

Phase transitions, anomalous baryon number violation, and electroweak multiplet dark matter

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(Received 23 May 2025; accepted 10 August 2025; published 12 September 2025)

We perform a comprehensive analysis of baryon number violation during an electroweak phase transition (EWPT) within the framework of a scalar electroweak multiplet extension of the Standard Model (SM). We classify the multiplet representations, topological properties, and corresponding thermal histories. Sphaleron or monopole topological field solutions emerge during the EWPT depending on the stage of the phase transition and the hypercharge of the new scalar multiplet. Furthermore, the monopole field solution pertains when the neutral component of the additional scalar multiplet is a viable dark matter candidate. We further analyze other formal considerations, including the construction of the “sphaleron matrix” for higher dimensional representations, computation of the sphaleron and monopole masses, and the choice of boundary conditions when solving the field equations of motion. We apply these considerations to the computation of sphaleron energy and monopole mass within the context of a multistep EWPT, employing the $SU(2)_L$ septuplet scalar extension to the SM as a case of study from the minimal dark matter paradigm. For the first step of a two-step EWPT, we delineate the relationship between the monopole mass and the parameters relevant to dark matter phenomenology.

DOI: [10.1103/lz6d-kn77](https://doi.org/10.1103/lz6d-kn77)

I. INTRODUCTION

The origin of baryon asymmetry of the universe (BAU) remains an open question at the frontier of particle physics and cosmology. In order to explain the BAU, Sakharov proposes three necessary conditions: (1) baryon-number violation; (2) C and CP violation; (3) departure from thermal equilibrium or CPT violation [1]. In principle, the Standard Model (SM) provides all the necessary ingredients for generation of the baryon asymmetry during the era of electroweak symmetry-breaking (EWSB), a scenario known as electroweak baryogenesis (EWBG).

Indeed, the first condition can be fulfilled by the non-perturbative weak sphaleron process. However, the SM fails to satisfy the second and third conditions. The CP violation associated with the Cabibbo-Kobayashi-Maskawa matrix is too weak to generate the observed BAU [2–4], and EWSB occurs through a smooth crossover transition due to the large Higgs mass [5–9], thereby missing the needed out of equilibrium requirement. Many beyond Standard Model (BSM) theories have been proposed to remedy these shortcomings and facilitate EWBG (see Refs. [10–12] for reviews). In this work, we focus on a key element of BSM EWBG: electroweak sphaleron and monopole dynamics. We do so in the context of a general class of BSM scenarios, namely, those involving an extended Higgs sector containing higher dimensional electroweak multiplets.

Electroweak baryogenesis requires a first-order electroweak phase transition (FOEWPT), during which bubbles of broken symmetry nucleate in the symmetric phase. The BSM CP-violating interactions at the bubble walls generate a left-handed fermion number density that biases

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symmetric phase sphaleron transitions into the generation of nonzero baryon plus lepton number ($B + L$) [10,13,14]. The asymmetry diffuses into the bubble interiors. A sufficiently “strong” FOEWPT leads to suppression of the broken phase sphaleron rate, thereby allowing preservation of the asymmetry [15]. A central question, therefore, pertains to the broken phase sphaleron rate: Is it sufficiently quenched so as to preserve the BAU?

While the most reliable approaches to answering this question are obtained using lattice computations, as a practical matter performing a broad survey of BSM scenarios and associated parameter choices relies on (semi) analytic methods and perturbation theory. The latter provides a baseline for comparison and validation against nonperturbative studies. The aim of the following study is to refine this baseline and clarify some formal considerations along the way. In doing so, we recall that the analytic result for the broken phase sphaleron rate, Γ_{sph} can be written as the product of a dynamical prefactor A and a statistical factor [16–18]:

$$\Gamma_{\text{sph}} = A e^{-E_{\text{sph}}/T}, \quad (1)$$

where E_{sph} is the energy associated with the semiclassical sphaleron solution. Our focus in the present study falls on the latter.

To further set the context, we recall that the thermal history of EWSB can entail either a single, direct transition to the present “Higgs phase” or a series of steps. In the presence of additional scalar fields Φ , a different vacuum associated with a nonzero vacuum expectation value (VEV) for one or more components of Φ may precede the Higgs phase. Alternatively, the Higgs phase may also involve a nonzero Φ VEV. While Φ may be either a SM gauge singlet or carry SM quantum numbers, in this study we consider the case where Φ is an $\text{SU}(3)_C$ singlet but charged under $\text{SU}(2)_L \times U(1)_Y$. We further specify that only the neutral component of Φ obtains a nonzero VEV. Three representative patterns of EWSB are illustrated in Fig. 1, where case (a), (b), and (c) represent the SM one-step electroweak

phase transition (EWPT), one-step EWPT to the mixed phase, and two-step EWPT, respectively.

Each case may accommodate EWBG. For the single-step transitions in (a) and (b), the presence of Φ will modify Γ_{sph} through thermal loops and, for (b), through additional contributions to the semiclassical sphaleron solution. Note that for (b), constraints from the electroweak ρ parameter place strong constraints on $\langle \Phi \rangle$, when Φ is neither a gauge singlet nor a second Higgs doublet. One may evade these constraints through a suitable choice of field content, as in the Georgi-Machacek model [19]. For (c), the first step may accommodate EWBG if (i) this step involves a FOEWPT; (ii) if BSM CP violation (CPV) interactions generate a sufficiently large asymmetry; (iii) the rate of baryon number violating (BNV) processes in the EWSB Φ vacuum is sufficiently suppressed; and (iv) the second step to the Higgs phase does not allow for re-excitation of the electroweak (EW) sphalerons. The viability of this possibility has been demonstrated in Refs. [20–22]. To our knowledge, BNV dynamics for these scenarios in the presence of Φ have not been explored in a unified and systematic way. In what follows we endeavor to do so, focusing on cases (b) and (c) wherein Φ can play an active role in the semiclassical solutions to the field equations associated with BNV. We investigate both the corresponding topological structure and relevant energy scale.

To that end, this study mainly consists of two parts. In the first part, we review, update, and clarify various formal aspects related to the semiclassical treatment of the BNV solutions in BSM theories, including topological properties associated with different representations, relationships between various treatments for the BNV configurations, a general construction of the one-form framework for a general scalar multiplet, and choice of boundary conditions when solving equations of motion. Most importantly, we show that when the Φ VEV alone breaks electroweak symmetry, the corresponding BNV configuration is the sphaleron for Φ having nonzero hypercharge, Y , or a monopole for $Y(\Phi) = 0$. Dark matter direct detection constraints imply that only a $Y = 0$ multiplet with

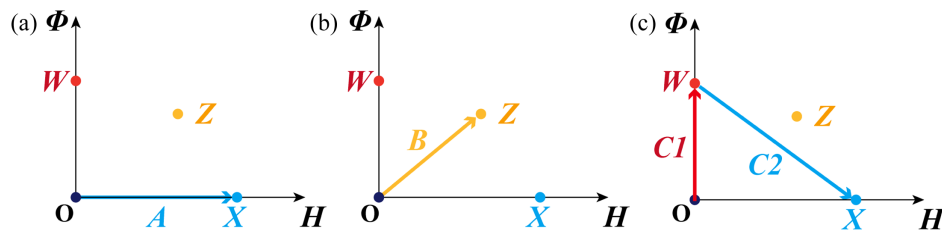


FIG. 1. Three possible EWSB patterns. In each plot, the horizontal axis represents the SM Higgs vacuum, and the vertical axis stands for the vacuum of BSM (electroweak multiplet in our study) particle. The O denotes the symmetric phase, while X , W , and Z represent the three stationary points of the total potential, i.e., the solutions for which the first derivative of the potential with respect to the fields vanishes. Panel (a) represents a one-step EWPT from symmetric phase to pure Higgs phase; panel (b) denotes a one-step EWPT to the mixed phase where the Higgs particle and scalar multiplet particle both obtain a VEV; panel (c) depicts a two-step EWPT scenario, where the first step moves from symmetric phase to the pure multiplet phase, and the second step moves from the multiplet phase to the Higgs phase.

vanishing VEV (point X in Fig. 1) can be a viable dark matter candidate [23,24]. Thus, phase W in the two-step scenario (c) would be associated with monopole-catalyzed BNV processes.

In the latter part, we compute the sphaleron energy for scenarios (b) and (c) for Φ being an electroweak septuplet, whose presence in the Higgs vacuum of scenario (c) can contribute to the dark matter (DM) relic density. In this instance, we delineate the dependence of the monopole mass on the parameters relevant to DM phenomenology: the DM mass, its self-interaction, and the coupling to SM fields that enters the annihilation and direct detection cross sections. In our current work, we primarily focus on the analysis of the zero-temperature model. Our aim is to provide a methodology for applying the sphaleron and monopole formalism to different EWSB patterns, where the zero-temperature model can provide a good approximation of physical quantities. The thermally corrected model can be analyzed in a parallel manner. We find that, depending on the values of these parameters, the monopole mass for step C1 of case (c) can be significantly larger than in the SM electroweak sphaleron energy for the single step transition in case (a), suggesting that the two-step scenario can be particularly conducive to EWBG.

Our discussion of these issues is organized as follows. In Sec. II, we present a detailed analysis of the theory formalism, applicable to both the SM and BSM scenarios. In Sec. III, we discuss an electroweak multiplet extension to the SM and present two possible types of EWPT involving this extension. In Sec. IV, we compute the corresponding sphaleron energy and monopole mass for this model as appropriate to the EWSB scenario. Our conclusions are summarized in Sec. V. In the Appendix, we present other sphaleron configurations, computation details of sphaleron energy, monopole mass, and model mass matrices.

II. THEORY FORMALISM

In this section, we demonstrate the properties of topological field solutions when the SM Higgs (H) is extended by a general-dimensional $SU(2)$ multiplet Φ . As depicted in Fig. 1, there are three notable scenarios: the SM Higgs acquiring a VEV, the multiplet acquiring a VEV, and both acquiring s VEVs, corresponding to vacuum X , W , and Z , respectively. Below are the key features demonstrated in this section:

- (1) The classification of topological field solutions when an $SU(2)$ multiplet Φ is added to the SM field content. If Φ acquires a VEV as shown at point W in Fig. 1, a monopole or sphaleron solution emerges depending on whether its hypercharge is zero or not. When the SM Higgs and Φ acquire VEVs simultaneously, the field solution is a sphaleron.
- (2) We construct the general-dimensional sphaleron and monopole matrices for the first time. The sphaleron matrix U_{sph} is constructed from the multiplication of

two Wigner matrices. Later, we demonstrate that the monopole matrix can be derived from the sphaleron matrix when the noncontractible loop parameter μ in U_{sph} equals $1/2$.

- (3) We provide a comprehensive comparison of the mechanisms for sphaleron and monopole-induced BNV. Although the mechanism for monopole-induced BNV has received less attention in EWBG discussions compared to the corresponding sphaleron-induced BNV, it is nevertheless relevant for the scenarios considered in this work.
- (4) We present a derivation of the sphaleron energy and monopole mass for a general electroweak multiplet and clarify the requirements on the corresponding boundary conditions used when obtaining the field equation solutions.

We also address other formal aspects in this section: a summary of the main choices of different sphaleron configurations and their relationships, and the computation of the $1/2$ baryonic charge of the sphaleron configuration.

A. Topological field solutions for a general-dimensional $SU(2)$ multiplet

We first discuss the situation when a single multiplet acquires a VEV after EWSB. The classification of field solutions is related to the mapping of static space at infinity (which is a two-sphere, S^2) to the vacuum manifold of the theory. The vacuum manifold, denoted by $\mathcal{M}$, is characterized by the coset space [25]:

$$\mathcal{M} = \frac{G}{H} = \frac{SU(2)_L \times U(1)_Y}{H}, \quad (2)$$

where G and H are the symmetry groups before and after EWSB, respectively. The residual symmetry group is related to the representation of the multiplet, specifically its hypercharge, Y . For a $Y=0$ multiplet, $H = U(1)_Y \times U(1)_{\text{em}}$, since the multiplet electric charge is the eigenvalue solely of T_3 . For a $Y \neq 0$ multiplet, $H = U(1)_{\text{em}}$, with generator $Q = T_3 + Y$, is the only remaining symmetry group. In the foregoing cases, the number of unbroken generators after EWSB are two and one, respectively.

For a $Y \neq 0$ multiplet, the resulting topological field solution is the sphaleron [26]:

$$\mathcal{M} = \frac{SU(2)_L \times U(1)_Y}{U(1)_{\text{em}}} \sim S^3, \quad \pi_3(\mathcal{M}) = \mathbb{Z}, \quad (3)$$

where the symbol $\sim$ denotes an isomorphism and $\pi_3(\mathcal{M})$ denotes the mapping of a three-sphere to $\mathcal{M}$ at spatial infinity. Such a three-sphere is formed by the noncontractible loop parameter μ and the spatial angular parameters θ and ϕ . Without the dependence on μ , the mapping would be $\pi_2(S^3)$, which is the null mapping associated with

continuous deformation to the vacuum configuration. Thus, the existence of the solution requires the existence of an additional parameter, μ to yield $\pi_3(S^3)$. As we will see when analyzing the impact of the sphaleron on topological charge, μ will play the role of time. For a given, fixed choice of μ , the sphaleron configuration can be continuously deformed into the vacuum configuration, since $\pi_2(S^3) = 0$, leading to the unstable nature of sphaleron.

Under these considerations, one may also write the sphaleron scalar and gauge field configuration, following the convention by [26]:

$$\Phi = \frac{v_\phi}{\sqrt{2}} \phi(\xi) U_{\text{sph}}(\mu, \theta, \phi) \cdot \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}, \quad (4)$$

$$A_i^a T^a dx^i = -\frac{i}{g} f(\xi) (\partial_i U_{\text{sph}}) U_{\text{sph}}^{-1} dx^i. \quad (5)$$

Here, v_ϕ denotes the VEV of the multiplet, while $\phi(\xi)$ and $f(\xi)$ are radial profile functions for the scalar and gauge field, with $\xi \equiv gv_\phi r$ being the dimensionless radial parameter. In the gauge field configuration, the i index in A_i^a , ∂_i , and dx^i is contracted, with $i \in [\theta, \phi]$ and $dx^i \in [d\theta, d\phi]$. The construction of U_{sph} for an arbitrary SU(2) representation J entails substantial technique details, which we defer to Eq. (18). For clarity, we first present the fundamental representation of U_{sph} ¹:

$$U_{\text{sph}}(\mu, \theta, \phi) = \begin{bmatrix} e^{i\mu}(\cos\mu - i\cos\theta\sin\mu) & e^{i\varphi}\sin\theta\sin\mu \\ -e^{-i\varphi}\sin\theta\sin\mu & e^{-i\mu}(\cos\mu + i\cos\theta\sin\mu) \end{bmatrix}. \quad (6)$$

For a $Y = 0$ multiplet, the resulting topological field solution is a monopole [25]:

$$\mathcal{M} = \frac{SU(2)_L \times U(1)_Y}{U(1)_{\text{em}} \times U(1)_Y} \sim S^2, \quad \pi_2(\mathcal{M}) = \mathbb{Z}, \quad (7)$$

with corresponding field profiles

$$\Phi = \frac{v_\phi}{\sqrt{2}} \phi(\xi) U_{\text{mon}}(\theta, \phi) \cdot \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}, \quad (8)$$

¹Strictly speaking, the sphaleron configuration is defined only at $\mu = \pi/2$. For other values of μ , U_{sph} gives a field ansatz for A_i^a .

$$A_i^a T^a dx^i = -\frac{i}{g} f(\xi) (\partial_i U_{\text{mon}}) U_{\text{mon}}^{-1} dx^i, \quad (9)$$

where $\pi_2(\mathcal{M})$ denotes the mapping of a two-sphere to $\mathcal{M}$ at spatial infinity. Such mapping is stable, implying that the monopole is stable. In contrast to the $Y \neq 0$ case, stability of the mapping requires no additional dependence on a variable that might play the role of time. Note that in Eq. (7) the $U(1)_Y$ factor is redundant since gauge invariance precludes renormalizable couplings of Φ to the fermion sector. Nevertheless, we include it here for pedagogical clarity.

