Hadrons in $\mathcal{N}=2$ supersymmetric QCD from non-Abelian string on 2D black hole

E. Ievlev,[1] A. Marshakov,[2,3] G. Sumbatian,[4] and A. Yung[4,5]

[1]*William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[2]*Skoltech, Moscow 121205, Russia*
[3]*Department of Mathematics, HSE University, Moscow 119048, Russia*
[4]*NRC "Kurchatov Institute"–PNPI, Gatchina, St. Petersburg 188300, Russia*
[5]*HSE University, St. Petersburg 194100, Russia*

(Received 24 May 2026; accepted 13 July 2026; published 10 August 2026)

We continue the study of non-Abelian vortex string in 4D $\mathcal{N}=2$ supersymmetric QCD (SQCD) as critical superstring, and extend this analysis to $U(N)$ gauge theory with arbitrary even N and $N_f=2N$ number of quarks. We introduce a special mass deformation and show that the SQCD hadron spectrum is still given by the string spectrum on the 2D $\mathcal{N}=2$ supersymmetric black hole. We perform a cross-check by computing the multiplicity of hadronic states of the high-energy part of the spectrum both from string and field theory pictures. We also clarify the spontaneous breaking of the global flavor symmetry by the vacuum expectation value of the massless baryon. We finally claim that phase diagram of $\mathcal{N}=2$ SQCD with $N_f=2N$ consists of the Higgs phase at weak coupling and string/hadronic phase at strong coupling, separated by phase transition, and is seen as a conifold transition from the string theory point of view.

DOI: 10.1103/m1nb-v2s4

I. INTRODUCTION

Non-Abelian vortex strings in four-dimensional (4D) $\mathcal{N}=2$ supersymmetric QCD (SQCD) [1–4], are flux tube configurations, saturating the Bogomol'nyi-Prasad-Sommerfield (BPS) bound; see Refs. [5–8] for reviews. Besides usual translational moduli, they also have (internal) orientational and size zero modes, comprising a two-dimensional world-sheet sigma model, the so-called $\mathcal{N}=(2,2)$ supersymmetric weighted $\mathbb{CP}$-model $\mathbb{WCP}(N,N_f-N)$.

It was shown in Ref. [9] that the non-Abelian solitonic string in $\mathcal{N}=2$ SQCD with gauge group U(2) and $N_f=2N=4$ quark flavors behaves as a critical superstring.[1] For $N_f=2N$, the world-sheet $\mathbb{WCP}(N,N)$ sigma model is conformal, and for $N=2$, the six orientational/size moduli together with four translational moduli form a ten-dimensional target space $\mathbb{R}^4\times Y_6$ required for a superstring to become critical [9,10]. The target geometry contains the conifold Y_6—the noncompact Calabi-Yau (CY) space [11,12]. The resulting critical vortex string is type IIA, and its low-lying closed-string spectrum was found in Refs. [10,13].

Most string modes are non-normalizable over Y_6, so they are not localized in 4D; in particular, there is no 4D graviton. An exception is the conifold complex-structure deformation modulus b, whose wave function is logarithmically normalizable and was interpreted as a massless baryon in 4D $\mathcal{N}=2$ SQCD [10].

Massive states were also analyzed using $\mathcal{N}=2$ Liouville theory [14,15], by an approach akin to that in little string theories [16], based on the conjectured equivalence [17–19] between the critical string on conifold and noncritical $c=1$ string with compact scalar on the self-dual radius plus a Liouville field. This equivalence was used in Refs. [13,20] to obtain the low-lying hadron spectrum of 4D $\mathcal{N}=2$ SQCD with U(2) and $N_f=4$.

More recently, in Ref. [21], a direct derivation was presented: a Coulomb branch opening up at strong coupling in the $\mathbb{WCP}(N,N)$ model on noncompact toric CY manifolds is described by $\mathcal{N}=2$ Liouville theory with an N-dependent background charge. Using this Liouville description, in Ref. [22], an interpolation program between SQCDs with different gauge groups/quark flavor numbers by turning on quark masses and decoupling (or integrating in) quark flavors was initiated. In particular, one can connect in this way the U(2) theory with $N_f=4$ with

[1]By "critical superstring," we mean a canonical superstring with $\mathcal{N}=1$ local supersymmetry in the world-sheet (super) gravity sector; the $\mathcal{N}=(2,2)$ supersymmetric world-sheet models (including $\mathcal{N}=2$ Liouville) belong to the matter sector.

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

U(4) and $N_f = 8$ SQCD by lowering the masses of four extra flavors.

Quark masses break the world-sheet conformal invariance, so the mass-deformed theory cannot be quantized directly; instead, one finds true string vacuum by solving the effective supergravity equations (vanishing of certain β functions in world-sheet theory) with initial conditions set by the mass deformation [22]. Since the relation $N_f = 2N$ is kept intact by our mass deformation, the coupling constant in the $\mathbb{WCP}(N,N)$ model does not run, and the world-sheet conformal invariance is then ensured by finding the effective gravity solution.

It was shown in Ref. [23] that the mass-deformed background found in Ref. [22] actually has the geometry of a trumpet, first discussed in Refs. [24,25]. It is T-dual to the 2D $\mathcal{N} = 2$ black hole with cigar metric, described exactly by the $\mathcal{N} = 2$ coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model [17,18,26–28]. Since $\mathcal{N} = 2$ Liouville theory is mirror to the $\mathcal{N} = 2$ cigar [29], it was argued in Ref. [23] that the theory with both superpotential and mass deformations included still yields the same cigar geometry, and only the black hole mass or entropy gets contributions from both deformations, being dominated by each of them in the corresponding regime.

This framework allowed us to extract the hadron spectrum of $\mathcal{N} = 2$ SQCD with gauge group U(4) and $N_f = 8$. Remarkably, the functional form of the string spectrum turned out to be insensitive to the mass deformation; only the multiplicities of stringy states on each energy level change, which was shown to be qualitatively consistent with SQCD-side field-theory arguments [23].

In this paper, we extend this line of research and generalize this construction to the case with an arbitrary even number of colors N. In other words, we take $\mathcal{N} = 2$ SQCD with gauge group $U(N)$ and $N_f = 2N$ and use a special interpolation procedure, generalizing that from Refs. [22,23]. We analyze the hadron spectrum both from the field theory side and from the string theory perspective and find excellent agreement. This serves as another strong check of our description of the non-Abelian flux tube as a critical superstring.

First we find that the SQCD hadron spectrum does not depend on N and is still given by the string spectrum of the cigar or $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset with the same Kač-Moody level $k = 1$. Then, we make a quantitative check of the multiplicities of states in large N limit, both from string theory and SQCD sides, and find an exact agreement. From the string theory side, the growth of the number of hadronic states with N is found from the near-Hagedorn behavior of the 2D black hole.

We also check that stringy massless b-baryons can be viewed as Goldstone states, present due to spontaneous breaking of the global symmetry by the vacuum expectation value (VEV) of the b-baryon. This check also confirms the self-consistency of our approach.

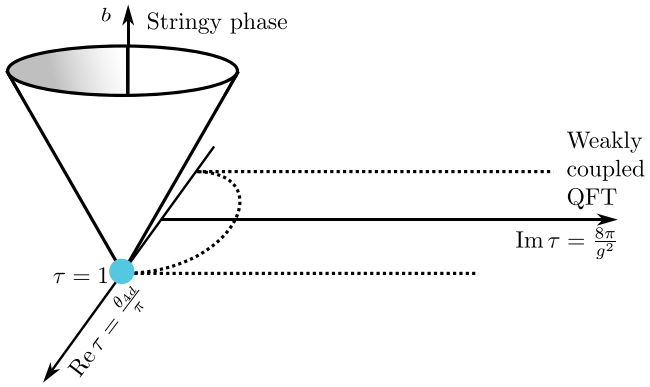


FIG. 1. Phase diagram for 4D SQCD. The fundamental domain of 4D coupling τ_{SW} is shown in the horizontal plane. SQCD is in the Higgs phase on this plane. The stringy phase where $\langle b \rangle \neq 0$ is schematically shown by the cone.

Qualitatively, the emergent picture can be summarized by the $\mathcal{N} = 2$ SQCD phase diagram shown in Fig. 1. Physics depends on the value of the complexified (by four-dimensional θ-angle θ_{4D}) non-Abelian gauge coupling constant [30]

$$\tau_{\mathrm{SW}} = i\frac{8\pi}{g^2} + \frac{\theta_{4D}}{\pi},$$

which does not run in $\mathcal{N} = 2$ SQCD with $N_f = 2N$.

At small g^2 (large $\mathrm{Im}\,\tau_{\mathrm{SW}}$), one has a perturbative Higgs phase, where the physical spectrum consists of a finite number of screened quarks and Higgsed gauge bosons together with their superpartners; see Sec. II A and Ref. [7] for details.

The region of $g^2 \sim 1$ is at strong coupling. In particular, at the "super strong" point $\tau_{\mathrm{SW}} = 1$, a nonperturbative Higgs branch is developed, as schematically shown in Fig. 1. On this nonperturbative branch, the massless stringy baryon b, associated with the complex structure modulus of the conifold, develops a VEV. Here, we have a stringy phase with towers of massive string states; see Sec. II C.

Many years ago, it was proposed in Ref. [31] that there is no phase transition between the Higgs phase at weak coupling and the confinement phase at strong coupling in scalar QCD.[2] Motivated by the dynamics of 4D $\mathcal{N} = 2$ SQCD, we challenge this conclusion. In our theory, the hadron phase at strong coupling can be described by string theory of non-Abelian vortices, responsible for the confinement of monopoles. We show that although both phases do have massless states, these states belong to different

[2]The reason (motivated by lattice studies) is that there is no order parameter to distinguish between these phases, since the Wilson loop in a theory with dynamical quarks in the fundamental representation of gauge group has a perimeter law in both phases, due to the quark pair creation which breaks the confining string.

representations of the global symmetry group, and therefore these two phases are not analytically connected. They become separated by a phase transition, seen from the string theory point of view as a famous conifold transition.

It is also worth noting that there is a qualitative distinction between the two phases. As we mentioned above, in the Higgs phase at weak coupling, the physical spectrum is formed by a finite number of screened quarks and Higgsed gluons together with their superpartners. In contrast, the stringy phase has an infinite tower of 4D hadron states, typical for the spectrum in string theory. The higher-energy states of this spectrum cannot be interpreted as composites of a finite number of light states.

One can even speculate that in real-world QCD, the hadronic phase is in the same universality class as the stringy phase of $\mathcal{N}=2$ SQCD. Historically, the difficulties of describing Regge behavior of hadron trajectories by field theory methods led to the creation of string theory. Here, the nonperturbative hadron spectrum of $\mathcal{N}=2$ SQCD in the stringy phase is determined and computed using the world-sheet theory of the critical superstring.

The paper is organized as follows: in Sec. II, we introduce 4D $\mathcal{N}=2$ SQCD and the world-sheet $\mathbb{WCP}(N,N)$ model on the non-Abelian string. Next, we discuss $\mathcal{N}=2$ Liouville theory describing the Coulomb branch of $\mathbb{WCP}(2,2)$ model, which opens up at strong coupling. We also review the string spectrum of the 2D $\mathcal{N}=2$ black hole, mirror to $\mathcal{N}=2$ Liouville theory. In Sec. III, we study the special interpolation procedure: consider the mass-deformed background with trumpet metric and its dual description in terms of the 2D black hole. We also discuss the resulting phase diagram of 4D $\mathcal{N}=2$ SQCD with $N_f=2N$. In Sec. IV, we consider the multiplicity of hadronic states from both string theory and SQCD sides in the large N limit. In Sec. V, we show that massless b-baryons can be viewed as Goldstone bosons upon global flavor symmetry breaking by the VEV of the b-baryon. Section VI contains our conclusions, and Appendixes present some details of our calculations.

II. NON-ABELIAN VORTEX STRING AND $\mathcal{N}=2$ LIOUVILLE THEORY

In this section, we introduce the 4D SQCD with a specific mass deformation. Then we review some basic facts about the world-sheet theory of the non-Abelian vortex string and the hadron spectrum in this context, before plunging into concrete computations in the rest of this paper.

A. Four-dimensional $\mathcal{N}=2$ SQCD

Our main theory of interest is the $\mathcal{N}=2$ SQCD in 4D with eight supercharges; see, for example, Ref. [7] for a detailed review of this theory. We focus on the case with the gauge group $U(N)$ and $N_f=2N$ matter hypermultiplets in the fundamental representation; with this choice, the 4D gauge coupling does not run. However, the conformal invariance of the 4D SQCD is explicitly broken by quark masses and the Fayet-Iliopoulos (FI) term [32], with FI parameter ξ, which is not renormalized and defines the VEVs of quarks.

At weak coupling $g^2 \ll 1$, this theory is in the Higgs phase. In a vacuum where the first N quark flavors are massless at zero ξ, the matrix of adjoint scalars of the $\mathcal{N}=2$ vector multiplet develops VEV of the form

$$\langle a\rangle = -\mathrm{diag}(m_1,\ldots,m_N), \tag{2.1}$$

where m_A $(A=1,\ldots,N_f)$ are bare quark masses. Adjoint condensates (2.1) break the $\mathrm{U}(N)$ gauge group down to $\mathrm{U}(1)^N$, with the masses of the off-diagonal gauge bosons given by $|m_k-m_l|$ $(k,l=1,\ldots,N)$, while the quark masses of q^{kA} and $\tilde{q}_{Ak}$ (two complex scalars of the $\mathcal{N}=2$ hypermultiplet) are equal to $|m_k-m_A|$.

At nonzero ξ, first N squarks also develop the VEVs as follows:

$$\langle q^{kA}\rangle = \sqrt{\xi}\delta^{kA}, \qquad \langle \tilde{q}_{Ak}\rangle = 0 \qquad k=1,\ldots,N, \qquad A=1,\ldots,N_f. \tag{2.2}$$

These condensates break the $\mathrm{U}(N)$ gauge group, Higgsing all gauge bosons. The quarks are screened, while the monopoles are confined by non-Abelian strings.[3] At weak coupling, these stringy hadrons are heavy and decay into perturbative states; however, at strong coupling, the theory enters the so-called instead-of-confinement regime [33,34]. In this regime, quarks and gluons decay into monopole-antimonopole pairs and we are left with hadrons formed by monopoles confined by non-Abelian strings.

