Spectral functions at nonzero temperature

V. P. Nair[1] and Robert D. Pisarski[2]

[1]*Department of Physics, City College of New York, CUNY, New York, New York 10031, USA*
[2]*Department of Physics, Brookhaven National Laboratory, Upton, New York 11973, USA*

(Received 16 April 2025; accepted 9 February 2026; published 16 April 2026)

We present a straightforward derivation of the spectral representation of a scalar field at nonzero temperature, assuming that the field is relativistically invariant in vacuum. This form was first derived by Bros and Buchholz.

DOI: 10.1103/m4st-sddq

I. INTRODUCTION

The analysis of field theories is of interest in a variety of physical problems, from the behavior of the early universe to the collisions of heavy ions. A fundamental quantity of interest is the behavior of spectral functions, from which transport quantities can be computed using the Kubo formula.

In vacuum, by relativistic invariance the spectral function is a function of a single variable, which can be chosen to be the Mandelstam variable s. At nonzero temperature, because relativistic invariance is lost, generally one expects that the spectral density is then a function of two variables, such as s and the spatial momentum, p^2.

For a general theory at nonzero temperature, one does not expect any specific relation in the spectral density between s and p^2. By developing a set of axioms, similar to the Wightman axioms, but for fields at nonzero temperature, Bros and Buchholz showed that *if* the theory is relativistically invariant in vacuum, then that constrains the form of the spectral density at nonzero temperature. The key point is that the difference between the vacuum correlations and those at nonzero temperature is in the choice of the state. If the underlying dynamics is relativistically invariant, then this informs the spectral representation even at nonzero temperature.

In this paper we present an alternate derivation of the Bros-Buchholz form in a more straightforward manner, without the use of axiomatic field theory. The utility of the Bros-Buchholz form has been developed recently by Lowdon *et al.* [1–7].

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

II. THE BROS-BUCHHOLZ REPRESENTATION

We construct the Bros-Buchholz representation at nonzero temperature along the lines of how the standard Källen-Lehmann representation is obtained in field theory. We consider a scalar field theory and start with the definition of the Wightman function at nonzero temperature,

$$W(x,y)=\frac{1}{\mathcal{Z}}\mathrm{Tr}[e^{-\beta H}\phi(x)\phi(y)]. \tag{1}$$

As usual, using the spacetime translational invariance at the level of operators we can write

$$\phi(x)=e^{iP\cdot x}\phi(0)e^{-iP\cdot x}, \tag{2}$$

where P_μ is the total 4-momentum operator. Using this result and a complete set of states $|a\rangle$ and $|b\rangle$,

$$W(x,y)=\frac{1}{\mathcal{Z}}\sum\!\!\!\!\!\!\int_{a,b} e^{-\beta E_b}|\phi_{ab}(0)|^2 e^{-i(p_a-p_b)\cdot(x-y)}. \tag{3}$$

The states $|a\rangle$ and $|b\rangle$ are characterized by their energy ϵ_α in its rest frame and the 3-momentum $\vec{p}_a$ so that $\phi_{ab}(0)=\langle\alpha,\vec{p}_a|\phi(0)|\beta,\vec{p}_b\rangle$. The sum-integral in Eq. (3) represents

$$\begin{aligned}\sum\!\!\!\!\!\!\int_a &= \sum_\alpha\int\frac{d^3p_a}{2p_a^0(2\pi)^3}, \qquad p_a^0=\sqrt{\epsilon_\alpha^2+\vec{p}_a^2}\\ &= \sum_\alpha\int_0^\infty d\sqrt{s'}\,\delta(\sqrt{s'}-\epsilon_\alpha)\int\frac{d^3p_a}{2p_a^0(2\pi)^3},\end{aligned} \tag{4}$$

where, in the last line, $p_a^0=\sqrt{s'+\vec{p}_a^2}$. We can thus write Eq. (3) as

$$W(x,y) = \frac{1}{\mathcal{Z}} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \sum_{\alpha,\beta} \int_0^\infty \frac{ds'}{2\sqrt{s'}} \delta(\sqrt{s'} - \epsilon_\alpha) \times \int_0^\infty \frac{ds''}{2\sqrt{s''}} \delta(\sqrt{s''} - \epsilon_\beta) \int \frac{d^3p_a}{2p_a^0(2\pi)^3} \times \int \frac{d^3p_b}{2p_b^0(2\pi)^3} e^{-\beta p_b^0} |\phi_{ab}(0)|^2 \delta^{(4)}(p - p_a + p_b), \tag{5}$$

where $p_b^0 = \sqrt{s'' + \vec{p}_b^{\,2}}$.

