

High-temperature massless scalar partition function on general stationary backgrounds

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(Received 13 September 2025; accepted 16 October 2025; published 10 November 2025)

The high-temperature equilibrium partition function of a massless real scalar field nonminimally coupled to the scalar curvature is computed at second order in the derivative expansion on a generic stationary background. Using covariant perturbation theory, the expression of the thermal partition function at second order in powers of curvatures is also obtained, including its nonlocal contributions. For conformal coupling, the Weyl anomaly at fourth order in derivatives and second order in curvatures is evaluated using both expansions and the results found to be consistent.

DOI: [10.1103/m5qv-q8hq](https://doi.org/10.1103/m5qv-q8hq)

I. INTRODUCTION

Since its inception in the 1970s [1–4], the Weyl anomaly has played a central role in physics with many realizations in various fields [5,6]. As with other types of anomalies [7,8], quantum violations of scale invariance also have implications for hydrodynamics and transport theory, some of which have been investigated in a number of works (see, for example, [9–13]). Many of these analyses rely on computing the theory’s one-loop effective action on a generic stationary background, in terms of which the equilibrium partition function at one loop is obtained [14,15], as well as the constitutive relations for the different currents and the associated transport coefficients.

The local part of the high-temperature partition function for a scalar field on static or stationary geometries has been the subject of attention in the literature (see, for example, [16–20]). Nonlocal contributions, on the other hand, have been evaluated so far only on ultrastatic backgrounds [18,19]. The task set in the present work is precisely to obtain an *explicit* expression of the finite-temperature equilibrium partition function for a massless scalar field on a *generic* stationary background and arbitrary non-minimal coupling to background curvature, including also the nonlocal terms.

Apart from its immediate intrinsic technical interest, the generalization of the massless scalar high-temperature partition function to arbitrary stationary metrics has interesting applications to the study of the hydrodynamics of scalar fluids and the rôle of the Weyl anomaly in this context. Comparing with the ultrastatic case studied so far in the literature, the expressions obtained here using covariant perturbation theory include the contributions of the gradients of four additional metric functions, related to fluid vorticity, acceleration, and spatial changes in the local Tolman temperature.

The plan of this paper is as follows. In Sec. II we present the calculation of the scalar high-temperature partition function on a general stationary geometry to second order in the derivative expansion, as well as of the associated Weyl anomaly. In order to analyze nonlocal contributions, in Sec. III we carry out an expansion to second order in curvatures of the partition function, finding the existence of extra form factors associated with fluid vorticity and local temperature gradients. We finish in Sec. IV with a brief discussion of our results and prospects for some future work.

II. EQUILIBRIUM SCALAR PARTITION FUNCTION TO SECOND ORDER IN DERIVATIVES

We consider a massless scalar field $\varphi(x) \equiv \varphi(t, \mathbf{x})$ on a curved spacetime with action

$$S = \frac{1}{2} \int d^4x \sqrt{-G} \varphi (\square - \xi \mathcal{R}) \varphi, \quad (2.1)$$

where $\square$ is the four-dimensional Laplace-Beltrami operator, $\mathcal{R}$ the four-dimensional scalar curvature, and $\xi \in \mathbb{R}$ the

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nonminimal coupling constant. We moreover assume a generic stationary background described by the line element

$$ds^2 = G_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu = -e^{2\sigma(\mathbf{x})}[dt + a_i(\mathbf{x})dx^i]^2 + g_{ij}(\mathbf{x})dx^i dx^j. \quad (2.2)$$

In the context of hydrodynamics, the metric functions $\sigma(\mathbf{x})$ and $a_i(\mathbf{x})$ are related to the fluid velocity, vorticity, acceleration, and local (Tolman) temperature, while the transverse metric $g_{ij}(\mathbf{x})$ is the source of the energy-momentum current [15]. The ultrastatic case considered in Refs. [18,19] corresponds to setting $\sigma(\mathbf{x}) = a_i(\mathbf{x}) = 0$.

Using the imaginary time formalism [21], the finite-temperature equilibrium partition function is computed as

$$\begin{aligned} \log Z(\beta_0) &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \text{tr} \log(-\square + \xi \mathcal{R}) \\ &= -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{ds}{s^{1-\epsilon}} \text{tr} e^{s(\square - \xi \mathcal{R})} \right] \Big|_{\epsilon=0}, \end{aligned} \quad (2.3)$$

where the sum is over bosonic Matsubara frequencies $\omega_n = \frac{2\pi n}{\beta_0}$ (with $n \in \mathbb{Z}$ and β_0 the period of the Euclidean time coordinate) and a regularization scale μ has been introduced. The four-dimensional Laplace-Beltrami operator $\square$ acting on Fourier-transformed fields $\varphi(\omega_n, \mathbf{x})$ reads

$$\begin{aligned} \square &= \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) - (2\omega_n a^i - \partial_i \sigma g^{ij}) \partial_j - \omega_n a^j \partial_j \sigma \\ &\quad + \omega_n^2 (a^i a_i - e^{-2\sigma}) - \frac{\omega_n}{\sqrt{g}} \partial_i (\sqrt{g} a^i), \end{aligned} \quad (2.4)$$

whereas the four-dimensional scalar curvature $\mathcal{R}$ of the stationary metric (2.2) can be written as

$$\mathcal{R} = R + \frac{1}{4} e^{2\sigma} f_{ij} f^{ij} - 2g^{ij} \partial_i \sigma \partial_j \sigma - \frac{2}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \sigma), \quad (2.5)$$

