

Chiral algebra, Wilson lines, and mixed Hodge structure of Coulomb branch

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We find an intriguing relation between the chiral algebra and the mixed Hodge structure of the Coulomb branch of four dimensional $\mathcal{N} = 2$ superconformal field theories. We identify the space of irreducible characters of the $\mathcal{N} = 4$ $SU(N)$ chiral algebra $\mathbb{V}[\mathcal{T}_{SU(N)}]$ by analytically computing the Wilson line Schur index, and imposing modular invariance. We further establish a map from the $\mathbb{V}[\mathcal{T}_{SU(N)}]$ characters to the characters of the $\mathcal{T}_{p,N}$ chiral algebra. We extract the pure part of the mixed Hodge polynomial PH_c of the Coulomb branch compactified on a circle, and prove that PH_c encodes the representation theory of $\mathbb{V}[\mathcal{T}_{SU(N)}]$. We expect this to be a new entry of the 4D mirror symmetry framework.

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Introduction. The rich structure of four dimensional (4D) $\mathcal{N} = 2$ superconformal field theories (SCFTs) is encoded in vastly different physical observables. Surprisingly, they often exhibit hidden relations. In this paper, we investigate two seemingly unrelated observables and their exact correspondence: the representation theory of the nonunitary associated chiral algebra [1], which to some extent is tied to the Higgs branch of vacua, and the mixed Hodge polynomial associated to the Coulomb branch compactified on a circle [2].

It is widely believed that nonvacuum modules of the associated chiral algebra $\mathbb{V}[\mathcal{T}]$ encodes the Bogomol'nyi–Prasad–Sommerfield (BPS) line and surface operators in the 4D theory $\mathcal{T}$ [3–7]. The physical picture is clear: operators in the chiral algebra may form bulk-defect operator product expansion with the defect Schur operators and generate defect Schur operators, thus giving rise to representations.

However, at the level of exactly computable physical observables, the correspondence is far from straightforward. The simplest entry is the identification of the Schur index $\mathcal{I}$ with the $\mathbb{V}[\mathcal{T}]$ vacuum character ch_0 . In [4,6], the vortex defect Schur index is shown to reproduce exactly the chiral algebra irreducible characters. On the other hand, in a few examples the line defect Schur index is shown to be linear combinations of chiral

algebra characters [3] with nonconstant coefficients. Schematically, the line index reads

$$\mathcal{I}_L(q, b) = \sum_k R_k(q, b) ch_k(q, b). \quad (1)$$

Here $q = e^{2\pi i \tau}$ denotes the standard parameter in defining a character or index, and b the flavor fugacity collectively. Due to the functional coefficients $R_k(q, b)$, the resulting index $\mathcal{I}_L$ is not at all a character of $\mathbb{V}[\mathcal{T}]$, irreducible or not, in the sense that it does not solve the required modular differential equations [7–9]. In fact, besides a class of Argyres–Douglas theories and a few sporadic cases, the representation theory of $\mathbb{V}[\mathcal{T}]$ is entirely unknown due to the highly nonrational, logarithmic nature, not to mention any quantitative relation to the 4D BPS nonlocal operators. Nonetheless, the existing observations strongly suggest that irreducible characters provide a complete basis for defect Schur index.

Another crucial observable is the Coulomb branch of vacua of an $\mathcal{N} = 2$ SCFT. For a class- $\mathcal{S}$ theory of type G engineered by a Riemann surface $\Sigma_{g,n}$, the Coulomb branch $\mathcal{M}_C$ of the 4D theory compactified on a circle is given by the hyperkähler Hitchin moduli space $\mathcal{M}_H$ of $\Sigma_{g,n}$. Alternatively, the Coulomb branch can also be described topologically as the character variety $\mathcal{M}$ of the Riemann surface $\Sigma_{g,n}$, the space of homomorphisms from the fundamental group $\pi_1(\Sigma_{g,n})$ to the chosen gauge group G , modulo conjugation. The two descriptions emphasize respectively the holomorphic-symplectic and the algebraic-topological feature of the Coulomb branch $\mathcal{M}_C$. The conical singularity together with the holomorphic-topological feature of $\mathcal{M}_C$

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gives rise to the *mixed Hodge structure*, often expressed in the form of mixed Hodge polynomials H_c . *A priori*, this cohomological structure seems entirely independent of the chiral algebra $\mathbb{V}[\mathcal{T}]$.