We now note that, since the Fourier transform of $\phi(0)$ is an integral over all momenta, the matrix element $\phi_{ab}(0)$ in Eq. (5) can be nonzero for any values of $p_a - p_b$, and so is a function of $\vec{p}_a - \vec{p}_b$, ϵ_α, and ϵ_β. Further, from the Lorentz transformation property of ϕ, we have

$$\phi(\Lambda^{-1}x) = U(\Lambda)\phi(x)U^{-1}(\Lambda). \tag{6}$$

Setting $x = 0$, we see that

$$\langle a|\phi(0)|b\rangle = \langle a|U(\Lambda)\phi(0)U^{-1}(\Lambda)|b\rangle = \langle a'|\phi(0)|b'\rangle, \tag{7}$$

where $|a'\rangle$, $|b'\rangle$ are the Lorentz transforms of $|a\rangle$, $|b\rangle$. We see that the value of the matrix element is unaltered under the Lorentz transformation of $|a\rangle$, $|b\rangle$. We can therefore take the matrix element to be a function of the invariants $(p_a - p_b)^2$ and ϵ_α, ϵ_β. So we can define an invariant function

$$f(s', s'', (p_a - p_b)^2) = \sum_{\alpha,\beta} \delta(\sqrt{s'} - \epsilon_\alpha)\delta(\sqrt{s''} - \epsilon_\beta) \frac{|\langle a|\phi(0)|b\rangle|^2}{4\sqrt{s'}\sqrt{s''}}. \tag{8}$$

Using this relation, we can now write Eq. (5) as

$$W(x,y) = \frac{1}{\mathcal{Z}} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \int \frac{d^3p_a}{2p_a^0(2\pi)^3} \times \int \frac{d^3p_b}{2p_b^0(2\pi)^3} \delta^{(4)}(p - p_a + p_b) \times \int_0^\infty ds' \int_0^\infty ds'' e^{-\beta p_b^0} f(s', s'', (p_a - p_b)^2). \tag{9}$$

The last argument of f can be written out as

$$(p_a - p_b)^2 = s' + s'' - 2p_a \cdot p_b = s' + s'' - 2\sqrt{\vec{p}_a^{\,2} + s'}\sqrt{\vec{p}_b^{\,2} + s''} + 2\vec{p}_a \cdot \vec{p}_b. \tag{10}$$

Consider now the integral

$$\mathcal{I} = \int_0^\infty ds' \int \frac{d^3p_a}{2p_a^0(2\pi)^3} \delta^{(4)}(p - p_a + p_b) \times f(s', s'', (p_a - p_b)^2) = \int_0^\infty ds' \int \frac{d^3p_a}{2p_a^0(2\pi)^3} \delta(p^0 - p_a^0 + p_b^0)\delta^{(3)}(\vec{q} - \vec{p}_a) \times f\left(s', s'', s' + s'' - 2\sqrt{\vec{p}_a^{\,2} + s'}\sqrt{\vec{p}_b^{\,2} + s''} + 2\vec{p}_a \cdot \vec{p}_b\right), \tag{11}$$

where $\vec{q} = \vec{p} + \vec{p}_b$. We can then use the δ-function to replace the integral over $\vec{p}_a$ by one over $\vec{q}$, so that $\mathcal{I}$ is a function of $\vec{p}_b$, $\vec{q}$, p^0, and s''.

