV}^2, \quad s'_0 \in [95, 99] \text{ GeV}^2. \quad (24)$$

Note that $\sqrt{s'_0}$ is restricted by the mass 9.999 GeV of the first radially excited $\eta_b(2S)$ meson.

The credible results for the form factor $g_1(q^2)$ in the sum rule context is obtained in the Euclidean region $q^2 < 0$. But the coupling g_1 has to be determined at $q^2 = m_{\eta_c}^2$. This problem can be avoided by introducing a variable $Q^2 = -q^2$ and employing the notation $g_1(Q^2)$ for the new function. Afterwards, we employ an extrapolating function $\mathcal{G}_1(Q^2)$ which for $Q^2 > 0$ coincides with the SR data, but can also be computed at $Q^2 < 0$. For these purposes, we use the function $\mathcal{G}_i(Q^2)$

$$\mathcal{G}_i(Q^2) = \mathcal{G}_i^0 \exp \left[a_i^1 \frac{Q^2}{m^2} + a_i^2 \left(\frac{Q^2}{m^2} \right)^2 \right], \quad (25)$$

where $\mathcal{G}_i^0$, a_i^1 , and a_i^2 are parameters extracted from comparison with the SR data.

In this article, the SR computations are carried out for $Q^2 = 2-20$ GeV². Relevant numerical predictions for $g_1(Q^2)$ are depicted in Fig. 3. Then, one easily determines the parameters $\mathcal{G}_1^0 = 0.13$ GeV⁻¹, $a_1^1 = 4.24$, and $a_1^2 = -9.62$ of the function $\mathcal{G}_1(Q^2)$: This function is shown in Fig. 3 as well.

Finally, we find g_1 ,

$$g_1 \equiv \mathcal{G}_1(-m_{\eta_c}^2) = (1.0 \pm 0.2) \times 10^{-1} \text{ GeV}^{-1}. \quad (26)$$

We calculate the width of the decay $\mathcal{M} \rightarrow \eta_b \eta_c$ by means of the formula

$$\Gamma[\mathcal{M} \rightarrow \eta_b \eta_c] = g_1^2 \frac{m_{\eta_b}^2 \lambda_1}{8\pi} \left(1 + \frac{\lambda_1^2}{m_{\eta_b}^2} \right), \quad (27)$$

where $\lambda_1 = \lambda(m, m_{\eta_b}, m_{\eta_c})$, and

$$\lambda(x, y, z) = \frac{\sqrt{x^4 + y^4 + z^4 - 2(x^2 y^2 + x^2 z^2 + y^2 z^2)}}{2x}. \quad (28)$$

As a result, we obtain

$$\Gamma[\mathcal{M} \rightarrow \eta_b \eta_c] = (48.4 \pm 15.0) \text{ MeV}. \quad (29)$$

A part ± 13.7 of the total errors in Eq. (29) is connected with the uncertainties of the coupling g_1 , whereas ± 6.1 is generated by the ambiguities of the particles' masses in Eq. (27).

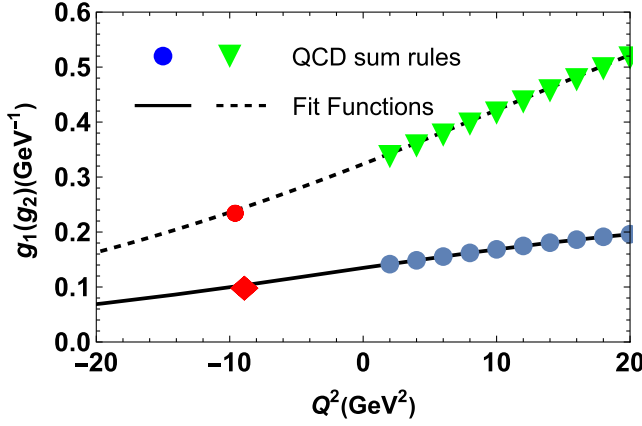


FIG. 3. QCD data and fit functions for the form factor $g_1(Q^2)$ (solid line) and $g_2(Q^2)$ (dashed line). The diamond and circle fix the points $Q^2 = -m_{\eta_c}^2$ and $Q^2 = -m_{J/\psi}^2$, respectively.

B. $\mathcal{M} \rightarrow J/\psi \Upsilon$

Here, we consider the decay $\mathcal{M} \rightarrow J/\psi \Upsilon$ and extract the strong coupling g_2 at the vertex $\mathcal{M}J/\psi \Upsilon$, which is equal to the form factor $g_2(q^2)$ at the mass shell $q^2 = m_{J/\psi}^2$.

The SR for the form factor $g_2(q^2)$ is derived by means of the correlation function

$$\Pi_{2\mu\nu}(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J_\mu^\Upsilon(y) \times J_\nu^{J/\psi}(0) J^\dagger(x) \} | 0 \rangle, \quad (30)$$

where $J_\mu^\Upsilon(x)$ and $J_\nu^{J/\psi}(x)$ are interpolating currents of the vector quarkonia Υ and J/ψ , respectively. They are defined as

$$J_\mu^\Upsilon(x) = \bar{b}_i(x) \gamma_\mu b_i(x), \quad J_\nu^{J/\psi}(x) = \bar{c}_j(x) \gamma_\nu c_j(x), \quad (31)$$

with i and j being the color indices.