Below, we concentrate on the theories with arbitrary even N, $N=2K$ with some integer K. We also consider a particular $\mathbb{Z}_K$-symmetric set of bare quark masses,

$$\{m_A\}_{A=1}^{N_f} = \{\underbrace{m_2,m_2,\ldots,m_{2K},m_{2K}}_{N=2K};\underbrace{\tilde{m}_2,\tilde{m}_2,\ldots,\tilde{m}_{2K},\tilde{m}_{2K}}_{N=2K}\}, \qquad \tilde{m}_{2k}=m_{2k}, \tag{2.3}$$

where we take $m_{2k-1}=m_{2k}$, $k=1,\ldots,K$ with the parameters m_{2k} evenly spaced on a circle of radius $m\in\mathbb{R}_{\geq 0}$, as follows:

[3] In fact, in $\mathrm{U}(N)$ theories, confined monopoles are junctions of two distinct elementary non-Abelian strings; see Ref. [7] for a review. In particular, the baryons represent closed "necklace" configurations of monopoles on a closed string.

$$m_{2k} = m \exp\left(\frac{2\pi i k}{K}\right), \quad k = 1, \ldots K. \tag{2.4}$$

This particular mass choice will be explored in Sec. III A.

At large m, the original SQCD splits into K sectors with mutual interactions suppressed by the energy scale $\Delta m \sim m/K$, as follows:

$$[U(2K) + (N_f = 4K)] \overset{m\to\infty}{\longrightarrow} [U(2) + (N_f = 4)]^{\otimes K} \tag{2.5}$$

(see Ref. [22] for details). Below in this paper, we often consider the large K limit; still, we assume that $m/K \to \infty$ at the initial step of our mass interpolation procedure.

For $m \to 0$ we recover the full $\mathrm{SU}(N = 2K)$ gauge group with $N_f = 4K$ fundamental flavors. In this case, the global flavor $\mathrm{SU}(N_f)$ is also broken down by quark VEVs to the so-called color-flavor locked group. The resulting global symmetry is

$$SU(N)_{C+F} \times SU(N) \times U(1)_B; \tag{2.6}$$

see Ref. [7] for more details. The unbroken global $\mathrm{U}(1)_B$ factor above is identified with a baryonic symmetry.

B. World-sheet theory of the non-Abelian string

In $\mathcal{N} = 2$ SQCD, the flux-tube strings are 1/2-BPS saturated and preserve $\mathcal{N} = (2, 2)$ supersymmetry with four supercharges on the world sheet. Their tension is determined by the 4D FI parameter

$$\tau = 2\pi\xi = \frac{1}{2\pi\alpha'}, \tag{2.7}$$

and in addition to translational moduli, the internal modes form an extra world-sheet sigma model, which we review below.

1. $\mathbb{WCP}(N, N)$ model

The effective theory on the string world sheet for the internal moduli is a 2D $\mathcal{N} = (2, 2)$ supersymmetric $\mathbb{WCP}(N, N)$ model, defined as a low-energy limit of the U(1) gauge theory [35] with twisted-mass deformation

$$\begin{aligned} S = \int d^2x \Big\{ & |\nabla_\alpha n^i|^2 + |\tilde{\nabla}_\alpha \rho^j|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} |\partial_\alpha \sigma|^2 \\ & + \frac{1}{2e_0^2} D^2 - \frac{\Theta}{2\pi} F_{01} - |\sqrt{2}\sigma + m_i|^2 |n^i|^2 \\ & - |\sqrt{2}\sigma + \tilde{m}_j|^2 |\rho^j|^2 + D(|n^i|^2 - |\rho^j|^2 - \mathrm{Re}\,\beta) \Big\}, \\ \alpha, \beta = 0, 1, \quad & i, j = 1, \ldots, N; \end{aligned} \tag{2.8}$$

see Ref. [7] for details. Here, the complex fields n^i and ρ^j describe the orientational and size moduli, respectively [1,4,36–39]. They have charges $+1$ and -1, so that

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \qquad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha.$$

Twisted masses of the n^i and ρ^j fields coincide with the masses m_i and $\tilde{m}_j$ of the 4D quarks.

The complexified inverse coupling in Eq. (2.8),

$$\beta = \mathrm{Re}\,\beta + i\frac{\Theta}{2\pi},$$

is defined via the 2D FI term (twisted superpotential)

$$\frac{\beta}{2} \int d^2\tilde{\theta}\, \sqrt{2}\, \Sigma = \frac{\beta}{2}(D - iF_{01}), \tag{2.9}$$

where Θ is the 2D θ-angle, while Σ is the twisted chiral superfield for the 2D gauge multiplet, which contains complex scalar σ, gauge field strength, and auxiliary field D as components [35].

The global symmetry group of the $\mathbb{WCP}(N, N)$ sigma model coincides with the unbroken global group of the 4D SQCD (2.6); see Ref. [10]. The fields n^i and ρ^j transform in the representations

$$\left(\mathbf{N}, \mathbf{1}, \frac{1}{2}\right)\left(\mathbf{1}, \mathbf{N}, \frac{1}{2}\right),$$

respectively.

The BPS spectrum of states in this 2D world-sheet theory coincides with the BPS spectrum of 4D states in the quark vacuum (2.2) given by the exact Seiberg-Witten solution [30] at $\xi = 0$. This coincidence was observed in Refs. [40,41] and explained later in Refs. [3,4] using the picture of 4D monopoles confined by the non-Abelian string that are seen as kinks in the 2D world-sheet theory. It is called 2D–4D correspondence.

The β function vanishes in the $\mathbb{WCP}(N, N)$ model, and this theory is conformal in the massless limit. Therefore, its target space is Ricci flat and Kähler, due to $\mathcal{N} = (2, 2)$ supersymmetry; i.e., it is a (noncompact) CY manifold.

Let us discuss how the interpolation procedure [discussed around Eq. (2.5)] looks from the world-sheet point of view. Consider classical vacua of the $\mathbb{WCP}(N, N)$ model. At $\mathrm{Re}\,\beta > 0$, we have N vacua of the world-sheet theory, parametrized by the VEV of σ, as follows:

$$\sqrt{2}\sigma = -m_{i_0}, \quad |n^{i_0}|^2 = \mathrm{Re}\beta, \quad i_0 = 1, \ldots, N, \tag{2.10}$$

where each choice of i_0 corresponds to a particular vacuum of the 2D world-sheet theory. Masses of the perturbative excitations $n^i, i \neq i_0$, and ρ^j are given by exact BPS formulas $|m_i - m_{i_0}|$ and $|\tilde{m}_j - m_{i_0}|$, respectively. The number of vacua remains intact in quantum theory, since it is protected by Witten index, equal to N. For the particular set of masses (2.3), all n^i and ρ^i fields (except for $i = i_0$ and one of the neighbors $i = i_0 \pm 1$) decouple in

the $m/K \to \infty$ limit, and the $\mathbb{WCP}(N,N)$ model (2.8) reduces to $\mathbb{WCP}(2,2)$ being the starting point of the interpolation procedure.

The target space of the $\mathbb{WCP}(2,2)$ is a noncompact CY_3 manifold, namely the conifold [10,12]. Combined with the flat Minkowski 4D space, it forms a ten-dimensional space required for a superstring to be critical [9,10]. The conifold Y_6 is a cone with a section $S^2 \times S^3$ at a fixed radial coordinate. The conifold singularity can be smoothed out in two distinct ways: by resolving the singularity with $\mathbb{CP}^1$ with a nonvanishing Kähler class or by deforming the complex structure; see Refs. [11,12]. The first option, called the resolved conifold, amounts to keeping a nonzero value of β in the D-term condition

$$|n^i|^2 - |\rho^j|^2 = \mathrm{Re}\,\beta;$$

see Eq. (2.8) for $N=2$. For $\rho^j = 0$, this gives exactly the $\mathbb{CP}^1$ part of the target space, and $\mathrm{Re}\,\beta$ defines the size of the "blown up" $\mathbb{CP}^1 \simeq S^2$. The resolved conifold has no normalizable zero modes. In particular, the Kähler modulus β, which, in principle, could become a massless scalar field in four dimensions, has a non-normalizable (quadratically divergent) wave function over the Y_6 and therefore is not dynamical [10].

However, we are interested in considering the model at "super strong" coupling $\beta = 0$, where the so-called "thin string conjecture" is applicable [9,10], which implies that at vanishing β, the world-sheet model can be treated as consistent fundamental string theory.

The conifold singularity $\det w^{ij} = 0$, where $w^{ij} = n^i \rho^j$ is the set of U(1) gauge-invariant "mesonic" variables, can be alternatively smoothed at $\beta = 0$ by the deformation of complex structure. Following Refs. [11,12], this deformation can be described as

$$\det w^{ij} = b, \tag{2.11}$$

where b is a complex parameter of deformed conifold. Promoted to a 4D field, parameter b was interpreted as a scalar component of a massless baryonic hypermultiplet of 4D $\mathcal{N}=2$ QCD in Ref. [10]. Its quantum numbers with respect to the global symmetry group (2.6) for $N=2$ are determined by Eq. (2.11), as follows:

$$(\mathbf{1},\mathbf{1},2);$$

i.e., it is a singlet with respect to both $SU(2)$ groups with $B(b)=2$ baryonic charge [10]. The massless field b can form a condensate. Thus, we have a new Higgs branch in 4D $\mathcal{N}=2$ SQCD, which develops only at the critical value of the 4D coupling constant $\tau_{\mathrm{SW}} = 1$ associated with $\beta = 0$ [42]; see Fig. 1 and discussion in the Introduction.

2. $\mathcal{N}=2$ *Liouville theory*

It was recently shown [21] that the Coulomb branch of 2D $\mathbb{WCP}(N,N)$ world-sheet model which opens up at the strong coupling point $\beta = 0$ can be described by $\mathcal{N}=2$ Liouville theory (see Ref. [43] for a review of Liouville theory in general). Its bosonic action reads

$$S_{\mathrm{eff}} = \frac{1}{4\pi}\int d^2x\sqrt{h}\left(\frac{1}{2}h^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) - \frac{Q}{2}\phi R^{(2)}\right), \tag{2.12}$$

for the real Liouville scalar field ϕ, supplemented by the real compact scalar $Y \sim Y + 2\pi$. Here, $R^{(2)}$ is the Ricci scalar for the world-sheet metric $h_{\alpha\beta}$, and $h = \det(h_{\alpha\beta})$. The linear dilaton term

$$\Phi = -\frac{Q}{2}\phi \tag{2.13}$$

contains the background charge $Q = \sqrt{2(N-1)}$, so for the conifold $N=2$ case, $Q = \sqrt{2}$. The $\mathcal{N}=2$ Liouville central charge is $c_L = 3(1+Q^2)$, so that in $N=2$ case, $c_L|_{N=2} = 9$ as required for criticality.

The $\mathcal{N}=2$ Liouville interaction superpotential (a marginal deformation of the $\mathcal{N}=2$ Liouville theory (2.12); see Ref. [43]) comes from the 2D FI term (2.9) in the $\mathbb{WCP}(2,2)$ model and has the form

$$L_{\mathrm{int}} = b\int d^2\tilde{\theta}\, e^{-\frac{\phi+iY}{Q}}, \tag{2.14}$$

where the scalars ϕ and Y are promoted to (twisted) chiral superfields, and b is the conifold complex structure deformation parameter (2.11). To get Eq. (2.14), we use the relation

$$\sigma = \gamma e^{-\frac{\phi+iY}{Q}}, \tag{2.15}$$

between the complex scalar σ from the U(1) gauge multiplet and the fields ϕ and Y in the Liouville theory, where $\gamma = -\sqrt{2}b/\beta$. Note that constant γ is singular at $\beta \to 0$ in order to keep the Liouville superpotential (2.14) finite; see Ref. [21] for details.

The string coupling constant $g_s = e^{\Phi}$, determined by linear dilaton (2.13), grows at large negative ϕ. On the other hand, at nonvanishing b, the Liouville wall prevents the field ϕ from penetrating into the region of large negative values. For $Q = \sqrt{2}$, the string coupling $g_s \sim 1/|b|$, and below we keep b large to ensure that the string perturbation theory is reliable; see Refs. [18,20]. In terms of 4D SQCD, at large b we are on the nonperturbative Higgs branch far away from the origin; see the SQCD phase diagram in Fig. 1.

C. SQCD hadrons from Liouville theory

The primary operators of the $\mathcal{N}=2$ Liouville theory for the $N=2$ ($Q=\sqrt{2}$) case describe physical string states, interpreted as hadrons in 4D SQCD; see Ref. [13] for details. The operators relevant for our 4D applications have the form [18,44,45]

$$T_{j,m} \simeq e^{iQmY} e^{Qj\phi}, \tag{2.16}$$

with the conformal dimensions

$$\Delta_{j,m} = \frac{Q^2}{2}\{m^2 - j(j+1)\} = \frac{1}{k}\{m^2 - j(j+1)\}.$$

The spectrum of allowed values of j and m in Eq. (2.16) is exactly determined, using the mirror description [29] of the theory as a $\mathrm{SL}(2,R)/U(1)$ coset with the level

$$k = \frac{2}{Q^2} \tag{2.17}$$

of the $\mathcal{N}=2$ supersymmetric version of the Kač-Moody algebra in Refs. [27,46–49]; see Ref. [50] for a review.

We are interested in the discrete spectrum

$$j = -\frac{1}{2}, -1; \qquad m = \pm\{j, j-1, j-2, \ldots\},$$

where the restriction to two values of j comes from unitarity bound for $k|_{Q=\sqrt{2}} = 1$ [13].

Dressing the vertex (2.16) with the 4D plane wave, one obtains the operators corresponding to 4D states. For the scalar and spin-2 states, we have

$$\mathcal{T}_{j,m} = e^{ip_\mu x^\mu} T_{j,m} \quad \text{and} \quad V^{\mu\nu}_{j,m} = \psi^\mu_L \psi^\nu_R e^{ip_\mu x^\mu} T_{j;m}, \tag{2.18}$$

where $\psi^\mu_{L,R}$ are world-sheet fermionic superpartners of x^μ. Imposing the mass-shell condition

$$\mathsf{M}^2_T = -p_\mu p^\mu = 2\Delta_{j,m} - 1, \qquad \mathsf{M}^2_V = 2\Delta_{j,m}, \tag{2.19}$$

and the Gliozzi–Scherk–Olive (GSO) projection [18], we find the masses [in dimensionless units, imposing normalization $4\pi\tau = 1$ or

$$\alpha' = 2 \tag{2.20}$$

for the string tension (2.7)] of the 4D scalar and spin-2 states [13,20], as follows:

$$\mathsf{M}^2_T = 2m^2 - \frac{1}{2}, \qquad j = -\frac{1}{2}, \qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$$
$$\mathsf{M}^2_V = 2m^2, \qquad j = -1, \qquad |m| = 1, 2, \ldots. \tag{2.21}$$

This stringy spectrum of 4D SQCD is shown in Fig. 2, taken from Ref. [13].