The essential point is then that both the arguments of f, and the integration measures, are Lorentz invariant. Thus, $\mathcal{I}$ must be a function of the Lorentz invariants constructed from the variables $\vec{p}_b$, $\vec{q}$, p^0, and s''. Since p^0 is the only time component, there are five possible invariants:

$$\alpha_1 = (p^0)^2 - \vec{q}^{\,2}, \qquad \alpha_2 = (p^0)^2 - \vec{p}_b^{\,2}, \qquad \alpha_3 = (p^0)^2 - \vec{p}_b \cdot \vec{q}, \qquad \alpha_4 = s'', \qquad \epsilon(p^0) = \begin{cases} +1 & p^0 > 0 \\ -1 & p^0 < 0 \end{cases}. \tag{12}$$

[We could consider $\theta(p^0)$ and $\theta(-p^0)$ as well, where θ denotes the step-function, but these are equivalent to what we have in Eq. (12) since $\theta(\pm p^0) = \frac{1}{2}(1 \pm \epsilon(p^0))$. Thus, we can write

$$\mathcal{I} = \mathcal{I}_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + \epsilon(p^0)\mathcal{I}_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4). \tag{13}$$

Now consider the special case when the state $|b\rangle$ is the vacuum state. In this case, $\vec{p}_b = 0$ and $\epsilon_\beta = 0$, so that, once the integration over s'' is carried out, it is set to zero. The corresponding $\mathcal{I}$ takes the form

$$\mathcal{I}|_{\vec{p}_b=0, s''=0} = \mathcal{I}_1((p^0)^2 - \vec{p}^{\,2}, (p^0)^2, (p^0)^2, 0) + \epsilon(p^0)\mathcal{I}_2((p^0)^2 - \vec{p}^{\,2}, (p^0)^2, (p^0)^2, 0). \tag{14}$$

If the state $|b\rangle$ is the vacuum state, the spectral representation should reduce to the usual Källen-Lehmann representation, which is of the form [8]

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \theta(p^0)\rho(p^2) \tag{15}$$

for some spectral density $\rho(p^2)$. Thus, functions on the right-hand side of Eq. (14) can only depend on $\epsilon(p^0)$ and $(p^0)^2 - \vec{p}^{\,2}$. The compatibility of the form of $\mathcal{I}$ in Eq. (13) with the form obtained for the particular case of $|b\rangle = |0\rangle$

shows that, in general, there should be no dependence on the variables in the second and third variables, α_2 and α_3, in $\mathcal{I}$. We therefore conclude that, at *any* temperature,

$$\mathcal{I} = \mathcal{I}_1((p^0)^2 - \vec{q}^2, s'') + \epsilon(p^0)\mathcal{I}_2((p^0)^2 - \vec{q}^2, s''). \quad (16)$$

Having argued that there is no dependence on α_2 and α_3, we can see that the functions $\mathcal{I}_1$ and $\mathcal{I}_2$ at finite temperature can be obtained from the corresponding vacuum ones with the shift $\vec{p} \to \vec{q} = \vec{p} + \vec{p}_b$. Physically, this just reflects that fact that in the trace which enters into the partition function at nonzero temperature, Eqs. (1) and (3), is a sum over all states, including those with momenta $\vec{p}_b$, and not just the vacuum.

The second point is that the Källen-Lehmann representation also shows that we need $\mathcal{I}_1((p^0)^2 - \vec{p}^2) = \mathcal{I}_2((p^0)^2 - \vec{p}^2)$, so that we obtain the $\theta(p^0)$ factor in Eq. (15) via $\theta(p^0) = \frac{1}{2}(1 + \epsilon(p^0))$. The result $\mathcal{I}_1((p^0)^2 - \vec{p}^2) = \mathcal{I}_2((p^0)^2 - \vec{p}^2)$ carries over to the finite temperature case with the shift $\vec{p} \to \vec{q} = \vec{p} + \vec{p}_b$, so that we may also write $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 = \theta(p^0)2\mathcal{I}_2$.