To find the phenomenological component $\Pi_{2\mu\nu}^{\text{Phys}}(p, p')$ of the sum rule, we need to rewrite Eq. (30) using the involved particles' physical parameters. By taking into account only contributions of the ground-level states, we recast the correlator $\Pi_{2\mu\nu}(p, p')$ into the form

$$\begin{aligned} \Pi_{2\mu\nu}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J_\mu^\Upsilon | \Upsilon(p', \varepsilon_1) \rangle \langle 0 | J_\nu^{J/\psi} | J/\psi(q, \varepsilon_2) \rangle}{p'^2 - m_\Upsilon^2} \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{q^2 - m_{J/\psi}^2} \\ &\times \langle \Upsilon(p', \varepsilon_1) J/\psi(q, \varepsilon_2) | \mathcal{M}(p) \rangle \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{p^2 - m^2} + \dots, \end{aligned} \quad (32)$$

where $m_\Upsilon = (9460.40 \pm 0.09 \pm 0.04)$ MeV and $m_{J/\psi} = (3096.900 \pm 0.006)$ MeV are masses of the Υ and J/ψ

mesons [49]. In the expression above, we denote by ε_1 and ε_2 the polarization vectors of these quarkonia, respectively.

To further simplify Eq. (32), it is convenient to employ the matrix elements of the mesons Υ and J/ψ ,

$$\begin{aligned} \langle 0 | J_\mu^\Upsilon | \Upsilon(p', \varepsilon_1) \rangle &= f_\Upsilon m_\Upsilon \varepsilon_{1\mu}, \\ \langle 0 | J_\nu^{J/\psi} | J/\psi(q, \varepsilon_2) \rangle &= f_{J/\psi} m_{J/\psi} \varepsilon_{2\nu}. \end{aligned} \quad (33)$$

Here, $f_\Upsilon = (708 \pm 8)$ MeV and $f_{J/\psi} = (411 \pm 7)$ MeV are decay constants of the mesons: Their experimental values are borrowed from Ref. [50].

Besides, one should specify the matrix element $\langle \Upsilon(p', \varepsilon_1) J/\psi(q, \varepsilon_2) | \mathcal{M}(p) \rangle$, which depends on the momenta and polarization vectors of the particles $\mathcal{M}$, Υ , and J/ψ ,

$$\begin{aligned} \langle \Upsilon(p', \varepsilon_1) J/\psi(q, \varepsilon_2) | \mathcal{M}(p) \rangle &= g_2(q^2) [q \cdot p' \\ &\times \varepsilon_1^* \cdot \varepsilon_2^* - q \cdot \varepsilon_1^* p' \cdot \varepsilon_2^*]. \end{aligned} \quad (34)$$

As a result, for $\Pi_{2\mu\nu}^{\text{Phys}}(p, p')$ we get the expression

$$\begin{aligned} \Pi_{2\mu\nu}^{\text{Phys}}(p, p') &= g_2(q^2) \frac{\Lambda f_\Upsilon m_\Upsilon f_{J/\psi} m_{J/\psi}}{(p^2 - m^2)(p'^2 - m_\Upsilon^2)(q^2 - m_{J/\psi}^2)} \\ &\times \left[\frac{(m^2 - m_\Upsilon^2 - q^2)}{2} g_{\mu\nu} - p'_\mu q_\nu \right] + \dots. \end{aligned} \quad (35)$$

The function $\Pi_{2\mu\nu}^{\text{OPE}}(p, p')$, i.e., the QCD component of the sum rule, is given by the expression

$$\begin{aligned} \Pi_{2\mu\nu}^{\text{OPE}}(p, p') &= \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr} [\gamma_\mu S_b^{ib}(y-x) \\ &\times \gamma_5 S_c^{bj}(x) \gamma_\nu S_c^{ja}(-x) \gamma_5 S_a^{ai}(x-y)]. \end{aligned} \quad (36)$$

We utilize the structures proportional to $g_{\mu\nu}$ in the correlators and use corresponding amplitudes $\Pi_2^{\text{Phys}}(p^2, p'^2, q^2)$ and $\Pi_2^{\text{OPE}}(p^2, p'^2, q^2)$ to find SR for the form factor $g_2(q^2)$. After standard operations the sum rule for $g_2(q^2)$ reads

$$\begin{aligned} g_2(q^2) &= \frac{2(q^2 - m_{J/\psi}^2)}{\Lambda f_\Upsilon m_\Upsilon f_{J/\psi} m_{J/\psi} (m^2 - m_\Upsilon^2 - q^2)} \\ &\times e^{m^2/M_1^2} e^{m_\Upsilon^2/M_2^2} \Pi_2(\mathbf{M}^2, \mathbf{s}_0, q^2). \end{aligned} \quad (37)$$

Requirements that should be satisfied by the auxiliary parameters $\mathbf{M}^2$ and $\mathbf{s}_0$ are universal for all SR computations and have been explained in the previous section. Numerical analysis shows that the regions in Eq. (13) for the parameters (M_1^2, s_0) and

$$M_2' \in [10, 12] \text{ GeV}^2, \quad s_0' \in [98, 100] \text{ GeV}^2, \quad (38)$$

for (M_2', s_0') satisfy all these requirements. Note that s_0' is limited by the mass $m_{Y(2S)} = (10023.4 \pm 0.5) \text{ MeV}$ of the radially excited state $Y(2S)$, i.e., $s_0' < m_{Y(2S)}^2$.

We calculate $g_2(Q^2)$ in the domain $Q^2 = 2\text{--}20 \text{ GeV}^2$ and fit these data by the function $\mathcal{G}_2(Q^2)$ with parameters $\mathcal{G}_1^0 = 0.32 \text{ GeV}^{-1}$, $a_1^1 = 4.71$, and $a_1^2 = -6.81$. The coupling g_2 is extracted at $q^2 = m_{J/\psi}^2$, i.e., at $Q^2 = -m_{J/\psi}^2$,

$$g_2 \equiv \mathcal{G}_2(-m_{J/\psi}^2) = (2.4 \pm 0.5) \times 10^{-1} \text{ GeV}^{-1}. \quad (39)$$