with R the three-dimensional scalar curvature associated with the transverse metric g_{ij} , which is the one used to raise and lower spatial indices, and $f_{ij} = \partial_i a_j - \partial_j a_i$ the field strength of the Kaluza-Klein (KK) gauge field. In fact, the operator $L \equiv -(\square - \xi \mathcal{R})$ appearing in Eq. (2.3) can be recast in the more suggestive form

$$\begin{aligned} L &= -\frac{1}{\sqrt{g}} (\partial_i + \mathcal{A}_i) [\sqrt{g} g^{ij} (\partial_j + \mathcal{A}_j)] + \mathcal{Q} \\ &\equiv -\Delta_{\mathcal{A}} + \mathcal{Q}, \end{aligned} \quad (2.6)$$

where the generalized connection $\mathcal{A}_i$ and potential $\mathcal{Q}$ are given by

$$\begin{aligned} \mathcal{A}_i &= -\omega_n a_i + \frac{1}{2} \partial_i \sigma, \\ \mathcal{Q} &= e^{-2\sigma} \omega_n^2 + \xi R + \frac{\xi}{4} e^{2\sigma} f_{ij} f^{ij} \\ &\quad - \left(2\xi - \frac{1}{4} \right) g^{ij} \partial_i \sigma \partial_j \sigma - \left(2\xi - \frac{1}{2} \right) \Delta \sigma. \end{aligned} \quad (2.7)$$

Here, we have also introduced a generalized covariant derivative $\nabla_i^{\mathcal{A}} \equiv \nabla_i + \mathcal{A}_i$, such that $\Delta_{\mathcal{A}} \equiv g^{ij} \nabla_i^{\mathcal{A}} \nabla_j^{\mathcal{A}}$ is its corresponding Laplace-Beltrami operator, and denoted $\Delta \equiv g^{ij} \nabla_i \nabla_j$. Notice as well that the field strength of the generalized connection

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i = -\omega_n f_{ij} \quad (2.8)$$

determines the commutator of the covariant derivatives acting on scalar fields

$$[\nabla_i^{\mathcal{A}}, \nabla_j^{\mathcal{A}}] \varphi = \mathcal{F}_{ij} \varphi. \quad (2.9)$$

This relation will be important when applying the results of Ref. [22] to the calculation presented in the next section.

The partition function (2.3) can then be written as

$$\log Z(\beta) = -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{ds}{s^{1-\epsilon}} \int d^3 \mathbf{x} \sqrt{g} \mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}; s) \right] \Big|_{\epsilon=0}, \quad (2.10)$$

where $\mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}'; s)$ is the heat kernel of the operator (2.6), solving the equation

$$\left(\frac{\partial}{\partial s} + L \right) \mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}'; s) = 0 \quad \text{with} \quad \lim_{s \rightarrow 0} \mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}'; s) = \frac{1}{\sqrt{g(\mathbf{x})}} \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \quad (2.11)$$

The derivative expansion of the partition function is implemented by the Seeley-DeWitt series of the heat kernel in the coincidence limit

$$\mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}; s) \sim (4\pi s)^{-\frac{3}{2}} e^{-s(\mathcal{Q} - \frac{1}{6}R)} \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} b_k(\mathbf{x}), \quad (2.12)$$

where the first few coefficients are given by [23–27]

$$\begin{aligned}
 b_0(\mathbf{x}) &= 1, \\
 b_1(\mathbf{x}) &= 0, \\
 b_2(\mathbf{x}) &= -\frac{1}{3}\Delta\mathcal{Q} + \frac{1}{6}\mathcal{F}_{ij}\mathcal{F}^{ij} + \frac{1}{15}\Delta R - \frac{1}{90}R_{ij}R^{ij} + \frac{1}{90}R_{ijk\ell}R^{ijk\ell}, \\
 b_3(\mathbf{x}) &= -\frac{1}{2}g^{ij}\partial_i\mathcal{Q}\partial_j\mathcal{Q} + \frac{1}{10}\Delta^2\mathcal{Q} - \frac{2}{15}g^{k\ell}\nabla_k f_{ij}\nabla_\ell f^{ij} - \frac{1}{30}\nabla_j f_i{}^j\nabla_k f^{ik} - \frac{1}{5}f_{ij}\Delta f^{ij} + \frac{1}{15}R^{ij}\nabla_i\nabla_j\mathcal{Q} + \frac{1}{5}g^{ij}\partial_i\mathcal{Q}\partial_j R \\
 &\quad + \frac{1}{5}f_{ij}f^{jk}f_k{}^i + \frac{1}{10}f^{ij}R_{ijk\ell}f^{k\ell} - \frac{1}{15}f^{ij}R_{jk}f^k{}_i + \dots
 \end{aligned} \tag{2.13}$$

Here, $R_{ijk\ell}$ and R_{ij} are, respectively, the Riemann and Ricci tensors of the transverse metric g_{ij} , while in the expression of $b_3(\mathbf{x})$ the ellipsis stands for terms with more than four derivatives. As shown in Ref. [28] (see also [29,30]), the exponential in Eq. (2.12) resums nonderivative contributions of $\mathcal{Q}$ and R to all orders in s , which are therefore absent from the coefficients shown in (2.13).

