

# Contributions of interference and noninterference components to $CP$ asymmetries in heavy meson decays

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In multibody decays of heavy mesons, conventional charge-parity ( $CP$ ) asymmetry observables obtained by integrating over the full phase space are insensitive to the higher-order wave expansion contributions in the decay amplitude squared and consequently fail to retain information on interference effects among different resonances. To overcome this limitation, one can introduce a phase-space partitioning scheme based on the zeros of Legendre polynomials, supplemented by a sign-function weighting procedure. On such a basis, two observables are defined, namely, an asymmetry observable  $\mathcal{A}_{\pm}^{\text{asy},l}$  and the corresponding  $CP$  asymmetry  $\mathcal{A}_{CP}^{\text{asy},l}$ . We further separate the observables into interference and noninterference parts and analyze their respective roles. As an application, the decay channel  $B^{\pm} \rightarrow \pi^{\pm}\pi^{+}\pi^{-}$  are analyzed in the region near the  $\rho^0(1450)$  resonance. Using the LHCb data, the results show that odd- $l$  schemes are particularly effective in isolating interference contributions, while even- $l$  schemes are more sensitive to noninterference terms. This new assignment scheme has the potential to be extended to other decay processes, thus enriching the available physical observables.

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## I. INTRODUCTION

Charge-parity ( $CP$ ) violation (CPV) was first discovered in 1964 through studies of the neutral kaon system [1]. It is an important feature of the weak interaction and is related to the matter-antimatter asymmetry in the Universe [2]. In the Standard Model (SM), CPV results from the weak complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix that reflects the transitions of different generations of quarks. CPV in decay processes requires at least two interfering amplitudes with distinct strong and weak phases. The weak phases, related to the complex elements of the CKM matrix, change the sign under the  $CP$  conjugation, whereas the strong phases, which may arise from hadronic final-state interactions, remain invariant. Experimental discovery of CPV has been established in many decays [1,3–10]. The direct CPV is the most widely

studied form of CPV. Some interference terms vanish after the differential decay widths are integrated over the whole phase space, yet they can appear in angular observables [11–15]. For a  $B$  or  $D$  three-body decay, within the partial-wave analysis, the general expression for such a  $CP$  asymmetry (CPA) observable is

$$\mathcal{A}_{CP} \propto \sum_{l=0}^n \int w_{-}^l(s_{12}) ds_{12} P_l(\cos \theta) d \cos \theta - \sum_{l=0}^n \int w_{+}^l(s_{12}) ds_{12} P_l(\cos \theta) d \cos \theta, \quad (1)$$

where  $w_{\pm}^l(s_{12})$  denote the weights in the Legendre expansion of the decay amplitude squared for a pair of  $CP$ -conjugate processes,  $P_l(\cos \theta)$  is the  $l$ th Legendre polynomial, and  $\theta$  is the angle between two same-sign pions or kaons in the rest frame of the intermediate resonance.

In the conventional treatment, once the angular variable  $\cos \theta$  is integrated over the unbroken interval  $[-1, 1]$ , all terms  $w_{\pm}^l$  with  $l \geq 1$  vanish because of the orthogonality of the Legendre polynomials, and only the zeroth-order terms  $w_{+}^0$  and  $w_{-}^0$  survive [see Eqs. (13) and (14)]. As a result, the contributions from higher-order wave expansion terms are completely lost. This implies, in particular, that the interference effects among resonances with different quantum

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numbers cannot be fully explored in such a treatment. To overcome this limitation, we adopt a phase-space partitioning approach that subdivides the angular integration domain according to the zeros of Legendre polynomials, with alternating signs assigned to adjacent subintervals [16–20]. Two novel physical quantities are introduced, the asymmetry and the corresponding  $CP$  asymmetry observables. Then, we can further investigate the contributions from both interference and noninterference components, respectively.

The LHCb collaboration has reported detail amplitude analyses of the  $B^\pm \rightarrow \pi^\pm \pi^+ \pi^-$  decays [21,22]. In the low  $\pi^+ \pi^-$  invariant mass region, CPV was established through the interference between the  $\pi^+ \pi^-$   $S$ -wave and  $P$ -wave amplitudes, which was further supported by [23,24]. In the higher  $\pi^+ \pi^-$  invariant mass region, significant CPAs were also observed in the  $\pi^+ \pi^-$   $S$ - and  $D$ -wave components, as well as in the interference between  $S$ - and  $P$ -wave resonances. The collaboration reported the first observation of a CPA involving the  $f_2(1270)$  resonance, consistent with the theoretical predictions [25–28]. Related studies had also been reported earlier by the BABAR collaboration [29,30]. However, the pronounced  $S$ -wave behavior observed around 1.5 GeV/ $c^2$  cannot be definitively linked to the  $f_0(1500)$  resonance, as expected in the  $K$ -matrix model [21]. Compared with the low resonance mass region, theoretical studies of the interference mechanisms between different resonance states in the higher mass region remain relatively limited, possibly because of the complex structure of the higher excited states. In this work, we will focus on the vicinity of the  $\rho(1450)^0$  resonance, where overlap with other resonances could be important. By testing different hypotheses, we aim to assess the possible role of the  $f_0(1500)$  and to study the two observables introduced above.