The partial width of the decay $\mathcal{M} \rightarrow J/\psi Y$ is determined by the expression

$$\Gamma[\mathcal{M} \rightarrow J/\psi Y] = g_2^2 \frac{\lambda_2}{16\pi m^2} [m^4 + m_Y^4 + m_{J/\psi}^4, \\ + 4m_Y^2 m_{J/\psi}^2 - 2m^2(m_Y^2 + m_{J/\psi}^2)], \quad (40)$$

where $\lambda_2 = \lambda(m, m_Y, m_{J/\psi})$. Our computations yield

$$\Gamma[\mathcal{M} \rightarrow J/\psi Y] = (35.3 \pm 14.4) \text{ MeV}. \quad (41)$$

C. $\mathcal{M} \rightarrow B_c^+ B_c^-$

The partial width of the decay $\mathcal{M} \rightarrow B_c^+ B_c^-$ is fixed by the coupling g_3 at the vertex $\mathcal{M} B_c^+ B_c^-$. In the framework of the QCD SR method the form factor $g_3(q^2)$ is calculated using the three-point correlation function

$$\Pi_3(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | T \{ J^{B_c^+}(y) \\ \times J^{B_c^-}(0) J^+(x) \} | 0 \rangle, \quad (42)$$

where $J^{B_c^+}(x)$ and $J^{B_c^-}(x)$ are the interpolating currents for the mesons B_c^+ and B_c^-

$$J^{B_c^+}(x) = \bar{b}_i(x) i\gamma_5 c_i(x), \quad J^{B_c^-}(x) = \bar{c}_j(x) i\gamma_5 b_j(x). \quad (43)$$

To compute the physical part of the SR, we make use of the expressions

$$\langle 0 | J^{B_c^\pm} | B_c^\pm \rangle = \frac{f_{B_c} m_{B_c}^2}{m_b + m_c} \quad (44)$$

and

$$\langle B_c^+(p') B_c^-(q) | \mathcal{M} \rangle = g_3(q^2) p \cdot p', \quad (45)$$

where $m_{B_c} = (6274.47 \pm 0.27 \pm 0.17) \text{ MeV}$ and $f_{B_c} = (371 \pm 37) \text{ MeV}$ are the parameters of the mesons $B_c^\pm$ [49, 51]. In terms of m_{B_c} and f_{B_c} the correlation function $\Pi_3(p, p')$ acquires the form

$$\Pi_3^{\text{Phys}}(p, p') = \frac{g_3(q^2) \Lambda f_{B_c}^2 m_{B_c}^4}{(m_b + m_c)^2 (p^2 - m^2) (p'^2 - m_{B_c}^2)} \\ \times \frac{m^2 + m_{B_c}^2 - q^2}{2(q^2 - m_{B_c}^2)} + \dots \quad (46)$$

The QCD component of SR for the form factor $g_3(q^2)$ is given by the expression

$$\Pi_3^{\text{OPE}}(p, p') = -i^4 \int d^4x d^4y e^{ip'y} e^{-ipx} \text{Tr}[\gamma_5 S_c^{ia}(y-x) \\ \times \gamma_5 S_b^{ai}(x-y)] \text{Tr}[\gamma_5 S_c^{bj}(x) \gamma_5 S_b^{jb}(-x)]. \quad (47)$$

In the process under consideration, the correlators $\Pi_3^{\text{Phys}}(p, p')$ and $\Pi_3^{\text{OPE}}(p, p')$ have simple Lorentz organizations. Having denoted by $\Pi_3^{\text{Phys}}(p^2, p'^2, q^2)$ and $\Pi_3^{\text{OPE}}(p^2, p'^2, q^2)$ corresponding amplitudes, we find the sum rule

$$g_3(q^2) = \frac{2(m_b + m_c)^2}{\Lambda f_{B_c}^2 m_{B_c}^4} \frac{q^2 - m_{B_c}^2}{m^2 + m_{B_c}^2 - q^2} \\ \times e^{m^2/M_1^2} e^{m_{B_c}^2/M_2^2} \Pi_3(\mathbf{M}^2, \mathbf{s}_0, q^2). \quad (48)$$

Here, $\Pi_3(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is the Borel transformed and subtracted amplitude $\Pi_3^{\text{OPE}}(p^2, p'^2, q^2)$.

The remaining operations are similar to ones explained above. Therefore, we present only the windows for M_2' and s_0' in the B_c^+ meson channel

$$M_2' \in [6.5, 7.5] \text{ GeV}^2, \quad s_0' \in [45, 47] \text{ GeV}^2. \quad (49)$$

The function $\mathcal{G}_3(Q^2)$ has the parameters $\mathcal{G}_3^0 = 0.21 \text{ GeV}^{-1}$, $a_3^1 = 1.79$, and $a_3^2 = -4.59$. Then, the strong coupling g_3 is equal to

$$g_3 \equiv \mathcal{G}_3(-m_{B_c}^2) = (1.0 \pm 0.2) \times 10^{-1} \text{ GeV}^{-1}. \quad (50)$$

Having employed g_3 , and Eq. (27) with evident replacements $m_{\eta_b} \rightarrow m_{B_c}$ and $\lambda_3 \rightarrow \lambda_3 = \lambda(m, m_{B_c}, m_{B_c})$, we find

$$\Gamma[\mathcal{M} \rightarrow B_c^+ B_c^-] = (17.7 \pm 6.7) \text{ MeV}. \quad (51)$$

IV. DECAYS INTO D MESON PAIRS

It has been noted above that the hadronic molecule $\mathcal{M}$ can decay to charmed mesons due to annihilation of the $b\bar{b}$ pair in $\mathcal{M}$. We study here the channels in which $\mathcal{M}$ transforms to mesons $D^{(*)+} D^{(*)-}$, $D^{(*)0} \bar{D}^{(*)0}$, and $D_s^{(*)+} D_s^{(*)-}$. In the context of the SR method these processes can be explored by means of the three-point correlation functions in which the heavy quark condensate $\langle \bar{b}b \rangle$ is replaced by $\langle \alpha_s G^2/\pi \rangle$.