A calculation of the partition function (2.10) using Eqs. (2.12) and (2.13) shows that, to second order in derivatives, only the coefficients $b_2(\mathbf{x})$ and $b_3(\mathbf{x})$ contribute, leading to the result¹

$$\begin{aligned}
 \log Z(\beta) &= \int d^3\mathbf{x}\sqrt{g}\left[-\frac{\pi^2}{90\beta^3} + \frac{1-6\xi}{144\beta}(4g^{ij}\partial_i\sigma\partial_j\sigma - R) \right. \\
 &\quad \left. + \frac{1+6\xi}{576\beta}e^{2\sigma}f_{ij}f^{ij} + \dots\right],
 \end{aligned} \tag{2.14}$$

with $\beta(\mathbf{x}) = \beta_0 e^{\sigma(\mathbf{x})}$ the inverse local equilibrium temperature and the ellipsis indicates higher-order terms in the derivative expansion. Notice that the partition function is explicitly invariant under KK gauge transformations, $a_i \rightarrow a_i(\mathbf{x}) + \partial_i\chi(\mathbf{x})$, corresponding to position-dependent time shifts $t \rightarrow t - \chi(\mathbf{x})$ in the background metric (2.2).

For conformal coupling, $\xi = \frac{1}{6}$, the second term in (2.14) vanishes and the thermal partition function to this quadratic derivative order remains invariant under Weyl rescalings $G_{\mu\nu}(\mathbf{x}) \rightarrow e^{2\lambda(\mathbf{x})}G_{\mu\nu}(\mathbf{x})$, acting as

$$\begin{aligned}
 \sigma(\mathbf{x}) &\longrightarrow \sigma(\mathbf{x}) + \lambda(\mathbf{x}), \\
 \beta(\mathbf{x}) &\longrightarrow e^{\lambda(\mathbf{x})}\beta(\mathbf{x}), \\
 g_{ij}(\mathbf{x}) &\longrightarrow e^{2\lambda(\mathbf{x})}g_{ij}(\mathbf{x}),
 \end{aligned} \tag{2.15}$$

while the KK field $a_i(\mathbf{x})$ and its field strength $f_{ij}(\mathbf{x})$ do not transform. Incidentally, the results for ultrastatic backgrounds in Ref. [19] are retrieved by setting $\sigma(\mathbf{x}) = a_i(\mathbf{x}) = 0$. Equation (2.14) also generalizes the partition function obtained in Ref. [31] for $\xi = 0$ to arbitrary ξ .

For any theory, whether conformal or not, the coefficients of the terms $\log(\beta\mu)$ in the partition function originate from the contributions in the heat kernel expansion (2.12) of order $s^{1/2}$, $s^{3/2}\omega_n^2$, and $s^{5/2}\omega_n^4$. Using the Seeley-DeWitt coefficients given in Eq. (2.13), we find the following logarithmic dependence on the renormalization scale μ for a generic stationary background²

$$\begin{aligned}
 \mu \frac{\partial}{\partial \mu} \log Z(\beta) &= -\frac{1}{32\pi^2} \int d^3\mathbf{x}\sqrt{g}\beta \left[\frac{1}{90}\Delta R - \frac{1}{90}R_{ij}R^{ij} + \frac{1}{90}R_{ijk\ell}R^{ijk\ell} + \frac{1}{3}\Delta\bar{P} \right. \\
 &\quad + \frac{1}{45}e^{2\sigma}\nabla_k f_{ij}\nabla^k f^{ij} + \frac{1}{180}e^{2\sigma}\nabla_j f^{ij}\nabla_k f_i{}^k + \frac{1}{30}e^{2\sigma}f_{ij}\Delta f^{ij} \\
 &\quad - \frac{1}{60}e^{2\sigma}\Delta^2 e^{-2\sigma} - \frac{1}{6}e^{2\sigma}g^{ij}\partial_i e^{-2\sigma}\partial_j \bar{P} - \frac{1}{180}e^{2\sigma}g^{ij}\partial_i e^{-2\sigma}\partial_j R \\
 &\quad - \frac{1}{90}e^{2\sigma}R^{ij}\nabla_i\nabla_j e^{-2\sigma} + \frac{3}{80}e^{4\sigma}(\Delta e^{-2\sigma})^2 + \frac{1}{20}e^{4\sigma}g^{ij}\partial_i e^{-2\sigma}\partial_j \Delta e^{-2\sigma} \\
 &\quad \left. - \frac{1}{6}e^{2\sigma}\bar{P}\Delta e^{-2\sigma} + \frac{1}{12}e^{2\sigma}\bar{P}f_{ij}f^{ij} + \frac{1}{8}e^{4\sigma}\bar{P}g^{ij}\partial_i e^{-2\sigma}\partial_j e^{-2\sigma} + \bar{P}^2 \right],
 \end{aligned} \tag{2.16}$$

¹In writing this equation, we exploited the freedom associated with the existence of the total derivative combination $\beta^{-1}[\partial_i(\sqrt{g}g^{ij}\partial_j\sigma) - \sqrt{g}g^{ij}\partial_i\sigma\partial_j\sigma] = \partial_i(\beta^{-1}\sqrt{g}g^{ij}\partial_j\sigma)$, taking the terms $\beta^{-1}R$, $\beta^{-1}\sqrt{g}g^{ij}\partial_i\sigma\partial_j\sigma$, and $f_{ij}f^{ij}$ as independent.