We now use this result back in Eq. (9) to carry out the integration over s', leaving the integration over $\vec{p}_b$:

$$W(x,y) = \frac{1}{\mathcal{Z}} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \int d^3p_b \times \int_0^\infty \frac{ds''}{2p_b^0(2\pi)^3} e^{-\beta p_b^0} \mathcal{I}((p^0)^2 - \vec{q}^2, s''). \quad (17)$$

At this point, we see that, apart from $\vec{q} = \vec{p} + \vec{p}_b$, the integrand can depend on $\vec{p}_b^2$ via p_b^0 as well, due to the exponential factor and the integration measure in Eq. (17). So we define a spectral function

$$\rho(\vec{p}_b^2, (p^0)^2 - \vec{q}^2) = \frac{1}{\mathcal{Z}} \int_0^\infty \frac{ds''}{2p_b^0(2\pi)^3} e^{-\beta p_b^0} 2\mathcal{I}_2((p^0)^2 - \vec{q}^2, s''). \quad (18)$$

We can then write $W(x,y)$ as

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \theta(p^0) \int d^3p_b \rho(\vec{p}_b^2, (p^0)^2 - \vec{q}^2). \quad (19)$$

Consider now the integration over p^0. For positive values of p^0, we define a variable s by $p^0 = \sqrt{s + \vec{q}^2}$ and change variables to s. For a function $f(p^0)$ which is even in p^0, the integral can then be written as

$$\begin{aligned} &\int_{-\infty}^\infty dp^0 e^{ip^0(x^0-y^0)} f(p^0) \\ &= \int_0^\infty dp^0 [e^{ip^0(x^0-y^0)} + e^{-ip^0(x^0-y^0)}] f(p^0) \\ &= \int_0^\infty \frac{ds}{2\sqrt{s+\vec{q}^2}} [e^{ip^0(x^0-y^0)} + e^{-ip^0(x^0-y^0)}] f(p^0) \\ &= \int_0^\infty ds \int_{-\infty}^\infty dp^0 \delta((p^0)^2 - \vec{q}^2 - s) e^{ip^0(x^0-y^0)} f(p^0). \end{aligned} \quad (20)$$

The second line where we consider the range of integration to be in the range $[0, \infty]$ is necessary to use s in place of p^0. In the third line of this equation, it is understood that p^0 in the exponent and the argument of $f(p^0)$ is $\sqrt{s + \vec{q}^2}$. In the last line, we introduce another p^0 as a free variable of integration. In a similar way

$$\begin{aligned} &\int_{-\infty}^\infty dp^0 e^{ip^0(x^0-y^0)} \epsilon(p^0) f(p^0) \\ &= \int_0^\infty ds \int_{-\infty}^\infty dp^0 \epsilon(p^0) \delta((p^0)^2 - \vec{q}^2 - s) e^{ip^0(x^0-y^0)} f(p^0). \end{aligned} \quad (21)$$

These equations can be combined to make a similar statement about an even function multiplied by $\theta(p^0)$. Going back to (19) and using these results, we get

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \int d^3p_b \int_0^\infty ds \times \theta(p^0)\delta(s - (p^0)^2 + (\vec{p} + \vec{p}_b)^2)\rho(\vec{p}_b^2, s). \quad (22)$$

The contribution from $\mathcal{I}_1$ is symmetric in $(x^0 - y^0)$ and does not contribute. The term involving $\mathcal{I}_2$ does contribute; it has a factor of $\epsilon(p^0)$ in the integrand as well. This is basically the spectral representation derived by Bros and Buchholz [9–12].

A more explicit form is obtained if we use the Kubo-Martin-Schwinger (KMS) condition for the Wightman function. In terms of its Fourier transform $\tilde{W}(p)$, the KMS condition implies that

$$\tilde{W}(p) = e^{\beta p^0} \tilde{W}(-p) = \frac{\tilde{W}(p) - \tilde{W}(-p)}{1 - e^{-\beta p^0}}. \quad (23)$$