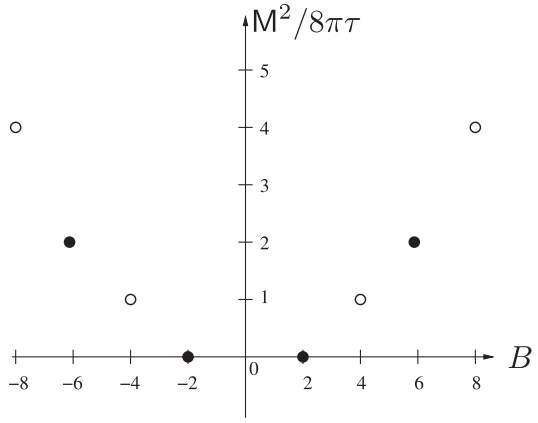


FIG. 2. Spectrum of spin-0 and spin-2 states (2.21) as a function of the baryonic charge (2.22). Closed and open circles denote spin-0 and spin-2 states, respectively.

In particular, the scalar vertex in Eq. (2.18) with $j=-1/2$, $m=\pm 1/2$ gives a massless state in strongly coupled 4D SQCD, corresponding to the complex structure modulus b for the string compactification on the conifold [cf. with Eq. (2.14), two values $m=\pm 1/2$ correspond to 2 real degrees of freedom of the complex scalar]. The associated string state has a logarithmically normalizable wave function in terms of the conifold radial coordinate [10,51]. On the Liouville side, this corresponds to the borderline normalization of the massless state (2.16) with $j=-\frac{1}{2}$, $m=\pm\frac{1}{2}$; see Ref. [13] for details.

For these states, m is related to the baryonic charge as [13,20]

$$B = 4m, \tag{2.22}$$

and all closed string states are baryons. All states in Eq. (2.18) are in fact lowest states of the corresponding linear Regge trajectories [20].

III. MASS DEFORMATION

In this section, we introduce the mass deformation (2.3) and study its response in the Liouville world-sheet theory.

A. Effective action and initial conditions

The derivation below follows closely Ref. [22], but generalizes the previous result obtained for $N=4$ to an arbitrary even $N=2K$. We start with the $\mathbb{WCP}(N,N)$ world sheet action (2.8) with nonvanishing mass parameters (2.3). Following Ref. [21], we integrate out the "matter" fields n_i, ρ_i, which become massive at the Coulomb branch of the $\mathbb{WCP}(N,N)$ model where field σ can take arbitrary values at $\beta=0$. To integrate out n^i and ρ^j, we first use the large N approximation developed by Witten in Ref. [52] and take

$$N = N_0 \cdot K$$

with fixed K, while $N_0 \to \infty$ plays the role of large-N parameter, to be set at the end to the critical conifold value $N_0 = 2$. The resulting kinetic part of the one-loop effective action for σ field is [22]

$$S^{\sigma}_{\text{eff}} = \frac{1}{8\pi}\int d^2x \sum_{A=1}^{2N} \frac{|\partial_\alpha \sigma|^2}{\left|\sigma + \frac{m_A}{\sqrt{2}}\right|^2}. \tag{3.1}$$

In the vicinity of any vacuum (2.10) [which are all equivalent due to a very special choice of $\mathbb{Z}_K$-symmetric mass deformation (2.3)], let us perform a change of variables, as follows:

$$\sigma + \frac{m_{i_0}}{\sqrt{2}} = \gamma_{i_0} e^{-\frac{\phi+iY}{Q}}; \tag{3.2}$$

c.f., Eq. (2.15), where $\gamma_{i_0} = -\sqrt{2}b_{i_0}/\beta$, while b_{i_0} is the complex structure deformation parameter of each conifold we start with at $m \to \infty$. Evaluating the sum over flavors, we obtain (see Appendix A)

$$S^{\sigma}_{\text{eff}} = \frac{2N_0}{4\pi Q^2}\int d^2x g_{cl}(\phi)\left[\frac{1}{2}(\partial_\alpha \phi)^2 + \frac{1}{2}(\partial_\alpha Y)^2\right]. \tag{3.3}$$

The "classical" warp factor of target space metric is given in the large ϕ limit by

$$g_{\text{cl}}(\phi) \approx 1 + \exp\left(-\frac{2\phi}{Q}\right)\sum_{i\neq i_0}\frac{|b_{i_0}|^2}{|M_{ii_0}|^2}, \tag{3.4}$$

where the radius of the compact dimension is

$$Q \overset{N_0\to\infty}{\approx} \sqrt{2N_0}. \tag{3.5}$$

In Eq. (3.7), we use the rescaled mass differences $M_{ii_0} = -\frac{\beta}{2}(m_i - m_{i0})$, kept finite at $\beta \to 0$, together with

$$b_{i_0} \sim \gamma_{i_0}\beta \underset{\gamma_{i_0}\to\infty,\beta\to 0}{=} \text{fixed}, \qquad M \sim m\beta \underset{m\to\infty,\beta\to 0}{=} \text{fixed}.$$

It has been shown in Ref. [21] that the equivalence of the Coulomb branch of the $\mathbb{WCP}(N,N)$ model with $\mathcal{N}=2$ Liouville theory is actually exact at vanishing bare masses, if the large-N result for the radius (3.5) is changed for

$$Q = \sqrt{2(N_0 - 1)}. \tag{3.6}$$

We follow this reasoning for the mass-deformed theory as well (see Ref. [22]), and extend the large-N_0 derivation to $N_0 = 2$, then $Q = \sqrt{2}$ from Eq. (3.6) brings us back to the critical non-Abelian string on the conifold.

Note that the resulting effective action (3.3) actually does not depend on a particular choice of vacuum i_0 in Eq. (3.2), due to $\mathbb{Z}_K$-symmetry of the proposed special deformation, and metric (3.4) is simplified to

$$g_{\text{cl}}(\phi) \approx 1 + \exp\left(-\frac{2\phi}{Q}\right)\frac{(K^2-1)}{12}\frac{|b|^2}{|M|^2}. \tag{3.7}$$

To ensure Eq. (3.7), we require the complex structure deformation parameters $b_{i_0} \sim \gamma_{i_0}\beta$ on each conifold, associated with a particular $U(2)$ factor in Eq. (2.5), to be the same:

$$b_i = b, \quad i = 1,\ldots,K = N/2. \tag{3.8}$$

Then the sum in Eq. (3.4) for the special choice Eqs. (2.4) and (3.8) is easily calculated (see Appendix A) and gives Eq. (3.7).

Hence, our interpolation procedure is universal: it relates $U(N=2K)$ SQCD with a $U(2)$ theory, whose spectrum at strong coupling was found due to relation with $\mathcal{N}=2$ Liouville, independently on a choice from the K factors we start with in Eq. (2.5). The mass deformation explicitly breaks the 2D conformal invariance, so the effective action (3.3) cannot be used directly for string quantization. Instead, the "classical" warp factor (3.7) can be used only as an initial condition for solving the effective supergravity equations [22].

B. Solutions of gravity equations and 2D black hole

The bosonic part of the effective action of the type II supergravity in string frame for the metric and the dilaton is given by

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}e^{-2\Phi}\{\mathcal{R} + 4G^{MN}\partial_M\Phi\partial_N\Phi + \cdots\}, \tag{3.9}$$

where G_{MN} is the D-dimensional metric ($M,N = 1,\ldots,D$) with scalar curvature $\mathcal{R}$, and the gravitational coupling can be written as $2\kappa^2 = \frac{\pi}{2}g_0^2(\alpha')^{\frac{D}{2}-1}$. As usual in string frame, the dependence on string coupling $g_s = e^{\Phi}$ comes from the dilaton exponent, therefore one can absorb the dimensionless g_0^2 into the dilaton constant, making in Eq. (3.9) a substitution:

$$2\kappa^2 \to 2\kappa_0^2 \equiv \frac{\pi}{2}(\alpha')^{\frac{D}{2}-1}, \qquad e^{-2\Phi} \to e^{-2(\Phi+\delta\Phi_0)},$$
$$\delta\Phi_0 = \log g_0, \tag{3.10}$$

where now $2\kappa_0^2$ is solely determined by α' or string tension (2.7).[4]

Einstein's equations, which ensure 2D conformal invariance by vanishing of the world-sheet β-functions, corresponding to the action (3.9), have the form

$$\mathcal{R}_{MN} + 2D_M D_N \Phi = 0,$$

and are supplemented by the equation for the dilaton,

$$4G^{MN}\partial_M\Phi\partial_N\Phi - 2G^{MN}D_M D_N\Phi + \frac{D-10}{2} = 0. \quad (3.11)$$

Assuming that our target space is a direct product of the flat 4D Minkowski space and internal 2D space with nontrivial metric, associated with mass-deformed $\mathcal{N}=2$ Liouville theory,

$$ds^2_{\rm int} = g(\phi)(d\phi^2 + dY^2), \quad (3.12)$$

so $D=6$ in Eq. (3.9), one can find [22] the metric warp factor and the dilaton,

$$g(\phi) = \frac{1}{1-e^{-Q(\phi-\phi_0)}},$$
$$\Phi(\phi) = -\frac{Q}{2}\phi - \frac{1}{2}\log\left[1-e^{-Q(\phi-\phi_0)}\right], \quad (3.13)$$

where one has to take critical $Q=\sqrt{2}$ to match the initial condition (3.7), which is satisfied with the choice of the parameter

$$\phi_0 = \frac{1}{Q}\log\frac{(K^2-1)|b|^2}{12|M|^2}, \quad (3.14)$$

which becomes dependent on the rank of gauge group $N=2K$.

At $M\to\infty$, solution (3.13) gives a flat metric with linear dilaton (2.13).[5] Note also that the first nontrivial term in the expansion of the warp factor (3.13) at large ϕ gives the deformation

$$(\partial_z\phi - i\partial_z Y)(\partial_{\bar z}\phi + i\partial_{\bar z}Y)e^{-Q\phi}$$

of free action, by marginal operator with $j=-1$, $m=0$, which is the bosonic part of the so-called nonchiral marginal deformation of $\mathcal{N}=2$ Liouville theory; see Ref. [43] for a review.

[4]We stress again, that in our theory of non-Abelian string both α' and κ_0^2 are of SQCD scale rather then string/Planck scale of fundamental string theory, and since our string theory contains no gravity in 4D, we cannot fix numeric constant g_0^2 by comparison with gravitational constant coming from Newton's law.

[5]Equation (3.11) is however satisfied only if $Q=\sqrt{2}$ is matched with $D=6$.

Hence, the bosonic part of the world-sheet action of the mass-deformed Liouville theory is

$$S_{\rm ws} = \frac{1}{4\pi}\int d^2x\sqrt{h}\Big\{\frac{1}{2}g(\phi)[(\partial_\alpha\phi)^2 + (\partial_\alpha Y)^2] + \Phi(\phi)R^{(2)} + L_{\rm int}\Big\}, \quad (3.15)$$

where the metric warp factor $g(\phi)$ and the dilaton $\Phi(\phi)$ are given by Eq. (3.13), while the Liouville superpotential still takes the form of Eq. (2.14), since it is not modified by the mass deformation [22]. The action (3.15) defines a continuous family of world-sheet conformal field theories, parametrized by parameter (3.14), and they all have the central charge $c_L = 3(1+Q^2) = 9$ (for our choice $Q=\sqrt{2}$).

It was shown in Ref. [23] that the mass-deformed string background (3.13) is actually equivalent to the "trumpet" geometry, T-dual to the 2D black hole's semi-infinite cigar (with radial coordinate $0\le\rho<\infty$ and angular coordinate $\theta\sim\theta+2\pi$), as follows:

$$S_{\rm BH} = \frac{k}{4\pi}\int d^2x\sqrt{h}\{(\partial_\alpha\rho)^2 + \tanh^2\rho(\partial_\alpha\theta)^2\} + \frac{1}{4\pi}\int d^2x\sqrt{h}\Phi(\rho)R^{(2)} \quad (3.16)$$

with the dilaton

$$\Phi(\rho) = \Phi_0 - \log\ \cosh\rho. \quad (3.17)$$

The supersymmetric version of the action (3.16) classically corresponds to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset theory [17,18,27,28] with the level (2.17) of supersymmetric Kač-Moody algebra.

Note that the target space geometry (3.16) in the supersymmetric case (in contrast to the bosonic 2D black hole) does not receive the stringy $\alpha'\sim 1/k$ corrections (see Ref. [53]); therefore, the effective action (3.9) we used in this section is a valid approximation.

C. Mass-deformed Liouville theory

As is commonly believed, $\mathcal{N}=2$ Liouville theory has a mirror description [29] in terms of the $\mathcal{N}=2$ supersymmetric version of a 2D black hole (3.16) with the Kač-Moody level $k=\frac{2}{Q^2}$, which determines the radius of the cigar $R=\sqrt{2k}$ at $\rho\to\infty$; see Eq. (3.16). The constant Φ_0 in the expression (3.17) (the value of the dilaton at the horizon $\rho=0$) is given by

$$\Phi_0 = \Phi_0^{(b)} + \log g_0, \qquad \Phi_0^{(b)} = -\frac{1}{k}\log|b| \quad (3.18)$$

in terms of the coefficient b (modulus of the conifold complex structure) in front of the superpotential by the "standard Liouville" argument with the shift

$$\phi \to \phi - Q \log|b| \equiv \phi - \phi_{\rm wall}, \tag{3.19}$$

where $\phi_{\rm wall}$ determines the position of the "Liouville wall." In Eq. (3.18), we also take into account the shift (3.10) of the dilaton related to the gravitational coupling.

As was shown in Ref. [26], the constant Φ_0 (3.17) determines the Arnowitt-Deser-Misner (ADM) mass of the black hole, as follows:

$$M_{\rm BH} = \frac{Q}{2} e^{-2\Phi_0} = \frac{1}{\sqrt{2k}} e^{-2\Phi_0}. \tag{3.20}$$

One can consider the compact dimension around the cigar in Eq. (3.16) as a thermal circle, where $T = \frac{1}{2\pi R} = \frac{Q}{4\pi}$ is the Hawking temperature. With this interpretation, Eq. (3.20) also can be understood as a definition of the black hole entropy[6]

$$s_{\rm BH} = \frac{M_{\rm BH}}{T} = 2\pi e^{-2\Phi_0}. \tag{3.21}$$

For the supersymmetric black hole associated with $\mathcal{N}=2$ Liouville theory, we have therefore

$$M^{(b)}_{\rm BH} = \frac{Q}{4\pi} s^{(b)}_{\rm BH} = \frac{Q}{2}\frac{1}{g_0^2}|b|^{2/k}, \tag{3.22}$$

due to Eq. (3.18).