²In fact, together with the expressions given in Eq. (2.13), we also need those four-derivative terms in $b_4(\mathbf{x})$ multiplying ω_n^4 , which read $\frac{3}{5}(\nabla e^{-2\sigma})^2 + \frac{4}{5}g^{ij}\partial_i e^{-2\sigma}\partial_j \Delta e^{-2\sigma}$ (see Refs. [25,27,32]).

where, to simplify, we introduced the notation

$$\begin{aligned}\bar{P} &\equiv -\left(\xi - \frac{1}{6}\right)R - \frac{\xi}{4}e^{2\sigma}f_{ij}f^{ij} + \left(2\xi - \frac{1}{4}\right)g^{ij}\partial_i\sigma\partial_j\sigma \\ &\quad + \left(2\xi - \frac{1}{2}\right)\Delta\sigma \\ &= -\left(\xi - \frac{1}{6}\right)R - \frac{\xi}{4}e^{2\sigma}f_{ij}f^{ij} \\ &\quad - \frac{1}{2}\left(3\xi - \frac{5}{8}\right)e^{4\sigma}g^{ij}\partial_ie^{-2\sigma}\partial_je^{-2\sigma} \\ &\quad + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)e^{2\sigma}\Delta e^{-2\sigma},\end{aligned}\quad (2.17)$$

that will also be used in the next section. All terms in (2.16) are of fourth order in the derivative expansion. It can be checked that each term inside the square bracket on the right-hand side of (2.16) scales as $e^{-4\lambda}$ under rigid Weyl transformations (2.15), thus canceling the change of the common factor $\sqrt{g}\beta \rightarrow e^{4\lambda}\sqrt{g}\beta$ and leaving the whole expression invariant.

For conformal coupling, $\xi = 1/6$, Eq. (2.16) renders the expression of the scale anomaly and must include the square of the Weyl tensor, as it behooves a type-B Weyl anomaly [4,33]. Its presence, in fact, can be read off Eq. (2.16). Indeed, in perturbation theory the nonlocal (Euclidean) zero-temperature effective action on a four-dimensional stationary backgrounds takes the form (cf. Refs. [1,4,33,34])

$$W_{\text{anom}} = c \int d^3x \sqrt{g(\mathbf{x})} \beta(\mathbf{x}) \mathcal{C}_{\mu\nu\alpha\beta}(\mathbf{x}) \log\left(\frac{\mu^2}{-\Delta}\right) \mathcal{C}^{\mu\nu\alpha\beta}(\mathbf{x}) + \dots, \quad (2.18)$$

where $\mathcal{C}_{\mu\nu\alpha\beta}^\mu$ is the four-dimensional Weyl tensor, c is the trace anomaly, and $\sqrt{g}\beta \equiv \int_0^{\beta_0} d\tau e^\sigma \sqrt{g}$ comes from the integration over the compact Euclidean time. In this expression, we have stressed that all quantities in the integrand are independent of the time coordinate and the ellipsis stands for terms of cubic order in the curvatures R , R_{ij} , f_{ij} , and $\bar{P}$, as well as possibly some terms proportional to $\square R$ or R^2 . Under a Weyl rescaling (2.15), the anomalous effective action (2.18) transforms as

$$\delta_\lambda W_{\text{anom}} = 2c \int d^3x \sqrt{g(\mathbf{x})} \lambda(\mathbf{x}) \beta(\mathbf{x}) \mathcal{C}_{\mu\nu\alpha\beta}(\mathbf{x}) \mathcal{C}^{\mu\nu\alpha\beta}(\mathbf{x}), \quad (2.19)$$

rendering the expression of the type-B Weyl anomaly.

At finite temperature, on the other hand, after summing over Matsubara frequencies the logarithmic part of the effective action exhibits the following structure³:

$$W_{\text{anom}} = \int d^3x \sqrt{g(\mathbf{x})} \beta(\mathbf{x}) \sum_{i,j} \mathfrak{R}_i(\mathbf{x}) \chi_{ij} \mathfrak{R}_j(\mathbf{x}) \log\left[\frac{\beta(\mathbf{x})\mu}{4\pi}\right], \quad (2.20)$$

where $\mathfrak{R}_i$ represents combinations of metric functions of second order in derivatives and χ_{ij} are appropriate differential form factors. This is heuristically seen by considering the simpler example of a stationary metric with $\sigma = 0$ and $g_{ij} = \delta_{ij}$. Then, we have

$$\begin{aligned}\frac{1}{2} \int d^4x \sqrt{-G} \mathcal{C}_{\mu\nu\alpha\beta} \mathcal{C}^{\mu\nu\alpha\beta} &= \int d^4x \sqrt{-G} \left(\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} - \frac{1}{3} \mathcal{R}^2 \right) \\ &= \frac{\beta_0}{2} \int d^3x \partial_j f^{ij} \partial_k f_i^k \\ &= -\frac{\beta_0}{4} \int d^3x f_{ij} \Delta f^{ij},\end{aligned}\quad (2.21)$$

where in the last line we have carried out various integrations by parts and dropped total derivative terms, with β_0 arising from the integration over the Euclidean compact time τ . When properly normalized, this reproduces the combined three terms in the second line of Eq. (2.16) with $\sigma = 0$ and $g_{ij} = \delta_{ij}$, up to boundary contributions.