Consider, now, the form of the “monopole matrix,” U_{mon} , and its relationship to U_{sph} . To that end, first note that U_{mon} depends on only the two spatial angular coordinates, θ and ϕ . Second, for the specific case when Φ is a real triplet (adjoint representation), we expect the monopole gauge field configuration to coincide with the 't Hooft-Polyakov monopole solution [27,28]:

$$A_0 = 0, \quad A_i^a = c_1 \epsilon_{iab} x^b \frac{F(\xi)}{r^2}, \quad (10)$$

where A_0 is the temporal gauge field and A_i^a is the spatial gauge field with spatial index i and isospin index a , c_1 is a normalization constant, and $F(\xi)$ is a dimensionless radial function. Below, we show explicitly how Eq. (10) follows from Eq. (9) for any $Y = 0$ scalar multiplet representation.

We will shortly show how to derive U_{mon} starting from U_{sph} . In doing so, it is also helpful to review the various equivalent choices of the sphaleron field configuration that have appeared in the literature. The configuration given in Eqs. (4) and (5) follows Manton and Klinkhamer’s convention, which we denote as the MK configuration. There is another widely used sphaleron configuration, proposed by Klinkhamer and Laterveer [29], which we denote as the KL configuration. We now show the equivalence of this KL field configuration with the MK configuration. Applying a unitary transformation $(U_{\text{sph}})^{-1}$ to the MK configuration in Eq. (4), the multiplet field becomes

$$\Phi \rightarrow \frac{v_\phi}{\sqrt{2}} \phi(\xi) \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}. \quad (11)$$

Similarly, applying a unitary transformation $(U_{\text{mon}})^{-1}$ to the monopole multiplet configuration, Eq. (8), yields the identical form of the multiplet configuration as in Eq. (11). In both cases, the gauge field transforms in the usual way: $A_\mu^a T^a \rightarrow U A_\mu^a T^a U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}$, implying that

$$\begin{aligned}
A_i^a T^a &\rightarrow U^{-1} \left(-\frac{i}{g} f(\xi) (\partial_i U) U^{-1} \right) U - \frac{i}{g} (\partial_i U^{-1}) U \\
&= \frac{i}{g} (1 - f(\xi)) U^{-1} \partial_i U,
\end{aligned} \tag{12}$$

where U denotes either U_{sph} or U_{mon} .

The forms for the scalar multiplet and gauge field configurations in Eqs. (11) and (12) are the same as the phase II KL configuration under the constraint that $f_0 = 0$ and $f_3 = f$, where f_0 and f_3 are the $U(1)$ field and A_i^3 field radial profile functions. We provide the full expressions of KL configurations and definitions of the three phases of noncontractible loop (NCL) later, in Sec. II C.

Despite this similarity of forms, the full parameter dependencies of U_{sph} and U_{mon} differ significantly, as dictated by the aforementioned topological considerations. The sphaleron matrix is parametrized by spherical angular parameters θ , ϕ , and the NCL parameter μ , while the monopole matrix carries no μ dependence. For each case we may write

$$iU^{-1} \partial_i U dx^i = \sum_{a=1}^3 F_a T^a, \tag{13}$$

where T^a are $SU(2)$ generators and F_a are scalar functions of the relevant parameters.

Focusing now on the sphaleron case, $U = U_{\text{sph}}$, for Φ in the fundamental representation the F_a are given as²

$$\begin{aligned}
F_1 &= -(2\sin^2 \mu \cos(\mu - \phi) - \sin 2\mu \cos \theta \sin(\mu - \phi)) d\theta \\
&\quad - (\sin 2\mu \cos(\mu - \phi) \sin \theta + \sin^2 \mu \sin 2\theta \sin(\mu - \phi)) d\phi, \\
F_2 &= -(2\sin^2 \mu \sin(\mu - \phi) + \sin 2\mu \cos \theta \cos(\mu - \phi)) d\theta \\
&\quad + (-\sin 2\mu \sin(\mu - \phi) \sin \theta + \sin^2 \mu \sin 2\theta \cos(\mu - \phi)) d\phi, \\
F_3 &= -\sin 2\mu \sin \theta d\theta + 2\sin^2 \theta \sin^2 \mu d\phi.
\end{aligned} \tag{14}$$

It is useful to provide a general construction of U_{sph} and the 1-form F_a applicable to a general scalar $SU(2)_L$ multiplet of arbitrary isospin J .³ To that end, it is natural to consider the parametrization of an arbitrary $SU(2)$ matrix in terms of the Wigner-D matrix. We consider the general sphaleron matrix:

$$U_{\text{sph}}(\alpha, \beta, \gamma) = \mathcal{D}^{1/2}(\alpha, \beta, \gamma) = e^{-i\alpha T^3} e^{-i\beta T^2} e^{-i\gamma T^3}, \tag{15}$$

²Analogous to Eq. (6), the sphaleron configuration is defined only at $\mu = \pi/2$.

³In passing, we note that Ahriche *et al.* [30] calculate the sphaleron energy for higher dimensional $SU(2)$ scalar representations, wherein they use but do not prove that the 1-form F_a is invariant concerning different representation dimensions. We will expand on their work by showing this invariance.

where the Euler angles α , β , and γ are representation independent. To determine their value, we first express this in the fundamental representation:

$$\begin{aligned}
\mathcal{D}^{1/2}(\alpha, \beta, \gamma) &= e^{-i\alpha \frac{\sigma_3}{2}} e^{-i\beta \frac{\sigma_2}{2}} e^{-i\gamma \frac{\sigma_3}{2}} \\
&= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \end{pmatrix}.
\end{aligned} \tag{16}$$

We then equate this matrix with the sphaleron matrix in Eq. (6); we can obtain the following relationships:

$$\begin{aligned}
\cos \left(\frac{\beta}{2} \right) \cos \left(\frac{\alpha}{2} + \frac{\gamma}{2} \right) &= 1 + \sin^2 \mu (\cos \theta - 1), \\
\sin \left(\frac{\beta}{2} \right) \sin \left(\frac{\alpha}{2} - \frac{\gamma}{2} \right) &= \sin \phi \sin \theta \sin \mu, \\
\sin \left(\frac{\beta}{2} \right) \cos \left(\frac{\alpha}{2} - \frac{\gamma}{2} \right) &= -\cos \phi \sin \theta \sin \mu, \\
\cos \left(\frac{\beta}{2} \right) \sin \left(\frac{\alpha}{2} + \frac{\gamma}{2} \right) &= \sin \mu \cos \mu (\cos \theta - 1).
\end{aligned} \tag{17}$$

We obtain these relations by first expanding Eqs. (6) and (16) as a linear combination of the basis matrices $I_{2 \times 2}$, σ_1 , σ_2 , and σ_3 , and then equating the corresponding expansion coefficients. While it is possible in principle to solve these equations and establish relationships between (α, β, γ) and (μ, θ, ϕ) , doing so in practice is cumbersome. Not only must we be careful with the sign of the final solution of three Euler angles, but also they have nonlinear dependence with μ, θ, ϕ , complicating the calculation of the F_a in Eq. (13). Therefore, although Eq. (15) looks quite intuitive, we seek an alternative method.

Instead, we can use the multiplication of multiple Wigner-D matrices to represent the sphaleron matrix. For a general representation J with matrix dimension $2J + 1$, we can write the sphaleron matrix as

$$\begin{aligned}
U_{\text{sph}}(\mu, \theta, \phi) &= D^J(\omega_-, -\theta, \mu) D^J(\mu, \theta, \omega_+) \\
&= e^{-i\omega_- T^3} e^{i\theta T^2} e^{-2i\mu T^3} e^{-i\theta T^2} e^{-i\omega_+ T^3},
\end{aligned} \tag{18}$$

with

$$\omega_{\pm} = -\mu \pm \left(\phi - \frac{\pi}{2} \right). \tag{19}$$

If we set $J = 1/2$, we can restore the standard sphaleron matrix Eq. (6). The higher-dimensional form of U_{sph} then follows directly from Eq. (18) by replacing the $SU(2)$ generators T^i with their higher-dimensional counterparts. For example, for $J = 1$, one obtains U_{sph} by substituting the following expressions for the generators T^2 and T^3 :

$$T^2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (20)$$

With this kind of parametrization method it is relatively straightforward to calculate the 1-form, since the Euler parameters are linearly related to μ , θ , and ϕ . The 1-form F_a can then be calculated through

$$F_a = [\text{Tr}(T^3)^2]^{-1} \text{Tr}[iU_{\text{sph}}^{-1} \partial_i U_{\text{sph}} \cdot T^a] dx^i. \quad (21)$$

Using this calculation method, we have verified explicitly that one obtains the same F_a for $J = [1, 3/2, 2, 5/2, 3]$ under the SU(2) generator representation [31].

Now consider the construction of the monopole matrix for a general SU(2) multiplet. We start by multiplying the sphaleron matrix with $\mu = \frac{\pi}{2}$ by a (θ, ϕ) -independent matrix U_R :

$$U_{\text{mon}}(\theta, \phi) = U_{\text{sph}}\left(\mu = \frac{\pi}{2}, \theta, \phi\right) \cdot U_R. \quad (22)$$

A key question is how we can test whether U_{mon} is correct for monopole field configurations. As derived in Eq. (12), the monopole gauge field configuration in KL convention reads as

$$A_i^a T^a dx^i = \frac{1 - f(\xi)}{g} iU_{\text{mon}}^{-1} (\partial_i U_{\text{mon}}) dx^i. \quad (23)$$

Substituting the expression (22) into (23), the monopole gauge field becomes

$$\begin{aligned} A_i^a T^a dx^i &= \frac{1 - f(\xi)}{g} U_R^{-1} \cdot [iU_{\text{sph}}^{-1} (\partial_i U_{\text{sph}}) dx^i] \cdot U_R \\ &= \frac{1 - f(\xi)}{g} U_R^{-1} \left[\sum_{i=1}^3 F_a T^a \right] U_R. \end{aligned} \quad (24)$$

Note that we do not introduce an additional matrix U_L multiplying $U_{\text{sph}}(\mu = \frac{\pi}{2}, \theta, \phi)$ on the left in Eq. (22), since $U_L^{-1} \cdot U_L = I$ always occurs in Eq. (23). We now take the specific form for U_R with⁴

$$U_R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & (-1)^{2J-1} & \cdots & 0 & 0 \\ (-1)^{2J} & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (25)$$

⁴The U_R matrix actually transforms the SU(2) generators into their complex conjugate representation, with $T^{a*} = -U_R^{-1} T^a U_R$, given in the Appendix of Ref. [32].

where J is the dimension of the scalar multiplet representation (i.e., $J = 1$ for the $Y = 0$ complex triplet). Using this choice for U_R , we find that Eq. (24) leads to the gauge field configuration:

$$A_i^a = \epsilon^{iab} x^b F, \quad (26)$$

where we have explicitly obtained this result for $J = [1, 3/2, 2, 5/2, 3]$, where

$$F = \frac{2(-1 + f(\xi))}{gr^2} \quad (27)$$

and where this form is exactly the 't Hooft-Polyakov monopole gauge field configuration.

To conclude this subsection, we discuss the field solution when both the SM Higgs and the multiplet obtain VEVs. We parametrize the Higgs field, H , and the multiplet field, Φ , as

$$H = \frac{v}{\sqrt{2}} h(\xi) \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}, \quad \Phi = \frac{v_\phi}{\sqrt{2}} \phi(\xi) \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad (28)$$

where v and v_ϕ represent the Higgs and multiplet VEVs, and $h(\xi)$ and $\phi(\xi)$ are the Higgs and multiplet radial profile functions, respectively. We impose model parameter constraints to ensure that the configuration in Eq. (28) minimizes the potential. In this way, Φ is locked to the Higgs doublet at infinity, and also at finite radius through the radial profile functions (which can differ but must share the same boundary conditions). Therefore, the topology of the vacuum manifold remains S^3 , ensuring that the sphaleron solution exists regardless of the hypercharge of Φ .

B. Mechanisms for sphaleron and monopole induced baryon number violation

In this section, we review the roles of sphalerons and monopoles in the process of BNV and the corresponding expressions for the BNV rate. The fundamental principle underlying both sphaleron- and monopole-induced BNV is the Adler-Bell-Jackiw anomaly [33,34]. For a theory that conserves the baryon minus lepton number (like the electroweak theory), and for each fermion generation, the observable variation of baryonic and leptonic currents is given by

$$\partial_\mu j_B^\mu = \partial_\mu j_L^\mu = \frac{g^2}{32\pi^2} F^{\mu\nu a} \tilde{F}_{\mu\nu}^a, \quad (29)$$

where g is the $SU(2)_L$ coupling constant and $\tilde{F}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta a}$. The right-hand side of Eq. (29) can be written as a total derivative,

$$\frac{g^2}{32\pi^2}F^{\mu\nu a}\tilde{F}_{\mu\nu}^a = \partial_\mu K^\mu, \quad (30)$$

where K^μ is defined as

$$K^\mu \equiv \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(F_{\nu\rho}^a A_\sigma^a - \frac{g}{3}f^{abc}A_\nu^a A_\rho^b A_\sigma^c\right), \quad (31)$$

with f^{abc} being the non-Abelian gauge group structure constant and $F_{\nu\rho}^a$ being the gauge field strength tensor

$$F_{\nu\rho}^a = \partial_\nu A_\rho^a - \partial_\rho A_\nu^a + gf^{ade}A_\nu^d A_\rho^e. \quad (32)$$

We further define a function $\mathcal{P}(t_a, t_b)$ of general initial and final time boundaries, denoted by t_a and t_b :

$$\begin{aligned} \mathcal{P}(t_a, t_b) &\equiv \int_{t_a}^{t_b} dt \int_{-\infty}^{+\infty} d^3x \partial_\mu K^\mu \\ &= \int d^3x K^0|_{t=t_a}^{t=t_b} + \int_{t_a}^{t_b} dt \int_S \vec{K} \cdot \overrightarrow{dS}, \end{aligned} \quad (33)$$

where the spatial part is integrated over all space, and where we applied the divergence theorem to K^i in the second line.

The winding number ω can be defined using $\mathcal{P}(t_a, t_b)$:

$$\omega \equiv \mathcal{P}(t_a = -\infty, t_b = +\infty), \quad (34)$$

which equals an integer for a dynamic process starting and ending at topologically distinct vacua.

1. The sphaleron induced baryon number violation

We start with the definition of the sphaleron baryonic charge. According to Eqs. (29), (30), and (33), the $\mathcal{P}(t_a, t_b)$ has the following relation with baryonic current j_B^μ :

$$\mathcal{P}(t_a, t_b) = \int d^3x j_B^0|_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \int_{-\infty}^{+\infty} d^3x \nabla \cdot \vec{j}_B. \quad (35)$$

The sphaleron baryonic charge Q_B is then defined as

$$\begin{aligned} Q_B &\equiv \int d^3x j_B^0|_{-\infty}^{t_0} \\ &= - \int_{-\infty}^{t_0} dt \int d^3x \nabla \cdot \vec{j}_B + \int_{-\infty}^{t_0} dt \int d^3x \partial_\mu K^\mu, \end{aligned} \quad (36)$$

where t_0 represents the sphaleron point corresponding to the top of the energy barrier in configuration space, connecting topologically distinct vacua. Assuming as in [35] that $\vec{j}_B$ exhibits only standard (topologically trivial) properties, we can neglect the $\nabla \cdot \vec{j}_B$ term in the baryonic

charge definition. Henceforth, in evaluating Q_B [Eq. (36)], we shall restrict our attention to the contribution arising from $\partial_\mu K^\mu$, which is topologically nontrivial. We will show that the Q_B takes half integer values.

In the sphaleron configuration, the noncontractible loop parameter μ is a dynamic variable, which takes different values along the time variable entering $\mathcal{P}(t_a, t_b)$. Regarding μ as a monotonically increasing function of t [36,37], we can replace the time boundaries in $\mathcal{P}(t_a, t_b)$ with the corresponding values of μ : $\mathcal{P}(\mu(t_a), \mu(t_b))$. Thus, the variation of μ in the sphaleron field configuration leads into a variation of winding number and sphaleron baryonic charge.