This paper is organized as follows. In Sec. II, we will introduce new observables which include higher-order wave expansion terms and the  $CP$  asymmetry induced by these observables. In Sec. III, we will apply the framework to  $B^\pm \rightarrow \pi^\pm \pi^+ \pi^-$ , with the two subsections presenting the detailed theoretical analysis and the numerical results, respectively. We will briefly give the summary in Sec. IV.

## II. THE HIGHER-ORDER WAVE EXPANSION TERM INDUCED ASYMMETRY AND ITS INDUCED $CP$ ASYMMETRY

The cascade decay approach is typically employed to deal with the multibody decay  $M \rightarrow M_1 M_2 M_3 \cdots$ , where  $M \rightarrow R_1 M_3 \cdots \rightarrow M_1 M_2 M_3 \cdots$ . When we focus on the mass region near  $R_1$ , the contributions from its neighboring resonances  $R_2, R_3, \cdots$  should also be taken into account. For three-body decays within the partial-wave framework, when the resonances corresponding to varying orbital angular momenta  $l$  are taken into account, the total

amplitude can be constructed in terms of the Legendre polynomials  $P_l(\cos \theta)$  as

$$\mathcal{M}_\pm^{\text{tot}}(s_{12}, \cos \theta) = \sum_l \mathcal{M}_\pm^l(s_{12}) P_l(\cos \theta), \quad (2)$$

where the subscripts “ $\pm$ ” represent the total decay amplitudes for the two  $CP$ -conjugate processes of the parent particle  $M$ ,  $p_i (i = 1, 2, \dots)$  are the four-momenta of the particles in the final states,  $s_{12} = (p_1 + p_2)^2$ , and  $\cos \theta = \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1| |\vec{p}_3|} = \frac{s_{23} - (s_{23,\text{max}} + s_{23,\text{min}})/2}{(s_{23,\text{max}} - s_{23,\text{min}})/2}$  with  $s_{23} = (p_2 + p_3)^2$  and  $s_{23,\text{min}}(\text{max})$  being the minimum (maximum) values of  $s_{23}$ , respectively, for fixed  $s_{12}$ .

The total amplitude squared can be expressed in the form of Legendre polynomials,

$$|\mathcal{M}_\pm^{\text{tot}}(s_{12}, \cos \theta)|^2 = \sum_l w_\pm^l(s_{12}) P_l(\cos \theta), \quad (3)$$

where  $w_\pm^l(s_{12})$  are the corresponding weights in the expansion and can be extracted by using the orthogonality of Legendre polynomials, which can be extracted using the  $w_\pm^l(s_{12}) = \int_{-1}^1 |\mathcal{M}_\pm^{\text{tot}}(s_{12}, \cos \theta)|^2 P_l(\cos \theta) d \cos \theta$ .

The orthogonality property of Legendre polynomials,

$$\int_{-1}^1 P_l(\cos \theta) P_{l'}(\cos \theta) d \cos \theta = \frac{2}{2l+1} \delta_{ll'}, \quad (4)$$

ensures that different partial waves can be separated. However, when integrating over the full angular range  $[-1, 1]$ , all terms with  $l \geq 1$  vanish due to

$$\int_{-1}^1 P_l(\cos \theta) d \cos \theta = 0 \quad \text{for } l \geq 1. \quad (5)$$

Consequently, conventional CPA observables are insensitive to interference effects between different partial waves. To recover this lost information and to probe the corresponding observable sensitivity, we subdivide the angular integration range using the zeros of Legendre polynomials. For the  $l$ th order Legendre polynomial  $P_l(\cos \theta)$ , there are  $l$  zeros in the interval  $[-1, 1]$ . We denote these zeros as  $x_1^{(l)} < x_2^{(l)} < \cdots < x_l^{(l)}$ , which divide the interval into  $l+1$  subintervals,

$$I_0 = [-1, x_1^{(l)}], \quad I_i = [x_i^{(l)}, x_{i+1}^{(l)}] \text{ for } i = 1, \dots, l-1, \quad I_l = [x_l^{(l)}, 1]. \quad (6)$$

Within each subinterval,  $P_l(\cos \theta)$  maintains a constant sign, while adjacent subintervals exhibit alternating signs. To quantify the effects of these behaviors, one can define an asymmetry observables as follows:

$$\mathcal{A}_{\pm}^{\text{asy},l} = \frac{\sum_{i=0}^l (-1)^{i+1} \int_{I_i} |\mathcal{M}_{\pm}^{\text{tot}}(s_{12}, \cos \theta)|^2 \tilde{d} \cos \theta ds_{12}}{\sum_{i=0}^l \int_{I_i} |\mathcal{M}_{\pm}^{\text{tot}}(s_{12}, \cos \theta)|^2 \tilde{d} \cos \theta ds_{12}}, \quad (7)$$

where  $\tilde{d} \cos \theta = (s_{23,\text{max}} - s_{23,\text{min}})/2d \cos \theta$ . A more intuitive description of the sign convention can be achieved by introducing the sign function  $\text{sgn}(z) (= \pm 1 \text{ when } z \gtrless 0)$ , and we can get

$$\mathcal{A}_{\pm}^{\text{asy},l} = \frac{\sum_j (\mathcal{N}_{\pm}^j) \text{sgn}(P_l(\cos \theta_j))}{\mathcal{N}_{\pm}}, \quad (8)$$

where  $\mathcal{N}_{\pm}^j$  are the event yields of the  $CP$ -conjugate processes in each bin  $j$ , which are

$$\mathcal{N}_{\pm}^j = R \int |\mathcal{M}_{\pm}^{\text{tot}}(s_{12}, \cos \theta)|_{s_{12}=s_{12,i}}^2 \tilde{d} \cos \theta, \quad (9)$$

where  $R$  is the phase-space factor, and  $s_{12,j}$  is the central value of  $s_{12}$  of bin  $j$ .