III. NONLOCAL TERMS

Our analysis so far has been based on expanding the high-temperature equilibrium partition function in the number of derivatives. In order to study nonlocal terms, on the other hand, we should rather consider an expansion in powers of the curvatures following the analysis of [22] (see also [35–38]). In order to apply the techniques presented in these references, we rewrite the operator (2.6) in the form

$$L = -\left(\Delta_{\mathcal{A}} - \frac{1}{6}R\right) - \omega_n^2 + P, \quad (3.1)$$

where we introduced

$$P \equiv \omega_n^2(1 - e^{-2\sigma}) + \bar{P} \quad (3.2)$$

and $\bar{P}$, defined in Eq. (2.17), is independent of the Matsubara frequencies. The constant term $-\omega_n^2$ in (3.1) gives rise to a prefactor $e^{-\omega_n^2 s}$ in the heat kernel, while the position-dependent term $\omega_n^2 e^{-2\sigma}$ is included in the

³As it will be discussed in the next section, the logarithmic Laplacian appearing in the zero-temperature term (2.18) is canceled by a similar contribution from the finite-temperature piece.

definition of P . The methods and expressions of [22] can now be applied to write the following solution to the heat kernel equation of L in powers of the curvatures R , R_{ij} , $\mathcal{F}_{ij}$, and the potential P :

$$\begin{aligned} & \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} \mathcal{K}(\omega_n; \mathbf{x}, \mathbf{x}; s) \\ &= (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} e^{-s\omega_n^2} \{1 + sP \\ &+ s^2[R_{ij}f_1(-s\Delta_{\mathcal{A}})R^{ij} + Rf_2(-s\Delta_{\mathcal{A}})R + Pf_3(-s\Delta_{\mathcal{A}})R \\ &+ Pf_4(-s\Delta_{\mathcal{A}})P + \mathcal{F}_{ij}f_5(-s\Delta_{\mathcal{A}})\mathcal{F}^{ij}] + \dots\}, \end{aligned} \quad (3.3)$$

where the ellipsis indicates terms of third and higher order, that we drop.⁴ The differential operators $f_i(-s\Delta_{\mathcal{A}})$ are defined by the integrals

$$\begin{aligned} f_1(-s\Delta_{\mathcal{A}}) &= \frac{1}{12} \int_0^1 d\alpha \alpha^4 \exp\left(\frac{1-\alpha^2}{4}s\Delta_{\mathcal{A}}\right), \\ f_4(-s\Delta_{\mathcal{A}}) &= \frac{1}{2} \int_0^1 d\alpha \exp\left(\frac{1-\alpha^2}{4}s\Delta_{\mathcal{A}}\right), \\ f_5(-s\Delta_{\mathcal{A}}) &= \frac{1}{4} \int_0^1 d\alpha \alpha^2 \exp\left(\frac{1-\alpha^2}{4}s\Delta_{\mathcal{A}}\right), \end{aligned} \quad (3.4)$$

together with the relations

$$\begin{aligned} f_2(-s\Delta_{\mathcal{A}}) &= \frac{1}{144}f_4(-s\Delta_{\mathcal{A}}) - \frac{1}{12}f_5(-s\Delta_{\mathcal{A}}) - \frac{1}{8}f_1(-s\Delta_{\mathcal{A}}), \\ f_3(-s\Delta_{\mathcal{A}}) &= \frac{1}{6}f_4(-s\Delta_{\mathcal{A}}) - f_5(-s\Delta_{\mathcal{A}}). \end{aligned} \quad (3.5)$$

Since the various form factors $f_i(-s\Delta_{\mathcal{A}})$ are connecting terms of first order in curvatures, in extracting the contribution of second order one can replace $\Delta_{\mathcal{A}}$ with Δ .

The equilibrium partition function is now obtained plugging the expansion (3.3) into (2.10), which reads

$$\begin{aligned} \log Z(\beta) &= \int d^3\mathbf{x} \sqrt{g} \left[-\frac{\pi^2}{90\beta_0^3} + \frac{\pi^2}{30\beta_0^3} \sigma - \frac{\pi^2}{30\beta_0^3} \sigma^2 + \frac{6\xi-1}{144\beta_0} R - \frac{8\xi-1}{96\beta_0} g^{ij} \partial_i \sigma \partial_j \sigma + \frac{\xi}{96\beta_0} f_{ij} f^{ij} + R_{ij} \chi_1(\beta_0 \sqrt{-\Delta}) R^{ij} \right. \\ &+ R \chi_2(\beta_0 \sqrt{-\Delta}) R + \bar{P} \chi_3(\beta_0 \sqrt{-\Delta}) R + \bar{P} \chi_4(\beta_0 \sqrt{-\Delta}) \bar{P} + f_{ij} \chi_5(\beta_0 \sqrt{-\Delta}) f^{ij} + 2\sigma \chi_6(\beta_0 \sqrt{-\Delta}) R \\ &\left. + 4\sigma \chi_7(\beta_0 \sqrt{-\Delta}) \bar{P} + 4\sigma \chi_8(\beta_0 \sqrt{-\Delta}) \sigma \right], \end{aligned} \quad (3.6)$$

where we kept only terms that are bilinears in curvatures including P and expanded $\beta = \beta_0 e^\sigma$ to second order in σ . This expansion is mandatory because σ has to be considered first order in curvatures, as follows from $P - \bar{P} = \omega_n^2(1 - e^{-2\sigma}) \approx 2\omega_n^2 \sigma$ [see Eq. (3.2)].