Now we will demonstrate the sphaleron half integer baryonic charge property. According to Eqs. (36) and (33), the sphaleron baryonic charge Q_B reads as

$$Q_B = \int d^3x K^0|_{t=-\infty}^{t=t_0} + \int_{-\infty}^{t_0} dt \int d^3x \nabla \cdot \vec{K}. \quad (37)$$

At $t = -\infty$, the gauge fields are in vacuum states, while at $t = t_0$ their configuration corresponds to being at the top of the field energy barrier separating two adjacent topologically distinct vacua. We will compute both the first K^0 term and the second surface term in Eq. (37) using the KL configuration, where the gauge field falls off as $1/r$. In the KL configuration, we choose

$$t = -\infty, \quad \mu = 0; \quad t = t_0, \quad \mu = \frac{\pi}{2}; \quad t = \infty, \quad \mu = \pi, \quad (38)$$

where the same relation of μ and t also applies for the MK configuration. Note that the definition of Q_B is only related with the gauge field, so phases I and III of the KL configuration, wherein only the scalar field varies, are not relevant (see Sec. II C below for a discussion of the KL configuration).

Now we label the sphaleron baryonic charge in the KL configuration as $Q_B(\text{KL})$. Under this convention, the gauge field is given by

$$\begin{aligned} A_i^a(\text{KL})T^a dx^i &= \frac{1}{g}(1 - f(\xi))[F_1 T^1 + F_2 T^2] \\ &\quad + \frac{1}{g}(1 - f_3(\xi))F_3 T^3, \end{aligned} \quad (39)$$

where $f(\xi)$ and $f_3(\xi)$ are the radial profile functions of the $SU(2)$ gauge fields, and the 1-form F_a are defined in Eq. (14). Substituting this gauge field configuration into K^μ [Eq. (31)] yields $K^0 = 0$. Consequently, the $\nabla \cdot \vec{K}$ term in Q_B alone generates the entire baryonic charge:

$$\begin{aligned}
Q_B(\text{KL}) &= \int_{-\infty}^{t_0} dt \int d^3x \nabla \cdot \vec{K} \\
&= - \int_{-\infty}^{t_0} dt \frac{4}{3\pi} \left[(-1 + f(\xi))^2 + \frac{1}{2} (-1 + f_3(\xi))^2 \right] \\
&\quad \times \sin^2(\mu(t)) \frac{d\mu(t)}{dt} \Big|_{r=0}^{r=\infty}. \quad (40)
\end{aligned}$$

Here, we have (i) expanded $\nabla \cdot \vec{K}$ in spherical coordinates, and (ii) substituted the field configuration [Eq. (39)] into K^i . By imposing the boundary conditions $f(0) = f_3(0) = 0$ and $f(\infty) = f_3(\infty) = 1$ [see Eq. (69)], one obtains

$$Q_B(\text{KL}) = \frac{1}{\pi} \left(\mu - \frac{1}{2} \sin(2\mu) \right) \Big|_{\mu=0}^{\mu=\frac{\pi}{2}} = \frac{1}{2}. \quad (41)$$

Reference [35] argues that Q_B is gauge invariant, so that its value should remain unchanged even in other sphaleron configurations, such as the MK configuration, that are related to the KL configuration through a gauge transformation. Through explicit computation, we find that this argument holds up to an overall sign, and that evidently only the absolute value of Q_B stays invariant, i.e., $|Q_B| = 1/2$. For example, in the MK configuration [see Eq. (5)], the value of Q_B reads (again, $K^0 = 0$) as

$$\begin{aligned}
Q_B(\text{MK}) &= \int_{-\infty}^{t_0} dt \int d^3x \nabla \cdot \vec{K} \\
&= - \int_{-\infty}^{t_0} dt \frac{2}{\pi} f^2(\xi) \times \sin^2(\mu(t)) \frac{d\mu(t)}{dt} \Big|_{r=0}^{r=\infty}, \quad (42)
\end{aligned}$$

where we follow the same computational strategy as for $Q_B(\text{KL})$, but now impose the MK constraint $f_3(\xi) = f(\xi)$. Finally, applying the boundary conditions $f(0) = 0$ and $f(\infty) = 1$, yields

$$Q_B(\text{MK}) = -\frac{1}{\pi} \left(\mu - \frac{1}{2} \sin(2\mu) \right) \Big|_{\mu=0}^{\mu=\frac{\pi}{2}} = -\frac{1}{2}. \quad (43)$$

Compare the second line of Eqs. (40) and (42); we see that the expression of Q_B is different in these two configurations.⁵ Moreover, its value depends explicitly on the choice of field boundary conditions. A complete analysis of the boundary conditions is carried out later in Sec. II C, where we show that our choice of boundary conditions used here is unique in order to obtain a finite sphaleron energy. Such a sign difference may or may not affect the BNV process, and we defer such implication to future studies.

We also note that, as demonstrated in Ref. [35], an additional gauge transformation (U_c) can be applied to

make the gauge field fall off faster than $1/r$ so that the surface term in Eq. (37) vanishes,

$$U_c = \exp((-1)^n i \Theta(r) \vec{T} \cdot \hat{x}), \quad (44)$$

where $n = 1$ or 2 depending on the convention of sign; where $\Theta(r)$ runs rapidly from 0 to π when r runs from 0 to ∞ ; where $\vec{T}$ are the SU(2) generators, and $\hat{x} = \vec{x}/r$. In such a gauge, the nonvanishing Q_B arises from the first term in Eq. (37). In this scenario (see Appendix B), the MK configuration produces $Q_B = +\frac{1}{2}$ for $n = 1$ and $Q_B = -\frac{1}{2}$ for $n = 2$.

To end this subsection, we now recall the rate for BNV violation associated with transitions between adjacent vacua that pass through the configuration with $Q_B = 1/2$. Following Ref. [15] (and references therein) we write

$$\frac{\partial \bar{n}_B}{\partial t} + 3H \bar{n}_B = -k(t) \bar{n}_B, \quad (45)$$

where $\bar{n}_B$ is the net baryon number density, $\bar{n}_B \equiv n_B - n_{\bar{B}}$, H is the Hubble constant, and

$$k(t) = \frac{13n_f \Gamma_{\text{sph}}}{2 VT^3}, \quad (46)$$

with n_f being the number of fermion generations, V being the space volume, and

$$\Gamma_{\text{sph}} = A_{\text{sph}} e^{-E_{\text{sph}}/T}. \quad (47)$$

Here, E_{sph}/T is the leading-order action evaluated at the sphaleron point; A_{sph} collects the quantum fluctuations and represents the higher-order corrections to the sphaleron rate. We will elaborate upon the discussion of E_{sph} after the corresponding discussion of the monopole-induced BNV.

2. Monopole induced baryon number violation

In contrast to the sphaleron, the monopole is a topologically stable configuration. In the monopole background, the following scattering process can occur⁶:

$$u + u \rightarrow \bar{d} + e^+, \quad (48)$$

where u , $\bar{d}$, and e^+ denote the up quark, antidown quark, and positron, respectively. This process changes the baryon plus lepton number by $\Delta(B + L) = -2$. As demonstrated by Rubakov and Callan [38–40], it is the nonstatic fluctuation fields around the static monopole solution that give rise to such $B + L$ violation scattering. We write these quantum fluctuations as $a_\mu(r, t)$, with

⁵For KL configuration with $f_3(\xi) = f(\xi)$, we will obtain a profile factor of $\frac{2}{\pi}(-1 + f(\xi))^2$ in the second line of $Q_B(\text{KL})$, which is different from that in the MK case, the $\frac{2}{\pi}f^2(\xi)$.

⁶To simplify the presentation, we have omitted the chirality and potential flavor quantum numbers of quarks and leptons.

$$\begin{aligned}
A_0 &= T^a \hat{x}^a a_0(r, t)/i, \\
A_i &= T^a \hat{x}^a \hat{x}_i a_1(r, t)/i + A_i^{\text{cl}}, \\
\Phi &= \Phi^{\text{cl}},
\end{aligned} \tag{49}$$

where T^a are the SU(2) generators, $\hat{x} = \vec{x}/r$, and A_i^{cl} and Φ^{cl} are the classical background monopole solutions following the notations in Refs. [39]. The fluctuation fields $a_0(r, t)$ and $a_1(r, t)$ obey

$$a_0(r, \pm\infty) = a_1(r, \pm\infty) = 0. \tag{50}$$

Importantly, there exist certain fluctuation configurations obeying these boundary conditions that (1) yield the integer winding number; and (2) lead the action functional to be arbitrary small. The second property implies that the fermion number nonconserving matrix element is not suppressed by a factor of $\exp(-\text{const}/\text{coupling}^2)$ [39].

The monopole induced BNV comes from the scattering process of the s -wave fermions and monopole, with the corresponding cross section denoted as $\sigma_{\Delta B \neq 0}$. In general, one would expect this cross section to depend on the monopole size, the scattering kinematics (e.g., center of mass scattering energy), and possibly a nonperturbative exponential function of the gauge coupling, g . References [38–40] show that $\sigma_{\Delta B \neq 0}$ is not suppressed either by the geometric size of the monopole or by a nonperturbative factor $\exp(-f(v)/g^2)$, with $f(v)$ being some constant function of the multiplet VEV, v . Denoting the remaining relevant energy scale as the “fermion energy,” E_f , one may thus write [41]

$$\sigma_{\Delta B \neq 0} \simeq v^{-1} \frac{c}{E_f^2}, \tag{51}$$

where v is the monopole-fermion relative velocity and c is an appropriate constant. This $\sigma_{\Delta B \neq 0}$ plays a similar BNV role as that of Γ_{sph} in the sphaleron case as encoded in the right-hand side of Eq. (45), although they have distinct features that we now delineate.

For a Boltzmann equation description of monopole-catalyzed BNV, analogous to Eq. (45), one requires in addition to the cross section the monopole and fermion number densities in the early Universe plasma. To arrive at the former, we observe that during the period of EWSB, monopoles can be created through the bubble collision mechanism [25] (if EWSB occurs via a first-order phase transition) or the monopole thermal pair production mechanism [20]. Henceforth, we assume a first-order EWPT. It has been estimated that the thermal monopole antimonopole pair production rate exceeds the bubble collision rate over a wide range of parameters [20]. The thermal monopole antimonopole pair production mechanism leads to an equilibrium monopole density of

$$\left. \frac{n_M}{T^3} \right|_{\text{eq.}} = \left(\frac{m_M(T)/T_{\text{nuc}}}{2\pi} \right)^{3/2} e^{-m_M(T)/T}, \tag{52}$$

where n_M is the monopole density, $m_M(T)$ is the monopole mass, and T_{nuc} is the nucleation temperature.

The resulting Boltzmann equation for monopole-catalyzed BNV is given by

$$\frac{d\bar{n}_B}{dt} + 3H\bar{n}_B = -\langle \sigma_{\Delta B \neq 0} v \rangle g_f \bar{n}_f n_M, \tag{53}$$

which is derived in Appendix D, where $\langle \dots \rangle$ denotes the thermal average of the enclosed quantity, g_f is the number of fermion degrees of freedom (d.o.f.), n_f is the net fermion number density, $\bar{n}_f \equiv n_f - n_{\bar{f}}$, and n_M is the monopole density. In this study, we assume the monopole density is in thermal equilibrium, given by Eq. (52), where an exponential suppression factor $e^{-m_M/T}$ appears.

In summary, for our current leading-order BNV rate computation, we focus only on the computation of the exponential factor, $e^{-E/T}$, where E refers to either sphaleron energy or monopole mass. We stress that the origin of $e^{-E/T}$ is different in these two cases. For the sphaleron case it comes from the sphaleron transition rate between adjacent vacua, while in the monopole case it arises from the equilibrium monopole density. We defer an exact computation of the monopole density as well as next-to-leading-order contribution to the sphaleron rate or monopole-fermion BNV scattering cross section to future work.

C. Sphaleron energy, monopole mass, and equations of motion

Our computation in this subsection is based on the KL sphaleron configuration, from which one may also derive the monopole configuration according to Eq. (22). To enhance the accuracy of the sphaleron energy computation, we include the contribution from the U(1) gauge field.⁷ For the monopole mass calculation, which pertains to a $Y = 0$ multiplet extension, the U(1) gauge field decouples and does not contribute at leading order. Nevertheless, we still include the U(1) field in the monopole configuration at the beginning as a more general setup and show explicitly how it decouples when $Y = 0$.

1. Sphaleron energy

The NCL in the KL configuration commences and terminates at topologically distinct vacua and comprises three phases [29]:

⁷An additional motivation for including the U(1) gauge field is its ability to impart an elliptic deformation to the otherwise spherical sphaleron explosion, thereby generating gravitational radiation [42].

- (1) I, $\mu \in [-\pi/2, 0]$: builds up the scalar field configuration;
- (2) II, $\mu \in [0, \pi]$: builds up and then destroys the gauge field configuration;
- (3) III, $\mu \in [\pi, 3\pi/2]$: destroys the scalar field configuration.

where $\mu = -\pi/2, 3\pi/2$ represent the vacuum state, while $\mu = \pi/2$ denotes the sphaleron state. The field *Ansätze* vary in different phases. In phase I and III, the profile functions are given by⁸

$$A_i^a T^a = a_i = 0, \quad (54)$$

$$\Phi = \frac{v_\phi(\sin^2\mu + \phi(\xi)\cos^2\mu)}{\sqrt{2}} \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}, \quad (55)$$

where a_i denotes the $U(1)$ gauge field. In phase II, the field configurations read as

$$\Phi = \frac{v_\phi}{\sqrt{2}} \phi(\xi) \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}, \quad (56)$$

$$A_i^a T^a dx^i = \frac{1}{g}(1-f(\xi))[F_1 T^1 + F_2 T^2] + \frac{1}{g}(1-f_3(\xi))F_3 T^3, \quad (57)$$

$$a_i dx^i = \frac{1}{g}(1-f_0(\xi))F_3, \quad (58)$$

where g' denotes the $U(1)$ gauge coupling constant; f and f_3 are the $SU(2)$ gauge field radial profile functions; f_0 is the $U(1)$ gauge field radial profile function; and F_1, F_2 , and F_3 are functions of θ and ϕ defined in Eq. (13).

In the following analysis, we construct the sphaleron energy using a single scalar multiplet extension to the SM, while referring to $V(H, \Phi)$ as a general object to demonstrate the main ideas. We define that $V(H, \Phi)$ vanishes in the limit $\xi \rightarrow \infty$. We will present the explicit interaction terms in the next section. The field energy can be written as

$$E = \frac{4\pi\Omega}{g} \int d\xi \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} f_{ij}^a f_{ij}^a + (D_i H)^\dagger (D_i H) + (D_i \Phi)^\dagger (D_i \Phi) + V(H, \Phi) \right], \quad (59)$$

where Ω is the zero temperature VEV for the SM Higgs field. The general expressions of the Yang-Mills term, $U(1)$ term, and covariant derivative term read as

$$\begin{aligned} \frac{1}{4} F_{ij}^a F_{ij}^a(\xi, \mu) &= \sin^2\mu \left(\frac{8}{3} f'^2 + \frac{4}{3} f_3'^2 \right) + \frac{8}{\xi^2} \sin^4\mu \left\{ \frac{2}{3} f_3'^2 (1-f)^2 + \frac{1}{3} \{f(2-f) - f_3\}^2 \right\}, \\ \frac{1}{4} f_{ij} f_{ij}(\xi, \mu) &= \frac{4}{3} \left(\frac{g}{g'} \right)^2 \left\{ f_0'^2 \sin^2\mu + \frac{2}{\xi^2} \sin^4\mu (1-f_0)^2 \right\}, \\ (D_i \Phi)^\dagger (D_i \Phi)(\xi, \mu) &= \frac{v_\phi^2}{\Omega^2} \left\{ \frac{1}{2} \xi^2 \phi'^2 + \frac{4}{3} \phi^2 \sin^2\mu \{ (J(J+1) - J_3^2)(1-f)^2 + J_3^2(f_0 - f_3)^2 \} \right\}. \end{aligned} \quad (60)$$

Note that the above expressions hold for phase II of the KL configurations, where $\mu \in [0, \pi]$. The Higgs covariant derivative term $(D_i H)^\dagger (D_i H)$ in Eq. (59) can be obtained from the expression of $(D_i \Phi)^\dagger (D_i \Phi)$ in Eq. (60) with the choice of $J = 1/2$ and $J_3 = 1/2$ and

the field notation $\phi \rightarrow h$. Thus, the sphaleron energy can be expressed as

$$\begin{aligned} E_{\text{sph}} &= E\left(\mu = \frac{\pi}{2}\right) \\ &= \frac{4\pi\Omega}{g} \int_0^\infty d\xi \left[\frac{1}{4} F_{ij}^a F_{ij}^a\left(\xi, \frac{\pi}{2}\right) + \frac{1}{4} f_{ij} f_{ij}\left(\xi, \frac{\pi}{2}\right) \right. \\ &\quad + (D_i H)^\dagger (D_i H)\left(\xi, \frac{\pi}{2}\right) + (D_i \Phi)^\dagger (D_i \Phi)\left(\xi, \frac{\pi}{2}\right) \\ &\quad \left. + V(H, \Phi)\left(\xi, \frac{\pi}{2}\right) \right]. \end{aligned} \quad (61)$$

⁸In general, the gauge fields need not vanish in phases I and III except at $\mu = -\frac{\pi}{2}$ and $\mu = \frac{3\pi}{2}$. Here, we impose that the gauge fields are nonzero only in phase II [29]. Although this restriction is not fully general, it simplifies the topological discussion in the preceding subsections and the computation of Q_B in Eq. (40), since the gauge fields then exist only for $\mu \in [0, \pi]$ under both the KL and MK conventions.