A. $\mathcal{M} \rightarrow D^0 \bar{D}^0$ and $D^+ D^-$

Let us analyze in a detailed form the decay $\mathcal{M} \rightarrow D^0 \bar{D}^0$. In this process the coupling $\tilde{g}_1$ that describes a strong interaction of particles $\mathcal{M}$, D^0 , and $\bar{D}^0$ at the vertex $\mathcal{M} D^0 \bar{D}^0$ can be obtained from the three-point correlation function

$$\tilde{\Pi}_1(p, p') = i^2 \int d^4 x d^4 y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J^{D^0}(y) \times J^{\bar{D}^0}(0) J^\dagger(x) \} | 0 \rangle. \quad (52)$$

Above $J^{D^0}(x)$ and $J^{\bar{D}^0}(x)$ are the interpolating currents for the mesons D^0 and $\bar{D}^0$:

$$J^{D^0}(x) = \bar{u}_j(x) i\gamma_5 c_j(x), \quad J^{\bar{D}^0}(x) = \bar{c}_i(x) i\gamma_5 u_i(x). \quad (53)$$

First, we find the physical side of the SR and write down $\tilde{\Pi}_1(p, p')$ in the following way:

$$\begin{aligned} \tilde{\Pi}_1^{\text{Phys}}(p, p') &= \frac{\langle 0 | J^{D^0} | D^0(p') \rangle \langle 0 | J^{\bar{D}^0} | \bar{D}^0(q) \rangle}{p'^2 - m_{D^0}^2} \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{q^2 - m_{D^0}^2} \\ &\times \langle D^0(p') \bar{D}^0(q) | \mathcal{M}(p) \rangle \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{p^2 - m^2} + \dots, \end{aligned} \quad (54)$$

where $m_{D^0} = (1864.84 \pm 0.05)$ MeV is the D^0 and $\bar{D}^0$ mesons' mass.

To simplify $\tilde{\Pi}_1^{\text{Phys}}(p, p')$, we rewrite it in terms of the involved particles' masses and current couplings (decay constants). The matrix element of the pseudoscalar D^0 ($\bar{D}^0$) meson is given by the expression

$$\langle 0 | J^{D^0} | D^0 \rangle = \frac{f_D m_{D^0}^2}{m_c}, \quad (55)$$

with $f_D = (211.9 \pm 1.1)$ MeV being its decay constant [52]. The vertex $\langle D^0(p') \bar{D}^0(q) | \mathcal{M}(p) \rangle$ is equal to

$$\langle D^0(p') \bar{D}^0(q) | \mathcal{M}(p) \rangle = \tilde{g}_1(q^2) p \cdot p'. \quad (56)$$

Here, $\tilde{g}_1(q^2)$ is the strong form factor that at the mass shell of the $\bar{D}^0$ meson, i.e., at $q^2 = m_{D^0}^2$, gives the strong coupling G_1 .

Afterwards, we recast $\tilde{\Pi}_1^{\text{Phys}}(p, p')$ into the following form:

$$\begin{aligned} \tilde{\Pi}_1^{\text{Phys}}(p, p') &= \tilde{g}_1(q^2) \frac{\Lambda f_D^2 m_D^4}{2m_c^2(p^2 - m^2)(p'^2 - m_{D^0}^2)} \\ &\times \frac{(m^2 + m_{D^0}^2 - q^2)}{(q^2 - m_{D^0}^2)} + \dots. \end{aligned} \quad (57)$$

Note that the whole expression in Eq. (57) is the invariant amplitude $\tilde{\Pi}_1^{\text{Phys}}(p^2, p'^2, q^2)$, which can be employed to get SR for the form factor $\tilde{g}_1(q^2)$.

The second component necessary to find SR for $\tilde{g}_1(q^2)$ is the correlator Eq. (52) calculated in terms of the quark propagators, which reads

$$\begin{aligned} \tilde{\Pi}_1^{\text{OPE}}(p, p') &= \frac{1}{3} \int d^4 x d^4 y e^{ip'y} e^{-ipx} \langle \bar{b} b \rangle \\ &\times \text{Tr}[\gamma_5 S_c^{ja}(y-x) S_c^{ai}(x) \gamma_5 S_u^{ij}(-y)], \end{aligned} \quad (58)$$

where $S_u(x)$ is the u quark propagator [48].

The correlation function $\tilde{\Pi}_1^{\text{OPE}}(p, p')$ depends on three propagators and vacuum condensate $\langle \bar{b} b \rangle$ of b quarks. This function differs from standard correlators, which contain four heavy quark propagators $S_{b(c)}(x)$. In fact, to calculate $\tilde{\Pi}_1^{\text{OPE}}(p, p')$ one contracts heavy and light quark fields, and because mesons $D^0 \bar{D}^0$ do not contain b quarks, gets an expression with the condensate $\langle \bar{b} b \rangle$.

For further studies, we make use of the relation between condensates

$$\langle \bar{b} b \rangle = -\frac{1}{12m_b} \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \quad (59)$$

derived in Ref. [43] from the sum rule analysis. This expression was obtained at the leading order of the perturbative QCD and is valid provided that the higher order corrections in m_b^{-1} are very small. Having activated this equality, we get

$$\begin{aligned} \tilde{\Pi}_1^{\text{OPE}}(p, p') &= -\frac{1}{36m_b} \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \int d^4 x d^4 y e^{ip'y} e^{-ipx} \\ &\times \text{Tr}[\gamma_5 S_c^{ja}(y-x) S_c^{ai}(x) \gamma_5 S_u^{ij}(-y)]. \end{aligned} \quad (60)$$

Stated differently, the correlator $\tilde{\Pi}_1^{\text{OPE}}(p, p')$ is suppressed by the factor $\langle \alpha_s G^2 / \pi \rangle$. We denote the corresponding invariant amplitude as $\tilde{\Pi}_1^{\text{OPE}}(p^2, p'^2, q^2)$ and carry out all required operations to fix SR for the form factor $\tilde{g}_1(q^2)$.