To describe the distinction of asymmetry between the pair of  $CP$ -conjugate processes, we define the CPA as the following:

$$\mathcal{A}_{\text{CP}}^{\text{asy},l} = \frac{1}{2} (\mathcal{A}_{-}^{\text{asy},l} - \mathcal{A}_{+}^{\text{asy},l}). \quad (10)$$

The observables defined in Eq. (10) probe the  $CP$  violation in the partial-wave weights normalized channel by channel, i.e., in the shape of the angular distribution, and are therefore complementary to the conventional  $CP$  asymmetry, which is not contained in this family of observables since  $\mathcal{A}_{\text{CP}}^{\text{asy},0}$  vanishes identically. The conventional  $CP$  asymmetry corresponds to the  $l = 0$  case of the normalized definition of  $\hat{A}_{\text{CP},l}$  in Ref. [20].

### III. APPLICATION TO $B^{\pm} \rightarrow \pi^{\pm} \pi^{+} \pi^{-}$ DECAY

#### A. Theoretical analysis

In this section, we apply the theoretical framework introduced above to the  $B^{\pm} \rightarrow \pi^{\pm} \pi^{+} \pi^{-}$  decay, focusing on the region near the  $\rho^0(1450)$  resonance, whose width is relatively large, i.e., about 400 MeV. This broad width necessitates a careful treatment of overlapping contributions from nearby resonances. We will restrict the analysis

to an observation window of one-half width around  $\rho^0(1450)$ , i.e.,  $s_{12} = [(m_{\rho^0} - \Gamma_{\rho^0}/2)^2, (m_{\rho^0} + \Gamma_{\rho^0}/2)^2]$ , and will systematically examine possible contributions from  $f_2(1270)$ ,  $f_0(1370)$ ,  $f_2(1430)$ ,  $f_0(1500)$ ,  $f_2'(1525)$ , and  $f_2(1565)$ , which appear in this window. Their classifications, masses, and decay widths are summarized in Table I. The LHCb collaboration observed a significant  $D$ -wave signal, which was identified as the  $f_2(1270)$  resonance. For the  $f_2(1430)$  ( $\Gamma_{f_2(1430)} = 46$  MeV) and  $f_2'(1525)$  ( $\Gamma_{f_2'(1525)} = 72$  MeV) resonances, both of them are too high in mass and too narrow to be likely to induce a significant effect in the region of interest. This observation is further supported for  $f_2'(1525)$  by its strongly suppressed  $\pi\pi$  branching fraction,  $\mathcal{B}(f_2'(1525) \rightarrow \pi\pi) \approx 8\%$ , compared with  $\mathcal{B}(f_2'(1525) \rightarrow KK) \approx 89\%$  and  $\mathcal{B}(f_2'(1525) \rightarrow \eta\eta) \approx 10\%$ . In addition, dedicated fits are performed, in which  $f_2(1270)$  is replaced by either  $f_2(1430)$  or  $f_2'(1525)$  within the two schemes given below, confirming that the contributions from  $f_2(1430)$  and  $f_2'(1525)$  can be neglected. For similar reasons, the tensor resonance  $f_2(1565)$  is also ignored, although its width  $\Gamma_{f_2(1565)} = 130$  MeV is comparable to  $\Gamma_{f_2(1270)} = 186.6$  MeV. To assess possible scalar contributions, we consider two representative schemes. Scheme 1 (S1) excludes the contributions from  $\pi\pi$  S-wave components and only includes the interference between  $P$ - and  $D$ -wave resonances,  $\rho^0(1450)(l=1)$  and  $f_2(1270)(l=2)$ . Scheme 2 (S2) additionally incorporates the contribution from  $f_0(1370)$  or  $f_0(1500)(l=0)$  into S1, to determine whether a scalar contribution is required and which one provides a more appropriate description.

The analysis is initially performed within the framework S2, because S1 represents a special case of S2 with  $\mathcal{M}_{\pm}^{f_0} = 0$ . In S2, the total amplitude can be expressed as

$$\mathcal{M}_{\pm}^{\text{tot}} = \mathcal{M}_{\pm}^{f_0} P_0(\cos \theta) + \mathcal{M}_{\pm}^{\rho^0} P_1(\cos \theta) + \mathcal{M}_{\pm}^{f_2} P_2(\cos \theta), \quad (11)$$

where  $f_0$  represents either the  $f_0(1370)$  or  $f_0(1500)$  resonance, and  $\rho^0$  and  $f_2$  are used as the abbreviated forms for  $\rho^0(1450)$  and  $f_2(1270)$ , respectively. We can abbreviate  $\mathcal{M}_{\pm}^{f_0}$ ,  $\mathcal{M}_{\pm}^{\rho^0}$ , and  $\mathcal{M}_{\pm}^{f_2}$  as  $\mathcal{M}_{\pm}^R$ , which can be factorized as