2. Monopole mass

Recall that for the monopole, its field configuration is stable, and there is no need for an NCL. According to Eq. (24), the monopole configuration can be constructed from the sphaleron configuration in phase II when $\mu = \pi/2$, viz.

$$\Phi = \frac{v_\phi}{\sqrt{2}} \phi(\xi) \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix},$$

$$\begin{aligned} A_i^a T^a dx^i &= \frac{1}{g} (1 - f(\xi)) U_R^{-1} \cdot [F_1 T^1 + F_2 T^2] \cdot U_R \Big|_{\mu=\frac{\pi}{2}} \\ &\quad - \frac{1}{g} (1 - f_3(\xi)) U_R^{-1} \cdot F_3 T^3 \cdot U_R \Big|_{\mu=\frac{\pi}{2}}, \\ a_i dx^i &= \frac{1}{g} (1 - f_0(\xi)) F_3 \Big|_{\mu=\frac{\pi}{2}}, \end{aligned} \quad (62)$$

where U_R is a constant matrix defined in Eq. (25); the $\mu = \pi/2$ condition is applied to the 1-form F_a .

Now we discuss the definition of monopole mass. When $\mu = \frac{\pi}{2}$, we find that the forms of the monopole field gauge configurations lead to Yang-Mills and $U(1)$ terms that are identical to those of the sphaleron case since the U_R rotation matrix has no effect on these two terms. On the other hand, for general values of J and J_3 , the presence of the U_R matrix would lead to a different scalar field kinetic term compared to the sphaleron case. However, the difference comes only from the coefficient of J_3 . For a $Y = 0$ multiplet, the neutral component that appears in the

monopole solution has $J_3 = 0$. For this case, one may also obtain the scalar kinetic term from the corresponding sphaleron case by setting $J_3 = 0$ and $Y = 0$. Thus, the monopole mass can be defined as

$$\begin{aligned} m_M &\equiv \frac{4\pi\Omega}{g} \int_0^\infty d\xi \left[\frac{1}{4} F_{ij}^a F_{ij}^a \left(\xi, \frac{\pi}{2} \right) + \frac{1}{4} f_{ij} f_{ij} \left(\xi, \frac{\pi}{2} \right) \right. \\ &\quad \left. + (D_i \Phi)^\dagger (D_i \Phi) \left(\xi, \frac{\pi}{2} \right) + V \left(H, \Phi \right) \left(\xi, \frac{\pi}{2} \right) \right]. \end{aligned} \quad (63)$$

In what follows, we present the equations of motion and corresponding boundary conditions as needed to obtain the sphaleron energy and monopole mass. Subsequently in Sec. IV, where we numerically solve the monopole field equations of motion and present the result in Fig. 4(b), we see that $f_0(\xi) = 1$ for all range of ξ , so the $U(1)$ field is totally decoupled as expected for the monopole case.

3. Equation of motion

For a general setup, we consider the computation of sphaleron equations of motion when both the SM Higgs and an additional multiplet obtain VEVs, which corresponds to sphaleron energy at point Z in Fig. 1. The sphaleron or monopole equations of motion at points X and W can be obtained through simplifications of this general setup.

The equations of motion (EOM) are differential equations governing the radial profile functions of the field ansatzes: f and f_3 for the $SU(2)$ gauge field, f_0 for the $U(1)$ gauge field, h for the SM Higgs field, and ϕ for the additional multiplet field. Similar to the case in Ahriche *et al.*'s work [30], the EOM reads as

$$\begin{aligned} f'' + \frac{2}{\xi^2} (1 - f) [f(f - 2) + f_3(1 + f_3)] + (1 - f) \left(\frac{v^2 h^2}{4\Omega^2} + \alpha \phi^2 \right) &= 0, \\ f_3'' - \frac{2}{\xi^2} [3f_3 + f(f - 2)(1 + 2f_3)] + \left(\frac{v^2}{4\Omega^2} h^2 + \beta \phi^2 \right) (f_0 - f_3) &= 0, \\ f_0'' + \frac{2}{\xi^2} (1 - f_0) - \frac{g^2}{g^2} \left(\frac{v^2}{4\Omega^2} h^2 + \beta \phi^2 \right) (f_0 - f_3) &= 0, \\ h'' + \frac{2}{\xi} h' - \frac{2}{3\xi^2} h [2(1 - f)^2 + (f_0 - f_3)^2] - \frac{1}{g^2 v^2 \Omega^2} \frac{\partial V[h, \phi]}{\partial h} &= 0, \\ \phi'' + \frac{2}{\xi} \phi' - \frac{8\Omega^2 \phi}{3v^2 \xi^2} [2\alpha(1 - f)^2 + \beta(f_0 - f_3)^2] - \frac{1}{g^2 v^2 \Omega^2} \frac{\partial V[h, \phi]}{\partial \phi} &= 0, \end{aligned} \quad (64)$$

where f' denotes $df/d\xi$ and f'' denotes $d^2f/d\xi^2$. In the zero temperature computation, we set $v = \Omega = 246.22$ GeV. However, at finite temperatures, v is a function of the temperature; and α and β are defined as

$$\alpha = \frac{[J(J + 1) - J_3^2] v_\phi^2}{2\Omega^2}, \quad \beta = \frac{J_3^2 v_\phi^2}{\Omega^2}, \quad (65)$$

where J denotes the multiplet representation dimension, and J_3 is the third component value. Since we put the

multiplet's VEV in its neutral component, J_3 equals to the opposite value of hypercharge Y . The only as yet unspecified term in the EOMs in Eq. (64) is the potential term $V[h, \phi]$, which depends on the specific BSM model and type of EWPT.

Before concluding this section, we will clarify some subtleties regarding the EOM boundary conditions. The boundary condition for fields at spatial infinity is clear: each field should approach its vacuum. On the other hand, at the spatial origin, some subtleties would appear, depending on the choice of coordinate system. Working with spherical-polar coordinates, the usual criteria for boundary conditions can be summarized as [26,35]

- (1) when $\xi \rightarrow 0$, the fields are singularity-free,
- (2) when $\xi \rightarrow \infty$, the fields asymptotically approach the “vacuum state.” The “vacuum state” refers to the gauge and scalar field configurations making the Yang-Mills term and potential term vanish, and any derivative terms in the Hamiltonian need to vanish asymptotically for a finite energy configuration.

The above two criteria work well in the MK sphaleron configuration. One might wonder whether they still hold in the KL configuration, as given in Eq. (57). In particular, the first singularity-free condition at the origin requires $f(\xi) = f_3(\xi) = 1$ as $\xi \rightarrow 0$. We will shortly show that this requirement conflicts with the sphaleron finite energy condition. Therefore, it is necessary to investigate the boundary conditions of the equations of motion in detail. In this work, we will show that the first boundary condition is not necessary; the only essential condition comes from the finite energy requirement.

Let us elaborate on the singularity issue. According to the MK configuration, Eq. (5), U_{sph} is a function that contains the angular parameters θ, ϕ . When $r \rightarrow 0$, if the field does not vanish, the field would have some preferred angular direction at the origin, which can lead to a rotational singularity. In the following, we will demonstrate that such singularity is removable. As demonstrated in Eq. (12), a unitary gauge transformation can connect following two field configurations under the zero weak mixing angle scenario

$$-f(\xi)(\partial_i U)U^{-1} \xleftrightarrow{U} [1 - f(\xi)]U^{-1}\partial_i U, \quad (66)$$

which means such a gauge transformation can interchange the boundary condition at the origin and spatial infinity. For example, following two sets of boundary conditions can be converted to each other by a gauge transformation:

- (1) $\xi \rightarrow 0, f(\xi) \rightarrow 0; \xi \rightarrow \infty, f(\xi) \rightarrow 1;$
- (2) $\xi \rightarrow 0, f(\xi) \rightarrow 1; \xi \rightarrow \infty, f(\xi) \rightarrow 0.$

Thus, when $\xi \rightarrow 0$, the free of singularity condition is not strict, since we can always make such gauge transformation to remove the singularity. In fact, the only criterion for sphaleron or monopole field boundary condition is that the sphaleron/monopole has finite energy.

In the following, we use the sphaleron example for illustration. The sphaleron energy is an integral over dimensionless radial parameter ξ , given in Eq. (59), with the energy density in different terms given in Eq. (60). To ensure a finite sphaleron energy, we require

- (1) when $\xi \rightarrow 0$, all energy density components have finite values, and
- (2) when $\xi \rightarrow \infty$, all energy density components asymptotically approach zero, decaying more rapidly than $1/\xi$.

Criterion (i) requires that the coefficient proportional to $1/\xi^2$ vanishes in Eq. (60). In other words, when $\xi \rightarrow 0$,

$$f_3(1-f) = 0, \quad f(2-f) - f_3 = 0, \quad 1 - f_0 = 0; \quad (67)$$

when $\xi \rightarrow \infty$, condition (ii) requires that all terms except $1/\xi^2$ vanish:

$$f' = f'_3 = f'_0 = \phi' = 0, \quad 1 - f = 0, \quad f_0 - f_3 = 0. \quad (68)$$

Combining Eqs. (67) and (68), we have

$$\begin{aligned} \xi \rightarrow 0, \quad & f = f_3 = \phi = 0, \quad f_0 = 1 \\ \xi \rightarrow \infty, \quad & f = f_3 = f_0 = \phi = 1. \end{aligned} \quad (69)$$

Requiring f and f_3 to have identical boundary conditions [since both f and f_3 characterize the $SU(2)$ gauge field], one can verify that all other combinations lead to a trivial solution or that a stable numerical solution does not exist. We present the sphaleron numerical solutions under boundary conditions in Eq. (69) in Fig. 2 for the pure Standard Model scenario.

III. ELECTROWEAK SEPTUPLET EXTENSION TO THE SM: MODEL ANALYSIS

In this section, we will analyze the scalar septuplet extension to the SM under different EWPT scenarios, using the formalism outlined in Sec. II. As a prelude, let us review the motivation for focusing on the scalar septuplet. In general, for an electroweak multiplet having isospin J , J cannot be arbitrarily large. When $J \geq 5$, the scale at which the gauge coupling Landau pole occurs would decrease to around $\Lambda_{\text{Landau}} \leq 10$ TeV [43]. Furthermore, the partial wave unitarity condition for tree-level scattering amplitude constrains $J \leq 7/2$ for a complex scalar multiplet and $J \leq 4$ for a real scalar multiplet [44,45]. We further focus on the possibility that the neutral component, having $J_3 + Y = 0$, contributes to the dark matter relic density. In order to avoid stringent experimental dark matter direct detection constraints, we require that the neutral field does not couple to the Z boson, which requires $Y = 0$. Since only a multiplet with integer J can have a $J_3 = 0$ component, we will focus on this scenario. Such an electroweak multiplet with vanishing VEV can be a dark matter candidate [32,46].

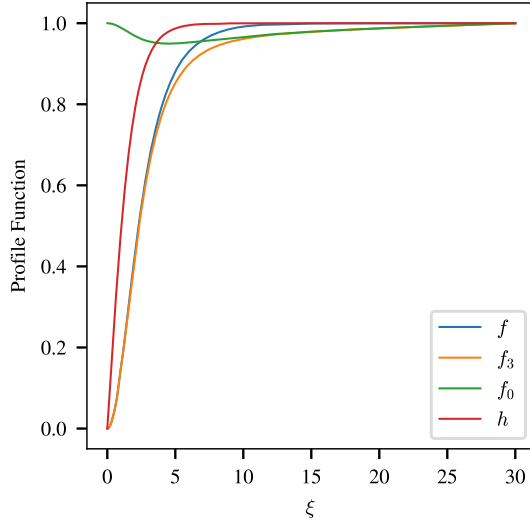


FIG. 2. The solutions of profile functions in the Standard Model. The set of equations of motion (64) is solved using the `solve_bvp` function from the open-source PYTHON package `scipy.integrate`. The horizontal axis is a dimensionless radial parameter $\xi = g\Omega r$, where $\Omega = 246$ GeV and g is the SU(2) gauge coupling constant. The corresponding sphaleron energy $B_{\text{sph}} = 1.900506$, where the formal expression $E_{\text{sph}} = B_{\text{sph}} \cdot (4\pi\Omega)/g$ is defined in Eq. (98).

Thus, the highest dimension for an electroweak multiplet satisfying the unitary condition and providing a viable dark matter candidate is the septuplet with $J = 3$ [46]. In addition, being a member in the minimal dark matter scenario [46], the neutral septuplet can be a viable DM candidate without imposing a Z_2 symmetry by hand. Depending on the various patterns of EWSB in the septuplet extension, the monopole and the sphaleron solutions can each emerge at different points in cosmic history. We, thus, compute the corresponding sphaleron energy and monopole mass.

As discussed in the introduction, we consider three patterns of EWSB, as shown in Fig. 1. Figure 1(a) displays the one-step EWPT to the pure Higgs phase, where the additional scalar can change the sphaleron energy of the Higgs phases through thermal loops. In principle, the thermal loop corrections should also be included when analyzing patterns (b) or (c), since the EWSB occurs at the hot early Universe. The three-dimensional effective field theory (3DEFT) is a powerful analytic method to organize the thermal corrections [47–49]. There are recent applications of 3DEFT to the nucleation rate computation [50,51], whose results show that the thermal correction would bias the zero temperature four-dimensional model parameters (including VEV) to some extent. However, zero temperature analysis can still provide a useful baseline for subsequent $T > 0$ analyses. In our present work, we mainly aim to provide a methodology for applying the sphaleron and monopole formalism to different EWSB patterns, so the zero temperature analysis is a useful starting point.

When nonzero temperature effects are included, a similar analysis strategy can be applied to the thermal potential.

For our current zero-temperature analysis, we are more interested in cases (b) and (c). We label the VEVs of the scalar potential stationary points in Fig. 1 as, $X(v, 0)$, $W(0, v_\phi)$, and $Z(v_z, v_{z\phi})$. In general, $v_z \neq v$ and $v_{z\phi} \neq v_\phi$. Furthermore, when we parametrize the scalar fields and perform a model analysis, we usually regard the field VEVs as input parameters. Thus, for patterns (b) and (c), we cannot use one single model analysis strategy, since the required input VEVs and model parameter relationships may differ in different EWSB patterns. We will show two analysis strategies separately after the introduction of the model.