The work in this paper is two-fold. First, we work out all the characters of the $\mathcal{N} = 4$ $SU(N)$ chiral algebra $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and its modularity. Second, we prove that the representation theory is actually encoded in the seemingly unrelated mixed Hodge polynomial of the Coulomb branch.

Concretely, we focus on the Wilson line operators corresponding to any irreducible representation of $SU(N)$, and compute the line defect index analytically. We extract the space of irreducible characters of the chiral algebra and show that the basis is in one-to-one correspondence with the integer partitions $\mathcal{P}$ of N [10]. The logarithmic nature of the algebra implies existence of logarithmic characters, which are constructed by imposing modular invariance of the character space $\mathcal{V}$.

We will prove that the structure of $\mathcal{V}$ is encoded in the Coulomb branch by computing the pure part of its mixed Hodge polynomial. Concretely, we find

$$PH_c(\mathcal{M}, q) = \sum_{\mathcal{P} \in \mathcal{P}(N)} q^{\dim \mathcal{V}_{\mathcal{P}} - 1}, \quad (2)$$

where $\mathcal{V}_{\mathcal{P}}$ contains all the logarithmic partners of the irreducible representations labeled by the partition $\mathcal{P}$.

The finding in this letter may be considered as a new entry of the four dimensional mirror symmetry [11–17], which reveals a deep relation between the representation-theoretic structure of the Higgs branch and the geometric structure of the Coulomb branch. The mixed Hodge polynomial computation is in general much easier to carry out than the fixed variety of the Hitchin moduli space, making it a powerful tool in studying 4D mirror symmetry and the relation to 3D mirror symmetry and symplectic duality.

The chiral algebra and modules. The associated chiral algebra $\mathbb{V}[\mathcal{T}_{SU(N)}]$ of the 4D $\mathcal{N} = 4$ $SU(N)$ theory $\mathcal{T}_{SU(N)}$ was proposed in [1] to be a $\mathcal{N} = 4$ super $\mathcal{W}$ -algebra with $N - 1$ chiral primary generators of conformal weight $d_i/2$, where d_i are the degrees of the Casimir invariants of $SU(N)$. In particular, the algebra contains the 2D $\mathcal{N} = 4$ small superconformal algebra.

In [18], a conjectural free field realization of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ associated to the Weyl group S_N using $N - 1$ $bc\beta\gamma$ systems is proposed. The conjecture is proven in [19], and more recently the universal $\mathcal{W}$ -algebra is shown to be unique [20,21].

The representation theory of the simplest case $N = 2$ is studied in [22]. There are two irreducible representations, the vacuum, and the representation M constructed as the quotient of one $bc\beta\gamma$ system by the vacuum: the $bc\beta\gamma$ system coincides with the one in [18], and is a reducible but

indecomposable representation of $\mathbb{V}[\mathcal{T}_{SU(N)}]$. The fact that there are only two irreducible representations can also be argued by working out the two-dimensional space of solutions to the system of flavored modular linear differential equations (MLDEs) [7,23] associated to the three null states in the chiral algebra [8].

For $N > 2$, the situation is less clear. In the following we will identify the space of characters by computing the Wilson line defect index of $\mathcal{T}_{SU(N)}$ for all $N \geq 2$.

The Wilson line index. There are BPS line defects preserving the four supercharges used to define the Schur index [3]. These nonlocal operators can be inserted as half or full line through the origin in the four dimensional (Euclidean) spacetime. The simplest line operators to consider are the Wilson line operators coupled to the gauge groups in a Lagrangian theory. The corresponding defect Schur index can be written as a contour integral with the insertion of the Lie group character(s) of the representation of the half or full Wilson line [24].