TABLE I. The classifications, masses (in MeV), and decay widths (in MeV) of the resonances [31].

| Scalar      | Mass          | Decay width  | Vector         | Mass          | Decay width  | Tensor       | Mass             | Decay width           |
|-------------|---------------|--------------|----------------|---------------|--------------|--------------|------------------|-----------------------|
| $f_0(1370)$ | 1200–1500     | 200–500      | $\rho^0(1450)$ | $1465 \pm 25$ | $400 \pm 60$ | $f_2(1270)$  | $1275.4 \pm 0.8$ | $185.8^{+2.8}_{-2.1}$ |
| $f_0(1500)$ | $1522 \pm 25$ | $108 \pm 33$ |                |               |              | $f_2(1430)$  | $\approx 1430$   | $46 \pm 15$           |
|             |               |              |                |               |              | $f_2'(1525)$ | $1517.3 \pm 2.4$ | $84.4 \pm 2.7$        |
|             |               |              |                |               |              | $f_2(1565)$  | $1571 \pm 13$    | $132 \pm 23$          |

TABLE II. The projection weights of every interference and noninterference component for the  $w_{\pm}^l$ .

| $l$     | Noninterference terms         |                                  |                               | Interference terms                                              |                                                                 |                                                              |
|---------|-------------------------------|----------------------------------|-------------------------------|-----------------------------------------------------------------|-----------------------------------------------------------------|--------------------------------------------------------------|
|         | $ \mathcal{M}_{\pm}^{f_2} ^2$ | $ \mathcal{M}_{\pm}^{\rho_0} ^2$ | $ \mathcal{M}_{\pm}^{f_0} ^2$ | $\text{Re}[\mathcal{M}_{\pm}^{\rho_0}\mathcal{M}_{\pm}^{f_0*}]$ | $\text{Re}[\mathcal{M}_{\pm}^{\rho_0}\mathcal{M}_{\pm}^{f_2*}]$ | $\text{Re}[\mathcal{M}_{\pm}^{f_0}\mathcal{M}_{\pm}^{f_2*}]$ |
| $l = 0$ | 1/5                           | 1/3                              | 1                             | —                                                               | —                                                               | —                                                            |
| $l = 1$ | —                             | —                                | —                             | 2                                                               | 4/5                                                             | —                                                            |
| $l = 2$ | 2/7                           | 2/3                              | —                             | —                                                               | —                                                               | 2                                                            |
| $l = 3$ | —                             | —                                | —                             | —                                                               | 6/5                                                             | —                                                            |
| $l = 4$ | 18/35                         | —                                | —                             | —                                                               | —                                                               | —                                                            |

$$\mathcal{M}_{\pm}^R(s_{12}) = c_{\pm}^R F_R^{\text{BW}} e^{i\delta_{\pm}^R} P_l(\cos \theta), \quad (12)$$

where  $F_R^{\text{BW}}$  is the Breit-Wigner line-shape of the resonance  $R$ , and  $c_{\pm}^R$  and  $\delta_{\pm}^R$  are the corresponding amplitude and the relative phase, respectively.

By combining Eq. (11) and the relationships between  $\cos^n \theta$  and  $P_l(\cos \theta)$ , the form of the total amplitude squared can be derived as

$$|\mathcal{M}_{\pm}^{\text{tot}}|^2 = \sum_{l=0}^4 w_{\pm}^l P_l(\cos \theta), \quad (13)$$

where the weights  $w_{\pm}^l$  include both interference and noninterference contributions, and the projection weights for every component are summarized in Table II. With these, we can get

$$\begin{aligned} w_{\pm}^0 &= \left( |\mathcal{M}_{\pm}^{f_0}|^2 + \frac{1}{5} |\mathcal{M}_{\pm}^{f_2}|^2 + \frac{1}{3} |\mathcal{M}_{\pm}^{\rho_0}|^2 \right), & w_{\pm}^1 &= \left( 2\text{Re}[\mathcal{M}_{\pm}^{\rho_0}\mathcal{M}_{\pm}^{f_0*}] + \frac{4}{5}\text{Re}[\mathcal{M}_{\pm}^{\rho_0}\mathcal{M}_{\pm}^{f_2*}] \right), \\ w_{\pm}^2 &= \left( \frac{2}{3} |\mathcal{M}_{\pm}^{\rho_0}|^2 + \frac{2}{7} |\mathcal{M}_{\pm}^{f_2}|^2 + 2\text{Re}[\mathcal{M}_{\pm}^{f_0}\mathcal{M}_{\pm}^{f_2*}] \right), & w_{\pm}^3 &= \frac{6}{5} \text{Re}[\mathcal{M}_{\pm}^{\rho_0}\mathcal{M}_{\pm}^{f_2*}], \\ w_{\pm}^4 &= \frac{18}{35} |\mathcal{M}_{\pm}^{f_2}|^2. \end{aligned} \quad (14)$$