The corresponding form factors are given by

$$\chi_i(\beta_0 \sqrt{-\Delta}) = -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \int_0^\infty ds s^{1+\epsilon} (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} e^{-s\omega_n^2} f_i(-s\Delta) \right] \Big|_{\epsilon=0}, \quad (3.7)$$

for $i = 1, \dots, 4$ and

$$\begin{aligned} \chi_5(\beta_0 \sqrt{-\Delta}) &= -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \int_0^\infty ds s^{1+\epsilon} (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} \omega_n^2 e^{-s\omega_n^2} f_5(-s\Delta) \right] \Big|_{\epsilon=0}, \\ \chi_6(\beta_0 \sqrt{-\Delta}) &= -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \int_0^\infty ds s^{1+\epsilon} (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} \omega_n^2 e^{-s\omega_n^2} f_3(-s\Delta) \right] \Big|_{\epsilon=0}, \\ \chi_7(\beta_0 \sqrt{-\Delta}) &= -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \int_0^\infty ds s^{1+\epsilon} (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} \omega_n^2 e^{-s\omega_n^2} f_4(-s\Delta) \right] \Big|_{\epsilon=0}, \\ \chi_8(\beta_0 \sqrt{-\Delta}) &= -\frac{1}{2} \frac{\partial}{\partial \epsilon} \left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)} \int_0^\infty ds s^{1+\epsilon} (4\pi s)^{-\frac{3}{2}} \int d^3\mathbf{x} \sqrt{g} \sum_{n \in \mathbb{Z}} \omega_n^4 e^{-s\omega_n^2} f_4(-s\Delta) \right] \Big|_{\epsilon=0}. \end{aligned} \quad (3.8)$$

⁴As shown in [22], the Riemann tensor R_{ijkl} can be eliminated inside the integral in terms of R and R_{ij} .

Moreover, in writing the first line of (3.6) we dropped total derivative terms in P that do not contribute to the partition function. Notice that the form factors χ_5 – χ_7 originate from those terms in P proportional to ω_n^2 , whereas χ_8 comes from the ones scaling as ω_n^4 .

To carry out the sums and integrations in the definitions of the form factors, we apply Poisson resummation to the Matsubara series

$$\begin{aligned}\sum_{n \in \mathbb{Z}} e^{-s\omega_n^2} &= \frac{\beta_0}{\sqrt{4\pi s}} \left(1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2 \beta_0^2}{4s}} \right), \\ \sum_{n \in \mathbb{Z}} \omega_n^2 e^{-s\omega_n^2} &= \frac{\beta_0}{4\sqrt{\pi s^{\frac{3}{2}}}} + \frac{\beta_0}{2\sqrt{\pi s^{\frac{3}{2}}}} \sum_{n=1}^{\infty} \left(1 - \frac{n^2 \beta_0^2}{2s} \right) e^{-\frac{n^2 \beta_0^2}{4s}}, \\ \sum_{n \in \mathbb{Z}} \omega_n^4 e^{-s\omega_n^2} &= \frac{3\beta_0}{8\sqrt{\pi s^{\frac{5}{2}}}} + \frac{3\beta_0}{4\sqrt{\pi s^{\frac{5}{2}}}} \left(\sum_{n=1}^{\infty} e^{-\frac{n^2 \beta_0^2}{4s}} - \frac{\beta_0^2}{s} \sum_{n=1}^{\infty} n^2 e^{-\frac{n^2 \beta_0^2}{4s}} + \frac{\beta_0^4}{4s^2} \sum_{n=1}^{\infty} n^4 e^{-\frac{n^2 \beta_0^2}{4s}} \right),\end{aligned}\quad (3.9)$$

after which the integration over s can be done in terms of modified Bessel functions which, in turn, are expressed using exponential integrals (see Ref. [19] for a somewhat similar calculation). After some lengthy manipulations, we find the following results for all eight thermal form factors depending on $z \equiv \beta_0 \sqrt{-\Delta}$:

$$\begin{aligned}\chi_1(z) &= -\frac{\beta_0}{512z} - \frac{\beta_0}{960\pi^2} \left[\gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] + \frac{\beta_0 \zeta(3) z^2}{107520\pi^4} - \frac{\beta_0 \zeta(5) z^4}{5160960\pi^6} + \frac{\beta_0 \zeta(7) z^6}{181665792\pi^8} + \mathcal{O}(z^8), \\ \chi_2(z) &= \frac{25\beta_0}{36864z} + \frac{\beta_0}{2880\pi^2} \left[\gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] - \frac{\beta_0 \zeta(3) z^2}{483840\pi^4} + \frac{\beta_0 \zeta(7) z^6}{544997376\pi^8} + \mathcal{O}(z^8), \\ \chi_3(z) &= \frac{\beta_0}{384z} + \frac{\beta_0 \zeta(3) z^2}{23040\pi^4} - \frac{\beta_0 \zeta(5) z^4}{430080\pi^6} + \frac{\beta_0 \zeta(7) z^6}{8257536\pi^8} + \mathcal{O}(z^8), \\ \chi_4(z) &= -\frac{\beta_0}{32z} - \frac{\beta_0}{32\pi^2} \left[\gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] + \frac{\beta_0 \zeta(3) z^2}{1536\pi^2} - \frac{\beta_0 \zeta(5) z^4}{40960\pi^6} + \frac{\beta_0 \zeta(7) z^6}{917504\pi^8} + \mathcal{O}(z^8), \\ \chi_5(z) &= \frac{1}{576\beta_0} + \frac{z^2}{3840\pi^2 \beta_0} \left[1 + \gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] - \frac{\zeta(3) z^4}{143360\pi^4 \beta_0} + \frac{\zeta(5) z^6}{4128768\pi^6 \beta_0} + \mathcal{O}(z^8), \\ \chi_6(z) &= \frac{z^2}{5760\pi^2 \beta_0} \left[1 + \gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] - \frac{\zeta(3) z^4}{107520\pi^4 \beta_0} + \frac{\zeta(5) z^6}{2064384\pi^6 \beta_0} + \mathcal{O}(z^8), \\ \chi_7(z) &= \frac{1}{96\beta_0} + \frac{z^2}{384\pi^2 \beta_0} \left[1 + \gamma_E + \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] - \frac{\zeta(3) z^4}{10240\pi^4 \beta_0} + \frac{\zeta(5) z^6}{229376\pi^6 \beta_0} + \mathcal{O}(z^8), \\ \chi_8(z) &= -\frac{\pi^2}{240\beta_0^3} - \frac{z^2}{1152\beta_0^3} - \frac{z^4}{7680\pi^2 \beta_0^3} \left[4 + 3\gamma_E + 3 \log\left(\frac{\beta_0 \mu}{4\pi}\right) \right] + \frac{\zeta(3) z^6}{57344\pi^4 \beta_0^3} + \mathcal{O}(z^8),\end{aligned}\quad (3.10)$$