We consider $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. The (unnormalized) Schur index in the presence of a half Wilson line $W(Y)$ transforming under the $SU(N)$ representation $\mathcal{R}_Y$ (corresponding to the Young diagram Y) is given by the multivariate contour integral

$$\mathcal{I}_{W(Y)}(q, b) = y^{-\frac{N^2-1}{2}} \oint \left[\prod_{A=1}^{N-1} \frac{da_A}{2\pi i a_A} \right] \mathcal{Z}(a) s_Y(a), \quad (3)$$

where $s_Y(a)$ denotes the Schur polynomial associated with the representation $\mathcal{R}_Y$, and the integrand

$$\begin{aligned} \mathcal{Z}(a) := & \frac{(-1)^{N+1}}{N!} \vartheta_4(\mathbf{b}) \eta(\tau)^{3(N-1)} \\ & \times \frac{\prod_{A,B=1}^N \vartheta_1(\mathbf{a}_A - \mathbf{a}_B)}{\prod_{A,B=1}^N \vartheta_4(\mathbf{a}_A - \mathbf{a}_B + \mathbf{b})}, \end{aligned} \quad (4)$$

The contour integral of $\mathcal{Z}(a)$ reproduces the Schur index $\mathcal{I}_{SU(N)}(q, b)$ of the theory without Wilson line. We use different fonts to relate variables, $a_A = e^{2\pi i \mathbf{a}_A}$, $b = e^{2\pi i \mathbf{b}}$, $y = e^{2\pi i \mathbf{y}}$, while $q = e^{2\pi i \tau}$. The integration variables a_A are constrained by $a_1 \cdots a_N = 1$. The prefactor $y^{-\frac{N^2-1}{2}}$ captures the flavor central charge $k_{2d} = -\frac{N^2-1}{2}$, and is often omitted except for when discussing modular transformations.

The ratio of the Jacobi ϑ functions is elliptic in all the variables $\mathbf{a}_A$, *i.e.*, invariant under shifting $\mathbf{a}_A$ by 1 or τ . The integral vanishes whenever the number of boxes $|Y|$ in Y is not divisible by N . On the other hand, when $|Y|$ is divisible by N , the constraint becomes unnecessary: one may treat $a_1, \dots, a_N$ as N independent integration variables and replace $s_Y(a) \rightarrow (a_1 \cdots a_N)^{-|Y|/N} s_Y(a)$ without changing the result.

Consequently, without loss of generality, we can focus on the unconstrained integrals of the form

$$\mathcal{I}(\mathbf{n}) := \oint \prod_{A=1}^N \frac{da_A}{2\pi i a_A} \mathcal{Z}(a) a_1^{n_1} \cdots a_N^{n_N}, \quad (5)$$

where $\mathbf{n} = (n_1, \dots, n_N)$ satisfying $\sum_{A=1}^N n_A = 0$. The Wilson line index $\mathcal{I}_{W(Y)}$ is simply a linear combination of $\mathcal{I}(\mathbf{n})$ with integer coefficients. Note that the ordering of n_i is not important, hence we may treat $\mathbf{n}$ simply as a set.

The integral is computed exactly using integration formula [25,26]: details, additional examples of the computations, and generalization to some other theories will be reported in a companion paper [27]. (and optionally the Fermi-gas method [28]). In particular, the original Schur index is given by $\mathcal{I} = \mathcal{I}(\mathbf{0})$. Each step of the integration always generates a factor of Jacobi form as the residue, a factor of rational function in $b, q^{1/2}$, and at most one factor of Eisenstein series. The final result takes the form of

$$\mathcal{I}(\mathbf{n}) = \sum_{\alpha} R_{\alpha}(\mathbf{n}) \mathcal{J}_{\alpha}, \quad (6)$$

where $R_{\alpha}(\mathbf{n})$ denotes a rational function in $b, q^{1/2}$, and the quasi-Jacobi form $\mathcal{J}_{\alpha}$ encodes the remaining Jacobi form and products of Eisenstein series. Below, we give an explicit description of the complete basis of $\mathcal{J}_{\alpha}$.

The quasi-Jacobi basis $\mathcal{J}_{\mathcal{E}}$. First we decompose the set $\mathbf{n} = \{n_1, \dots, n_N\}$ into w disjoint subsets $\mathbf{n} = \bigcup_{j=1}^w \{n_{j1}, \dots, n_{j\ell_j}\}$ such that each subset sums to zero while containing no proper zero-sum subset. Hence,

$$\sum_{j=1}^w \ell_j = N \quad \text{and} \quad \sum_{k=1}^{\ell_j} n_{jk} = 0. \quad (7)$$

We call the data $\mathcal{E} = [\ell_1, \dots, \ell_w]$ the *cycle type* of $\mathbf{n}$ and ℓ_i *parts*, $w = w(\mathcal{E})$ the *width* of $\mathcal{E}$. Without loss of generality, we may assume $\ell_1 \geq \ell_2 \geq \dots \geq \ell_w$. Another useful notation for $\mathcal{E}$ is $[1^{m_1} 2^{m_2} \dots N^{m_N}]$ with multiplicities m_A .