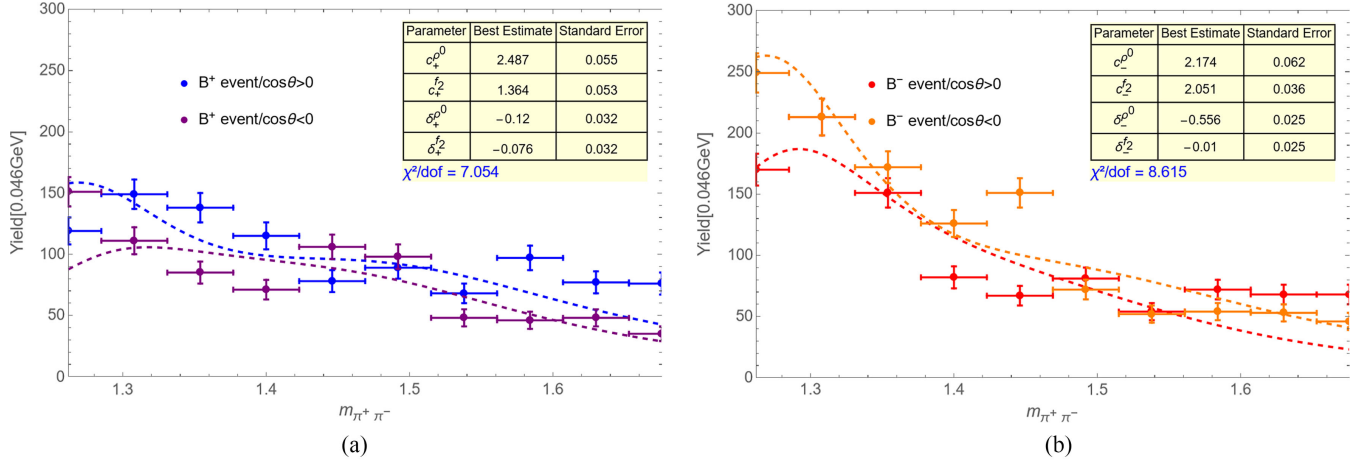
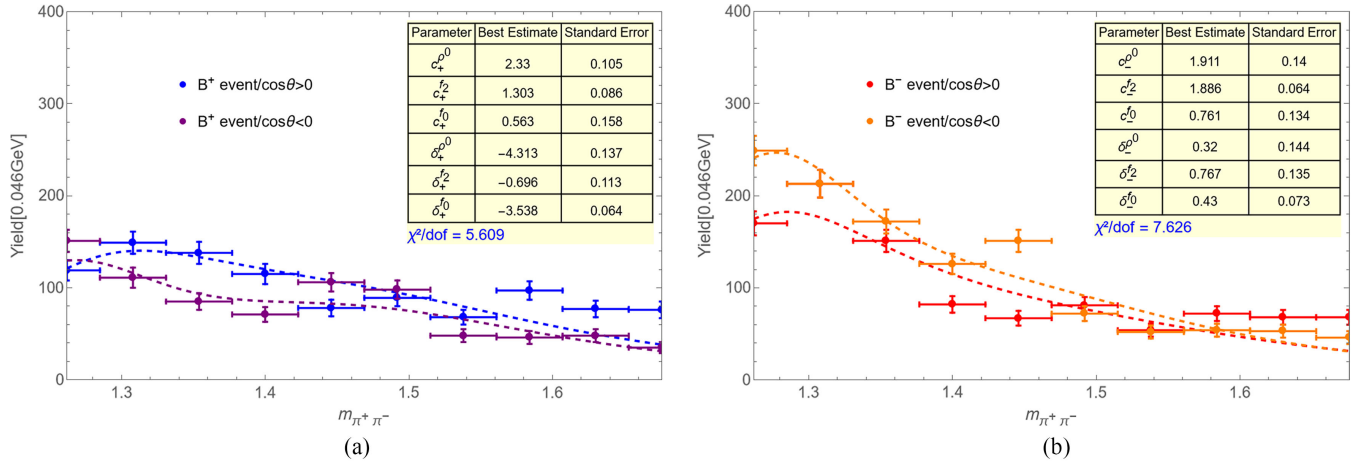
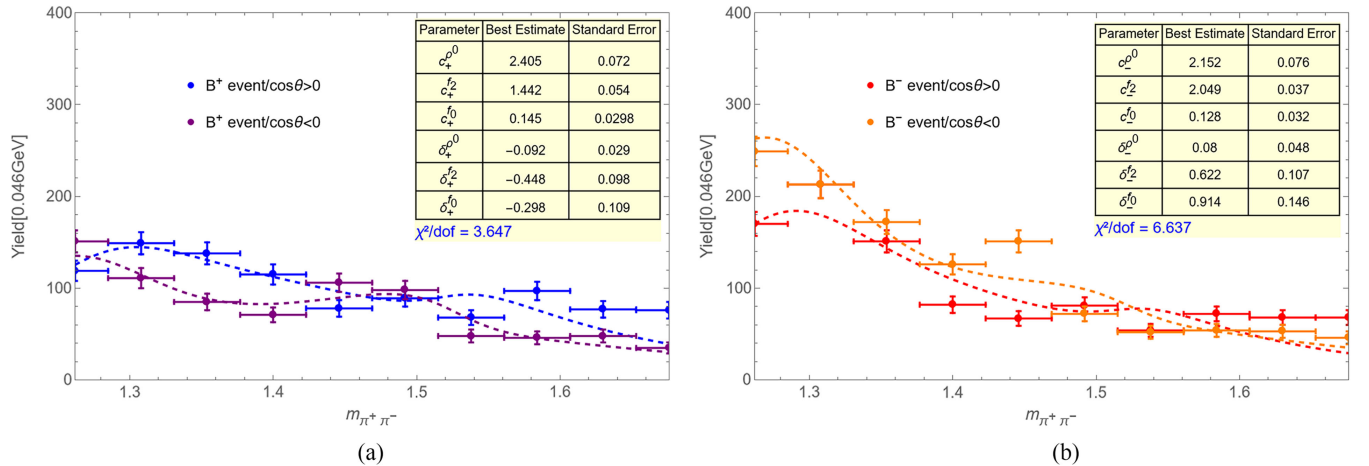
## B. Results