On the other hand, as was discussed in the previous section, the mass-deformed world-sheet background (3.13) is T-dual to *the same* theory (3.16) with the same $k = 2/Q^2$, where now, again taking into account Eq. (3.10), the dilaton constant

$$\Phi_0 = \Phi_0^{(M)} + \log g_0,$$
$$\Phi_0^{(M)} = -\frac{Q}{2}\phi_0 = -\frac{1}{2}\log\frac{(K^2-1)|b|^2}{12|M|^2} \tag{3.23}$$

is determined by location of the singularity (3.14) in the solution (3.13); i.e., is actually determined by the rank of gauge group and mass deformation scale M. The associated black hole mass and entropy is therefore[7]

$$M^{(M)}_{\rm BH} = \frac{Q}{4\pi} s^{(M)}_{\rm BH} = \frac{Q}{2}\frac{(K^2-1)|b|^2}{12 g_0^2 |M|^2}. \tag{3.24}$$

It was conjectured in Ref. [23] that when both marginal deformations are switched on [see Eq. (3.15)], one still gets the same dual black hole theory (3.16) with only the modified "total" value Φ_0 of the dilaton constant. We proposed that the total mass/entropy of the black hole upon switching on both deformations is just the sum[8] of the corresponding masses/entropies (3.22) and (3.24), i.e.,

$$\exp\left(-2\Phi_0^{\rm total}\right) = \frac{1}{g_0^2}\left\{\exp\left(-2\Phi_0^{(b)}\right) + \exp\left(-2\Phi_0^{(M)}\right)\right\}, \tag{3.25}$$

which gives for the total mass/entropy of the black hole in the theory with both deformations,

$$\frac{M^{\rm total}_{\rm BH}}{Q/2} = \frac{1}{2\pi} s^{\rm total}_{\rm BH} = \frac{1}{g_0^2}\left(|b|^{2/k} + \frac{(K^2-1)|b|^2}{12|M|^2}\right) \approx \overset{K\gg 1}{\approx} \frac{1}{g_0^2}\left(|b|^{2/k} + \frac{N^2|b|^2}{48|M|^2}\right). \tag{3.26}$$

Let us stress here again, that in the world-sheet theory on the cigar (3.16), parameter k (fixed to be $k=1$ for $Q=\sqrt{2}$ for critical string), which determines the spectrum of string states via 2D conformal dimensions (2.19), does not depend on the rank $N=2K$ of the gauge group in $U(N)$ SQCD. Dependence on the gauge group comes from the dilaton constant Φ_0, related to the black hole mass/entropy (3.26), or to the string coupling. It means that the hadron spectrum of 4D $\mathcal{N}=2$ SQCD with $U(N)$ gauge group, which emerges as we reduce the mass parameter M, does not depend on N and is still given by Eq. (2.21), but the multiplicity of hadron states is expected to grow with N. In Sec. IV B, we use the field theory arguments to show that this surprising behavior is quite reasonable in QCD with $\mathcal{N}=2$ supersymmetry.

D. $\mathcal{N}=2$ SQCD: stringy versus Higgs phase

As we discussed in the Introduction, the phase diagram of $\mathcal{N}=2$ SQCD with even N and $N_f = 2N$ has two phases: the Higgs phase at weak coupling and the stringy phase on the nonperturbative Higgs branch, where the massless stringy baryon b develops VEV. This branch is developed at SQCD "super strong" coupling point $\tau_{\rm SW}=1$, which corresponds to $\beta=0$ in the world-sheet model [34]; see Fig. 1. This is also a point of transition from

[6]Note that this 2D temperature appears only due to the existence of compact direction in the Euclidean internal part (3.12) of our string background, and has nothing in common with real temperature of 4D theory. We will also see in Sec. IV that Eq. (3.21) is only a part of the whole black hole entropy, surviving at $k\to\infty$.

[7]Expressions (3.23) and (3.24) are just generalizations to arbitrary K of the $K=2$ results, obtained in Ref. [23].

[8]As was already pointed out in Ref. [23], one can only conjecture this formula in nonlinear 2D theory (3.15), and any other function with each term in the rhs of Eq. (3.25) dominating at large negative values of Eq. (3.18) or Eq. (3.23), say tropical sum $e^{-2\Phi_0} = e^{-2\Phi_0^{(b)}} \oplus e^{-2\Phi_0^{(M)}} = \max(e^{-2\Phi_0^{(b)}}, e^{-2\Phi_0^{(M)}})$, is equally possible.

the resolved conifold with $\beta \neq 0$, $\langle b \rangle = 0$ to the deformed conifold with $\beta = 0$, $\langle b \rangle \neq 0$.

Now let us show that these two phases of 4D SQCD are not analytically connected. The point is that although both phases do have massless states, these states have different quantum numbers with respect to the global symmetry group (2.6).

The massless states in the Higgs phase are "extra" quark flavors q^{kA}, $\tilde{q}_{Ak}$ with $A = (N+1), \ldots 2N = N_f$; see Sec. II A and Ref. [7] for more details. Their masses are given by $|m_k - m_A| = |m_k - \tilde{m}_{A-N}|$. In the limit of interest, when all m_k, m_A vanish, the hypermultiplets $q^{kA}, \tilde{q}_{Ak}$ become exactly massless. Their masses are BPS-protected, and they remain exactly massless even if we increase the coupling g^2, going toward the strongly coupled region. In particular, these are the BPS states that belong to short $\mathcal{N} = 2$ multiplets (hypermultiplets), and their masses are not affected by the Higgs mechanism, which gives non-BPS masses of order $\sim g\sqrt{\xi}$ to gauge fields and the first N quark flavors upon their condensation; see Eq. (2.2) and Ref. [7] for a review. These quarks are screened in the Higgs phase and belong to the bifundamental representations

$$(\bar{\mathbf{N}}, \mathbf{N}), \qquad (\mathbf{N}, \bar{\mathbf{N}}) \tag{3.27}$$

of the two $SU(N)$ groups in Eq. (2.6), since their color index k gets "locked" to a flavor index from the set $B = 1, \ldots, N$.

From the point of view of the world-sheet theory (see Sec. II. 2. 1), the 4D bifundamental quarks are imprinted (due to the 2D–4D correspondence) as the size moduli ρ^j, which have the same masses $|m_k - \tilde{m}_{A-N}|$ for $j = A - N$ and also become exactly massless in the limit $m_A \to 0$. In terms of the 2D coupling constant, this phase corresponds to $\beta \neq 0$.

Instead, in the stringy phase, the bifundamental quarks discussed above become heavy and decouple. This phase corresponds to the point $\beta = 0$ in the 2D theory on the string; at this point, a Coulomb branch in the world-sheet theory is developed, where the 2D field σ acquires an arbitrarily large VEV, making the fields ρ^j massive. This signals that the corresponding 4D states (namely, extra quarks with flavor index $A = (N+1), \ldots, 2N$) also become heavy.

The only massless state in the stringy phase is the lightest baryon, which develops a VEV $\langle b \rangle \neq 0$ on the nonperturbative Higgs branch. This state corresponds to the complex structure modulus b of the conifold [see Eq. (2.11)] and therefore, as any other Calabi-Yau modulus, is associated with a massless state in 4D; see Ref. [54]. This argument guarantees that the mass of the b-baryon is exactly zero throughout the entire stringy phase (the order parameter parametrizing this phase is the VEV of b). In terms of the world-sheet Liouville theory, this baryon is a string state with $j = -1/2$, $m = \pm 1/2$; see Sec. II C, and it belongs to the $\frac{1}{2}N(N-1)$-dimensional antisymmetric representation

(3.28)

of $SU(N)$ [equally charged with respect to both $SU(N)$ factors of the global group (2.6)], where we denote $SU(N)$ representations by their Young diagrams; see Sec. IV B 2. Clearly, Eqs. (3.27) and (3.28) are different, so the two phases cannot be analytically connected. They are separated by a phase transition, which in the world-sheet theory on the non-Abelian string is seen as a conifold transition.

As already mentioned in the Introduction, we also have a dramatic qualitative distinction between the massive spectrum in the two phases. In the Higgs phase at weak coupling, the physical spectrum is formed by a finite number of screened quarks and Higgsed gluons together with their superpartners. In contrast, the stringy phase has infinite towers of 4D hadron states (2.21) typical of the spectrum of string theory. The higher energy states of this spectrum cannot be interpreted as composites of a finite number of the lightest states. This qualitatively resembles the hadron spectrum in real-world QCD.

IV. DENSITY OF STATES AND BLACK HOLE ENTROPY

As was discussed above, the string spectrum does not change as we reduce the mass deformation parameter M, but we expect the number of states (i.e., the multiplicity) on each energy level to increase. New states, which were separated by infinite barriers in the limit $M \to \infty$, emerge as we reduce M, since the system goes through a certain number of crossovers, similar to walls of marginal stability for BPS states, where multiplicity of states jumps.

In this section, we are going to confirm these expectations, first calculating the black hole entropy from the string theory side, and then comparing the result to the field theory calculation done in the large N limit. It extends the analysis of Ref. [23] from $N = 4$ colors to arbitrary even N, which allows us to gain some additional insight.

A. Black hole entropy and spectral density

To calculate the number of states, one can interpret a compact dimension around the cigar as a thermal cycle with temperature $T = (2\pi R)^{-1}$, given by the asymptotic cigar radius $R = \sqrt{2k}$, and calculate the entropy of the 2D black hole. We stress again, that our real time is one of the 4D Minkowski coordinates and has nothing to do with the compact dimension around the cigar, but we use this trick

with the Euclidean 2D black hole to estimate the number of states in 4D theory.

However, it turns out that our value $k=1$ for 2D black hole theory exactly corresponds to the Hagedorn temperature [55]. At Hagedorn temperature T_H and beyond, the high-energy levels are no longer suppressed by the Boltzmann factor $\exp(-E/T)$ due to the exponential growth of the density of string states

$$\omega(E) \underset{E\to\infty}{\sim} E^{\alpha} e^{\frac{E}{T_H}} \tag{4.1}$$

(we come back to the power factor E^{α} and discuss it in detail below), and the partition function

$$Z = \int_0^{\infty} dE\, \omega(E) e^{-\frac{E}{T}} \tag{4.2}$$

becomes divergent[9] at $T \geq T_H$.

For black holes, the Hagedorn behavior is related to the black hole/excited strings transition [57,58], and in particular the expression for entropy (3.21) works only at low temperatures, where one has a well-defined black hole geometry with small α' corrections ($\alpha' \sim 1/k$ for our theory, and they are actually absent in the supersymmetric case). As we increase the temperature, the theory enters a strong coupling regime, and at a certain critical value, the black hole turns into excited string. This type of behavior was found also for our 2D black hole with asymptotically linear dilaton (3.16) in Ref. [59]. Moreover, it is argued in Ref. [59] that the Hagedorn temperature is given by

$$T_H = T|_{k=1} = \left.\frac{1}{2\pi\sqrt{2k}}\right|_{k=1} = \frac{1}{2\pi\sqrt{2}}; \tag{4.3}$$

i.e., exactly corresponds to the value $k=1$ we are interested in for the critical non-Abelian string in 4D SQCD.

In Euclidean formulation, the Hagedorn behavior is described by a thermal scalar—a winding string mode around the thermal circle [60]. As the temperature approaches T_H, this mode becomes massless (and presumably tachyonic at $T > T_H$) leading to an instability; see also Refs. [59,61,62] for the nonsupersymmetric version.

The thermal scalar for the 2D black hole (3.16) is just our b-baryon, associated with the conifold complex structure b, being a massless state in 4D SQCD. It corresponds to the minimal winding state [23], i.e., its quantum numbers are

$$m = \pm\frac{k}{2}, \qquad j = -\frac{k}{2}.$$

Its contribution to the entropy of the 2D black hole (normalized to the unit 4D volume) was computed[10] in Ref. [23], as follows:

$$s = \frac{e^{-2\Phi_0^{\text{total}}}}{k-1} = \frac{\sqrt{2k}}{k-1} M_{\text{BH}}^{\text{total}}, \tag{4.4}$$

following the calculation from Ref. [63] in the nonsupersymmetric case.

In contrast to Eq. (3.21), becoming now (unlike the region $k\to\infty$) the subleading contribution into entropy, expression (4.4) takes into account the contribution of thermal scalar (leading at $T\to T_H$; see Ref. [60]) and diverges[11] at critical $k\to 1$. However, it is also proportional to the total ADM mass (3.26): at large $M\to\infty$ it does not feel the mass deformation parameter, $s \underset{M\to\infty}{\sim} |b|^2(k-1)^{-1}$, but small M enhances the divergence in Eq. (4.4), as follows:

$$s \approx \frac{1}{k-1}\frac{(K^2-1)|b|^2}{12 g_0^2 |M|^2}, \tag{4.5}$$

and, moreover, makes it dependent on the rank of gauge group.

Formulas (4.4) and (4.5) give not only the total entropy or the total number of states, they also allow us to understand that multiplicity at each energy level (or the number of states with fixed baryonic charge B) increases dramatically as we reduce M. To estimate the number of states at each high-energy level (filled now since we meet the Hagedorn behavior) or hadron mass, given by Eq. (2.21) [and depending on the baryon number B via Eq. (2.22)], let us first compute the integral (4.2) for the spectral density (4.1). We arrive at

$$Z = [2\pi(R-R_H)]^{-(\alpha+1)}\Gamma(1+\alpha),$$

so calculating the entropy,

$$s = \left(1 - R\frac{\partial}{\partial R}\right)\log Z \overset{k\to 1}{\approx} \frac{2(\alpha+1)}{k-1}, \tag{4.6}$$

[9]Similarly to Ref. [56], the string theory Green functions can acquire singularities in a finite spacetime domain (of the order of the string length).

[10]It comes from the effective action

$$S_{\text{baryon}} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-G} e^{-2\Phi}\{-G^{MN}\partial_M \bar{\mathcal{T}}_b \partial_N \mathcal{T}_b + |\mathcal{T}_b|^2\}$$

for the thermal scalar in our background, to be added to Eq. (3.9).