The integral $\mathcal{I}(\mathbf{n})$ can be written as a linear combination of a set of *quasi-Jacobi basis* forms $\mathcal{J}_{\mathcal{E}}$ of weight $w(\mathcal{E}) - 1$,

$$\mathcal{I}(\mathbf{n}) = \sum_{\mathcal{E}' \in \text{Children}(\mathcal{E})} R_{\mathbf{n}}(\mathcal{E}') \mathcal{J}_{\mathcal{E}'}, \quad (8)$$

The coefficients $R_{\mathbf{n}}(\mathcal{E}')$ are rational functions in b and $q^{1/2}$, whose form is not important for this paper. The set $\text{Children}(\mathcal{E})$ is determined by $\mathcal{E}$: a child $\mathcal{E}'$ of $\mathcal{E}$ is obtained by merging parts ℓ_i 's, with the only restriction that it is forbidden to merge even number of unit-valued parts $\ell_i = 1$ into one part. We denote the relation as parent $\succeq$ child, with $\mathcal{E} \succeq \mathcal{E}'$ trivially. The set $\text{Children}(\mathcal{E})$ contains $\mathcal{E}$ and all the children of $\mathcal{E}$. This sum structure is simply the

result of $\prod_A a_A^{n_A}$ changing at each step of integration as some a_A takes values at some pole.

The quasi-Jacobi forms $\mathcal{J}_{\mathcal{E}}$ can be written in closed form,

$$\begin{aligned} \mathcal{J}_{\mathcal{E}} &= y^{-\frac{N^2-1}{2}} \frac{\vartheta_4(\mathbf{b})}{i^{N^2-1} \vartheta_{1 \text{ or } 4}(N\mathbf{b})} \hat{\mathcal{J}}_{\mathcal{E}} \\ &= y^{-\frac{N^2-1}{2}} \frac{\vartheta_4(\mathbf{b})}{i^{N^2-1} \vartheta_{1 \text{ or } 4}(N\mathbf{b})} \\ &\quad \times \sum_{\sigma \in \mathcal{P}_{w-1}(\{1, \dots, w\})} \prod_i \left(-(k_i - 1)! E_{k_i} \left[\frac{(-1)^{|\ell_{\sigma_i}|}}{b^{|\ell_{\sigma_i}|}} \right] \right), \end{aligned} \quad (9)$$

where the $\vartheta_{1 \text{ or } 4}$ equals ϑ_1 when N is even, and ϑ_4 when N is odd. The set $\mathcal{P}_{w-1}(\{1, \dots, w\})$ consists of partitions σ of all $(w-1)$ -element subset of $\{1, \dots, w\}$,

$$\cup_i \{\sigma_{i1} \cdots \sigma_{ik_i}\} \subset \{1, 2, \dots, w\}, \quad (10)$$

such that $k_1 + k_2 + \dots = w - 1$. The symbol $|\ell_{\sigma_i}| := \ell_{\sigma_{i1}} + \dots + \ell_{\sigma_{ik_i}}$.

We expect this complete basis $\{\mathcal{J}_{\mathcal{E}}\}$ span the full space of irreducible characters of $\mathbb{V}[\mathcal{T}]$. In particular, the vacuum character reads

$$ch_0(q, b) = \mathcal{I}_{SU(N)}(q, b) = \sum_{\text{odd } \mathcal{E}} R_0(\mathcal{E}) \mathcal{J}_{\mathcal{E}}, \quad (11)$$

where an odd $\mathcal{E}$ is a partition with only odd parts, and the coefficients $R_0(\mathcal{E})$ are rational numbers,

$$R_0(\mathcal{E}) = \frac{1}{\prod_{A=1}^N m_A!} \prod_{j=1}^{w(\mathcal{E})} \left(\frac{(\ell_j - 2)!!}{(2i)^{(\ell_j - 1)/2}} \right)^2 \frac{1}{\ell_j!}, \quad (12)$$

using the notations $\mathcal{E} = [\ell_1, \dots, \ell_w] \sim [1^{m_1} \dots N^{m_N}]$.