A. The model

The general potential of the SM Higgs H and another SU(2) multiplet Φ can be written as [32]

$$\begin{aligned}
 V = & M_A^2(\Phi^\dagger\Phi) + \{M_B^2(\Phi\Phi)_0 + \text{H.c.}\} \\
 & - \mu^2 H^\dagger H + \lambda(H^\dagger H)^2 + \lambda_1(H^\dagger H)(\Phi^\dagger\Phi) \\
 & + \lambda_2((\bar{H}H)_1(\bar{\Phi}\Phi)_1)_0 + [\lambda_3(\bar{H}H)_0(\Phi\Phi)_0 + \text{H.c.}] \\
 & + V_{\text{self}}(\Phi, \bar{\Phi}), \quad (70)
 \end{aligned}$$

with

$$\begin{aligned}
 V_{\text{self}}(\Phi, \bar{\Phi}) = & \sum_{J=0}^{2J} \kappa_k((\Phi\Phi)_k(\bar{\Phi}\bar{\Phi})_k)_0 \\
 & + \sum_{k=0}^{2J} \{\kappa'_k((\Phi\Phi)_k(\Phi\Phi)_k)_0 \\
 & + \kappa''_k((\bar{\Phi}\Phi)_k(\Phi\Phi)_k)_0 + \text{H.c.}\}, \quad (71)
 \end{aligned}$$

where J is the multiplet isospin index, and $J = 3$ is the septuplet case. The scalar multiplet self-interaction potential $V_{\text{self}}(\Phi, \bar{\Phi})$ may be important in solving the core-cusp problem [52,53]. The $\bar{H}$ and $\bar{\Phi}$ are the complex conjugate representation of H and Φ . As pointed out in [32], the terms $(\Phi\Phi)_1$, $(\Phi\Phi)_3$, and $(\Phi\Phi)_5$ vanish due to the property of Clebsch-Gordan coefficients. Therefore, for the self-interaction potential, only terms with $k \in [0, 2, 4, 6]$ have non-zero contributions. Furthermore, only terms with $k = 0, 2$ are independent for our septuplet example [32,54], which simplifies our model analysis.

B. One-step EWPT to the mixed phase

In this pattern, we parametrize the general complex Higgs field (H), septuplet field (Φ), and their complex conjugate representation ($\bar{H}, \bar{\Phi}$) as

$$H = \begin{pmatrix} \omega^+ \\ \frac{1}{\sqrt{2}}(v_z + h + i\pi) \end{pmatrix}; \quad \bar{H} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_z + h - i\pi) \\ -\omega^- \end{pmatrix}, \quad (72)$$

$$\Phi = \begin{pmatrix} \phi_{3,3} \\ \phi_{3,2} \\ \phi_{3,1} \\ \frac{1}{\sqrt{2}}(v_{z\phi} + \phi + i\pi_\phi) \\ \phi_{3,-1} \\ \phi_{3,-2} \\ \phi_{3,-3} \end{pmatrix}; \quad \bar{\Phi} = \begin{pmatrix} \phi_{3,-3}^* \\ -\phi_{3,-2}^* \\ \phi_{3,-1}^* \\ \frac{-1}{\sqrt{2}}(v_{z\phi} + \phi - i\pi_\phi) \\ \phi_{3,1}^* \\ -\phi_{3,2}^* \\ \phi_{3,3}^* \end{pmatrix}, \quad (73)$$

where v_z and $v_{z\phi}$ are VEVs of the Higgs field and septuplet field, respectively. We use the subscript z to denote the VEVs in the one-step EWPT (point Z of Fig. 1), while for the two-step EWPT analysis in Sec. III C, the VEVs at point X and W of Fig. 1 are denoted as v and v_ϕ , respectively.

We put the septuplet VEV into its neutral component. Additional constraints need to be applied if we put the Higgs and septuplet's VEVs both into real neutral components. This can be fulfilled by requiring that all the fluctuation fields (inside Higgs or septuplet) have non-negative mass-squared eigenvalues. Of course, one important constraint follows from the tadpole condition

$$\left. \frac{\partial V}{\partial x_i} \right|_{\forall x_i=0} = 0, \quad (74)$$

where $x_i \in [h, \pi, \omega^\pm, \phi, \pi_\phi, \phi_{3,j}, \phi_{3,j}^*]$; j denotes the various subscripts that appear in Φ ; and $\forall x_i = 0$ means set all the field fluctuations equal to zero after the partial derivative. Subsequently, we can obtain five parameter constraints:

$$\begin{aligned} \text{Im}(M_B^2) &= 0, \\ \text{Im}(\lambda_3) &= 0, \\ \text{Im}(\kappa_0'') - 2 \text{Im}(\kappa_0') + \frac{4[\text{Im}(\kappa_2'') - 2 \text{Im}(\kappa_2')]}{3\sqrt{5}} &= 0, \\ \mu^2 &= \lambda v_z^2 + \lambda_{13} v_{z\phi}^2, \\ M_A^2 - \frac{2}{\sqrt{7}} \text{Re}(M_B^2) &= -\lambda_s v_{z\phi}^2 - \lambda_{13} v_z^2, \end{aligned} \quad (75)$$

where the first three constraints arise from a single tadpole condition, which we will explain now. The tadpole condition $\partial V / \partial \pi = 0$ gives us

$$c_1 \text{term}_1 + c_2 \text{term}_2 + c_3 \text{term}_3 = 0, \quad (76)$$

where c_1 , c_2 , and c_3 are constants depending on the VEV of the Higgs field; term_1 , term_2 , and term_3 read as

$$\begin{aligned} \text{term}_1 &= \text{Im}(M_B^2), & \text{term}_2 &= \text{Im}(\lambda_3), \\ \text{term}_3 &= \text{Im}(\kappa_0'') - 2 \text{Im}(\kappa_0') + \frac{4[\text{Im}(\kappa_2'') - 2 \text{Im}(\kappa_2')]}{3\sqrt{5}}. \end{aligned} \quad (77)$$

Thus, we have decomposed the single constraint in Eq. (76) into three separate constraints in Eq. (75), as to eliminate the mixing between h and π_ϕ and simplify our analysis. In addition, λ_{13} and λ_s are two combined parameters,

$$\lambda_{13} = \frac{1}{2} \lambda_1 - \frac{1}{\sqrt{14}} \lambda_3 \quad (78)$$

and

$$\begin{aligned} \lambda_s &= \frac{1}{7} [\kappa_0 + 2 \text{Re}(\kappa_0') - 2 \text{Re}(\kappa_0'')] \\ &+ \frac{4}{21\sqrt{5}} [\kappa_2 + 2 \text{Re}(\kappa_2') - 2 \text{Re}(\kappa_2'')]. \end{aligned} \quad (79)$$

As discussed in Ref. [32], λ_{13} enters the DM annihilation and direct detection rates, while λ_s characterizes DM self-interactions.

The mass terms in total potential can be expressed as a series addition of mass-squared matrices:

$$\begin{aligned} V(H, \Phi) &\supset \frac{1}{2} (h \quad \phi) \text{H}_{2 \times 2} \begin{pmatrix} h \\ \phi \end{pmatrix} + \frac{1}{2} (\pi \quad \pi_\phi) \text{Pi}_{2 \times 2} \begin{pmatrix} \pi \\ \pi_\phi \end{pmatrix} \\ &+ (\omega^+ \quad \phi_{3,1} \quad \phi_{3,-1}^*) \text{C1}_{3 \times 3} \begin{pmatrix} \omega^- \\ \phi_{3,1}^* \\ \phi_{3,-1} \end{pmatrix} \\ &+ (\phi_{3,2} \quad \phi_{3,-2}^*) \text{C2}_{2 \times 2} \begin{pmatrix} \phi_{3,2}^* \\ \phi_{3,-2} \end{pmatrix} \\ &+ (\phi_{3,3} \quad \phi_{3,-3}^*) \text{C3}_{2 \times 2} \begin{pmatrix} \phi_{3,3}^* \\ \phi_{3,-3} \end{pmatrix}, \end{aligned} \quad (80)$$

where we put the explicit expressions of mass-squared matrices into Appendix F. As expected, we observe a massless pseudoscalar particle and a massless charged Higgs particle after the computation of matrix eigenvalues. The matrix $\text{Pi}_{2 \times 2}$ has one nonzero eigenvalue, and $\text{C1}_{3 \times 3}$ has two nonzero eigenvalues.

Let us now enumerate the constraints that we need to apply. If this number plus the quantity of input parameters is less than or equal to the d.o.f. of model parameters, and all the parameters are involved in these constraints, we are free to move on. On the one hand, non-negative mass-squared matrix eigenvalues require at most nine constraints: one from $\text{Pi}_{2 \times 2}$, and two from each of the other four matrices. Also, we have five tadpole constraints, so we have 14 parameter constraints. On the other hand, we have 19 d.o.f. from the model, Eq. (70) (note that the complex parameter contributes 2 d.o.f., and we need to count the SM

two parameters, since the SM constraints are counted). So in total we can set at least five independent input parameters for this model. In principle, we might take these five input parameters to be v_z , $v_{z\phi}$, λ , λ_{13} , and λ_s , as they will appear in our later potential analysis. However, since the Higgs field mixes with the septuplet field, it is more convenient to use the mass eigenstate and mixing angle as input parameters. The two mass eigenstates h_1 and h_2 are given by

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} h \\ \phi \end{pmatrix}, \quad (81)$$

with the mass-squared eigenvalues m_1^2 and m_2^2 . We assign h_2 as a SM-like Higgs particle with $m_2 = 125.25$ GeV. The λ , λ_{13} , and λ_s can be expressed in terms of $\cos \theta$, m_1^2 , and m_2^2 . The relationships between these quantities are provided in Appendix F.

According to our analysis in Sec. II A, the formed topological field solution in the one-step EWPT to the mixed-VEV phase [scenario (c)] is the sphaleron, whose energy will be computed below. According to the sphaleron scalar field configuration, Eq. (56), we need to set all the fluctuation fields in Eqs. (72) and (73) equal to zero. Making the following replacement,

$$v_z \rightarrow h[\xi]v_z, \quad v_{z\phi} \rightarrow \phi[\xi]v_{z\phi}, \quad (82)$$

we can obtain the final potential formula in one-step EWPT as

$$\begin{aligned} V_{\text{One}}\left(\xi, \mu = \frac{\pi}{2}\right) &= \frac{1}{2}v_{z\phi}^2\phi[\xi]^2[\lambda_{13}v_z^2h(\xi)^2 - (v_{z\phi}^2\lambda_s + \lambda_{13}v_z^2)] \\ &\quad + \frac{1}{4}v_z^2h[\xi]^2[\lambda v_z^2h(\xi)^2 - 2(\lambda v_z^2 + \lambda_{13}v_{z\phi}^2)] \\ &\quad + \frac{1}{4}v_{z\phi}^4\lambda_s\phi[\xi]^4, \end{aligned} \quad (83)$$

where V_{One} represents the one-step EWPT to the mixed phase. The ground state potential reads as

$$V_{\text{One,ground}} = -\frac{1}{4}(v_{z\phi}^4\lambda_s + \lambda v_z^4 + 2\lambda_{13}v_{z\phi}^2v_z^2). \quad (84)$$

Thus far, we have finished the last task needed to solve the EOM and compute the sphaleron energy. Equations (83) and (84) multiplied by the normalization factor $\frac{g^2}{g^2\Omega^4}$ constitute the potential that appears in Eq. (61). However, for the potential term $V[h(\xi), \phi(\xi)]$ that appears in the EOMs, Eq. (64), we should directly use Eq. (83) without any such normalization factors.

C. Two-step EWPT

First, we expand the Higgs and septuplet fields around their extremal scalar field configuration,

$$H = \frac{h}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi = \frac{\phi}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (85)$$

Then, substituting Eq. (85) into Eq. (70), we can obtain a general potential expression, V_{general} . Secondly, applying the tadpole criteria

$$\frac{\partial V_{\text{general}}}{\partial h} = \frac{\partial V_{\text{general}}}{\partial \phi} = 0, \quad (86)$$

we can obtain nine extremal points, which have a $\mathbb{Z}_2$ symmetry. These nine extremal points can be shown by mirroring Fig. 1(c) to all four quadrants. The value and Hessian determinant of the X , W , and Z point in Fig. 1(c) are summarized in Table I. The VEV relations and defined parameters in Table I read as

$$\begin{aligned} v &= \frac{\mu}{\sqrt{\lambda}}, \quad v_\phi = \frac{\sqrt{2\sqrt{7}M_B^2 - 7M_A^2}}{\sqrt{7\lambda_s}}, \\ v_z^2 &= \frac{\lambda_s(\lambda_{13}v_\phi^2 - \lambda v^2)}{\lambda_{13}^2 - \lambda\lambda_s}, \\ v_{z\phi}^2 &= \frac{\lambda(\lambda_{13}v^2 - v_\phi^2\lambda_s)}{\lambda_{13}^2 - \lambda\lambda_s}, \\ V_z &= v_\phi^4\lambda_s + \lambda v^4 - 2\lambda_{13}v_\phi^2v^2, \end{aligned} \quad (87)$$

where v and v_ϕ denote VEVs at points X and W of Fig. 1 in two-step EWPT, while v_z and $v_{z\phi}$ denote VEVs of point Z in one-step EWPT, where the definitions of λ_{13} and λ_s are the same as with Eqs. (78) and (79). We notice that the relationship between VEVs and model parameters is different from the one-step EWPT to mixed phase, as shown in

TABLE I. Parameter table for the two-step EWPT. In the two-step EWPT, it is advantageous to express all quantities in terms of v and v_ϕ . Accordingly, both the potential in Eq. (89) and the associated Hessian determinant have been expressed in terms of these two VEVs.

	h	ϕ	V	Hessian determinant
O	0	0	0	$\lambda\lambda_s v^2 v_\phi^2$
X	v	0	$-\frac{1}{4}\lambda v^4$	$2\lambda v^2(\lambda_{13}v^2 - v_\phi^2\lambda_s)$
W	0	v_ϕ	$-\frac{1}{4}\lambda_s v_\phi^4$	$2v_\phi^2\lambda_s(\lambda_{13}v_\phi^2 - \lambda v^2)$
Z	v_z	$v_{z\phi}$	$\frac{\lambda\lambda_s V_z}{4(\lambda_{13}^2 - \lambda\lambda_s)}$	$\frac{4\lambda\lambda_s(\lambda v^2 - \lambda_{13}v_\phi^2)(\lambda_{13}v^2 - v_\phi^2\lambda_s)}{\lambda_{13}^2 - \lambda\lambda_s}$

Eq. (75). One can verify that, inside Eq. (87), if we put the expression of v and v_ϕ (the first line) into the v_z^2 (the second line) and $v_{z\phi}^2$ (the third line) definitions, the v_z^2 and $v_{z\phi}^2$ have the following relation:

$$\mu^2 = \lambda v_z^2 + \lambda_{13} v_{z\phi}^2, \quad M_A^2 - \frac{2\text{Re}(M_B^2)}{\sqrt{7}} = -\lambda_s v_{z\phi}^2 - \lambda_{13} v_z^2. \quad (88)$$

These are just the last two relations in Eq. (75), so the two analysis methods are consistent with each other.

Let us elaborate further on the mass matrix in the two-step EWPT. The computational methods should be quite parallel with the one-step scenario, where we need to start from the general field parametrization, Eqs. (72) and (73), while the difference comes from the relationship between VEVs and model parameters. Therefore, we can obtain the various mass matrices as displayed in Eq. (80), but with different parameter relationships.

Returning to our potential analysis, we can express the potential as

$$V_{\text{general}} = \frac{1}{4}[\phi^2(2h^2\lambda_{13} - 2v_\phi^2\lambda_s) + h^2(h^2\lambda - 2\lambda v^2) + \phi^4\lambda_s]. \quad (89)$$

Now we comment about the topological field solutions in the two-step EWPT. According to our analysis in Sec. II A and our $Y = 0$ septuplet setup, the topological solution at point W of Fig. 1(c) is the monopole, while the final electroweak vacuum gives rise to the sphaleron. We convert these two situations into different values of the VEVs:

- (1) W : Higgs VEV = 0, Septuplet VEV $\neq 0$, monopole.
 - (2) X : Higgs VEV $\neq 0$, Septuplet VEV = 0, sphaleron.
- In the following, the Higgs VEV and Septuplet VEV are denoted as v and v_ϕ , respectively.

For the purposes of deriving and solving the EOM and computing the sphaleron energy or monopole mass, we need to make the substitution $h \rightarrow h[\xi]v$, $\phi \rightarrow \phi[\xi]v_\phi$. Then the potential reads as

$$V_{\text{Two}}\left(\xi, \mu = \frac{\pi}{2}\right) = \frac{1}{4}h[\xi]^2 v^2 (h[\xi]^2 v^2 \lambda - 2\lambda v^2) + \frac{1}{4}\phi[\xi]^4 v_\phi^4 \lambda_s + \frac{1}{2}\phi[\xi]^2 v_\phi^2 (h[\xi]^2 v^2 \lambda_{13} - v_\phi^2 \lambda_s), \quad (90)$$

where V_{Two} represents potential in the two-step EWPT scenario, which has the identical property with Eq. (83) in the sphaleron energy computation.