[11]This divergence comes from the IR region $\rho\to\infty$, due to the non-normalizable behavior of the thermal scalar near the Hagedorn point $k\to 1$.

and comparing it with Eqs. (4.4) and (4.5) allows us to fix the parameter α in Eq. (4.1). It determines the growth of the multiplicity of states in the subleading order at large E. Namely, comparing Eqs. (4.5) and (4.6) gives

$$\alpha + 1 = \frac{1}{24 g_0^2} \frac{(K^2-1)|b|^2}{|M|^2}.$$

Plugging this back into Eq. (4.1), we finally arrive at the spectral density,

$$\omega(E) \approx \exp\left\{ \frac{E}{T_H} + \frac{(K^2-1)|b|^2}{24 g_0^2 |M|^2} \log E + \cdots \right\}$$
$$\approx \underset{K\gg 1}{\approx} \exp\left(\frac{E}{T_H} + \frac{N^2}{96} \frac{|b|^2}{\Lambda^2} \log E + \cdots \right), \qquad (4.7)$$

where we have introduced $\Lambda \equiv g_0|M|$. Here and below we use the large N limit; explicitly for high energies, we assume

$$E \gg N \gg 1 \qquad (4.8)$$

in the units where $\alpha' = 2$; see Eq. (2.20) in Sec. II C.

To avoid the singularity at $M \to 0$ in the $\log E$ term in Eq. (4.7), not expected in 4D SQCD, we require in the limit $M \to 0$ that

$$g_0 \to \infty, \qquad M \to 0, \qquad \Lambda = g_0|M| = \text{fixed}. \qquad (4.9)$$

At high energy, the main contribution to the number of states is due to the linear in E term in the exponent of Eq. (4.7), with the coefficient determined by Hagedorn temperature, which is just a constant (4.3). However the subleading logarithmic term $\sim \log E$ in Eq. (4.7) grows as N^2 with the rank of the SQCD gauge group. The coefficient in front of $\log E$ can be also written as

$$\frac{e^{-2\Phi_0^{\text{total}}}}{2} = \frac{1}{2g_s^2}, \qquad g_s = \frac{4\sqrt{3}}{N|b|}\Lambda, \qquad (4.10)$$

showing that this is a nonperturbative contribution in string coupling constant. We also see that actually Λ, introduced in Eq. (4.9), determines the string coupling for SQCD with $U(N)$ gauge group.[12] If $\Lambda \sim 1$ [in our units $\alpha' = 2$; see Eq. (2.7)] string coupling g_s is small at large b (this is enhanced in the large N limit) so the classical approximation to string theory we used in Sec. III B is indeed reliable.

Below, we are going to discuss in detail the field theory side and find a quantitative agreement of the N dependence of the $\log E$ term in Eq. (4.7) with the result from the SQCD side in the large N limit.

B. Field theory computation

The string argument above says that the spectrum of the 4D $\mathcal{N} = 2$ SQCD [by spectrum we mean solely the 4D energy levels/masses of states; see Eq. (2.21)] does *not* change as we reduce M interpolating from U(2) to $U(2K)$ theories (with $N_f = 2N = 4K$ flavors). The *only* effect is growth of the multiplicity of states (4.7).

This result from the string theory side seems to run into a contradiction with intuitive expectations on the field theory side. Naively, as we reduce m, one would expect emergence of new hadronic states in SQCD with masses of order m. However, it was argued in Ref. [23] that the behavior predicted by string theory is actually natural for $\mathcal{N} = 2$ SQCD. Below, we generalize these arguments to arbitrary even $N = 2K$ and estimate the density of hadron states from the SQCD side at large N.

Certainly, we cannot strictly calculate the hadron masses directly from SQCD, since the theory is in extremely strong-coupling regime $g^2 \to \infty$ (associated with $\beta = 0$). However, in $\mathcal{N} = 2$ SQCD, one can make some plausible assumptions about the dependence of hadron masses on bare quark masses and quantum numbers of allowed states.

The hadron mass naturally receives two contributions: one comes from the constituent monopole masses (playing the role of the constituent quarks; see Ref. [7]), while the other is from the energy of confining string. The first contribution is determined by BPS central charges $Z_{\text{BPS}} = Z_{\text{BPS}}(m_i, \tilde{m}_j)$, given by exact formula [30,34], while the second one is determined solely by string tension (2.7). We have proposed therefore in Ref. [23] the following formula:

$$m_H^2 = m^2_{\text{non-BPS}}(\xi) + |Z_{\text{BPS}}(m_i, \tilde{m}_j)|^2 \qquad (4.11)$$

for hadron masses, where the BPS part,

$$Z_{\text{BPS}} = i\vec{m}_n \vec{q}_n - i\vec{m}_\rho \vec{q}_\rho, \qquad (4.12)$$

can be found using the 2D–4D correspondence, briefly reviewed in Sec. II B. The point is that the BPS spectrum of 4D SQCD in the quark vacuum (2.2) at $\xi = 0$ actually coincides with the BPS spectrum of the 2D world-sheet $\mathbb{WCP}(N,N)$ theory of non-Abelian vortex. Hence, in Eq. (4.12), $\vec{m}_n = \{m_1, \ldots, m_N\}$ and $\vec{m}_\rho = \{\tilde{m}_1, \ldots, \tilde{m}_N\}$ are the masses of n and ρ fields (they coincide with the masses of $2N$ flavors of quarks; see Sec. II B). Moreover, we introduce the $\mathfrak{gl}_N \supset \mathfrak{u}_N$ charges (they form the basis of the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$) $\vec{q}_n$ and $\vec{q}_\rho$ (instead of $\mathfrak{sl}_N \supset \mathfrak{su}_N$ charges), since both n^i and ρ^j fields transform under the fundamental representation $\mathbf{N}$ of $\mathfrak{gl}_N$.

[12]Notice, that at $M \to \infty$ (i.e., for $U(2)$ gauge theory with massless quarks) the string coupling is rather given by $g_s \sim g_0/|b|^{1/k}\underset{k\to 1}{=} g_0/|b|$; i.e., large g_s is suppressed here only by the Liouville wall (3.19).

For example, the quark q^{iA} with the flavor $A=N+j$ has the same mass $|m_i-\tilde{m}_j|$ as the gauge-invariant "meson" $w^{ij}=n^i\rho^j$ with the charges

$$\vec{q}_{n^i}=\overrightarrow{e_i},\quad \vec{q}_{\rho^j}=\overrightarrow{e_j},\quad (\vec{e}_i)_j\equiv\delta_{ij},\quad i,j=1,\ldots,N \tag{4.13}$$

following from

$$Z_{n^i\rho^j}=i(m_i-\tilde{m}_j),$$

after substituting Eq. (4.13) into the BPS formula (4.12).

Actually, the mass formula (4.11) combines evidence both from field theory and string theory sides. In particular, $\mathcal{N}=2$ supersymmetry ensures that lower bounds $|Z_{\rm BPS}(m_i,\tilde{m}_j)|$ for masses of all physical states are given by corresponding central charges of supersymmetry algebra, so the only assumption in Eq. (4.11) is that the non-BPS part depends only on string scale α' (or ξ) and the baryonic charge

$$B=\pm\sum_j q_j, \tag{4.14}$$

where $\vec{q}=\vec{q}_n=\vec{q}_\rho$. At weak coupling, this assumption is supported by observation, that for all perturbative states $m^2_{\rm non-BPS}(\xi)\sim g^2\xi$ does not depend on quark masses at the leading order in g^2; see Ref. [7] for details. At strong coupling, the same assumption is supported by our string theory calculation, which shows that the hadron spectrum does not depend on the mass deformation parameter m.

1. $SU(2)$ case

In order to determine the BPS masses of the nonperturbative states, we need to write down their global charges and then use Eqs. (4.11) and (4.12). Let us start from the limit $M\to\infty$, where the non-Abelian vortex supports the $\mathbb{WCP}(2,2)$ model with the global symmetry group (2.6) given by[13]

$$SU(2)\times SU(2)\times U(1)_B. \tag{4.15}$$

The lightest of the nonperturbative states is the massless baryon, associated with the conifold complex structure parameter b on the CY side. Equation (2.11) shows that the b-baryon transforms with respect to Eq. (4.15) as $\det w^{ij}$; namely, it is a singlet with respect to both $SU(2)$ factors and has baryonic charge (4.14) $B=2$. The $\mathfrak{u}_2$ charges for this state are found from Eq. (4.13), i.e.,

$$\vec{q}_b=(1,0)+(0,1)=(1,1),$$

so that both $SU(2)$ charges $(\vec{q}_b)_1-(\vec{q}_b)_2=0$ vanish, while $B=(\vec{q}_b)_1+(\vec{q}_b)_2=2$ and Eq. (4.12) gives the corresponding central charge [34],

$$Z_b=i(m_1+m_2-\tilde{m}_1-\tilde{m}_2). \tag{4.16}$$

Note that within our interpolation procedure, the mass of this state in the $\mathbb{WCP}(2,2)$ model vanishes even at finite M because $\tilde{m}_i=m_i$.

All massive string states in $N=2$ theory are also singlets with respect to both $SU(2)$ factors in Eq. (4.15) (see Sec. II C), so that they differ only by their baryonic charges B. From the point of view of transformation properties, these states should behave as powers of $\det w$ in order to exist on the deformed conifold[14]; see Eq. (2.11). Therefore, their central charges are just multiples of Eq. (4.16), and the BPS part of the mass for these states always vanish. They become massive only due to the first non-BPS term in Eq. (4.11), depending only on the string tension $\tau=2\pi\xi$.

2. $SU(2K)$ with $K\geq 2$

In the general $SU(N)$ case (with even N), some extra points have to be clarified. The global symmetry group (2.6)

$$SU(N)_{C+F}\times SU(N)\times U(1)_B=SU(N)_n\times SU(N)_\rho\times U(1)_B \tag{4.17}$$

at $M\to\infty$ is broken down to

$$SU(N)\to SU(2)_{1,2}\times SU(2)_{3,4}\times\ldots\times SU(2)_{N-1,N}, \tag{4.18}$$

at each of the $SU(N)$ factors in Eq. (4.17). Following Ref. [23], in order to determine the allowed quantum numbers of the nonperturbative stringy states with respect to to the group (4.17), we propose the following generalization of the *conifold rule*:

(1) First, we assume that all allowed stringy states in the limit $M\to\infty$ are singlets with respect to all $SU(2)_{2i-1,2i}$, $i=1,\ldots,K$ factors in Eq. (4.18) for both $SU(N)$ groups associated with n and ρ fields.

(2) Second, a generalized conifold rule requires that the charges of stringy states with respect to the $SU(N)_\rho$ and $SU(N)_n$ in Eq. (4.17) coincide, i.e.,

$$\vec{q}_\rho=\vec{q}_n. \tag{4.19}$$

(3) As we reduce M and the global group (4.17) is restored, all hadron states should belong to

[13]Each $SU(2)$ factor here is broken down to U(1) by the nonzero quark mass differences.

[14]We call this below as *conifold rule*.

irreducible representations of this group.[15] To select the allowed representations, we take all states, satisfying two requirements above, as highest weight states of corresponding representations.

The first rule above determines the form of the highest weight vectors $\vec{q}$ [$\vec{q} \equiv \vec{q}_n = \vec{q}_\rho$; see Eq. (4.19)] of allowed irreducible baryonic representations $\lambda = \lambda_{\vec{q}}$. Namely, they should be orthogonal to the roots of corresponding $SU(2)_{2k-1,2k} \subset SU(N)$ subgroups, as follows:

$$\vec{\alpha}_k \vec{q} = 0, \qquad \vec{\alpha}_k = \vec{e}_{2k-1} - \vec{e}_{2k}, \qquad k = 1, \ldots, K. \quad (4.20)$$

The solution of Eq. (4.20) is obviously given by

$$\vec{q} = \sum_{k=1}^{K} \tilde{\lambda}_k (\vec{e}_{2k-1} + \vec{e}_{2k}), \quad (4.21)$$

where $\{\tilde{\lambda}_k\}_{k=1}^K$ is a set of non-negative integers satisfying

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_K$$

to ensure that the representation λ is irreducible.

The massless baryon (see footnote 15) corresponds to $\tilde{\lambda}_k = \delta_{k1}$, i.e., [at small M, when the global group (4.17) is restored] it belongs to the antisymmetric representation (3.28) which has totally

$$\dim \lambda_b = \dim \lambda_{\vec{e}_1 + \vec{e}_2} = \frac{N(N-1)}{2} \quad (4.22)$$

baryonic states with the charges

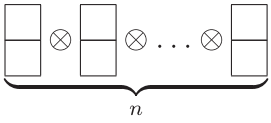
$$\vec{q}_b = \vec{e}_i + \vec{e}_j, \qquad i, j = 1, \ldots, N, \qquad i < j. \quad (4.23)$$

The non-BPS contribution to Eq. (4.11) vanishes for $j = -1/2$, $m = \pm 1/2$; see Eq. (2.21), or for the baryonic charge (2.22), (4.14) $|B| = 4|m| = 2$. The BPS part of their masses vanish when all bare masses $\mathbf{m} = 0$, and actually on the whole trajectory of our mass deformation (2.3) due to $m = \tilde{m}$ and $\vec{m}_n = \vec{m}_\rho$. For arbitrary quark masses, this multiplet splits with the central charges given by the exact formula (4.12),

$$Z^{(b)}_{\rm BPS} = i(m_i + m_j - \tilde{m}_i - \tilde{m}_j), \qquad i, j = 1, \ldots, N, \qquad i < j,$$

and baryon masses are given by $m_b = |Z^{(b)}_{\rm BPS}|$; c.f., with Eq. (4.16) for $N = 2$.

All massive baryonic states (which can be thought of as composites made of n elementary b-baryons) belong to the irreducible representations, arising in decomposition of the tensor product

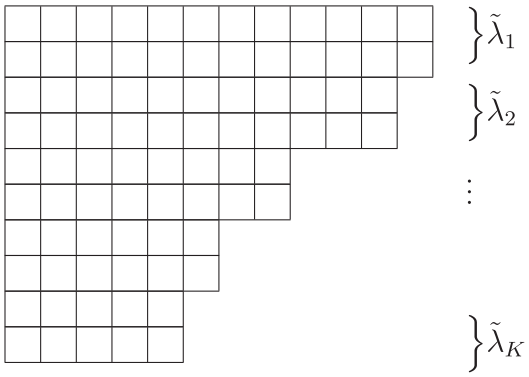


(4.24)

where arbitrary integer $n > 1$ is determined by the baryonic charge

$$|B| = \sum_{i=1}^{N} q_i = 2 \sum_{j=1}^{K} \tilde{\lambda}_j = 2n. \quad (4.25)$$

However, the previous analysis shows that one should extract from Eq. (4.24) the irreducible representations with the highest weights (4.21); i.e., corresponding to the Young diagrams[16]:



(4.26)

having the form of a "double-step staircase," with each ith pair of steps having the length of $\tilde{\lambda}_i$.