Map to $\mathcal{T}_{p,N}$ characters. We can establish a map from $\{\mathcal{J}_{\mathcal{E}}\}$ to known characters in other theory. Starting from the strongly coupled Argyres-Douglas theories $D_p(SU(N))$, one can construct new 4D SCFTs $\mathcal{T}_{p,N}$ [29,30]. It is known that the 4D central charges of $\mathcal{T}_{p,N}$ satisfy the relation $a_{4d} = c_{4d}$ just like the $\mathcal{T}_{SU(N)}$ theories. More importantly, the vacuum character of the $\mathcal{T}_{p,N}$ chiral algebra is simply a special limit of that of $\mathcal{T}_{SU(N)}$,

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2d}[\mathcal{T}_{p,N}] - p c_{2d}[\mathcal{T}_{SU(N)}]}{24}} \mathcal{I}_{SU(N)}\left(q^p, q^{\frac{p}{2}-1}\right).$$

This relation for the case $p = 3, N = 2$ was reinterpreted as an vector space isomorphism between the vacuum representation of the two chiral algebras [31].

Interestingly, the exact same limit

$$b \rightarrow q^{\frac{p}{2}-1}, \quad q \rightarrow q^p, \quad (13)$$

actually extends to a map between the representation theory of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$, where all the characters of the latter theory are all solutions by a unflavored modular differential equation. We verify for $N = 2, \dots, 7$, that under the specialization (13) all the quasi-Jacobi forms $\mathcal{J}_\ell$ map to the characters of the $\mathcal{T}_{p,N}$ chiral algebra. Concretely, under (13) $\mathcal{J}_\ell$ give solutions to the unflavored MLDE. For example, when $N = 5$

$$\mathcal{J}_{[3,1,1]} \xrightarrow{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} -3E_2(\tau), \quad (14)$$

$$\mathcal{J}_{[1^5]} \xrightarrow{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} 15E_2(\tau)^2 - 30E_4(\tau), \quad (15)$$

giving two nonlogarithmic solutions to the 25th order MLDE for $\mathcal{T}_{2,5}$ [32]. Alternatively, the limit can be written as modular transformations acting on the $\mathcal{T}_{p,N}$ chiral algebra vacuum character.

Modular property. We consider the following modular transformations,

$$S: (\tau, \mathfrak{b}, \mathfrak{y}) \rightarrow \left(-\frac{1}{\tau}, \frac{\mathfrak{b}}{\tau}, \mathfrak{y} - \frac{\mathfrak{b}^2}{\tau} \right), \quad (16)$$

$$T: (\tau, \mathfrak{b}, \mathfrak{y}) \rightarrow (\tau + 1, \mathfrak{b}, \mathfrak{y}). \quad (17)$$

Under $\Gamma^0(2) \subset SL(2, \mathbb{Z})$ generated by STS, T^2 , the overall ϑ -ratio factor in $\mathcal{J}_\ell$ transforms as a weight-zero Jacobi form. The remaining piece $\hat{\mathcal{J}}_\ell$ is a sum of products of Eisenstein series, which is a weight- $(w-1)$ quasi-Jacobi form, whose $\Gamma^0(2)$ -orbit spans a w -dimensional linear space $\mathcal{V}_\ell$ of quasi-Jacobi forms, which contains logarithmic terms. Whenever $\ell \neq \ell'$, the corresponding spaces do not overlap, except for the origin 0. It is easy to pick some basis to compute explicitly the modular transformation matrix. In particular, we choose $\hat{\mathcal{J}}_\ell, STS\hat{\mathcal{J}}_\ell, (T^2-1)^k STS\hat{\mathcal{J}}_\ell$, $k = 1, \dots, w-2$. The T^2 matrix can be written

$$T^2 = i^{N^2-1} X \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} X^{-1}, \quad (18)$$

where X is a $w \times w$ square matrix. The Jordan normal form of T^2 is precisely given by a Jordan matrix $J_{i^{N^2-1}, w}$. Since

each ℓ can be viewed as a partition of integer N , we obtain the full space

$$\mathcal{V} = \bigoplus_{\ell \in \mathcal{P}(N)} \mathcal{V}_\ell, \quad \dim \mathcal{V} = \sum_{\ell \in \mathcal{P}(N)} w(\ell). \quad (19)$$