The parameter estimations are performed by minimizing the  $\chi^2$  function using the binned event yields as the observables to be fitted. For S1, Eqs. (13), (12), and (9) are adopted with the  $\cos \theta$  being restricted to either  $[-1, 0]$  or  $[0, 1]$ . We set  $c_{\pm}^{f_0} = 0$  in Eq. (12) to remove the

contributions from the  $f_0(1370)$  or  $f_0(1500)$  resonance. Theoretical results for the event yields in each bin are obtained by matching the data in Ref. [21] and are shown in Table III. The best-fit parameters  $c_{\pm}^{\rho_0}$ ,  $c_{\pm}^{f_2}$ ,  $\delta_{\pm}^{\rho_0}$ , and  $\delta_{\pm}^{f_2}$ , together with the corresponding fitted yield distributions, are shown in Figs. 1(a) and 1(b) for the  $B^+$  and  $B^-$  decay

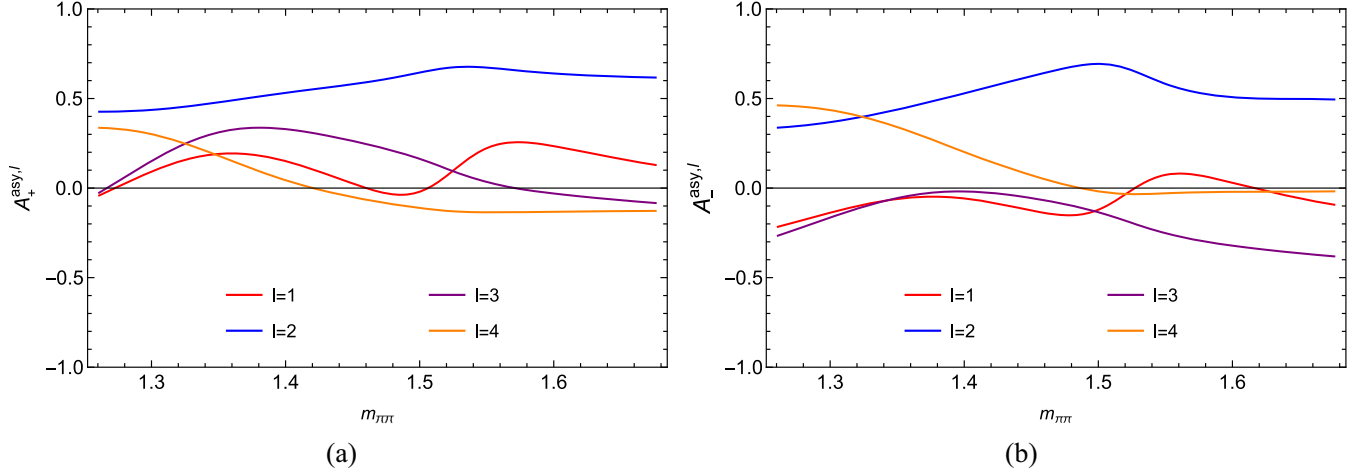
 TABLE III. The event yields for  $B^{\pm} \rightarrow \pi^{\pm}\pi^+\pi^-$  with  $m_{\pi\pi}$  ranging from 1.262 GeV to 1.676 GeV based on Fig. 12 in Ref. [21], denoted by  $N_i(B^-)$  and  $N_i(B^+)$  for  $B^- \rightarrow \pi^-\pi^+\pi^-$  and  $B^+ \rightarrow \pi^+\pi^+\pi^-$  decays, respectively, where the error propagation formula is adopted. Note that the helicity angle  $\theta_{\text{hel}}$  and  $\theta$  mentioned in this paper satisfy the relation  $\theta_{\text{hel}} + \theta = 180^\circ$ .