As was just discussed, within our interpolation procedure due to the conifold rule (4.19) the BPS contribution (4.12) to the masses of all hadrons vanishes, and they become massive due to the first non-BPS term in Eq. (4.11). This ensures that the hadron spectrum does not depend on bare masses $\mathbf{m}$, which matches the string theory result. The requirement $m = \tilde{m}$ is also important, since it ensures that coupling β does not run and no Λ-scale is generated in the $\mathbb{WCP}(N, N)$ model, so its Coulomb branch can really be reduced to the conformal (deformed) $\mathcal{N} = 2$ Liouville theory.

3. High-energy asymptotics

To compare the multiplicity of allowed hadron states at small M with the string result (4.7), we consider the

[15] Stronger requirement for all states to be singlets of the whole $SU(N)$ would kill, for example, the b-baryon state, which is expected to survive the transition to a theory with a larger gauge group. In contrast to the $N = 2$ case, the massless b-baryon now belongs to a nontrivial (second fundamental) representation $\square\!\!\square = \square \wedge \square$ of dimension $\frac{1}{2}N(N-1)$ (where $\square$ denotes the fundamental representation $\mathbf{N}$).

[16] The total number of cells (4.25) in this diagram is $2n$.

high-energy limit. Notice that at large energies E (where E is the 4D mass of the state) it follows from Eqs. (2.21) and (2.22) that $E \sim |B| = 2n$. Hence, the conditions (4.8) turn into

$$n \gg N \gg 1, \tag{4.27}$$

in the representation theory context, or in other words, we should consider the $n \to \infty$ asymptotic at large, but fixed N.

To calculate the dimension of the irreducible representation, one can use the Weyl formula:

$$\dim \lambda = \prod_{1\le i<j\le N} \frac{\lambda_i - \lambda_j + j - i}{j-i}, \tag{4.28}$$

where λ_i, $i = 1, \ldots, N$ is the length of the ith row. For the double-step ladder (4.26), we put $\tilde{\lambda}_k = \lambda_{2k-1} = \lambda_{2k}$, $k = 1, \ldots, K$ and get

$$\dim \lambda_B = \prod_{1\le i<j\le K} \prod_{\epsilon,\epsilon'=0,1} \frac{\tilde{\lambda}_i - \tilde{\lambda}_j + 2(j-i) + \epsilon - \epsilon'}{2(j-i) + \epsilon - \epsilon'} \tag{4.29}$$

from Eq. (4.28).

To estimate the number of baryonic states at high energy, we need to understand the behavior of the dimension of an irreducible representation in the limit (4.27). For generic representations, it is easy to calculate dimensions from scaling properties of the Weyl formula (4.28), the rhs of which scales under $\{\tilde{\lambda}_i \to t\tilde{\lambda}_i\}$ as $t^{\frac{1}{2}N(N-1)}$. The number of rows in the Young diagram (4.26) is restricted by large, but fixed N, so at $n \to \infty$ the diagram "grows to the right," and one can take into account this growth by scaling the row lengths as

$$\tilde{\lambda}_i \to t\tilde{\lambda}_i, \qquad B \to t\sum_i \tilde{\lambda}_i, \tag{4.30}$$

where $t = O(n) = O(B)$ is an overall scaling parameter, and all $\tilde{\lambda}_i$ are kept finite after rescaling. For generic double-step diagrams (4.26), we get ($N = 2K$) after rescaling of Eq. (4.29) as follows:

$$\begin{aligned}\dim \lambda_B &\simeq \prod_{1\le i<j\le K} \frac{t^4(\tilde{\lambda}_i - \tilde{\lambda}_j)}{4(j-i)^2[4(j-i)^2-1)} \\ &\underset{t\to\infty}{\sim} t^{4\frac{1}{2}K(K-1)} \sim B^{\frac{1}{2}N^2-N}.\end{aligned} \tag{4.31}$$

The leading behavior is actually universal[17] and is given by

$$\log \dim \lambda_B \sim \frac{1}{2}N^2 \log n \sim \frac{1}{2}N^2 \log B \sim \frac{1}{2}N^2 \log E. \tag{4.32}$$

This is the asymptotics of the multiplicity of baryonic states in given irreducible representation with mass $E \sim n \to \infty$ at large N.

The asymptotics (4.32) can possibly be corrected by growth of the *number* of allowed irreducible representations $p(B, N)$ at $B \to \infty$. It is well known, however (see, e.g. Ref. [64]), that the number of partitions of B into up to N parts $p(B, N)$ can be analyzed from their generating function

$$\sum_{B\ge 0} p(B,N)x^B = \prod_{k=1}^{N} \frac{1}{1-x^k}.$$

At large B with fixed N, the asymptotic[18] is

$$p(B,N) \underset{B\to\infty}{\sim} \frac{B^{N-1}}{N!(N-1)!} \sim e^{N\log B},$$

just as Nth power of B. It follows, say, from a standard "residue argument"

$$\begin{aligned}p(B,N) &= \mathrm{res}_{x=0} \frac{dx}{x^{B+1}\prod_{k=1}^{N}(1-x^k)} \\ &= -\mathrm{res}_{x=1} \frac{dx}{x^{B+1}\prod_{k=1}^{N}(1-x^k)} + \ldots \\ &\sim -\mathrm{res}_{x=1} \frac{dx}{x^{B+1}(1-x)^N\prod_{k=1}^{N}(1+x+\cdots+x^k)} + \ldots \\ &\sim \frac{1}{(N-1)!}\frac{d^{N-1}}{dx^{N-1}} \frac{1}{x^{B+1}\prod_{k=1}^{N}(1+x+\cdots+x^k)}\bigg|_{x=1} \\ &= \frac{(B+1)\ldots(B+N-1)}{N!(N-1)!} + \ldots \underset{B\to\infty}{\sim} \frac{B^{N-1}}{N!(N-1)!}.\end{aligned}$$

The number of double-step representations (4.26) is therefore given by

$$p(n,K) = p(B/2,K) \sim e^{K\log B} = e^{\frac{N}{2}\log B},$$

so that the total number $\#_{\mathrm{lad}}$ of possible double-step ladders (4.26) is

[17] As we already mentioned, for generic Young diagrams, the scaling gives

$$\dim \lambda \simeq t^{\frac{1}{2}N(N-1)} C_\lambda = B^{\frac{1}{2}N(N-1)} \tilde{C}_\lambda,$$

where the constant $\tilde{C}_\lambda = O(1)$ at large n and leading asymptotic behavior is the same (see also Appendix B 2 for some numeric tests of this scaling behavior).

[18] As well as asymptotic of partition in exactly N parts, in contrast with the famous Hardy-Ramanujan asymptotic of the number of all Young diagrams

$$p(B) \underset{B\to\infty}{\sim} \frac{1}{4\sqrt{3}B} e^{\pi\sqrt{\frac{2B}{3}}},$$

which actually leads to the Hagedorn behavior (4.1) in string theory.

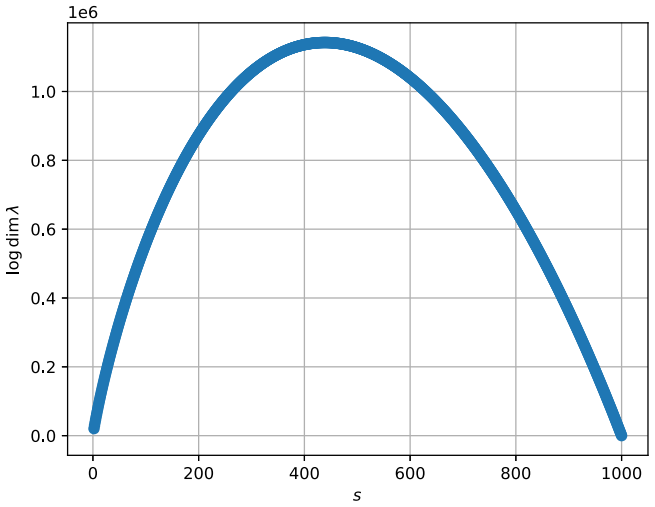


FIG. 3. Numerical plot of the dependence of log dim λ on s with $n = 10000000$, $N = 1000$ (source: Ref. [65]).

$$\log \#_{\text{lad}} \approx \frac{N}{2}\log B \sim \frac{N}{2}\log E. \tag{4.33}$$

The coefficient in front of $\log E$ is of the order $O(N)$ and does not change the main $O(N^2)$ asymptotics in Eq. (4.32), though it corrects the next term, depending on chosen class of Young diagrams.

We therefore conclude that the group theory analysis from the SQCD side confirms the N dependence of *the* $\log E$ term in string theory asymptotics (4.7) at $N \gg 1$. However, the group theory result (4.32) certainly cannot reproduce the nonperturbative coefficient (4.10) in front of $\log E$ term, arising from string theory dynamics.

4. Example: "Box" diagrams

In order to demonstrate some features of the asymptotic formulas, let us present here (see also Appendix B) an explicit example of the box-shaped diagrams, defined by

$$\begin{cases} \lambda_1 = \lambda_2 = \ldots = \lambda_s, & \text{for some even } s \in [1, N] \\ \lambda_i = 0, & i = s+1, \ldots, N. \end{cases} \tag{4.34}$$

These are not the generic diagrams, and the previous scaling analysis [see Eq. (4.31) and footnote 17] is not applicable. Indeed, it is easy to see from the Weyl formula (4.28), that the dimensions of "box" representations (4.34) scale under $\lambda_s \to t\lambda_s$ at $t \to \infty$ as $t^{s(N-s)}$. Hence, when s goes from 1 to N, the dimension dim λ first grows and then decreases; see the result of numeric analysis on Fig. 3, which actually corrects the naïve scaling parabola.

For the thin or thick strips with either $s = 2$ or $s = N - 2$, the dimension formula (4.28) gives scaling

$$\log \dim \lambda^{(s=2)} \underset{n\to\infty}{\sim} 2N \log n,$$
$$\log \dim \lambda^{(s=N-2)} \underset{n\to\infty}{\sim} 2N \log n;$$

see more details in the Appendix B. So, for the diagrams with either $s \ll N$ or $s \gg 1$, the leading asymptotics $N^2 \log E$ is absent, and they give contribution of the same order as the overall number of diagrams (4.33).

The naïve maximum for the box diagrams is at $s = N/2 = K$ (square-shaped diagram), where Eq. (4.28) gives

$$\log \dim \lambda^{(s=N/2)} \sim \frac{N^2}{4}\log n.$$

We see, therefore, that among the box diagrams, which are a special case of double-step ladders for even s, those with $s \sim N/2$ have the same behavior of asymptotic dimension

$$\log \dim \lambda^{(s\sim N/2)} \sim N^2 \log n$$

as generic diagrams, which match the $\log E \approx \log n$ term calculated from the string theory; see Eq. (4.7). However, the numerical coefficient in front of $N^2 \log n$ is not universal, and cannot be determined just by representation theory analysis.

V. MASSLESS BARYONS AS GOLDSTONES

As we already discussed, in strongly coupled $\mathcal{N} = 2$ SQCD, a new nonperturbative Higgs branch grows up, where stringy massless baryon b develops VEV (see Introduction and Fig. 1). At $M \to \infty$ in the world-sheet theory of non-Abelian string, this VEV is related to unique complex structure parameter of $K = N/2$ conifolds, associated with each of noninteracting $U(2)$ SQCDs; see Eq. (3.8).

As we reduce M, the global flavor group (4.17) is restored and massless b-baryons form the antisymmetric representation (3.28) of dimension (4.22) (see also footnote 15). Below, we show that these massless states can be viewed as Goldstone bosons resulting from the spontaneous symmetry breaking of the flavor group by the VEV of b.

The basis vectors $\{e_i \wedge e_j\}$ [with the weights (4.23)] in representation $\square = \square\wedge\square$ can be identified with the skew-symmetric matrices $\{e_i \wedge e_j^T = E_{ij} - E_{ji}\}$. In this basis, the b-baryon VEV at $M \to \infty$ is given by

$$\langle b \rangle = bJ, \qquad J \equiv \oplus_{j=1}^{K=N/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_j \equiv \begin{pmatrix} \epsilon & 0 & \ldots & 0 \\ 0 & \epsilon & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \epsilon \end{pmatrix}. \tag{5.1}$$

It is invariant under $\mathbb{Z}_K$ and unbroken $\oplus_{j=1}^{K=N/2}\mathfrak{sl}(2)_j$ action, since $\epsilon=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ is the $\mathfrak{sl}(2)\simeq\mathfrak{sp}(2)$-invariant, and b is a unique [see Eq. (3.8)] complex parameter, which determines the value of the b-condensate.[19]

The baryon VEV (5.1) breaks flavor group $SU(N)$ down to its (compact) $Sp(K)\subset SU(N)$ subgroup, which preserves a symplectic[20] form J, which satisfies $J+J^T=0$ and $J^2=-1$.

As mentioned above, b belongs to the second fundamental representation (3.28), implying the transformation law

$$b'=UbU^T,\qquad U\in SU(N),\tag{5.2}$$

and VEV (5.1) breaks this symmetry down to Sp(K). The $\mathfrak{sp}(K)\subset\mathfrak{sl}(N)$ subalgebra corresponds to invariant $\sigma(X)=X$ part of the automorphism

$$\sigma(X)=JX^TJ,\qquad X\in\mathfrak{sl}(N),$$

i.e., to the matrices satisfying

$$JX+X^TJ=0,\qquad X\in\mathfrak{sl}(N),$$

whose action preserves J.

The coset space $SU(N)/Sp(K)$ corresponds to the Goldstone modes, arising due to spontaneous symmetry breaking. Apart from that, the VEV of b also breaks the overall $U(1)_B$. The number of Goldstone modes is, therefore,

$$\begin{aligned}\#_{\text{Gold}}&=1+\dim\,\mathfrak{sl}(N)-\dim\,\mathfrak{sp}(K)\\&=1+(N^2-1)-\frac{N(N+1)}{2}=\frac{N(N-1)}{2}.\end{aligned}\tag{5.3}$$

This confirms our expectations, since Eq. (5.3) exactly coincides with Eq. (4.22). Note that b-baryon form a hypermultiplet in $\mathcal{N}=2$ SQCD [10] so the multiplicity in Eq. (5.3) should be multiplied by 4 to account for all real scalar superpartners.

Embedding $\mathfrak{sp}(K)\subset\mathfrak{sl}(N)$ (and corresponding form of the matrices in coset) can be described explicitly, the simplest way is to use the basis coming from the folding of Dynkin diagrams; see Appendix C 1. Dimension (5.3) can be also checked numerically, by explicit analysis of the formula (5.2) in its infinitesimal form; see Appendix C 2.

[19]Note that we can consider only one of two $SU(N)$ groups in Eq. (4.17) due to the conifold rule (4.19).

[20]Note that the symplectic form J from Eq. (5.1) can be brought to more conventional form, e.g., $\left(\begin{smallmatrix}0&I\\-I&0\end{smallmatrix}\right)$, by permutation of the basis elements.