The vacuum potential in the two-step EWPT reads as

$$V_{\text{Two,ground}} = -\frac{1}{4}(v_\phi^4 \lambda_s + \lambda v^4 - 2\lambda_{13} v_\phi^2 v^2). \quad (91)$$

The specific potential expressions at point W or X of Fig. 1(c) can be obtained from Eqs. (90) and (91) with the simplification rules that we listed in the paragraph below Eq. (89).

To fulfill a two-step EWPT, additional parameter constraints should be applied. As shown in Fig. 1(c), we require that our Universe undergoes the transition pattern $O \rightarrow W \rightarrow X$. In principle, such constraints should be considered in a thermal bath with thermal corrections included in the effective potential. Our zero- T analysis here only serves as an estimation. Here are the requirements:

- (1) O must be a secondary local minimum; this requires

$$\lambda_s > 0. \quad (92)$$

- (2) $V(W) > V(X)$; this implies

$$\lambda_s v_\phi^4 < \lambda v^4. \quad (93)$$

- (3) The Hessian determinant $\text{Hess}(X) > 0$; this requires

$$\lambda_{13} v^2 - v_\phi^2 \lambda_s > 0. \quad (94)$$

- (4) The Hessian determinant $\text{Hess}(W) > 0$; this implies

$$\lambda_{13} v_\phi^2 - \lambda v^2 > 0. \quad (95)$$

- (5) If we require the point Z exists, we need to solve the equations

$$v_z^2 = \frac{\lambda_s(\lambda_{13} v_\phi^2 - \lambda v^2)}{\lambda_{13}^2 - \lambda \lambda_s}, \quad v_{z\phi}^2 = \frac{\lambda(\lambda_{13} v^2 - v_\phi^2 \lambda_s)}{\lambda_{13}^2 - \lambda \lambda_s}, \quad (96)$$

with the constraints Eqs. (94) and (95); the conditions $v_z^2 > 0$ and $v_{z\phi}^2 > 0$ require

$$\lambda_{13}^2 - \lambda \lambda_s > 0. \quad (97)$$

One would observe that the Hessian determinant $\text{Hess}(Z) < 0$ under all of the above criteria, so the mixed point is not a stationary point.

These constraints are not totally independent, since the constraint Eq. (94) can be derived out from Eqs. (93) and (95), and the latter two conditions are of crucial importance. Overall, we have five input parameters: v , v_ϕ , λ , λ_s , and λ_{13} . The parameter ranges that satisfy the two-step EWPT are shown in Fig. 3. In this plot, the lower bound of the aquamarine region is constrained by Eq. (94). The vertical bound refers to the lines for constant λ_s , which gives the maximum value bound of v_ϕ according to Eq. (93). The smaller the value of λ_s , the larger an unconstrained

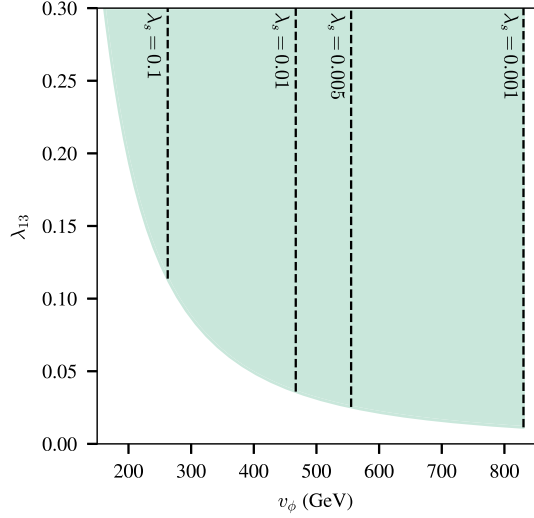


FIG. 3. Parameter ranges that satisfy the two-step EWPT. The intersection of the aquamarine color region and left-hand side of the λ_s vertical dashed line represents the feasible parameter region. The larger value of λ_s , the smaller the viable parameter region.

parameter region we would have. At the end of the first step, constrained by the effective portal coupling, the septuplet VEV cannot be arbitrarily small.

IV. SPHALERON ENERGY AND MONOPOLE MASS IN DIFFERENT EWPT SCENARIOS

The formal sphaleron energy and monopole mass can be defined as [55]

$$E_{\text{sph}}/m_{\text{mon}} = B_{\text{sph/mon}} \cdot \frac{4\pi\Omega}{g}, \quad (98)$$

where $\Omega = 246.22$ GeV is the zero temperature Higgs VEV and g is the weak coupling constant. The value of B_{sph} and B_{mon} is defined as the radial integral of Eqs. (61) and (63), respectively. In the SM, where the EWPT is shown in pattern (a) in Fig. 1, the sphaleron $B_{\text{sph}} = 1.900506$. We will compute the value of the B_{sph} in Fig. 1(b) at point Z and the value of B_{mon} in Fig. 1(c) at point W in this section.

Before presenting the results for E_{sph} and m_{mon} , we first comment on the constraints of the septuplet model. For one-step EWPT to the mixed phase, we consider the ρ parameter constraint, which imposes a significant limit on the septuplet VEV. In the two-step EWPT scenario, since the neutral septuplet field can contribute to the dark matter relic density, we focus primarily on constraints from the dark matter direct detection experiments. Additionally, there are constraints from the collider experiments, such as the $h \rightarrow \gamma\gamma$ channel and the electroweak precision tests (the oblique S , T , and U parameters). A detailed analysis of these constraints would diverge from the main focus of this

work, and we plan to conduct a comprehensive phenomenological study in future research.

A. One-step EWPT to the mixed phase

In this situation, both Higgs field and septuplet field obtain VEV after the phase transition, while the v_ϕ should be constrained by the ρ parameter. The ρ parameter under multiple electroweak scalars is defined as

$$\rho = \sum_i \frac{[J_i(J_i + 1) - Y_i^2]v_i^2}{2Y_i^2v_i^2}, \quad (99)$$

where J_i is the total isospin and Y_i denotes the hypercharge. In our situation, we have two scalar fields; one is the Higgs field with $J = \frac{1}{2}$ and $Y = \frac{1}{2}$, and another is the additional multiplet with a general J and $Y = 0$. Then, the ρ parameter is given by

$$\rho = 1 + 2J(J + 1) \frac{v_\phi^2}{v^2}; \quad (100)$$

the larger the isospin of the multiplet, the stronger a constraint is imposed on v_ϕ . The value of the ρ parameter is analyzed in Ref. [56] with $\rho = 1.00031 \pm 0.00019$. Within 95% significance level, v_ϕ is constrained to

$$v_\phi^2 \lesssim \frac{20.648}{J(J + 1)} \text{ GeV}^2. \quad (101)$$

For our septuplet case ($J = 3$), we are safe to take $v_\phi = 1$ GeV.

The computation of sphaleron energy can be separated into two parts: (i) obtain the solutions of the field profile function from the EOMs in Eq. (64); (ii) put the solutions into the sphaleron energy expression of Eq. (61). For the first step, we present the solutions of the field's profile function in Fig. 4(a), with the choice of parameters $\lambda_{13} = 0.05$ and $\lambda_s = 0.005$. Under this parameter choice, the $B_{\text{sph}} = 1.900535$, which is quite close to the SM B_{sph} value. Furthermore, we perform a parameter scan of B_{sph} and present the result in Fig. 5, where the input parameters are chosen according to the analysis of Sec. III B. Since the Higgs VEV is overwhelmingly larger than the septuplet VEV, the sphaleron energy differs little from the pure SM case. Moreover, the light scalar mass has almost no impact on the sphaleron energy, while as the mixing angle parameter $\cos\theta$ increases, the sphaleron energy initially increases before subsequently decreasing. In summary, in the one-step EWPT scenario, if we only consider one scalar multiplet extension, constrained by the ρ parameter, the additional multiplet has negligible influence on the SM sphaleron energy.

We would like to make some comments about the Georgi-Machacek model [19] where the VEVs for more

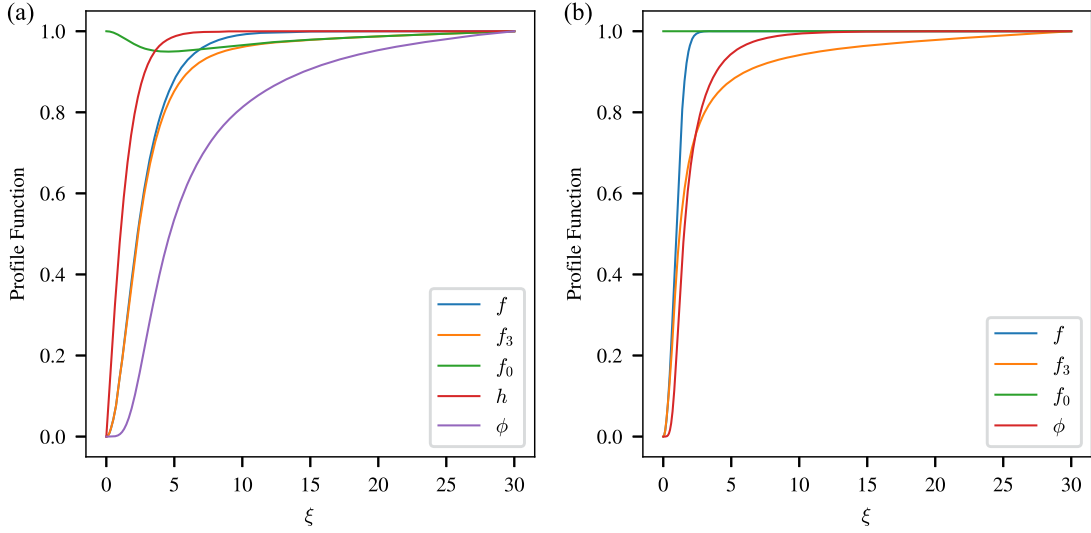


FIG. 4. The radial profile functions in different EWSB patterns. Panel (a) denotes the EWPT to the mixed phase, where we take $v_\phi = 1$ GeV, $\lambda_{13} = 0.05$, and $\lambda_s = 0.005$. In this scenario, both the Higgs field h and multiplet field ϕ obtain VEV, and the sphaleron energy value $B_{\text{sph}} = 1.900535$. Panel (b) represents the two-step EWPT [at point W of pattern (c) in Fig. 1], where we take $v = 0$ GeV, $v_\phi = 500$ GeV, $\lambda_{13} = 0.05$, and $\lambda_s = 0.005$. At this stage, the Higgs field's profile function does not appear due to its vanishing VEV, and the value of the monopole mass in this stage is $B_{\text{mon}} = 5.001145$.

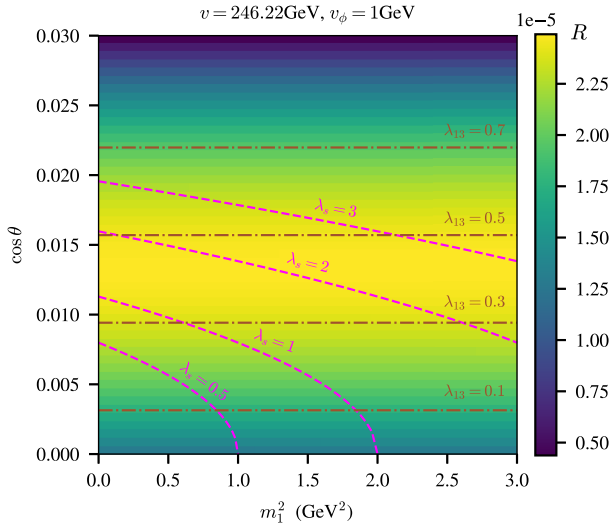


FIG. 5. Sphaleron energy in one-step EWPT to mixed phase. The model free parameters are chosen as $v_z, v_{z\phi}, m_2, \cos\theta$, and m_1 . These parameters represent the Higgs VEV ($v_z = 246.22$ GeV), the septuplet VEV ($v_{z\phi} = 1$ GeV), the Higgs mass ($m_2 = 125.25$ GeV), the mixing angle ($\cos\theta$), and the light scalar mass eigenvalue m_1 , where the last two parameters are regarded as scan input parameters. The color bar value denotes the relative change of sphaleron energy to the SM scenario, which is defined as $R = (E_{\text{Mixed}} - E_{\text{SM}})/E_{\text{SM}}$, where E_{Mixed} and E_{SM} are the sphaleron energy in the mixed phase and SM scenarios, respectively. The SM sphaleron energy has a B_{sph} value of 1.900506, where B_{sph} is defined in Eq. (98). Note that there is a multiplication factor of 10^{-5} in the color bar. The dashed line and dot-dashed line represent the contour lines of λ_s and λ_{13} , respectively. The contour line of λ is not plotted, as its absolute maximum relative change with respect to the SM value is smaller than $|-4 \times 10^{-4}|$. The SM scalar self-interaction coefficient is $\lambda = 0.129383$.

than one additional new multiplet can be large, while satisfying the ρ parameter constraint. The formalism to analyze this case will be the same as discussed here, but then including one additional field VEV. We might anticipate a significantly different result for the sphaleron energy in this case. We defer a detailed study to future work.

B. Two-step EWPT

As shown in Fig. 1(c), the first step is $C1: O \rightarrow W$, and the second step is $C2: W \rightarrow X$. The VEV of the multiplet at point W is unconstrained, since the ρ parameter is measured at point X in a zero temperature universe. Thus, the monopole mass at point W can reach a sizable value, and we can anticipate that B_{mon} could differ significantly from the SM value of B_{sph} . Parallel to the one-step EWPT analysis, we show the solutions of the field profile function in the right panel of Fig. 4. Also, we perform a parameter scan over B_{mon} and present the result in Fig. 6.

In Fig. 6, the intersection between the orange region and the right-hand side of the vertical dashed λ_{13} line represents the unconstrained monopole mass domain. From Eq. (90), we observe that the portal effective coupling λ_{13} does not affect the potential term V_{Two} under the vanishing Higgs VEV ($v = 0$) scenario, so that λ_{13} does not alter the monopole mass at W . On the other hand, the greater value of λ_s , the higher value of the monopole mass. Therefore, the influence of λ_{13} and λ_s to B_{mon} at two-step EWPT differs from their influence on B_{sph} in the one-step case. This difference can be deduced from the different potential configuration in the one-step case with Eq. (83) and two-step case with Eq. (90).

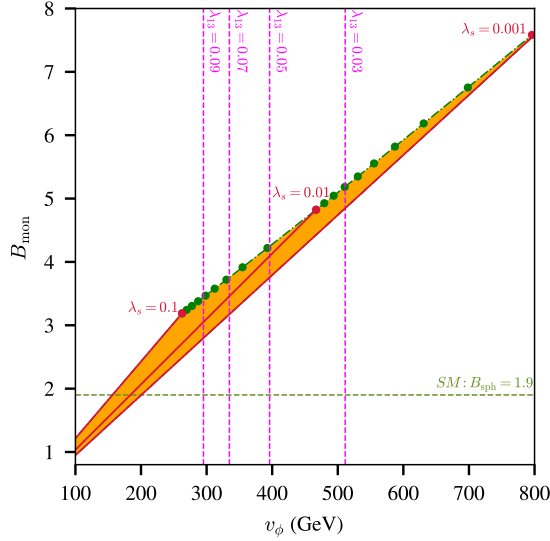


FIG. 6. Monopole mass B_{mon} in the first step (point W) of two-step EWPT. The intersection of the orange region and right-hand side of the dashed vertical λ_{13} line represents the viable monopole mass region. The bottom olive drab dashed line denotes the SM sphaleron B_{sph} value, which is approximately equal to 1.9. The orange region represents the monopole mass B_{mon} under a feasible choice of v_ϕ and λ_s . The bottom red line represents $\lambda_s = 0.001$, while the upper red line denotes $\lambda_s = 0.1$; λ_s continuously increases counterclockwise; the red or green dots represent the truncated points limited by the v_ϕ in Eq. (93). The constraint from λ_{13} in Eq. (95) rejects the left-hand side of the vertical λ_{13} line.