| Bin (GeV)   | $\cos \theta_{\text{hel}} > 0$ |              |              | $\cos \theta_{\text{hel}} < 0$ |              |              |
|-------------|--------------------------------|--------------|--------------|--------------------------------|--------------|--------------|
|             | $N_i(B^-) - N_i(B^+)$          | $N_i(B^-)$   | $N_i(B^+)$   | $N_i(B^-) - N_i(B^+)$          | $N_i(B^-)$   | $N_i(B^+)$   |
| 1.262–1.285 | $98 \pm 20$                    | $249 \pm 16$ | $151 \pm 12$ | $51 \pm 17$                    | $170 \pm 13$ | $119 \pm 11$ |
| 1.285–1.331 | $102 \pm 18$                   | $213 \pm 15$ | $111 \pm 11$ | $64 \pm 19$                    | $213 \pm 15$ | $149 \pm 12$ |
| 1.331–1.377 | $87 \pm 16$                    | $172 \pm 13$ | $85 \pm 9$   | $13 \pm 17$                    | $151 \pm 12$ | $138 \pm 12$ |
| 1.377–1.423 | $55 \pm 14$                    | $126 \pm 11$ | $71 \pm 8$   | $-33 \pm 14$                   | $82 \pm 9$   | $115 \pm 11$ |
| 1.423–1.469 | $45 \pm 16$                    | $151 \pm 12$ | $106 \pm 10$ | $-11 \pm 12$                   | $67 \pm 8$   | $78 \pm 9$   |
| 1.469–1.515 | $-26 \pm 13$                   | $72 \pm 8$   | $98 \pm 10$  | $-8 \pm 13$                    | $81 \pm 9$   | $89 \pm 9$   |
| 1.515–1.561 | $4 \pm 10$                     | $52 \pm 7$   | $48 \pm 7$   | $-14 \pm 11$                   | $54 \pm 7$   | $68 \pm 8$   |
| 1.561–1.607 | $8 \pm 10$                     | $54 \pm 7$   | $46 \pm 7$   | $-25 \pm 13$                   | $72 \pm 8$   | $97 \pm 10$  |
| 1.607–1.653 | $5 \pm 10$                     | $53 \pm 7$   | $48 \pm 7$   | $-9 \pm 12$                    | $68 \pm 8$   | $77 \pm 9$   |
| 1.653–1.676 | $11 \pm 9$                     | $46 \pm 7$   | $35 \pm 6$   | $-8 \pm 12$                    | $68 \pm 8$   | $76 \pm 9$   |

FIG. 1. Fitted results for  $B^+ \rightarrow \pi^+ \pi^+ \pi^-$  (left) and  $B^- \rightarrow \pi^- \pi^+ \pi^-$  (right) decays in S1.FIG. 2. Same as Fig. 1 except for adding the contribution from the  $f_0(1370)$  resonance in S2.FIG. 3. Same as Fig. 1 except for adding the contribution from the  $f_0(1500)$  resonance in S2, which is also presented in Ref. [20].

channels, respectively. For S2, we perform the fits with the  $f_0(1370)$  and  $f_0(1500)$  resonances being added separately, and the corresponding results are shown in Figs. 2 and 3. Comparing Figs. 2 and 3 suggests that including  $f_0(1500)$

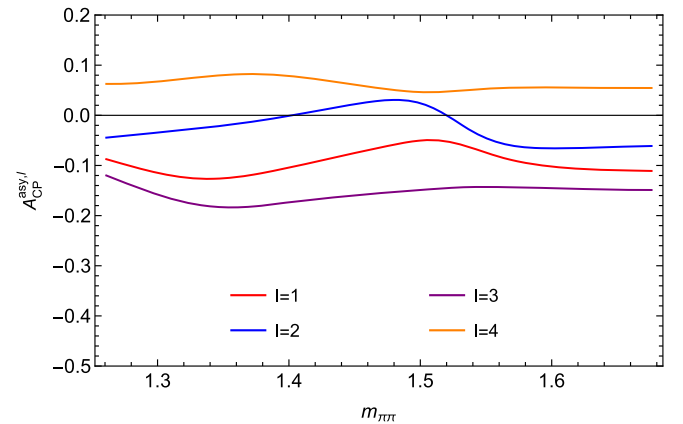
may provide a better phenomenological description than including  $f_0(1370)$ . In addition, we have performed a joint fit, including both the  $f_0(1370)$  and the  $f_0(1500)$ , in which the  $S$ -wave amplitude is taken as the coherent


 FIG. 4. Results of  $\mathcal{A}_{\pm}^{asy,l}$  with adding the contribution from  $f_0(1500)$  in S2.

sum  $\mathcal{M}_{\pm}^{f_0(1370)} + \mathcal{M}_{\pm}^{f_0(1500)}$ , introducing four additional free parameters relative to scenario S2. It finds that the  $\chi^2/\text{d.o.f.}$  changes slightly, i.e., once the reduction of the number of degrees of freedom is taken into account, the improvement is not statistically significant. Moreover, since the  $f_0(1370)$  is very broad and overlaps strongly with the  $f_0(1500)$  in the region under study, the two scalar components exhibit strong parameter correlations. This indicates that the present binned data cannot significantly distinguish between the single- and two-resonance descriptions of the  $S$ -wave. Besides, the fitted parameters of the  $f_0(1500)$  component in the joint fit are essentially consistent with those of scenario S2. We therefore adopt S2 as the baseline in the following analysis.

Substituting Eqs. (14) and (13) into Eq. (9) shows explicitly that only the  $l = 0$  partial-wave expansion term survives, whereas all higher components ( $l = 1, 2, 3, 4$ ) vanish. Consequently, both the noninterference contributions from  $w_2$  and  $w_4$  and the interference terms from  $w_1$ ,  $w_2$  and  $w_3$  in Eq. (14) do not contribute to the event yields after the integration. To recover their contributions, we subdivide the phase space according to the zeros of the Legendre polynomials and assign alternating signs to adjacent intervals by introducing the  $\text{sgn}(x)$  function in Eq. (8). For S2, using the fitted parameters in Fig. 3 and Eqs. (11)–(14), we evaluate Eq. (8) and obtain the  $\mathcal{A}_{\pm}^{asy,l}$  observables. The results lie in the range about  $[-40\% - 70\%]$  as shown in Fig. 4 for  $B^+$  and  $B^0$  decays. As the  $l$  varies, distinct resonance signals become clear, which are associated with  $f_0(1500)$ ,  $\rho^0(1450)$ , and  $f_2(1270)$ , and appear when  $l = 2, 3$ , and  $4$ , respectively, while a signal from the  $f_0(1500)$  state is also visible when  $l = 1$ . The observables  $\mathcal{A}_{\pm}^{asy,l}$  can be combined to obtain the corresponding CPA observables  $\mathcal{A}_{\text{CP}}^{asy,l}$ , as defined in Eq. (10). The theoretical results for  $\mathcal{A}_{\text{CP}}^{asy,l}$  are in the range about  $[-20\% - 9\%]$ , as shown in Fig. 5.