VI. CONCLUSIONS

In this paper, we studied the hadronic phase of strongly coupled 4D $\mathcal{N}=2$ SQCD using the special quark mass deformation for interpolation from SQCD with $U(2)$ gauge group and $N_f=4$ quark flavors to $U(N)$ SQCD with $N_f=2N$ flavors for arbitrary even N. Implying a stringy interpretation of hadrons of $\mathcal{N}=2$ SQCD, we found the description of the critical non-Abelian string in terms of mass-deformed $\mathcal{N}=2$ Liouville theory. Turning on simultaneously the Liouville superpotential and the mass deformation on the world sheet, we used the dual Euclidean 2D black hole picture to find the mass spectrum of string states. We were also able to find the growth of the multiplicity of states at high energies from analyzing the entropy of a 2D black hole.

A key outcome is that, while the positions of the stringy energy levels do not change along the interpolation in the mass deformation parameter M, the multiplicity on each level grows, and becomes parametrically large as we reduce M, as reflected by enhancement of the black hole entropy; cf. Eq. (4.5).

Extracting the distribution of states from the behavior of black hole entropy near the Hagedorn temperature, we fixed the asymptotic spectral density at high energies in the form

$$\omega(E)\sim E^{\alpha}e^{\frac{E}{T_H}},$$

with the universal leading Hagedorn exponent and a subleading power correction, whose power depends nontrivially on the deformation data; see Eq. (4.7). The coefficient $\alpha\sim 1/g_s^2\sim N^2|b|^2/\Lambda^2$ is nonperturbative in string coupling, determined in this regime by N and the parameter (4.9). This provides a quantitative prediction for how the multiplicity of baryonic string states depends on the rank N of the gauge group in SQCD.

We then justified these results directly in the 4D language by analyzing the spectrum of hadrons given by Eq. (4.11). It separates the non-BPS contribution given by string theory results (2.21) from the BPS part, given by exact formula depending on quark masses and global hadron charges with respect to the flavor group (4.17). The allowed baryons populate a restricted family of irreducible representations of Eq. (4.17) characterized by double-step (ladder) Young diagrams. Using the Weyl formula for the dimensions of irreducible representations and calculating the number of allowed Young diagrams, we found the characteristic behavior of the logarithm of the baryon multiplicity to be

$$\sim N^2\log B$$

at fixed large rank $N\gg 1$ and baryon number $B\to\infty$ (4.27), confirming the string theory result (4.7) for N^2-dependence of the $\log E$-term.

Finally, we showed that the massless b-baryon multiplet in $U(N)$ SQCD can be understood as Goldstone modes upon spontaneous breaking of the global flavor group (4.17) by b-baryon condensate. The corresponding VEV in the representation (3.28) breaks $SU(N)\times U(1)_B$ down to $Sp(N/2)\subset SU(N)$, yielding exactly $\frac{N(N-1)}{2}$ massless states, in agreement with dimension of the massless baryon representation (4.22).

To summarize the dynamics of $\mathcal{N}=2$ SQCD with $N_f=2N$ flavors, we presented its phase diagram; see Fig. 1. The theory has two phases, the Higgs phase at weak coupling and the hadronic/stringy phase at strong coupling where the massless stringy b-baryon develops VEV forming the nonperturbative Higgs branch. These two phases are separated by a phase transition; see Sec. III D.

Physics is dramatically different in these two phases. The hadron phase has towers of stringy states [see Eq. (2.21)] typical for the spectrum of the string theory, while in the Higgs phase at weak coupling the physical spectrum is formed by a finite number of screened quarks and Higgsed gluons together with their superpartners.

It would be interesting to sharpen the matching of multiplicities of hadronic states calculated from string theory and field theory sides beyond the asymptotic regime by incorporating finite-N corrections and to study dynamical observables (e.g., correlation functions and decay amplitudes) that probe the interactions among the resulting stringy hadrons.

ACKNOWLEDGMENTS

The authors are grateful to A. Artemev and A. Litvinov for useful discussions of conformal backgrounds, and to M. Bershtein and E. Feigin for illuminating some issues in representation theory. The work of E. I. was supported in part by U.S. Department of Energy Grant No. de-sc0011842. The work of A. M. was partially supported by the Basic Research Program HSE-BR-2025-84 of HSE University (Sec. IV B and Sec. V), and by the Russian Science Foundation under Grant No. 26-11-00342 (Sec. III and Sec. IV A). The work of G. S. was also partly supported by the Gribov Scholarship for works in the field of theoretical physics.

DATA AVAILABILITY

The code and data used to generate the results presented in this article are available in Ref. [65].

APPENDIX A: LARGE-N DERIVATION OF EFFECTIVE ACTION

We start with the one-loop kinetic term Eq. (3.1). For the special mass choice (2.3), the total number $N\equiv N_0K$ of quark flavors (here we consider large N_0, but set $N_0=2$ at the end) form K groups with equal masses in each group. Then the sum over masses in Eq. (3.1) can be rewritten as

$$\sum_{A=1}^{2N}\frac{1}{\left|\sigma+\frac{m_A}{\sqrt{2}}\right|^2}=\sum_{A=1}^{2KN_0}\frac{1}{\left|\sigma+\frac{m_A}{\sqrt{2}}\right|^2}=2N_0\sum_{i=1}^{K}\frac{1}{\left|\sigma+\frac{m_i}{\sqrt{2}}\right|^2},$$

and we get

$$\begin{aligned}S_{\text{eff}}&=\frac{2N_0}{8\pi}\int d^2x|\partial_\alpha\sigma|^2\sum_{i=1}^{K}\frac{1}{|\sigma+\frac{m_i}{\sqrt{2}}|^2}\\&=\frac{2N_0}{8\pi}\int d^2x\frac{|\partial_\alpha\sigma|^2}{|\sigma+\frac{m_{i_0}}{\sqrt{2}}|^2}\left(1+\sum_{i\neq i_0}^{K}\frac{\left|\sigma+\frac{m_{i_0}}{\sqrt{2}}\right|^2}{|\sigma+\frac{m_i}{\sqrt{2}}|^2}\right).\end{aligned}\tag{A1}$$

Changing variables according to Eq. (3.2), one gets

$$\begin{aligned}|\partial_\alpha\sigma|^2&=\frac{1}{Q^2}\gamma_{i_0}^2e^{-\frac{2\phi}{Q}}((\partial_\alpha\phi)^2+(\partial_\alpha Y)^2)\\&=\frac{1}{Q^2}\left|\sigma+\frac{m_{i_0}}{\sqrt{2}}\right|^2((\partial_\alpha\phi)^2+(\partial_\alpha Y)^2).a\end{aligned}$$

Furthermore, the sum in Eq. (A1) can be simplified as

$$\begin{aligned}\sum_{i\neq i_0}^{K}\frac{|\sigma+\frac{m_{i_0}}{\sqrt{2}}|^2}{|\sigma+\frac{m_i}{\sqrt{2}}|^2}&=\sum_{i\neq i_0}^{K}\frac{|\sigma+\frac{m_{i_0}}{\sqrt{2}}|^2}{|\sigma+\frac{m_{i_0}}{\sqrt{2}}+\frac{m_i-m_{i_0}}{\sqrt{2}}|^2}\\&=\sum_{i\neq i_0}^{K}\frac{1}{|1+\frac{m_i-m_{i_0}}{\sqrt{2}\gamma_{i_0}}\exp(\frac{\phi+iY}{Q})|^2}\\&\overset{\phi\to\infty}{\approx}\sum_{i\neq i_0}^{K}\frac{\exp(-\frac{2\phi}{Q})}{\left|\frac{m_i-m_{i_0}}{\sqrt{2}\gamma_{i_0}}\right|^2}=\exp\left(-\frac{2\phi}{Q}\right)\sum_{i\neq i_0}^{K}\frac{|b_{i_0}|^2}{|M_{ii_0}|^2},\end{aligned}$$

$$M_{ii_0}=-\frac{(m_i-m_{i_0})\beta}{2}.$$

The $\mathbb{Z}_K$ symmetry makes this sum independent of the choice of a particular i_0. Indeed, with an explicit set of masses (2.3)–(2.4) and universal (3.8), we have

$$\sum_{i\neq i_0}^{K}\frac{1}{|M_{ii_0}|^2}=\sum_{j=1}^{K-1}\frac{1}{|A|^2m^2\left|e^{\frac{2\pi ij}{K}}-1\right|^2},$$

where $A=-\beta/2$. The rhs can be further rewritten as

$$\begin{aligned}\frac{1}{2m^2|A|^2}\sum_{j=1}^{K-1}\frac{1}{\left(1-\cos[\frac{2\pi ij}{K}]\right)}&=\frac{1}{2m^2|A|^2}\sum_{j=1}^{K-1}\frac{1}{2\sin^2[\frac{\pi j}{K}]}\\&=\frac{1}{2m^2|A|^2}\left(\frac{K^2-1}{6}\right).\end{aligned}$$

In the large-K limit, we therefore obtain

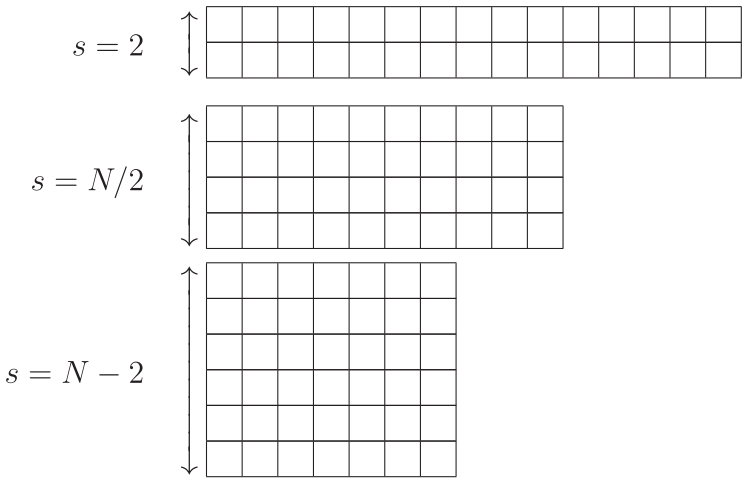
$$\sum_{i=2}^{K}\frac{1}{|M_{ii_0}|^2}=\frac{K^2-1}{12|M|^2}\underset{K\gg 1}{\approx}\frac{K^2}{12|M|^2},\qquad |M|^2\equiv\left|\frac{m\beta}{2}\right|^2,$$

which gives Eq. (3.7).

APPENDIX B: SOME ALLOWED REPRESENTATIONS AND ASYMPTOTICS

1. Box diagrams

For illustrating the results of Secs. IV B 2–IV B 4, we consider the box Young diagram of height s, as follows:



and demonstrate application of the Weyl formula (4.28) to computation of their dimensions. From Eq. (4.34) together with $\sum_{i=1}^{N}\lambda_i=2n$, we have

$$\lambda_i=\frac{2n}{s},\qquad 1\le i\le s;\qquad \lambda_j=0,\qquad s<j\le N.$$

Let us start with the $s=2$ case, where only nonvanishing $\lambda_1=\lambda_2=n$, and the Weyl formula (4.28) just gives

$$\dim\lambda^{(s=2)}=\frac{\lambda_1-\lambda_2+2-1}{2-1}\prod_{3\le j\le N}\frac{\lambda_1+j-1}{j-1}\times\prod_{3\le j\le N}\frac{\lambda_2+j-1}{j-1}$$
$$\overset{n\gg N}{\approx}1\cdot(\lambda_1)^{N-2}\lambda_2^{N-2}\overset{N\gg 1}{\approx}n^{2N}.$$

For the conjugated diagram with $s=N-2$, the dimension of the representation is the same.

As an opposite example, consider the box with exactly $s=N/2$. In this case, $\lambda_1=\lambda_2=\ldots\lambda_{N/2}=\frac{4n}{N}$, and plugging this into the Weyl formula yields

$$\dim\lambda^{(s=N/2)}=\prod_{N/2<j\le N}\prod_{1\le i\le N/2}\frac{\lambda_i+j-i}{j-i}$$
$$\overset{n\gg N}{\approx}\prod_{N/2<j\le N}\left(\frac{4n}{N}\right)^{\frac{N}{2}}\approx n^{\frac{N^2}{4}}.$$

2. Numeric illustration of scaling formula

Here, we present some numeric illustrations of the scaling formula (4.31). For the special cases of double-ladder diagrams with equal $\delta=\tilde{\lambda}_{i-1}-\tilde{\lambda}_i$ for $i=2\ldots K$, we choose:

(i) $n=400000$, $K=500$, $\delta=2$;
(ii) $n=500000$, $K=500$, $\delta=4$;
(iii) $n=1000000$, $K=1000$, $\delta=2$,

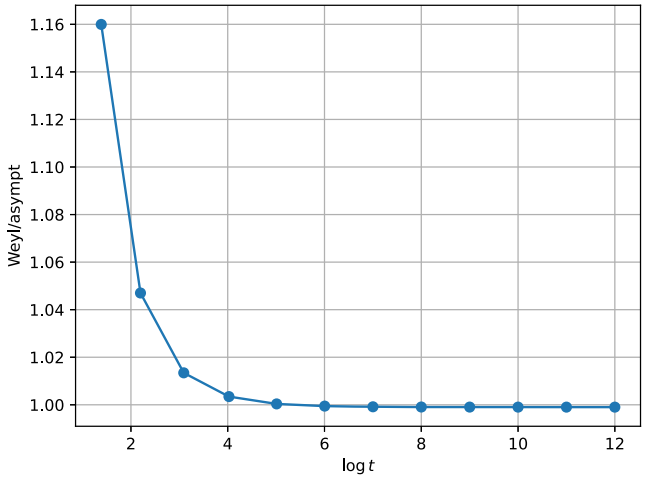


FIG. 4. $\log\dim\lambda$ to $\frac{1}{2}N^2\log t$ ratio for $n=1000000$, $K=1000$, $\delta=2$ (source: Ref. [65]).

and, making rescaling Eq. (4.30), numerically calculate the ratio $\frac{\log \dim \lambda(t)}{\frac{1}{2}N^2 \log t}$ of the direct application of the Weyl formula (4.29) with its main asymptotic (4.31); as an example, see Fig. 4.