It is interesting to observe that there is a sizeable orange region with B_{mon} greater than the SM value of B_{sph} . If this pattern persists at $T > 0$; if our Universe undergoes a first-order EWPT during the first step ($C1$); and if there exists sufficient BSM CPV to create the baryon asymmetry, this asymmetry can be well preserved at point W . For a demonstration in the real triplet extension, see Refs. [20–22,57]. In general, the second step $C2$ to the Higgs phase could either preserve or erase this baryon asymmetry. If the second step is first order and if the sphaleron energy at point X is sufficiently large, then this asymmetry can be preserved in the final Higgs phase. A complete analysis of this possibility for the $T > 0$ general electroweak multiplet case will appear in a future study.

Finally, we comment on model constraints implied by DM phenomenology. The CP -even or CP -odd neutral septuplet fields can serve as the dark matter candidate. (See Appendix G for the mass matrices and analyses.) The work in [32] analyzes the corresponding dark matter relic density and direct detection constraints. For the latter, the DM-nucleus spin independent cross section can be written as $\sigma_{\text{SI}} \propto g_{\text{eff}}^2$, with

$$g_{\text{eff}} = f_N \frac{2\lambda_{13}}{m_h^2} + \frac{3}{4} f_T f_N^{\text{PDF}}, \quad (102)$$

where $f_N \approx 0.287$ for proton; $f_N^{\text{PDF}} = 0.526$ denotes the second momentum of the nucleon parton distribution function (PDF); f_T is the coefficient function of a twist-two quark bilinear operator [32]; and m_h represents the Higgs particle mass. The first term in g_{eff} is the dominant contribution of σ_{SI} . In our parameter scan, we take λ_{13} smaller than 0.1 both in Figs. 5 and 6. For dark matter mass in 10^2 – 10^4 GeV, we have confirmed that our parameter choices with $\lambda_{13} = 0.03$ are unlimited by the newest dark matter direct search constraint [58,59].

V. CONCLUSION

Determining the origin of the cosmic baryon asymmetry remains an important challenge at the interface of particle and nuclear physics with cosmology. Among various possible baryogenesis mechanisms, we focus on EWBG, which naturally connects with the Higgs mechanism. In this context, we perform a detailed study of the formalism for baryon number dynamics—a key EWBG ingredient—in the context of a first-order EWPT catalyzed by the extension of the SM with a scalar electroweak multiplet. Furthermore, we choose the complex septuplet as a case of study to compute the sphaleron energy and monopole mass.

For the theory formalism, we first classify the topological field solutions. For two-step electroweak symmetry breaking, wherein the new multiplet VEV alone breaks electroweak symmetry during the first step, representations having nonzero or zero hypercharge admit sphaleron or monopole solutions, respectively. When a nonvanishing Higgs doublet VEV contributes to the symmetry breaking, either in the second step or in a one-step symmetry breaking scenario, only sphaleron solutions emerge. Experimental dark matter direct detection constraints imply that only zero hypercharge multiplets with vanishing VEVs in the present broken symmetry phase may contribute to the dark matter relic density. Such multiplets would, thus, be associated with monopole-catalyzed baryon number violation during the penultimate era of electroweak symmetry breaking. Then, we discuss other formal features of sphaleron and monopole field solutions. For the sphaleron induced BNV, we show that two previous configurations, MK and KL, are equivalent up to a gauge transformation. Furthermore, we construct the general dimensional sphaleron and monopole matrices, define the general sphaleron energy and monopole mass expressions, and discuss the choices of boundary conditions for field equations of motion.

For the sphaleron or monopole computation, we first analyze the multiplet extension model's parameter constraint in the one-step EWPT to the mixed phase and two-step EWPT scenario separately. In both scenarios, we have five potential related independent parameters: the Higgs and septuplet VEV, the Higgs and septuplet effective self couplings, and the Higgs-septuplet effective portal coupling. In the scenario of a single-step EWPT leading to a mixed-VEV phase, we find that the additional multiplet's

contribution to the sphaleron energy is negligible, primarily due to the constraint imposed by the electroweak ρ parameter. Conversely, in a two-step EWPT scenario, the monopole mass can achieve sufficiently large values so as to suppress the monopole density and allow for preservation of baryon asymmetry generated during this step, assuming a first-order EWPT and the requisite CP violation. For the latter case, we analyze the constraints on model parameters from the dark matter phenomenology. On the other hand, the DM self-interaction coupling, λ_s , is less constrained from the direct search experiments. We find that the monopole mass exhibits complementary dependencies on these two parameters: λ_{13} does not directly affect the value of the monopole mass, while the monopole mass increases monotonically with λ_s .

In the future, numerous studies can be conducted based on this work, for instance, the computation of sphaleron energy under thermal corrections and the computation of one-step EWPT with the Georgi-Machacek model, the computation of the next-to-leading-order BNV rate, the accurate estimation of the monopole density, etc.

ACKNOWLEDGMENTS

We thank Wei Chao, Tobias Diez, Yucheng Qiu, Xu-Xiang Li, Nicholas Manton, Tudor Stefan RATIU, Tuomas V.I. Tenkanen, Bingrong Yu, Michael Zantedeschi, and Jiang Zhu for useful discussions during this work. We thank the anonymous referee for providing insightful feedback. M.J.R.-M., Y.W., and W.Z. were supported in part by the National Natural Science Foundation of China under Grants No. 12375094 and No. 11975150 and by the Ministry of Science and Technology of China under Grant No. WQ20183100522. M.J.R.-M. also gratefully acknowledges support under the Double First Class Plan of the Shanghai Jiao Tong University and sponsorship from Shanghai Tang Junyuan Education Foundation. W.Z. is supported by the National Natural Science Foundation of China under Grant No. 12405120 and Start-up Funds for Young Talents of Hebei University (No. 521100224226).

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: RELATIONSHIP BETWEEN CHERN-SIMONS NUMBER AND $\mathcal{P}(t_a, t_b)$

In the main text, Eq. (33), we define an integral $\mathcal{P}(t_a, t_b)$ as

$$\begin{aligned} \mathcal{P}(t_a, t_b) &\equiv \int_{t_a}^{t_b} dt \int_{-\infty}^{+\infty} d^3x \partial_\mu K^\mu \\ &= \int d^3x K^0 \Big|_{t=t_a}^{t=t_b} + \int_{t_a}^{t_b} dt \int_S \vec{K} \cdot \overrightarrow{dS}, \end{aligned} \quad (\text{A1})$$

from which we defined the winding number and sphaleron baryonic charge, as shown in Eqs. (34) and (36), respectively. Now we want to connect $\mathcal{P}(t_a, t_b)$ into the definition of the Chern-Simons number.

The Chern-Simons number N_{CS} is defined as

$$N_{\text{CS}}(t = t_b) - N_{\text{CS}}(t = t_a) = \mathcal{P}(t_a, t_b), \quad (\text{A2})$$

where we can take $N_{\text{CS}}(t = -\infty) = 0$ for the vacuum configuration. If we further assign $t_b = t_0$, one has

$$\begin{aligned} N_{\text{CS}}(t = t_0) &= Q_B = \mathcal{P}(-\infty, t_0) \\ &= \int d^3x K^0 \Big|_{t=-\infty}^{t=t_0} + \int_{-\infty}^{t_0} dt \int_S \vec{K} \cdot \overrightarrow{dS}. \end{aligned} \quad (\text{A3})$$

Suppose the gauge field A_i^a falls off faster than $1/r$; the second term in the second line of Eq. (A3) equals zero, so that

$$\begin{aligned} N_{\text{CS}}(t = t_0) &= \int d^3x K^0 \Big|_{t=t_0} \\ &= \frac{g^2}{16\pi^2} \int d^3x \epsilon^{ijk} \left((\partial_i A_j^a) A_k^a + \frac{g}{3} f^{abc} A_i^a A_j^b A_k^c \right). \end{aligned} \quad (\text{A4})$$

Define $\mathcal{A}_i \equiv ig A_i^a \frac{\sigma^a}{2}$, where σ^a is the Pauli matrix. One can verify that the following relation holds:

$$\begin{aligned} \text{Tr} \left((\partial_i \mathcal{A}_j) \mathcal{A}_k - \frac{1}{3} \mathcal{A}_i [\mathcal{A}_j, \mathcal{A}_k] \right) \\ = -\frac{g^2}{2} \left((\partial_i A_j^a) A_k^a + \frac{g}{3} f^{abc} A_i^a A_j^b A_k^c \right), \end{aligned} \quad (\text{A5})$$

so that $N_{\text{CS}}(t = t_0)$ reads as

$$\begin{aligned} N_{\text{CS}}(t = t_0) \\ = -\frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} \left((\partial_i \mathcal{A}_j) \mathcal{A}_k - \frac{1}{3} \mathcal{A}_i [\mathcal{A}_j, \mathcal{A}_k] \right). \end{aligned} \quad (\text{A6})$$

Under a gauge transformation, $\mathcal{A}'_i = U \mathcal{A}_i U^{-1} + (\partial_i U) U^{-1}$, we have

$$\begin{aligned} N(\mathcal{A}', t = t_0) \\ = N(\mathcal{A}, t = t_0) \\ + \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} \partial_i \text{Tr}(U \mathcal{A}_j (\partial_k U^{-1})) \\ + \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr}(U (\partial_i U^{-1}) U (\partial_j U^{-1}) U (\partial_k U^{-1})), \end{aligned} \quad (\text{A7})$$

where the second term is a total derivative, which will vanish as long as the gauge field vanishes at the boundaries. The last term is important and has a topological meaning, which characterizes the number of times the $SU(2)$ is covered by a three-sphere.

APPENDIX B: GAUGE CHOICE WITH A_i^a FALLS OFF FASTER THAN $1/r$

We comment on the gauge transformation matrix U_c used in the MK configuration [35] which causes the gauge field A_i^a to decay more rapidly than $1/r$. We start from the gauge field configuration when $\mu = \pi/2$ (with an additional rotation in Ref. [35]):

$$A_i^a = -\frac{2f(\xi)}{gr^2} \epsilon_{iab} x_b, \quad (\text{B1})$$

applying the gauge transformation

$$U_c = \exp((-1)^n i\Theta(r) \vec{T} \cdot \hat{x}), \quad (\text{B2})$$

where $n = 1$ or 2 regarding on the convention of sign; where $\Theta(r)$ runs rapidly from 0 to π when r runs from 0 to ∞ ; and where $\vec{T}$ are the $SU(2)$ generators, and $\hat{x} = \vec{x}/r$. The gauge field configuration is transformed into

$$\begin{aligned} A_i^a = & \frac{(1 - 2f(\xi)) \cos \Theta(r) - 1}{gr^2} \epsilon_{iab} x_b \\ & + (-1)^n \frac{(1 - 2f(\xi)) \sin \Theta(r)}{gr^3} (\delta_{ia} r^2 - x_i x_a) \\ & + \frac{(-1)^n}{g} \frac{d\Theta(r)}{dr} \frac{x_i x_a}{r^2}, \end{aligned} \quad (\text{B3})$$

where the resulting A_i^a falls faster than $1/r$.

We compute the sphaleron baryonic charge based on the gauge field configuration, Eq. (B3), with general n values:

$$\begin{aligned} Q_B = N_{CS}(t = t_0) \\ = \int d^3x K^0|_{t=t_0} = \begin{cases} \frac{1}{2} & \text{when } n = 1 \\ -\frac{1}{2} & \text{when } n = 2 \end{cases}. \end{aligned} \quad (\text{B4})$$

Note that there is a typo in Ref. [35], where they obtained $Q_B = \frac{1}{2}$ with $n = 2$ in their definition of Eq. (B2).

APPENDIX C: OTHER SPHALERON CONFIGURATIONS

1. AKY configuration

Under the general spherically symmetric *Ansatz*, the gauge field configuration is written as [60]

$$\begin{aligned} A_j^a(x) = & \frac{1}{g} [D(r) \epsilon_{jam} x_m + B(r) (r^2 \delta_{ja} - x_j x_a) \\ & + C(r) x_j x_a]. \end{aligned} \quad (\text{C1})$$

The Higgs field is written as

$$H(x) = \frac{v}{\sqrt{2}} \left[H(r) + iK(r) \frac{\vec{\sigma} \cdot \vec{\hat{r}}}{2} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{C2})$$

where $D(r)$, $B(r)$, $C(r)$, $H(r)$, and $K(r)$ are radial functions. Usually, the radial gauge condition sets $C(r) = 0$. A recent systematic investigation on Akiba-Kikuchi-Yanagida (AKY) configuration is presented in Ref. [61].

2. KKB configuration

Start from a set of orthonormal vectors [62]

$$\begin{aligned} \mathbf{u}_1(\phi) &= (\cos \phi, \sin \phi, 0), \\ \mathbf{u}_2(\phi) &= (0, 0, 1), \\ \mathbf{u}_3(\phi) &= (\sin \phi, -\cos \phi, 0). \end{aligned} \quad (\text{C3})$$

The fields are expanded as follows:

$$\begin{aligned} A_i^a(\mathbf{r}) &= u_j^i(\phi) u_k^a(\phi) w_j^k(\rho, z), \\ a_i(\mathbf{r}) &= u_j^i(\phi) a_j(\rho, z), \\ H(\mathbf{r}) &= \tau^i u_j^i(\phi) h_j(\rho, z) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (\text{C4})$$

where we have changed the field labels to make them consistent with the convention in this study.

APPENDIX D: BOLTZMANN EQUATION OF MONOPOLE FERMION SCATTERING

Consider a particle species a undergoing the scattering process

$$a + X \rightarrow Y, \quad (\text{D1})$$

where X and Y denote multiparticle states. Assuming that the final state Y is in thermal equilibrium, the Boltzmann equation for species a is given by [63]

$$\dot{n}_a + 3Hn_a = -\langle \sigma v \rangle (n_a n_X - n_a^{\text{eq}} n_X^{\text{eq}}), \quad (\text{D2})$$

where n_a is the number density of a (with n_a^{eq} its equilibrium value), the dot denotes the time derivative, and H is the Hubble constant. The thermal average of the scattering cross section is defined as

$$\langle \sigma v \rangle = \sum_{aX \leftrightarrow Y} \int d\pi_a d\pi_X d\pi_Y (2\pi)^4 \delta^4(p_a + p_X - p_Y) \times |\mathcal{A}(aX \rightarrow Y)|^2 \frac{f_a^{\text{eq}} f_X^{\text{eq}}}{n_a^{\text{eq}} n_X^{\text{eq}}}, \quad (\text{D3})$$

where $d\pi_X$ and $d\pi_Y$ represent the phase space integrals as in Ref. [63]. Here, $\mathcal{A}$ denotes the transition amplitude, and we have neglected the CP violation, such that

$$|\mathcal{A}(aX \rightarrow Y)|^2 = |\mathcal{A}(Y \rightarrow aX)|^2. \quad (\text{D4})$$

We now extend the analysis to baryon washout induced by the BNV scattering between monopoles and fermions. The Boltzmann equations for the baryon and antibaryon are given by [41,64]

$$\dot{n}_B + 3Hn_B = -\langle \sigma_{\Delta B \neq 0} v \rangle_f g_f (n_f n_M - n_f^{\text{eq}} n_M^{\text{eq}}), \quad (\text{D5})$$

$$\dot{\bar{n}}_B + 3H\bar{n}_B = -\langle \sigma_{\Delta B \neq 0} v \rangle_{\bar{f}} g_f (\bar{n}_{\bar{f}} n_{\bar{M}} - n_f^{\text{eq}} n_M^{\text{eq}}), \quad (\text{D6})$$

where g_f is the number of fermion d.o.f.; n_B , n_f , and n_M denote the number density of baryons, fermions, and monopoles, respectively; while $\bar{n}_B$, $\bar{n}_{\bar{f}}$ and $n_{\bar{M}}$ denote those of antibaryons, antifermions and antimonopoles. We assume that (i) the CP violating effect in the monopole-fermion scattering process is negligible, $\langle \sigma v \rangle_f = \langle \sigma v \rangle_{\bar{f}}$; and (ii) monopoles are in thermal equilibrium, i.e., $n_M = n_{\bar{M}} = n_M^{\text{eq}}$. We define the net baryon and net fermion densities as

$$\bar{n}_B \equiv n_B - n_{\bar{B}}, \quad (\text{D7})$$

$$\bar{n}_f \equiv n_f - n_{\bar{f}}. \quad (\text{D8})$$

From Eqs. (D5) and (D6), we obtain the Boltzmann equation for the net baryon number density,

$$\dot{\bar{n}}_B + 3H\bar{n}_B = -\langle \sigma_{\Delta B \neq 0} v \rangle_f g_f \bar{n}_f n_M. \quad (\text{D9})$$

APPENDIX E: SPHALERON ENERGY COMPUTATION

In this Appendix, we provide the computation details of the sphaleron energy for the Yang-Mills term and the covariant derivative term.