Example. As an example, when $N = 2$, there are only two quasi-Jacobi forms

$$\mathcal{J}_{[2]} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \quad \mathcal{J}_{[1,1]} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_2 \begin{bmatrix} -1 \\ b \end{bmatrix}. \quad (20)$$

Any spin- j Wilson line index is spanned by this basis,

$$\mathcal{I}_j = \delta_{j \in \mathbb{Z}} \left(\mathcal{J}_{[1,1]} - \frac{i}{2} \mathcal{J}_{[2]} \sum_{\substack{m=-j \\ m \neq 0}}^{+j} \frac{b^m - b^{-m}}{q^{m/2} - q^{-m/2}} \right). \quad (21)$$

The two $\mathcal{J}$ precisely span the space of irreducible characters of the $\mathcal{N} = 4$ $SU(2)$ chiral algebra [22], where $\mathcal{J}_{[1,1]}$ simply gives the Schur index $\mathcal{I}_{SU(2)}$. There is one logarithmic partner of $\mathcal{J}_{[1,1]}$. Hence $\dim \mathcal{V}_{[1,1]} = 2 = w([1, 1])$. On the other hand, $\mathcal{J}_{[2]}$ is a Jacobi form, and hence $\dim \mathcal{V}_{[2]} = 1 = w([2])$.

When $N = 3$ and $\mathbf{n} = (n, -n, 0)$,

$$\begin{aligned} \mathcal{I}(\mathbf{n}) = & -\frac{i(b^n - b^{-n})[(b^{\frac{n}{2}} + b^{-\frac{n}{2}}) - (q^{\frac{n}{2}} + q^{-\frac{n}{2}})]}{12(q^{\frac{n}{2}} - q^{-\frac{n}{2}})^3} \mathcal{J}_{[3]} \\ & - \frac{i(b^n + b^{-n})(q^{\frac{n}{2}} + q^{-\frac{n}{2}}) - 2(b^n + b^{-n})}{24(q^{\frac{n}{2}} - q^{-\frac{n}{2}})^3} \mathcal{J}_{[2,1]} \\ & - \frac{i(b^n - b^{-n})}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} \mathcal{J}_{[1,1,1]}. \end{aligned}$$

The three $\mathcal{J}$ span $SU(N)$ all Wilson line index, and under $\Gamma^0(2)$, $\dim \mathcal{V}_{[3]} = 1$, $\dim \mathcal{V}_{[2,1]} = 2$, $\dim \mathcal{V}_{[1,1,1]} = 3$.

Mixed Hodge polynomials. The 4D $\mathcal{N} = 4$ $SU(N)$ theory is engineered by compactifying the 6D (0,2) SCFT on the genus-one Riemann surface with a puncture of type $[N-1, 1]$. When the theory is further compactified on S^1 , the Coulomb branch $\mathcal{M}_N$ can be described as the following character variety of $PGL(N, \mathbb{C})$ which is the Langlands dual of $SU(N)$

$$\begin{aligned} \mathcal{M}_N := & \{(a, b, x) \in PGL(N, \mathbb{C})^2 \times C \\ & \times |x[a, b] = id_N\} // PGL(N, \mathbb{C}). \end{aligned} \quad (22)$$

Here C denotes a conjugacy class in $PGL(N, \mathbb{C})$ corresponding to the partition $[N-1, 1]$, $[a, b] := aba^{-1}b^{-1}$, and $//PGL(N, \mathbb{C})$ denotes quotient by conjugation. Supersymmetry endows the Coulomb branch $\mathcal{M}_N$ with a hyperkähler structure. A smooth Kähler structure would

imply standard decomposition of differential forms into holomorphic-antiholomorphic (p, q) -forms, such that the degree of differential the differential form is $k = p + q$. However, the Coulomb branch $\mathcal{M}_N$ is noncompact and singular, allowing nontrivial mixed Hodge structure on its cohomology. Loosely speaking, the structure captures the possibility of k corresponding to different $p + q$, and is termed “mixing.” The matching between the mixed Hodge polynomial with the chiral algebra representation theory was first observed in [16,17] for the A_1 class- $\mathcal{S}$ theories, and here we prove the correspondence concretely for the $\mathcal{N} = 4$ theories.