We omit standard details and present only final results for the process $\mathcal{M} \rightarrow D^0 \bar{D}^0$. Numerical computations of the form factor $\tilde{g}_1(q^2)$ are performed using

$$M_2^2 \in [1.5, 3] \text{ GeV}^2, \quad s'_0 \in [5, 5.2] \text{ GeV}^2, \quad (61)$$

which are the Borel and continuum subtraction parameters in the D^0 channel. The coupling $\tilde{g}_1$ is fixed at the mass shell $q^2 = m_{D^0}^2$ and is equal to

$$\tilde{g}_1 \equiv \tilde{\mathcal{G}}_1(-m_{D^0}^2) = (2.5 \pm 0.4) \times 10^{-2} \text{ GeV}^{-1}. \quad (62)$$

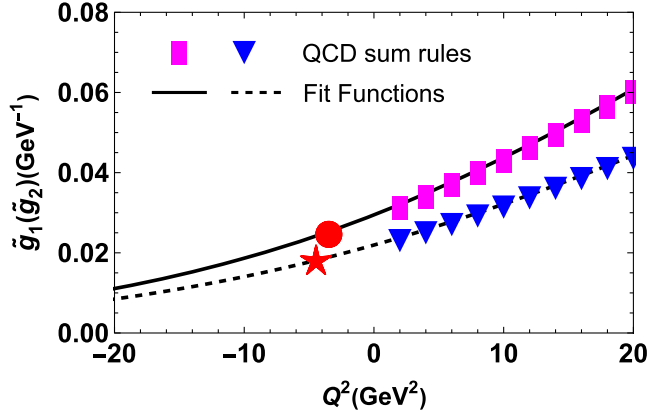


FIG. 4. Sum rules results and extrapolating functions for the form factors $\tilde{g}_1(Q^2)$ (solid line) and $\tilde{g}_2(Q^2)$ (dashed line). The circle and star mark the points $Q^2 = -m_{D^0}^2$ and $Q^2 = -m_{\bar{D}^0}^2$, respectively.

To find $\tilde{g}_1$, we have utilized the SR data in the interval $Q^2 = 2-20 \text{ GeV}^2$ and the fit function with parameters $\tilde{G}_1^0 = 0.03 \text{ GeV}^{-1}$, $\tilde{a}_1^1 = 6.89$, and $\tilde{a}_1^2 = -8.36$. The SR data and extrapolating function $\tilde{G}_1(Q^2)$ are plotted in Fig. 4.

The width of this channel is computed using the formula

$$\Gamma[\mathcal{M} \rightarrow D^0 \bar{D}^0] = \tilde{g}_1^2 \frac{m_{D^0}^2 \tilde{\lambda}_1}{8\pi} \left(1 + \frac{\tilde{\lambda}_1^2}{m_{D^0}^2}\right), \quad (63)$$

where $\tilde{\lambda}_1 = \lambda(m, m_{D^0}^2, m_{\bar{D}^0}^2)$. Finally, we get

$$\Gamma[\mathcal{M} \rightarrow D^0 \bar{D}^0] = (6.1 \pm 1.4) \text{ MeV}. \quad (64)$$

The strong coupling and width of the process $\mathcal{M} \rightarrow D^+ D^-$ are close to ones of the decay $\mathcal{M} \rightarrow D^0 \bar{D}^0$. Modifications connected with the mass difference of the mesons D^0 and $D^\pm$ are small and neglected in the present paper: We set $\Gamma[\mathcal{M} \rightarrow D^+ D^-] \approx \Gamma[\mathcal{M} \rightarrow D^0 \bar{D}^0]$.

B. $\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}$ and $D^{*-} D^{*+}$

The decay that we are going to consider in this subsection is the process $\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}$. Analysis of this channel differs from our previous studies only by some technical details. In the case of the decay $\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}$ we start with exploration of the correlator

$$\begin{aligned} \tilde{\Pi}_{2\mu\nu}(p, p') &= i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J_\mu^{D^{*0}}(y) \\ &\quad \times J_\nu^{\bar{D}^{*0}}(0) J^\dagger(x) \} | 0 \rangle. \end{aligned} \quad (65)$$

The correlation function $\tilde{\Pi}_{2\mu\nu}(p, p')$ contains the interpolating currents $J_\mu^{D^{*0}}(x)$ and $J_\nu^{\bar{D}^{*0}}(x)$ of the vector mesons D^{*0} and $\bar{D}^{*0}$, respectively:

$$J_\mu^{D^{*0}}(x) = \bar{u}_i(x) \gamma_\mu c_i(x), \quad J_\nu^{\bar{D}^{*0}}(x) = \bar{c}_j(x) \gamma_\nu u_j(x). \quad (66)$$

In terms of the parameters of the ground-level particles $\tilde{\Pi}_{2\mu\nu}(p, p')$ is given by the expression

$$\begin{aligned} \tilde{\Pi}_{2\mu\nu}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J_\mu^{D^{*0}} | D^{*0}(p', \varepsilon_1) \rangle \langle 0 | J_\nu^{\bar{D}^{*0}} | \bar{D}^{*0}(q, \varepsilon_2) \rangle}{p'^2 - m_{D^{*0}}^2} \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{q^2 - m_{\bar{D}^{*0}}^2} \\ &\quad \times \langle D^{*0}(p', \varepsilon_1) \bar{D}^{*0}(q, \varepsilon_2) | \mathcal{M}(p) \rangle \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{p^2 - m^2} + \dots, \end{aligned} \quad (67)$$

where $m_{D^{*0}} = (2006.85 \pm 0.05) \text{ MeV}$ is the mass of the mesons D^{*0} and $\bar{D}^{*0}$. To continue let us utilize the matrix elements