APPENDIX C: MORE ON THE GOLDSTONE BOSONS

1. Folding of Dynkin diagrams and breaking of $SU(N)$

Changing the basis by permutation of even vectors $P = P_{2,N}P_{4,N-2}\ldots P_{N/2-2,N/2+2}$, one brings the VEV of b in Eq. (5.1) to the antidiagonal form

$$\Omega = PJP = \begin{pmatrix} 0 & \ldots & 0 & \epsilon \\ 0 & \ldots & \epsilon & 0 \\ \vdots & \iddots & & \vdots \\ \epsilon & \ldots & & 0 \end{pmatrix},$$

also satisfying $\Omega + \Omega^T = 0$ and $\Omega^2 = -1$. In this basis, the $\mathfrak{sp}(K)$ subalgebra

$$\Omega X + X^T\Omega = 0, \qquad X \in \mathfrak{sl}(N)$$

is given by $A_{2K-1} \to C_K$ folding, i.e., is generated by invariant under involution $\sigma(x_i) = \sigma(x_{2K-i})$ for the Cartan-Chevalley $x = \{h, e, f\}$ generators combinations

$$x_i + x_{2K-i}, \quad i = 1, \ldots, K-1,$$

generating, together with $x_K = \sigma(x_K)$, the $\mathfrak{sp}(K)$ subalgebra of the matrices

$$X = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix} \in \mathfrak{sp}(K), \tag{C1}$$

where $K \times K$ blocks satisfy ${}^tB = B$ and ${}^tC = C$, and ${}^tA$ means transposition with respect to antidiagonal. The dimension of Eq. (C1) is exactly

$$2\frac{K(K+1)}{2} + K^2 = K(2K+1) = \frac{N(N+1)}{2} = \dim \mathfrak{sp}(K = N/2),$$

where K^2 comes from the $A \in \mathfrak{gl}(K) \subset \mathfrak{sp}(K)$ subalgebra, corresponding to "A_K-part" of the C_K Dynkin diagram; see Fig. 5, while $K(K+1)/2$ is the number of extra positive (or negative) roots, generated in block B by commutators with e_K (or with f_K in block C).

The coset space $SU(N)/Sp(K)$ corresponds to the Goldstone modes, arising due to spontaneous symmetry breaking. Its dimension (together with extra broken $U(1)_B$) is

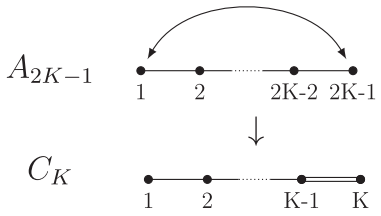


FIG. 5. Folding of the Dynkin diagrams corresponding to the $\mathfrak{sp}(K) \subset \mathfrak{sl}(2K)$ embedding.

$$N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}. \tag{C2}$$

Indeed, the matrices from the coset space $\mathfrak{sl}(N)/\mathfrak{sp}(K = N/2)$ satisfy the anti-involution constraint $\sigma(x_i) = -\sigma(x_{2K-i})$ and therefore are given by

$$\tilde{X} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & -{}^t\tilde{A} \end{pmatrix} \in \mathfrak{sl}(N)/\mathfrak{sp}(K), \tag{C3}$$

with ${}^t\tilde{B} = -\tilde{B}$ and ${}^t\tilde{C} = -\tilde{C}$, and there is an extra condition for the matrix

$$\tilde{A} = \begin{pmatrix} * & \ldots & * & * \\ * & \ldots & * & * \\ \vdots & \iddots & & \vdots \\ * & \ldots & & 0 \end{pmatrix}$$

coming from factoring out h_K by $\sigma(h_K) = -h_K$. The dimension of the space (C3) is (up to $U(1)_B$) the number of Goldstone modes (C2), as follows:

$$\begin{aligned} 2\frac{K(K-1)}{2} + K^2 - 1 = 2K^2 - K - 1 &= \frac{N(N-1)}{2} - 1 \\ &= \dim \mathfrak{sl}(N) - \dim \mathfrak{sp}(K) \\ &= \#_{\text{Gold}} - 1. \end{aligned} \tag{C4}$$

2. Numeric check

Let us finally present the results of numeric verification of Eq. (C4). Consider Eq. (5.2) in the infinitesimal form

$$b' = UbU^T \approx b + i\delta b = b + i\sum_{i=1}^{N^2-1} \alpha^a \{T^a b + b(T^a)^T\},$$

where $\{T^a\}$ are generators of $\mathfrak{su}_N$. To count the number of Goldstone modes, one has to find the number of linearly independent parameters, solving

$$\delta b = \sum_{i=1}^{N^2-1} \{T^a b + b(T^a)^T\}\alpha^a = 0, \tag{C5}$$

which is a system of N^2-1 linear equations for $\{\alpha^a\}$.

Calculating the rank of matrix in Eq. (C5) gives the number of Goldstones. This algorithm was implemented in the *Mathematica* package [65] and confirms Eq. (C4) for every even N (calculations for $N=\{2,4,6,8,10,12,50,\ldots\}$ give the Goldstone mode's numbers $\#_{\text{Gold}}-1=\{0,5,14,27,44,65,1224,\ldots\}$, correspondingly).

[1] A. Hanany and D. Tong, Vortices, instantons and branes, J. High Energy Phys. 07 (2003) 037.
[2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Non-Abelian superconductors: Vortices and confinement in $\mathcal{N}=2$ SQCD, Nucl. Phys. **B673**, 187 (2003).
[3] M. Shifman and A. Yung, Non-Abelian string junctions as confined monopoles, Phys. Rev. D **70**, 045004 (2004).
[4] A. Hanany and D. Tong, Vortex strings and four-dimensional gauge dynamics, J. High Energy Phys. 04 (2004) 066.
[5] D. Tong, TASI lectures on solitons, arXiv:hep-th/0509216.
[6] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Solitons in the Higgs phase: The moduli matrix approach, J. Phys. A **39**, R315 (2006).
[7] M. Shifman and A. Yung, Supersymmetric solitons and how they help us understand non-Abelian gauge theories, Rev. Mod. Phys. **79**, 1139 (2007); for an expanded version see M. Shifman and A. Yung, *Supersymmetric Solitons* (Cambridge University Press, Cambridge, England, 2009), 10.1017/9781009402200.
[8] D. Tong, Quantum vortex strings: A review, Ann. Phys. (Amsterdam) **324**, 30 (2009).
[9] M. Shifman and A. Yung, Critical string from non-Abelian vortex in four dimensions, Phys. Lett. B **750**, 416 (2015).
[10] P. Koroteev, M. Shifman, and A. Yung, Non-Abelian vortex in four dimensions as a critical string on a conifold, Phys. Rev. D **94**, 065002 (2016).
[11] P. Candelas and X. C. de la Ossa, Comments on conifolds, Nucl. Phys. **B342**, 246 (1990).
[12] A. Neitzke and C. Vafa, Topological strings and their physical applications, arXiv:hep-th/0410178.
[13] M. Shifman and A. Yung, Critical non-Abelian vortex in four dimensions and little string theory, Phys. Rev. D **96**, 046009 (2017).
[14] E. Ivanov and S. Krivonos, U(1) supersymmetric extension of the Liouville equation, Lett. Math. Phys. **7**, 523 (1983).
[15] D. Kutasov and N. Seiberg, Noncritical superstrings, Phys. Lett. B **251**, 67 (1990).
[16] D. Kutasov, Introduction to little string theory, published in *Superstrings and Related Matters 2001*, Proceedings of the ICTP Spring School of Physics, edited C. Bachas, J. Maldacena, K. S. Narain, and S. Randjbar-Daemi (ICTP Publications & Printing, Trieste, Italy, 2002), pp. 165–209.
[17] D. Ghoshal and C. Vafa, c = 1 string as the topological theory of the conifold, Nucl. Phys. **B453**, 121 (1995).
[18] A. Giveon and D. Kutasov, Little string theory in a double scaling limit, J. High Energy Phys. 10 (1999) 034.
[19] A. Giveon, D. Kutasov, and O. Pelc, Holography for noncritical superstrings, J. High Energy Phys. 10 (1999) 035.
[20] M. Shifman and A. Yung, Hadrons of $\mathcal{N}=2$ supersymmetric QCD in four dimensions from little string theory, Phys. Rev. D **98**, 085013 (2018).
[21] P. Gavrylenko, E. Ievlev, A. Marshakov, I. Monastyrskii, and A. Yung, 2D sigma models on non-compact Calabi-Yau and $\mathcal{N}=2$ Liouville theory, Phys. Rev. D **111**, 106003 (2025).
[22] A. Yung, Flowing between string vacua for the critical non-Abelian vortex with a deformation of N = 2 Liouville theory, Phys. Rev. D **110**, 025004 (2024).
[23] E. Ievlev, A. Marshakov, G. Sumbatian, and A. Yung, Critical non-Abelian vortex string and 2D N = 2 black hole, Phys. Rev. D **112**, 105010 (2025).
[24] A. Giveon, Target space duality and stringy black holes, Mod. Phys. Lett. A **06**, 2843 (1991).
[25] R. Dijkgraaf, H. Verlinde, and E. Verlinde, String propagation in a black hole geometry, Nucl. Phys. **B371**, 269 (1992).
[26] E. Witten, String theory and black holes, Phys. Rev. D **44**, 314 (1991).
[27] S. Mukhi and C. Vafa, Two-dimensional black hole as a topological coset model of c = 1 string theory, Nucl. Phys. **B407**, 667 (1993).
[28] H. Ooguri and C. Vafa, Two-dimensional black hole and singularities of CY manifolds, Nucl. Phys. **B463**, 55 (1996).
[29] K. Hori and A. Kapustin, Duality of the fermionic 2-D black hole and $N=2$ Liouville theory as mirror symmetry, J. High Energy Phys. 08 (2001) 045.
[30] N. Seiberg and E. Witten, Monopoles, duality and chiral symmetry breaking in $N=2$ supersymmetric QCD, Nucl. Phys. **B431**, 484 (1994).
[31] E. H. Fradkin and S. H. Shenker, Phase diagrams of lattice gauge theories with Higgs fields, Phys. Rev. D **19**, 3682 (1979).
[32] P. Fayet and J. Iliopoulos, Spontaneously broken supergauge symmetries and Goldstone spinors, Phys. Lett. **51B**, 461 (1974).
[33] M. Shifman and A. Yung, Lessons from supersymmetry: "Instead-of-Confinement" mechanism, Int. J. Mod. Phys. A **29**, 1430064 (2014).

[34] E. Ievlev, M. Shifman, and A. Yung, String baryon in four-dimensional N = 2 supersymmetric QCD from the 2D-4D correspondence, Phys. Rev. D **102**, 054026 (2020).
[35] E. Witten, Phases of N = 2 theories in two dimensions, Nucl. Phys. **B403**, 159 (1993).
[36] For a review see e.g. A. Achucarro and T. Vachaspati, Semilocal and electroweak strings, Phys. Rep. **327**, 347 (2000).
[37] M. Shifman and A. Yung, Non-Abelian semilocal strings in $\mathcal{N} = 2$ supersymmetric QCD, Phys. Rev. D **73**, 125012 (2006).
[38] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, On the moduli space of semilocal strings and lumps, Phys. Rev. D **76**, 105002 (2007).
[39] M. Shifman, W. Vinci, and A. Yung, Effective world-sheet theory for non-Abelian semilocal strings in $\mathcal{N} = 2$ supersymmetric QCD, Phys. Rev. D **83**, 125017 (2011).
[40] N. Dorey, The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms, J. High Energy Phys. 11 (1998) 005.
[41] N. Dorey, T. J. Hollowood, and D. Tong, The BPS spectra of gauge theories in two and four dimensions, J. High Energy Phys. 05 (1999) 006.
[42] E. Ievlev and A. Yung, Critical non-Abelian vortex and holography for little string theory, Phys. Rev. D **104**, 114033 (2021).
[43] Y. Nakayama, Liouville field theory: A decade after the revolution, Int. J. Mod. Phys. A **19**, 2771 (2004).
[44] J. Teschner, Operator product expansion and factorization in the H_3^+ WZNW model, Nucl. Phys. **B571**, 555 (2000).
[45] O. Aharony, A. Giveon, and D. Kutasov, LSZ in LST, Nucl. Phys. **B691**, 3 (2004).
[46] L. J. Dixon, M. E. Peskin, and J. D. Lykken, N = 2 superconformal symmetry and SO(2, 1) current algebra, Nucl. Phys. **B325**, 329 (1989).
[47] P. M. S. Petropoulos, Comments on SU(1, 1) string theory, Phys. Lett. B **236**, 151 (1990).
[48] S. Hwang, Cosets as gauge slices in SU(1, 1) strings, Phys. Lett. B **276**, 451 (1992).
[49] J. M. Evans, M. R. Gaberdiel, and M. J. Perry, The no ghost theorem for AdS(3) and the stringy exclusion principle, Nucl. Phys. **B535**, 152 (1998).
[50] J. M. Evans, M. R. Gaberdiel, and M. J Perry, The no-ghost theorem and strings on AdS_3, *Nonperturbative Aspects of Strings, Branes and Supersymmetry*, edited by M. J. Duff *et al.* (World Scientific, Singapore, 1999), pp. 435–444.
[51] A. Strominger, Massless black holes and conifolds in string theory, Nucl. Phys. **B451**, 96 (1995).
[52] E. Witten, Instantons, the quark model, and the 1/N expansion, Nucl. Phys. **B149**, 285 (1979).
[53] T. G. Mertens, Hagedorn string thermodynamics in curved spacetimes and near black hole horizons, Ph.D. thesis, Gent U., 2015, arXiv:1506.07798.
[54] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).
[55] R. Hagedorn, Statistical thermodynamics of strong interactions at high-energies, Nuovo Cimento Suppl. **3**, 147 (1965).
[56] V. Fainberg and A. Marshakov, A propagator for the fermionic string, Phys. Lett. B **211**, 81 (1988).
[57] L. Susskind, Some speculations about black hole entropy in string theory, arXiv:hep-th/9309145.
[58] G. T. Horowitz and J. Polchinski, A correspondence principle for black holes and strings, Phys. Rev. D **55**, 6189 (1997).
[59] A. Giveon, D. Kutasov, E. Rabinovici, and A. Sever, Phases of quantum gravity in AdS_3 and linear dilaton backgrounds, Nucl. Phys. **B719**, 3 (2005).
[60] J. J. Atick and E. Witten, The Hagedorn transition and the number of degrees of freedom of string theory, Nucl. Phys. **B310**, 291 (1988).
[61] D. Kutasov, Accelerating branes and the string/black hole transition, arXiv:hep-th/0509170.
[62] V. Kazakov, I. K. Kostov, and D. Kutasov, A matrix model for the two-dimensional black hole, Nucl. Phys. **B622**, 141 (2002).
[63] Y. Chen and J. Maldacena, String scale black holes at large D, J. High Energy Phys. 01 (2022) 095.
[64] George E. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, England, 1976).
[65] G. Sumbatian, `uN_from_BH`, https://github.com/Defa777/uN_from_BH.