The compactly supported mixed Hodge polynomial $H_c(\mathcal{M}_N; q, t) = \sum_{i,j} h_c^{i,j}(\mathcal{M}_N) q^i t^j$ was computed in [2]:

$$H_c(\mathcal{M}_N; q, t) = (t\sqrt{q})^{d_N} \mathbb{H}^{[N]} \left(\sqrt{q}, -\frac{1}{t\sqrt{q}} \right). \quad (23)$$

Here d_N is the dimension of $\mathcal{M}_N$:

$$d_N = 2(N-1), \quad (24)$$

and the polynomial $\mathbb{H}^{[N]}(z, w)$ is generated by

$$\sum_{N \geq 0} (z-w)^2 \mathbb{H}^{[N]}(z, w) T^N = \prod_{n \geq 1} \frac{(1 - zwT^n)^2}{(1 - z^2T^n)(1 - w^2T^n)}. \quad (25)$$

From H_c , one can construct the Poincaré polynomial PH_c of the pure part of the cohomology of $\mathcal{M}_N$,

$$PH_c(\mathcal{M}_N; q) := q^{d_N/2} \mathbb{H}^{[N]}(\sqrt{q}, 0). \quad (26)$$

The factor $(z-w)^2$ arises because we consider the character variety of $PGL(N, \mathbb{C})$ rather than that of $GL(N, \mathbb{C})$. Hence, the Poincaré polynomial $PH_c(\mathcal{M}_N; q)$ can be obtained directly from the contour integral,

$$PH_c(\mathcal{M}_N; q) = \oint \frac{dT}{2\pi iT} T^{-N} \frac{1-q}{(q; T)}. \quad (27)$$

Here $(q; T) := \prod_{n \geq 0} (1 - qT^n)$ is the T -Pochhammer symbol. In other words,

$$PH_c(\mathcal{M}_N; q) = \sum_{[1^{m_1} \dots N^{m_N}] \in \mathcal{P}(N)} q^{m_1 + \dots + m_N - 1}. \quad (28)$$

Identifying $[1^{m_1} \dots N^{m_N}]$ with $\mathcal{E} = [\ell_1, \dots, \ell_N]$, we have

$$m_1 + \dots + m_N = w(\mathcal{E}), \quad (29)$$

and therefore,

$$PH_c(\mathcal{M}_N; q) = \sum_{\mathcal{E} \in \mathcal{P}(N)} q^{w(\mathcal{E})-1} = \sum_{\mathcal{E} \in \mathcal{P}(N)} q^{\dim \mathcal{V}_{\mathcal{E}}-1}.$$

Hence, each term in the polynomial corresponds to an irreducible character in $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and the number of logarithmic partners as encoded by the dimension $\dim \mathcal{V}_{\mathcal{E}}$.

Discussion. To summarize, this paper identifies the space of characters of the $\mathcal{N} = 4$ $SU(N)$ chiral algebra $\mathbb{V}[\mathcal{T}_{SU(N)}]$ by computing the Wilson line index of the 4D theory, systematically identifying the rational-functional coefficients to reveal the hidden characters. The irreducible characters are in one-to-one correspondence with the integer partitions $\mathcal{E}$ of N . The logarithmic nature of the algebra implies existence of logarithmic characters, which are constructed by imposing modular invariance of the character space $\mathcal{V}$. We prove that the structure of $\mathcal{V}$ is encoded in the Coulomb branch by computing the pure part of its mixed Hodge polynomial.

The representation theory of chiral algebra is a rich and challenging problem. Our results show that it is nontrivially encoded in the nonlocal operator spectrum of the 4D theory, and can be extracted from exact computation. It would be highly desirable to match the chiral algebra data with the fixed varieties of the Coulomb branch on S^1 , which is the standard entry in the 4D mirror symmetry dictionary. Besides line operators, Gukov-Witten type surface operators [33,34] and its index is another important subject to study, which are also determined by partitions of N , and naively one would expect nontrivial relation to the representation theory of $\mathbb{V}[\mathcal{T}_{SU(N)}]$. It would also be interesting to generalize our results to $\mathcal{N} = 4$ theories with other gauge groups, higher rank class- $\mathcal{S}$ theories and Argyres-Douglas theories with exactly marginal deformations [35], establishing concrete relations between the physics from the Higgs branch and the Coulomb branch of 4d $\mathcal{N} = 2$ theories.

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