$$\begin{aligned} \langle 0 | J_\mu^{D^{*0}} | D^{*0}(p', \varepsilon_1) \rangle &= f_{D^*} m_{D^{*0}} \varepsilon_{1\mu}, \\ \langle 0 | J_\nu^{\bar{D}^{*0}} | \bar{D}^{*0}(q, \varepsilon_2) \rangle &= f_{D^*} m_{D^{*0}} \varepsilon_{2\nu}, \end{aligned} \quad (68)$$

with $f_{D^*} = (00 \pm 0.05) \text{ MeV}$ and $\varepsilon_1, \varepsilon_2$ being the decay constant and polarization vectors of the mesons D^{*0} and $\bar{D}^{*0}$. The vertex $\mathcal{M} D^{*0} \bar{D}^{*0}$ has the following form:

$$\begin{aligned} \langle D^{*0}(p', \varepsilon_1) \bar{D}^{*0}(q, \varepsilon_2) | \mathcal{M}(p) \rangle &= \tilde{g}_2(q^2) [q \cdot p' \\ &\quad \times \varepsilon_1^* \cdot \varepsilon_2^* - q \cdot \varepsilon_1^* p' \cdot \varepsilon_2^*]. \end{aligned} \quad (69)$$

Then, $\tilde{\Pi}_{2\mu\nu}(p, p')$ expressed using parameters of the hadronic molecule $\mathcal{M}$, and mesons D^{*0} and $\bar{D}^{*0}$, is given by the formula

$$\begin{aligned} \tilde{\Pi}_{2\mu\nu}^{\text{Phys}}(p, p') &= \frac{\tilde{g}_2(q^2) \Lambda f_{D^*}^2 m_{D^{*0}}^2}{(p^2 - m^2)(p'^2 - m_{D^{*0}}^2)(q^2 - m_{\bar{D}^{*0}}^2)} \\ &\quad \times \left[\frac{1}{2} (m^2 - m_{D^{*0}}^2 - q^2) g_{\mu\nu} - q_\mu p'_\nu \right] + \dots \end{aligned} \quad (70)$$

The correlator $\tilde{\Pi}_{2\mu\nu}(p, p')$ computed by employing the quark propagators reads

$$\begin{aligned} \tilde{\Pi}_{2\mu\nu}^{\text{OPE}}(p, p') &= i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle \bar{b} b \rangle \\ &\quad \times \text{Tr}[\gamma_\mu S_c^{ja}(y-x) S_c^{ai}(x) \gamma_\nu S_u^{ij}(-y)]. \end{aligned} \quad (71)$$

The correlator $\tilde{\Pi}_{2\mu\nu}(p, p')$ in both cases contains the two Lorentz structures $g_{\mu\nu}$ and $q_\mu p'_\nu$. We use invariant amplitudes that correspond to structures proportional to $g_{\mu\nu}$ and derive a required sum rule.

Numerical analysis has been carried out using the following input parameters: For the Borel and continuum subtraction parameters in the channel of the meson D^{*0} we use

$$M_2^2 \in [2, 3] \text{ GeV}^2, \quad s'_0 \in [5.7, 5.8] \text{ GeV}^2. \quad (72)$$

The decay constant f_{D^*} of the mesons D^{*0} is fixed as $(252.2 \pm 22.66) \text{ MeV}$. The extrapolating function $\tilde{\mathcal{G}}_2(Q^2)$ has the parameters $\tilde{\mathcal{G}}_2^0 = 0.022 \text{ GeV}^{-1}$, $\tilde{a}_2^1 = 6.70$, and $\tilde{a}_2^2 = -8.18$ (see Fig. 4). Then, the strong coupling $\tilde{g}_2$ amounts to

$$\tilde{g}_2 \equiv \tilde{\mathcal{G}}_2(-m_{D^{*0}}^2) = (1.8 \pm 0.4) \times 10^{-2} \text{ GeV}^{-1}. \quad (73)$$

The partial width of the decay $\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}$ can be found by means of the formula

$$\Gamma[\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}] = \frac{\tilde{g}_2^2 \lambda_2}{16\pi} \left(m^2 - 4m_{D^{*0}}^2 + 6 \frac{m_{D^{*0}}^4}{m^2} \right), \quad (74)$$

where $\lambda_3 = \lambda(m, m_{D^{*0}}, m_{D^{*0}})$. The partial decay width of this decay is

$$\Gamma[\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}] = (5.7 \pm 1.8) \text{ MeV}. \quad (75)$$

The width of the process $\mathcal{M} \rightarrow D^{*+} D^{*-}$ equals approximately $\Gamma[\mathcal{M} \rightarrow D^{*0} \bar{D}^{*0}]$.

C. $\mathcal{M} \rightarrow D_s^{(*)+} D_s^{(*)-}$

The analysis of the decay $\mathcal{M} \rightarrow D_s^+ D_s^-$ is done using the scheme that has been explained and applied above. There are modifications due to the parameters of the $D_s^\pm$ mesons such as their masses $m_{D_s} = (1969.0 \pm 1.4) \text{ MeV}$ and decay constant $f_{D_s} = (249.9 \pm 0.5) \text{ MeV}$. The SR data are calculated by employing the Borel and continuum subtraction parameters in the D_s^+ channel

$$M_2^2 \in [2.5, 3.5] \text{ GeV}^2, \quad s'_0 \in [6, 8] \text{ GeV}^2. \quad (76)$$

The extrapolating function $\tilde{\mathcal{G}}_3(Q^2)$ is determined by $\tilde{\mathcal{G}}_3^0 = 0.024 \text{ GeV}^{-1}$, $\tilde{a}_3^1 = 6.88$, and $\tilde{a}_3^2 = -8.42$. The strong coupling $\tilde{g}_3$ is found by means of this function and is equal to

$$\tilde{g}_3 \equiv \tilde{\mathcal{G}}_3(-m_{D_s}^2) = (2.1 \pm 0.4) \times 10^{-2} \text{ GeV}^{-1}. \quad (77)$$