From Eq. (22), we see that

$$\tilde{W}(p) - \tilde{W}(-p) = \int d^3p_b \int_0^\infty ds \epsilon(p^0) \times \delta(s - (p^0)^2 + (\vec{p} + \vec{p}_b)^2)\rho(\vec{p}_b^2, s), \quad (24)$$

so that we can write the spectral representation for the Fourier transform of $W(x,y)$ as

$$\tilde{W}(p) = \frac{\epsilon(p^0)}{1 - e^{-\beta p^0}} \int_0^\infty ds \int d^3 p_b \delta(s - (p^0)^2 + (\vec{p} - \vec{p}_b)^2) \rho(\vec{p}_b^{\,2}, s). \tag{25}$$

In this equation, we have changed $\vec{p}_b$ to $-\vec{p}_b$ to facilitate comparison with [9–12]. Notice that, as $\beta \to \infty$, $(1 - e^{-\beta p^0})^{-1} \to \theta(p^0)$ and $\rho(\vec{p}_b^{\,2}, s) \to \rho(0, s)$ because of the $e^{-\beta p_b^0}$ factor. Thus, from Eq. (25) we correctly recover the vacuum spectral representation in this limit.

While we have used the Lorentz invariance of the underlying dynamics to bring the spectral representation to the form given above, the final result obviously is not Lorentz invariant, since the density matrix represents the choice of state in the rest frame of the medium. This is evident from the term $e^{-\beta p_b^0}$ and the partition function $\mathcal{Z}$ in the expression for the spectral function.

It is straightforward to generalize the result to a moving medium. We replace

$$\mathrm{e}^{-\beta H} \to \mathrm{e}^{-\beta(H u^0 - \vec{u}\cdot\vec{P})}, \tag{26}$$

where

$$u^\mu = \frac{1}{\sqrt{1 - v^2}} (1, v^i) \tag{27}$$

is the 4-velocity of the medium. Similarly,

$$\mathrm{e}^{-\beta p_b^0} \to \mathrm{e}^{-\beta(p_b^0 u^0 - \vec{p}_b\cdot\vec{u})}, \tag{28}$$

and likewise for the partition function. With these changes, Eq. (22) gives the spectral representation of $W(x, y)$ in a moving medium.

III. CONCLUSIONS

In this paper we have given a (relatively) simple derivation of the Bros-Buchholz form of the spectral density for a scalar field at nonzero temperature. While relativistic invariance in vacuum constrains this form, it is still a function of two variables.

It is worth comparing this form, Eq. (25), with the standard dispersion relation,

$$\tilde{W}(p) = \frac{\epsilon(p^0)}{1 - e^{-\beta p^0}} \tilde{\rho}(\vec{p}^{\,2}, p^0). \tag{29}$$

This is clearly a more general form than Eq. (25). It does not involve the relativistically invariant combination $s - (p^0)^2 + (\vec{p} - \vec{p}_b)^2$, nor the integral over the spatial momentum $\vec{p}_b$. That is, Eq. (29) does not have Lorentz invariance in the vacuum built in, as does Eq. (25).

The general dispersion relation of Eq. (29) has been extensively studied in weak coupling; see Refs. [13–17]. In weak coupling, one finds that the pole structure in vacuum is subsumed under cuts from thermal discontinuities. At one loop order this is in a limited region of p^0 and p space, from Landau damping. At two loop order and higher, there are cuts for all $p^0 > 0$ and $p \geq 0$. This is the standard understanding of thermal broadening. As emphasized by Lowden and Philipsen [4–6], with the Bros-Buchholz spectral density, it is necessary to adopt a much more restrictive approach to the possible forms of the spectral density.

It is also possible to compute transport coefficients in strong coupling; this includes both AdS/CFT models, e.g. Refs. [18,19], and at large N, Refs. [20,21]. For AdS/CFT models, in strong coupling the analytic structure is completely unexpected, being an infinite set of poles [18]. This suggests a phase transition at intermediate coupling to the behavior seen in weak coupling [19]. It would be interesting to compute the spectral densities for these models in the Bros-Buchholz form. This is not obvious, since the spectrum in vacuum is conformally invariant.

In any case, we hope that our derivation of the Bros-Buchholz spectral representation will inspire others to use it in related problems at nonzero temperature.