1. Yang-Mills term

Under a general SU(2) representation, the Yang-Mills term transforms

$$F^{aij} F_{ij}^a = F^{aij} F_{ij}^b \frac{1}{2S(R)} \text{Tr}[\{T^a, T^b\}] = \frac{1}{S(R)} \text{Tr}[F^{aij} T^a \cdot F_{ij}^b T^b], \quad (\text{E1})$$

where $S(R)$ is the Dynkin index, and we use $\text{Tr}[\{T^a, T^b\}] = 2S(R)\delta^{ab}$. Since

$$F_{ij}^a T^a = \partial_i A_j^a T^a - \partial_j A_i^a T^a + g\epsilon^{abc} A_i^b A_j^c T^a, \quad (\text{E2})$$

and

$$\begin{aligned} \epsilon^{abc} A_i^b A_j^c T^a &= \epsilon^{bca} T^a A_i^b A_j^c, \\ &= \frac{1}{i} [T^b, T^c] A_i^b A_j^c, \\ &= \frac{1}{i} [A_i^b T^b A_j^c T^c - A_j^c T^c A_i^b T^b], \end{aligned} \quad (\text{E3})$$

where we have used the fact that $[T^b, T^c] = i\epsilon^{bca} T^a$ for all SU(2) multiplets. We can deduce that the Yang-Mills term is invariant for different SU(2) multiplet representations.

2. Kinetic term

For a general SU(2) multiplet, its covariant derivative reads as

$$(D_i \Phi) = \partial_i \Phi - ig A_i^a T^a \Phi - ig' a_i X \Phi. \quad (\text{E4})$$

Since our sphaleron construction occurs in spherical coordinates, the index $i \in [r, \theta, \phi]$. The kinetic term in the second phase of the KL sphaleron configuration reads as

$$\begin{aligned} (D_i \Phi)^\dagger (D_i \Phi) &= (\partial_i \Phi)^\dagger (\partial_i \Phi) + g^2 \langle \Phi^\dagger | J^b J^a | \Phi \rangle A_i^a A_i^b + g'^2 \langle \Phi^\dagger | X^2 | \Phi \rangle a^i a^i + 2gg' A_i^3 a_i J^3 X \Phi^\dagger \Phi, \\ &= (\partial_i \Phi)^\dagger (\partial_i \Phi) + h^2 g^2 \left[\frac{v^2}{4} (J(J+1) - (J^3)^2) A_\mu^+ A^{\mu-} + \frac{v^2}{2} (J^3)^2 A_\mu^3 A^{\mu 3} \right] \\ &\quad + g'^2 (J^3)^2 h^2 \frac{v^2}{2} (a_r^2 + a_\theta^2 + a_\phi^2) - gg' (J^3)^2 v^2 h^2 \left(\frac{a_\theta A_\theta^3}{r^2} + \frac{a_\phi A_\phi^3}{(r \sin \theta)^2} \right), \end{aligned} \quad (\text{E5})$$

where

$$A_\mu^+ A^{\mu-} = \frac{(A_\theta^1)^2 + (A_\theta^2)^2}{r^2} + \frac{(A_\phi^1)^2 + (A_\phi^2)^2}{r^2 \sin^2 \theta},$$

$$A_\mu^3 A^{\mu 3} = \frac{(A_\theta^3)^2}{r^2} + \frac{(A_\phi^3)^2}{r^2 \sin^2 \theta}, \quad (\text{E6})$$

where we need to know the explicit expression of A_i^a , with $i \in [r, \theta, \phi]$ being the spherical coordinates' label

and $a \in [1, 2, 3]$ being the SU(2) generators' label. The expressions of A_i^a can be computed through Eq. (57).

3. General energy form

The U(1) field sphaleron energy computation is straightforward, so we do not list the result here. Finally, we scale the sphaleron energy in the following way [55]:

$$\int d^3x \left(\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} f_{ij} f_{ij} + (D_i \Phi)^\dagger (D_i \Phi) \right) \rightarrow \frac{4\pi\Omega}{g} \int d\xi \left(\frac{1}{4} F_{ij}^a F_{ij}^a(\xi) + \frac{1}{4} f_{ij} f_{ij}(\xi) + (D_i \Phi)^\dagger (D_i \Phi)(\xi) \right), \quad (\text{E7})$$

where we add the dimensionless radial parameter (ξ) to each component to label the differences before and after the transformation. The sphaleron energy expressions read as

$$\begin{aligned} \frac{1}{4} F_{ij}^a F_{ij}^a(\xi) &= \sin^2 \mu \left(\frac{8}{3} f'^2 + \frac{4}{3} f_3'^2 \right) + \frac{8}{\xi^2} \sin^4 \mu \left\{ \frac{2}{3} f_3'^2 (1-f)^2 + \frac{1}{3} \{f(2-f) - f_3\}^2 \right\}, \\ \frac{1}{4} f_{ij} f_{ij}(\xi) &= \frac{4}{3} \left(\frac{g}{g'} \right)^2 \left\{ f_0'^2 \sin^2 \mu + \frac{2}{\xi^2} \sin^4 \mu (1-f_0)^2 \right\}, \\ (D_i \Phi)^\dagger (D_i \Phi)(\xi) &= \frac{v_\phi^2}{\Omega^2} \left\{ \frac{1}{2} \xi^2 \phi'^2 + \frac{4}{3} \phi'^2 \sin^2 \mu \{ (J(J+1) - J_3^2)(1-f)^2 + J_3^2 (f_0 - f_3)^2 \} \right\}. \end{aligned} \quad (\text{E8})$$

APPENDIX F: MASS MATRICES IN THE ONE-STEP EWPT UNDER SU(2) HIGGS PLUS SEPTUPLT MODEL

In this Appendix, we list the explicit mass matrices that appear in Eq. (80).

1. Higgs matrix

$$H_{2 \times 2} = \begin{pmatrix} 2\lambda v_z^2 & 2\lambda_{13} v_z v_{z\phi} \\ 2\lambda_{13} v_z v_{z\phi} & 2v_{z\phi}^2 \lambda_s \end{pmatrix}, \quad (\text{F1})$$

where λ_{13} and λ_s are two combined parameters that are defined in Eqs. (78) and (79). The two mass eigenstates h_1 and h_2 can be obtained via a rotation,

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} h \\ \phi \end{pmatrix}; \quad (\text{F2})$$

the mass eigenvalues for h_1 and h_2 are m_1 and m_2 , respectively, which read as

$$\begin{aligned} m_1^2 &= \lambda v_z^2 + \lambda_s v_{z\phi}^2 - \sqrt{(\lambda v_z^2 - \lambda_s v_{z\phi}^2)^2 + 4\lambda_{13}^2 v_z^2 v_{z\phi}^2}, \\ m_2^2 &= \lambda v_z^2 + \lambda_s v_{z\phi}^2 + \sqrt{(\lambda v_z^2 - \lambda_s v_{z\phi}^2)^2 + 4\lambda_{13}^2 v_z^2 v_{z\phi}^2}. \end{aligned} \quad (\text{F3})$$

We will assign h_2 as a SM-like Higgs particle, and take $m_2 = 125.25$ GeV. The λ, λ_{13} , and λ_s can be expressed by

$$\begin{aligned} \lambda &= \frac{m_1^2 \cos^2 \theta + m_2^2 \sin^2 \theta}{2v_z^2}, \\ \lambda_s &= \frac{m_1^2 \sin^2 \theta + m_2^2 \cos^2 \theta}{2v_{z\phi}^2}, \\ \lambda_{13} &= \frac{(m_2^2 - m_1^2) \sin 2\theta}{4v_z v_{z\phi}}. \end{aligned} \quad (\text{F4})$$

Since $v_z = 246.22$ GeV and we choose $v_{z\phi} = 1$ GeV, the input parameters are $\cos \theta, m_1^2$.

2. Pseudo-scalar matrix

$$P_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{4\text{Re}(M_{\tilde{H}}^2)}{\sqrt{7}} + v_{z\phi}^2 \kappa_\pi + \frac{2\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} \end{pmatrix}, \quad (\text{F5})$$

$$\begin{aligned} \kappa_\pi &= \frac{2}{7} [\text{Re}(\kappa_0'') - 4\text{Re}(\kappa_0')] \\ &+ \frac{8}{21\sqrt{5}} [\text{Re}(\kappa_2'') - 4\text{Re}(\kappa_2')]. \end{aligned} \quad (\text{F6})$$

3. Charged Higgs matrices

$$C1_{3 \times 3} = \begin{pmatrix} 0 & -\frac{v_z v_{z\phi} \lambda_2}{2\sqrt{14}} & -\frac{v_z v_{z\phi} \lambda_2}{2\sqrt{14}} \\ -\frac{v_z v_{z\phi} \lambda_2}{2\sqrt{14}} & M_A^2 + v_{z\phi}^2 \kappa_{122} + v_z^2 \left(\frac{\lambda_1}{2} + \frac{\lambda_2}{4\sqrt{42}} \right) & \frac{2\text{Re}(M_B^2)}{\sqrt{7}} + \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} - v_{z\phi}^2 \kappa_{123} \\ -\frac{v_z v_{z\phi} \lambda_2}{2\sqrt{14}} & \frac{2\text{Re}(M_B^2)}{\sqrt{7}} + \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} - v_{z\phi}^2 \kappa_{123} & M_A^2 + v_{z\phi}^2 \kappa_{122} + v_z^2 \left(\frac{\lambda_1}{2} - \frac{\lambda_2}{4\sqrt{42}} \right) \end{pmatrix}, \quad (\text{F7})$$

where

$$\begin{aligned} \kappa_{122} &= \frac{\kappa_2 - 4\text{Re}(\kappa_2'')}{21\sqrt{5}} - \frac{\text{Re}(\kappa_0'')}{7}, \\ \kappa_{123} &= \frac{1}{7}(\kappa_0 + 2\text{Re}(\kappa_0') - \text{Re}(\kappa_0'')) + \frac{1}{21\sqrt{5}}(3\kappa_2 + 8\text{Re}(\kappa_2') - 4\text{Re}(\kappa_2'')). \end{aligned} \quad (\text{F8})$$

The three eigenvalues of matrix $C1_{3 \times 3}$ are difficult to obtain. However, we can numerically calculate them, and we find that one of these eigenvalues is equal to zero. This zero eigenvalue corresponds to the massless charged Higgs particle:

$$C2_{2 \times 2} = \begin{pmatrix} M_A^2 + v_{z\phi}^2 \kappa_{211} + v_z^2 \left(\frac{\lambda_1}{2} + \frac{\lambda_2}{2\sqrt{42}} \right) & -\frac{2\text{Re}(M_B^2)}{\sqrt{7}} - \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} + v_{z\phi}^2 \kappa_{212} \\ -\frac{2\text{Re}(M_B^2)}{\sqrt{7}} - \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} + v_{z\phi}^2 \kappa_{212}^* & M_A^2 + v_{z\phi}^2 \kappa_{211} + v_z^2 \left(\frac{\lambda_1}{2} - \frac{\lambda_2}{2\sqrt{42}} \right) \end{pmatrix}, \quad (\text{F9})$$

where

$$\begin{aligned} \kappa_{211} &= \frac{2\sqrt{5}}{21}[\kappa_2 - \text{Re}(\kappa_2'')] - \frac{\text{Re}(\kappa_0'')}{7}, \\ \kappa_{212} &= \frac{1}{7}(\kappa_0 + 2\kappa_0' - \kappa_0'') + \frac{2\sqrt{5}}{21}(2\kappa_2' - \kappa_2''); \end{aligned} \quad (\text{F10})$$

$$C3_{2 \times 2} = \begin{pmatrix} M_A^2 + v_{z\phi}^2 \kappa_{311} + v_z^2 \left(\frac{\lambda_1}{2} + \frac{1}{4} \sqrt{\frac{3}{14}} \lambda_2 \right) & \frac{2\text{Re}(M_B^2)}{\sqrt{7}} + \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} + v_{z\phi}^2 \kappa_{312} \\ \frac{2\text{Re}(M_B^2)}{\sqrt{7}} + \frac{\text{Re}(\lambda_3) v_z^2}{\sqrt{14}} + v_{z\phi}^2 \kappa_{312}^* & M_A^2 + v_{z\phi}^2 \kappa_{311} + v_z^2 \left(\frac{\lambda_1}{2} - \frac{1}{4} \sqrt{\frac{3}{14}} \lambda_2 \right) \end{pmatrix}, \quad (\text{F11})$$

where

$$\begin{aligned} \kappa_{311} &= -\frac{\text{Re}(\kappa_0'')}{7} + \frac{\sqrt{5}\text{Re}(\kappa_2'')}{21}, \\ \kappa_{312} &= -\frac{1}{7}(\kappa_0 + 2\kappa_0' - \kappa_0'') + \frac{\sqrt{5}}{21}(\kappa_2 + 2\kappa_2' - \kappa_2''). \end{aligned} \quad (\text{F12})$$

APPENDIX G: MASS MATRICES IN THE HIGGS VACUUM OF TWO-STEP EWPT UNDER SU(2) HIGGS PLUS SEPTUPLET MODEL

In this Appendix, we present the Higgs plus septuplet model's mass matrices at the end of the two-step EWPT [point X of Higgs vacuum at Fig. 1(c)]. The structure of the mass matrices is different from Eq. (80) in the one-step scenario. The mass matrix part of the potential can be written as

$$\begin{aligned} V(H, \Phi) &\supset \frac{1}{2} \begin{pmatrix} h & \phi \end{pmatrix} H'_{2 \times 2} \begin{pmatrix} h \\ \phi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \pi & \pi_\phi \end{pmatrix} \text{Pi}'_{2 \times 2} \begin{pmatrix} \pi \\ \pi_\phi \end{pmatrix} \\ &\quad + \begin{pmatrix} \phi_{3,i} & \phi_{3,-i}^* \end{pmatrix} \text{Ci}'_{2 \times 2} \begin{pmatrix} \phi_{3,i}^* \\ \phi_{3,-i} \end{pmatrix}, \end{aligned} \quad (\text{G1})$$

where the charged Higgs field $\omega^\pm$ is massless, and therefore not included in the list and where the matrix for the CP -even neutral scalar fields is given as follows:

$$H'_{2 \times 2} = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & \lambda_{13} v^2 \end{pmatrix}, \quad (\text{G2})$$

which corresponds to the SM Higgs field mass ($2\lambda v^2$) and CP -even neutral septuplet field mass ($\lambda_{13} v^2$). The CP -odd neutral field mass matrix reads as

$$Pi'_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 2M_A^2 + \frac{1}{14} v^2 (7\lambda_1 + \sqrt{14}\lambda_3) \end{pmatrix}, \quad (\text{G3})$$

where only the septuplet fields obtain a nonzero mass eigenvalue. The CP -even or CP -odd neutral scalar fields of the septuplet field, as described in Eqs. (G2) and (G3), may contribute to the dark matter relic density. The mass matrix of the charged component of the septuplet field is given by

$$Ci'_{2 \times 2} = \begin{pmatrix} M_A^2 + \frac{1}{2} \lambda_1 v^2 + \frac{1}{4\sqrt{42}} i \lambda_2 v^2 & (-1)^{i+1} \left(M_A^2 + \frac{\text{Re}(\lambda_3) v^2}{\sqrt{14}} \right) \\ (-1)^{i+1} \left(M_A^2 + \frac{\text{Re}(\lambda_3) v^2}{\sqrt{14}} \right) & M_A^2 + \frac{1}{2} \lambda_1 v^2 - \frac{1}{4\sqrt{42}} i \lambda_2 v^2 \end{pmatrix}, \quad (\text{G4})$$

where the index $i \in [1, 2, 3]$ represents different charged fields in Eq. (G1). The charged matrix $Ci'_{2 \times 2}$ has a similar structure with the septuplet charged matrix in Ref. [32], while our relationship between M_A^2 and M_B^2 is different with Ref. [32]. The difference comes from the model tadpole conditions, which are distinct for the different EWSB patterns (for example, two-step EWPT analysis in our case).

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