with γ_E the Euler-Mascheroni constant. Higher corrections can be computed systematically using the technique described to arbitrary order in z . It should be pointed out that all contributions proportional to $\log z$ cancel out among the vacuum and finite-temperature contributions, so the only nonlocal terms remaining are simple poles in z .

It is important to stress at this point the fundamental difference between the expansion (3.6) in powers of the curvatures R , R_{ij} , f_{ij} , and $\bar{P}$ and the one in the number of derivatives leading to Eq. (2.14). In fact, the two expressions cannot be compared straightforwardly. As an example, the coefficient of the term proportional to $f_{ij} f^{ij}$ in Eq. (2.14) results from adding to the corresponding

contribution in the first line of Eq. (3.6) the leading local piece of $f_{ij} \chi_5(\beta_0 \sqrt{-\Delta}) f^{ij}$ in (3.10):

$$\frac{\xi}{96\beta_0} f_{ij} f^{ij} + \frac{1}{576\beta_0} f_{ij} f^{ij} = \frac{1+6\xi}{576\beta_0} f_{ij} f^{ij}. \quad (3.11)$$

Similarly, the part proportional to σ^2 from the leading term in $4\sigma \chi_8(\beta_0 \sqrt{-\Delta}) \sigma$ combines with the powers of σ in the first line of (3.6), to give

$$\begin{aligned}
& -\frac{\pi^2}{90\beta_0^3} + \frac{\pi^2}{30\beta_0^3}\sigma - \frac{\pi^2}{30\beta_0^3}\sigma^2 - \frac{\pi^2}{60\beta_0^3}\sigma^2 \\
& = -\frac{\pi^2}{90\beta_0^3} + \frac{\pi^2}{30\beta_0^3}\sigma - \frac{\pi^2}{20\beta_0^3}\sigma^2. \quad (3.12)
\end{aligned}$$

This coincides with the expansion of the first term in the partition function (2.14) in powers of σ to quadratic order:

$$-\frac{\pi^2}{90\beta_0^3}e^{-3\sigma} = -\frac{\pi^2}{90\beta_0^3} + \frac{\pi^2}{30\beta_0^3}\sigma - \frac{\pi^2}{20\beta_0^3}\sigma^2 + \dots \quad (3.13)$$

As for gradients of σ and the curvature term, collecting the relevant contributions in (3.6), we find, modulo total derivatives,

$$\begin{aligned}
& \frac{6\xi-1}{144\beta_0}R - \frac{8\xi-1}{96\beta_0}g^{ij}\partial_i\sigma\partial_j\sigma + 4\sigma\chi_7(\beta_0\sqrt{-\Delta})\bar{P} + 4\sigma\chi_8(\beta_0\sqrt{-\Delta})\sigma \\
& = \frac{6\xi-1}{144\beta_0}(1-\sigma)R - \frac{6\xi-1}{36\beta_0}g^{ij}\partial_i\sigma\partial_j\sigma + \dots, \quad (3.14)
\end{aligned}$$

reproducing the expansion in powers of σ of the local inverse temperature in Eq. (2.14):

$$\begin{aligned}
& \frac{1-6\xi}{144\beta} (4g^{ij}\partial_i\sigma\partial_j\sigma - R) = \frac{6\xi-1}{144\beta_0}(1-\sigma)R - \frac{6\xi-1}{36\beta_0}g^{ij}\partial_i\sigma\partial_j\sigma \\
& + \dots. \quad (3.15)
\end{aligned}$$

This match of the expansions (2.14) and (3.6) provides a nontrivial test of our results. Accounting for higher orders in σ requires including contributions beyond those in Eq. (3.3).