ACKNOWLEDGMENTS

We thank P. Lowdon for discussions and suggestions, especially for pointing out the simplification using the KMS condition. R. D. P. also thanks O. Philipsen for numerous discussions. V. P. N. was supported in part by the U.S. National Science Foundation Grants No. PHY-2112729 and No. PHY-2412479. R. D. P. was supported by the U.S. Department of Energy under Contract No. DE-SC0012704 and thanks the Alexander v. Humboldt Foundation for their support.

DATA AVAILABILITY

No data were created or analyzed in this study.

[1] P. Lowdon, R.-A. Tripolt, J. M. Pawlowski, and D. H. Rischke, Spectral representation of the shear viscosity for local scalar QFTs at finite temperature, Phys. Rev. D **104**, 065010 (2021).
[2] P. Lowdon and R.-A. Tripolt, Real-time observables from Euclidean thermal correlation functions, Phys. Rev. D **106**, 056006 (2022).
[3] P. Lowdon, Euclidean thermal correlation functions in local QFT, Phys. Rev. D **106**, 045028 (2022).
[4] P. Lowdon and O. Philipsen, Pion spectral properties above the chiral crossover of QCD, J. High Energy Phys. 10 (2022) 161.
[5] P. Lowdon and O. Philipsen, Goldstone bosons at nonzero temperature, Phys. Rev. D **112**, L061701 (2025).
[6] P. Lowdon and O. Philipsen, Goldstone bosons across thermal phase transitions, J. High Energy Phys. 02 (2026) 090.
[7] D. Bala, O. Kaczmarek, P. Lowdon, O. Philipsen, and T. Ueding, Pseudo-scalar meson spectral properties in the chiral crossover region of QCD, J. High Energy Phys. 05 (2023) 332.
[8] C. Itzykson and J. B. Zuber, Quantum Field Theory, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1980).
[9] J. Bros and D. Buchholz, Particles and propagators in relativistic thermo field theory, Z. Phys. C **55**, 509 (1992).
[10] J. Bros and D. Buchholz, Towards a relativistic KMS condition, Nucl. Phys. **B429**, 291 (1994).
[11] J. Bros and D. Buchholz, Axiomatic analyticity properties and representations of particles in thermal quantum field theory, Ann. Inst. Henri Poincare Phys. Theor. **64**, 495 (1996).
[12] J. Bros and D. Buchholz, Asymptotic dynamics of thermal quantum fields, Nucl. Phys. **B627**, 289 (2002).
[13] S. Jeon, Hydrodynamic transport coefficients in relativistic scalar field theory, Phys. Rev. D **52**, 3591 (1995).
[14] G. D. Moore, Transport coefficients in large N(f) gauge theory: Testing hard thermal loops, J. High Energy Phys. 05 (2001) 039.
[15] G. Aarts and J. M. Martinez Resco, Transport coefficients, spectral functions and the lattice, J. High Energy Phys. 04 (2002) 053.
[16] G. Aarts and J. M. Martinez Resco, Shear viscosity in the O(N) model, J. High Energy Phys. 02 (2004) 061.
[17] M. Bajec, S. Grozdanov, and A. Soloviev, Spectra of correlators in the relaxation time approximation of kinetic theory, J. High Energy Phys. 08 (2024) 065.
[18] S. A. Hartnoll and S. P. Kumar, AdS black holes and thermal Yang-Mills correlators, J. High Energy Phys. 12 (2005) 036.
[19] J. Casalderrey-Solana, S. Grozdanov, and A. O. Starinets, Transport peak in the thermal spectral function of $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma at intermediate coupling, Phys. Rev. Lett. **121**, 191603 (2018).
[20] P. Romatschke, Finite-temperature conformal field theory results for all couplings: O(N) model in $2 + 1$ dimensions, Phys. Rev. Lett. **122**, 231603 (2019); **123**, 209901(E) (2019).
[21] P. Romatschke, Shear viscosity at infinite coupling: A field theory calculation, Phys. Rev. Lett. **127**, 111603 (2021).