The width of the decay $\mathcal{M} \rightarrow D_s^+ D_s^-$ amounts to

$$\Gamma[\mathcal{M} \rightarrow D_s^+ D_s^-] = (4.1 \pm 1.2) \text{ MeV}. \quad (78)$$

In the case of the second decay $\mathcal{M} \rightarrow D_s^{*+} D_s^{*-}$ the corresponding strong coupling $\tilde{g}_4$ is

$$\tilde{g}_4 \equiv \tilde{\mathcal{G}}_4(-m_{D_s^*}^2) = (1.6 \pm 0.3) \times 10^{-2} \text{ GeV}^{-1}. \quad (79)$$

Here, the parameters of the function $\tilde{\mathcal{G}}_4(Q^2)$ are $\tilde{\mathcal{G}}_4^0 = 0.019 \text{ GeV}^{-1}$, $\tilde{a}_4^1 = 6.703$, and $\tilde{a}_4^2 = -8.18$. For the width of this process we find

$$\Gamma[\mathcal{M} \rightarrow D_s^{*+} D_s^{*-}] = (4.4 \pm 1.2) \text{ MeV}. \quad (80)$$

In calculating the coupling $\tilde{g}_4$ and width of the decay $\mathcal{M} \rightarrow D_s^{*+} D_s^{*-}$ as the mass and the decay constant of the mesons $D_s^{*\pm}$ we have utilized $m_{D_s^*} = (2112.2 \pm 0.4) \text{ MeV}$ and $f_{D_s^*} = (221 \pm 7) \text{ MeV}$, respectively.

V. DECAYS INTO $B^{(*)+} B^{(*)-}$, $B^{(*)0} \bar{B}^{(*)0}$, AND $B_s^{(*)0} \bar{B}_s^{(*)0}$ MESONS

Decays of the hadronic molecule $\mathcal{M}$ into pairs of the B and B_s mesons can be studied by the same way. These processes are triggered by $c\bar{c}$ annihilation in the molecule and lead to creations of BB meson pairs with appropriate charges and quantum numbers. One can see that $\mathcal{M} \rightarrow B^{(*)+} B^{(*)-}$, $B^{(*)0} \bar{B}^{(*)0}$, and $B_s^{(*)0} \bar{B}_s^{(*)0}$ are possible decay channels of the hadronic molecule $\mathcal{M}$.

The correlation function required to estimate the strong coupling G_1 at the vertex $\mathcal{M} B^+ B^-$ is given by the formula

$$\hat{\Pi}_1(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{-ipx} \langle 0 | \mathcal{T} \{ J^{B^-}(y) \times J^{B^+}(0) J^\dagger(x) \} | 0 \rangle, \quad (81)$$

where $J^{B^+}(x)$ and $J^{B^-}(x)$ are the interpolating currents of the mesons B^+ and B^- :

$$J^{B^+}(x) = \bar{b}_i(x) i\gamma_5 u_i(x), \quad J^{B^-}(x) = \bar{u}_j(x) i\gamma_5 b_j(x). \quad (82)$$

The contribution of the ground-state particles $\mathcal{M}$, B^+ , and B^- to $\hat{\Pi}_1(p, p')$ is

$$\hat{\Pi}_1^{\text{Phys}}(p, p') = \frac{\langle 0 | J^{B^-} | B^-(p') \rangle \langle 0 | J^{B^+} | B^+(q) \rangle}{p'^2 - m_B^2} \frac{1}{q^2 - m_B^2} \times \langle B^-(p') B^+(q) | \mathcal{M}(p) \rangle \frac{\langle \mathcal{M}(p) | J^\dagger | 0 \rangle}{p^2 - m^2} + \dots. \quad (83)$$

To compute the phenomenological side of the SR $\hat{\Pi}_1^{\text{Phys}}(p, p')$, we make use of the matrix elements

$$\langle 0 | J^{B^\pm} | B^\pm \rangle = \frac{f_B m_B^2}{m_b} \quad (84)$$

and

$$\langle B^-(p') B^+(q) | \mathcal{M} \rangle = G_1(q^2) p \cdot p'. \quad (85)$$

In the formulas above, $m_B = (2112.2 \pm 0.4)$ MeV and $f_B = (206 \pm 7)$ MeV are the mass and decay constant of the mesons $B^\pm$.

Then, for $\hat{\Pi}_1^{\text{Phys}}(p, p')$, we get

$$\begin{aligned} \hat{\Pi}_1^{\text{Phys}}(p, p') &= \frac{G_1(q^2) \Lambda f_B^2 m_B^4}{m_b^2(p^2 - m^2)(p'^2 - m_B^2)} \\ &\times \frac{m^2 + m_B^2 - q^2}{2(q^2 - m_B^2)} + \dots \end{aligned} \quad (86)$$

The correlation function $\hat{\Pi}_1(p, p')$ in terms of the quark propagators reads

$$\begin{aligned} \hat{\Pi}_1^{\text{OPE}}(p, p') &= \frac{1}{3} \int d^4x d^4y e^{ip'y} e^{-ipx} \langle \bar{c}c \rangle \\ &\times \text{Tr}[\gamma_5 S_b^{ja}(y-x) S_b^{ai}(x) \gamma_5 S_u^{ij}(-y)]. \end{aligned} \quad (87)$$

The heavy quark condensate $\langle \bar{c}c \rangle$ with some accuracy can be replaced by [43]

$$\langle \bar{c}c \rangle = -\frac{1}{12m_c} \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle. \quad (88)$$