A second important test is provided by the computation of the coefficients of the logarithmic terms in the form factors (3.10), giving the result

$$\begin{aligned}
\mu \frac{\partial}{\partial \mu} \log Z(\beta) = & -\frac{\beta_0}{32\pi^2} \int d^3\mathbf{x} \sqrt{g} \left(\frac{1}{30} R_{ij} R^{ij} + \frac{1-20\xi+60\xi^2}{60} R^2 + \frac{1}{120} f_{ij} \Delta f^{ij} \right. \\
& \left. + \frac{7-60\xi+120\xi^2}{30} \sigma \Delta R + \frac{6-60\xi+180\xi^2}{45} \sigma \Delta^2 \sigma \right), \quad (3.16)
\end{aligned}$$

where we have substituted the expression of $\bar{P}$ given in Eq. (2.17), keeping only those terms bilinear in curvatures [see the remarks after Eq. (3.6)]. Their apparent differences notwithstanding, this expression is in full agreement with the general result provided in Eq. (2.16). Indeed, expanding there all exponentials to linear order in σ and applying some Riemann and Ricci tensor identities, we retrieve the expression (3.16) after a number of integration by parts, retaining only those terms up to second order in curvatures. For conformal coupling $\xi = 1/6$, we obtain the Weyl anomaly

$$\begin{aligned}
\mathcal{A}_{\text{Weyl}} = & -\frac{\beta_0}{1920\pi^2} \int d^3x \sqrt{g} \left[2 \left(R_{ij} R^{ij} - \frac{1}{3} R^2 \right) \right. \\
& \left. + \frac{1}{2} f_{ij} \Delta f^{ij} + \frac{2}{3} \sigma \Delta R + \frac{4}{3} \sigma \Delta^2 \sigma \right], \quad (3.17)
\end{aligned}$$

where the quantity between the square brackets is the four-dimensional Weyl tensor squared of the stationary metric (2.2) and we omitted total derivative terms.

A comparison of (3.6) with the result for ultrastatic backgrounds obtained in [19] shows the existence of four additional form factors χ_5 – χ_8 acting on σ , R and f_{ij} . Interestingly, none of them give rise to nonlocal terms, which are still associated solely with χ_1 – χ_4 :

$$\begin{aligned}
\log Z(\beta)_{\text{nonlocal}} = & -\frac{1}{32} \int d^3\mathbf{x} \sqrt{g} \left[\frac{1}{16} R_{ij} \frac{1}{\sqrt{-\Delta}} R^{ij} \right. \\
& - \frac{25}{1152} R \frac{1}{\sqrt{-\Delta}} R \\
& \left. - \frac{1}{12} \bar{P} \frac{1}{\sqrt{-\Delta}} R + \bar{P} \frac{1}{\sqrt{-\Delta}} \bar{P} \right]. \quad (3.18)
\end{aligned}$$

Despite replicating the structure found in [19], these nonlocal terms incorporate through $\bar{P}$ the effect of Tolman temperature gradients, $T^{-1}\partial_i T = -\partial_i \sigma$, as well as of the fluid vorticity encoded in the KK field strength, $\omega^i \sim \frac{1}{2} \epsilon^{ijk} f_{jk}$, none of them present for ultrastatic metrics. The new form factors, however, do contribute to the Weyl anomaly, inducing the last three terms on the right-hand side of Eq. (3.17).

Finally, let us point out that the structure of the nonlocal terms (3.18) we obtained using functional methods can also be recovered from a calculation of three scalar one-loop diagrams at finite temperature where we, respectively, insert two gravitons $h_{\mu\nu}$, two external sources J coupling to φ^2 , and one graviton $h_{\mu\nu}$ and one external source J (cf. Refs. [39,40]). All these three diagrams contain contributions scaling as $|\mathbf{p}|^{-1}$, with $\mathbf{p}$ the incoming three-momentum, corresponding to the four nonlocal terms shown in Eq. (3.18). No nonlocal terms arise, however, from tadpole diagrams.

IV. CLOSING REMARKS

In this paper we have computed the high-temperature equilibrium partition function of a massless scalar field on a general stationary geometry nonminimally coupled to the background scalar curvature, using expansions in the number of derivatives as well as in powers of curvatures. The results found in both cases display a dependence on local temperature gradients and fluid vorticity, encoded in the metric functions $\sigma(\mathbf{x})$ and $a_i(\mathbf{x})$, thus generalizing the results found for ultrastatic metrics to general stationary geometries and arbitrary curvature coupling. The general expression of the Weyl anomaly at second order in curvatures has also been obtained from the logarithmic terms in both forms of the partition function. In a hydrodynamical context, our expressions can be used to

construct the fluid equilibrium partition function at finite temperature, including the effects of the local temperature gradients, vorticity, and acceleration.

These implications of our results for hydrodynamics and transport are one of the various interesting problems waiting to be addressed. Another concerns the extension of these results to physical situations in which conformal invariance is spontaneously broken [41,42], as well as the possible application of the techniques introduced in Ref. [43] to incorporate the effect of spontaneous gauge symmetry breaking to the case of the Weyl anomaly. These and other questions will be investigated elsewhere.

ACKNOWLEDGMENTS

The work of M. V. is supported by Spanish Science Ministry Grants No. PID2021-123703NB-C21 (MCIU/AEI/FEDER, EU) and No. PID2024-156016NB-I00 (MCIU/AEI/FEDER, EU). M. A. V.-M. acknowledges financial support from Spanish Science Ministry Grants No. PID2021-123703NB-C22 (MCIU/AEI/FEDER, EU) and No. PID2024-160856NB-I00 (MCIU/AEI/FEDER, EU). Both authors are also funded by Basque Government Grant No. IT979-16.

DATA AVAILABILITY

No data were created or analyzed in this study.

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