This leads to the correlator

$$\begin{aligned} \hat{\Pi}_1^{\text{OPE}}(p, p') &= -\frac{1}{36m_c} \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \int d^4x d^4y e^{ip'y} e^{-ipx} \\ &\times \text{Tr}[\gamma_5 S_b^{ja}(y-x) S_b^{ai}(x) \gamma_5 S_u^{ij}(-y)]. \end{aligned} \quad (89)$$

Numerical computations of the form factor $G_1(q^2)$ are performed for $-q^2 = Q^2 = 2-10$ GeV². The coupling G_1 is extracted at the $Q^2 = -m_B^2$ by utilizing the fit function $\hat{G}_1(Q^2)$ with parameters $\hat{G}_1^0 = 0.045$ GeV⁻¹, $\hat{a}_2^1 = 2.82$, and $\hat{a}_4^2 = -2.61$:

$$G_1 \equiv \hat{G}_1(-m_B^2) = (2.6 \pm 0.4) \times 10^{-2} \text{ GeV}^{-1}. \quad (90)$$

Then, the width of the process $\mathcal{M} \rightarrow B^+ B^-$ is equal to

$$\Gamma[\mathcal{M} \rightarrow B^+ B^-] = (3.8 \pm 0.9) \text{ MeV}. \quad (91)$$

In Table I we provide brief information about the remaining decay modes of the molecule $\mathcal{M}$.

Information on the partial widths of the channels considered in the two last sections permit us to evaluate the full width of the hadronic molecule

$$\Gamma[\mathcal{M}] = (155 \pm 23) \text{ MeV}. \quad (92)$$

TABLE I. Decay modes of the hadronic molecule $\mathcal{M}$ generated by $c\bar{c}$ annihilation, strong couplings G_i , and partial widths Γ_i .

i	Channels	$G_i \times 10^2$ [GeV ⁻¹]	Γ_i [MeV]
1	$B^+ B^-$	2.6 ± 0.4	3.8 ± 0.9
2	$B^{*+} B^{*-}$	2.8 ± 0.3	4.2 ± 1.3
3	$B_s^0 \bar{B}_s^0$	2.0 ± 0.3	2.2 ± 0.5
4	$B_s^{*0} \bar{B}_s^{*0}$	2.6 ± 0.5	3.5 ± 1.0

VI. CONCLUDING REMARKS

In this article, we have studied the scalar hadronic molecule $\mathcal{M} = B_c^+ B_c^-$ in the context of the QCD sum rule method. Our prediction for the mass of this molecule $m = (12725 \pm 85)$ MeV provides important information about this state. It turns out that the molecular structure $\mathcal{M}$ is unstable against two-meson fall-apart processes.

The fully heavy hadronic molecules $B_c^{(*)+} B_c^{(*)-}$ were explored in the framework of the coupled-channel unitary approach in Ref. [24]. Results in this method depend on the parameter Λ used to regularize some integrals. In the case of the scalar structure $B_c^+ B_c^-$ at $\Lambda = 600$ MeV, the authors found the pole position $E_p = (12503.3 - i126.8)$ MeV. In other words, the mass of the hadronic molecule $B_c^+ B_c^-$ is 12503.3 MeV, which is smaller than the one obtained in the present work. Nevertheless, the molecule $B_c^+ B_c^-$ in this picture is also unstable and can decay strongly into the pair of mesons $\eta_b \eta_c$.

The mass of $\mathcal{M}$ has enabled us to fix its possible decay channels. We have computed partial widths of the two-meson dissociations $\mathcal{M} \rightarrow \eta_b \eta_c, J/\psi \Upsilon$, and $B_c^+ B_c^-$, which are dominant decay modes of the molecule $\mathcal{M}$. The annihilations of the $b\bar{b}$ and $c\bar{c}$ pairs in $\mathcal{M}$ give rise to other two-meson strong decays: We have considered numerous processes generated by this second mechanism. To calculate the widths of two-meson decays, we have employed the method of the three-point sum rule. This is necessary to estimate strong couplings of particles at various $\mathcal{M}$ -meson-meson vertices. The full width of $\mathcal{M}$ saturated by these decay modes amounts to $\Gamma[\mathcal{M}] = (155 \pm 23)$ MeV: The fall-apart processes constitute 66% of this parameter.

In our previous publications [40–42], we performed comprehensive investigations of the four-quark mesons $0^+, 1^+$, and 2^+ with the quark content $b\bar{c}b\bar{c}$. These states were treated as diquark-antidiquark structures. The scalar particle $T_{b\bar{c}b\bar{c}}$ was modeled as a tetraquark composed of the scalar diquark and antidiquark [40]. The mass and full width of this tetraquark were found to be equal to $m = (12697 \pm 90)$ MeV and $\Gamma[T_{b\bar{c}b\bar{c}}] = (142.4 \pm 16.9)$ MeV, respectively.

Comparing the masses of these two exotic structures, we see that the hadronic molecule $\mathcal{M}$ is heavier than the diquark-antidiquark state, though differences between their masses and widths are small: In fact, within the errors of computations, one may state that these two scalar

compounds have approximately the same masses. Their full widths are also comparable with each other, though there are identical decay channels with considerably different partial widths. For instance, the partial width of the decays into $B_c^+ B_c^-$ mesons in the case of the diquark-antidiquark and molecule structures are equal to 33.9 MeV and 17.7 MeV, respectively. Even lower and upper limits for these parameters, obtained by taking into account the errors of calculations, generate only negligible small overlapping regions. Therefore, such decay modes may be used to

distinguish the molecular structure of the exotic state $bc\bar{b}\bar{c}$ from its diquark organization.

Detailed studies of the diquark-antidiquark and hadronic molecule structures $bc\bar{b}\bar{c}$ with different spin-parities are necessary to calculate their parameters and reveal reactions where they may be observed.

DATA AVAILABILITY

No data were created or analyzed in this study.

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