To isolate the interference effects and compare them with the noninterference contributions in  $\mathcal{A}_{\pm}^{asy,l}$  and  $\mathcal{A}_{\text{CP}}^{asy,l}$ , we redefine two observables as  $\mathcal{A}_{\pm}^{asy,\text{int}/\text{non}}$  and  $\mathcal{A}_{\text{CP}}^{asy,\text{int}/\text{non}}$ , where the superscripts “int” and “non” denote contributions arising solely from the interference and noninterference terms, respectively, and the index  $l$  is hidden for brevity. To facilitate the comparison, the quantities in Figs. 4 and 5 containing the summed contributions from both sources are labeled as  $\mathcal{A}_{\pm}^{asy,\text{tot}}$  and  $\mathcal{A}_{\text{CP}}^{asy,\text{tot}}$ . Specifically, each term in Eq. (14) is separated into interference and noninterference components and then multiplied by the corresponding Legendre polynomial. Adopting Eqs. (8) and (10), we compute  $\mathcal{A}_{\pm}^{asy,\text{int}/\text{non}}$  and  $\mathcal{A}_{\text{CP}}^{asy,\text{int}/\text{non}}$ , and the results are shown in Figs. 6–8, including the results of  $\mathcal{A}_{\pm}^{asy,\text{tot}}$  and  $\mathcal{A}_{\text{CP}}^{asy,\text{tot}}$ . It can be seen from Figs. 6 and 7 that for  $l = 1$  and  $3$ , the interference contributions dominate, while the noninterference contributions vanish. For  $l = 2$  and  $4$ , the opposite situation is observed. From our analysis, this behavior primarily reflects the adopted region-subdivision scheme, defined by the zeros of the Legendre polynomials,


 FIG. 5. Results of the  $\mathcal{A}_{\text{CP}}^{asy,l}$  obtained in S2 for the  $B^{\pm} \rightarrow \pi^{\pm} \pi^+ \pi^-$  decays.

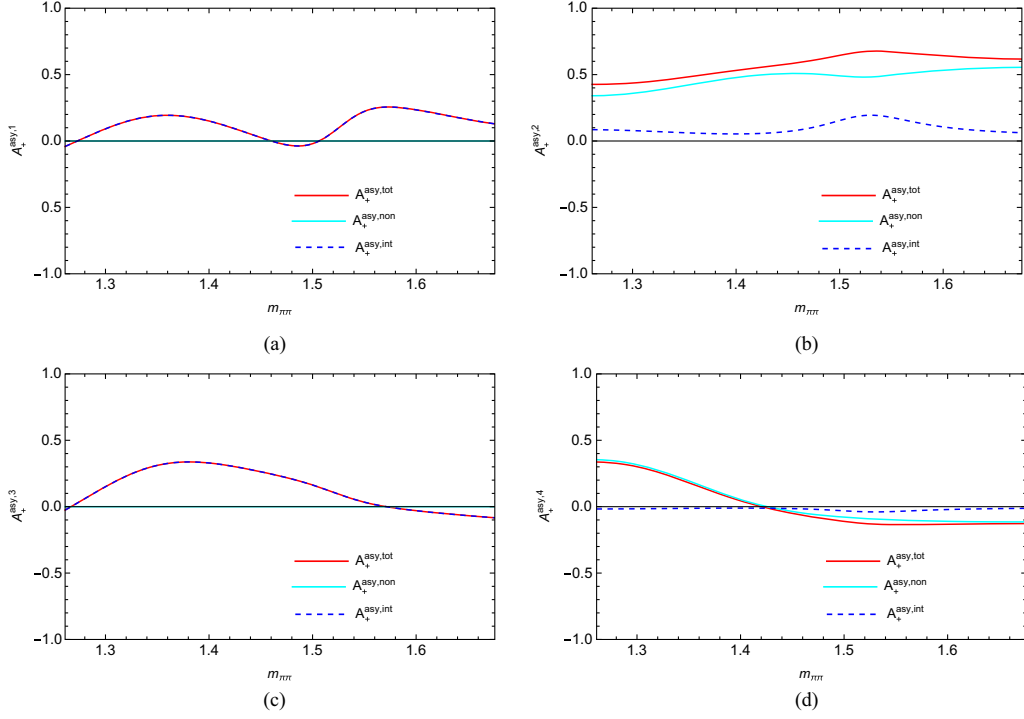


FIG. 6. Results of  $\mathcal{A}_+^{\text{asy,int}}$ ,  $\mathcal{A}_+^{\text{asy,non}}$ , and  $\mathcal{A}_+^{\text{asy,tot}}$  with adding the contribution from  $f_0(1500)$  in S2.

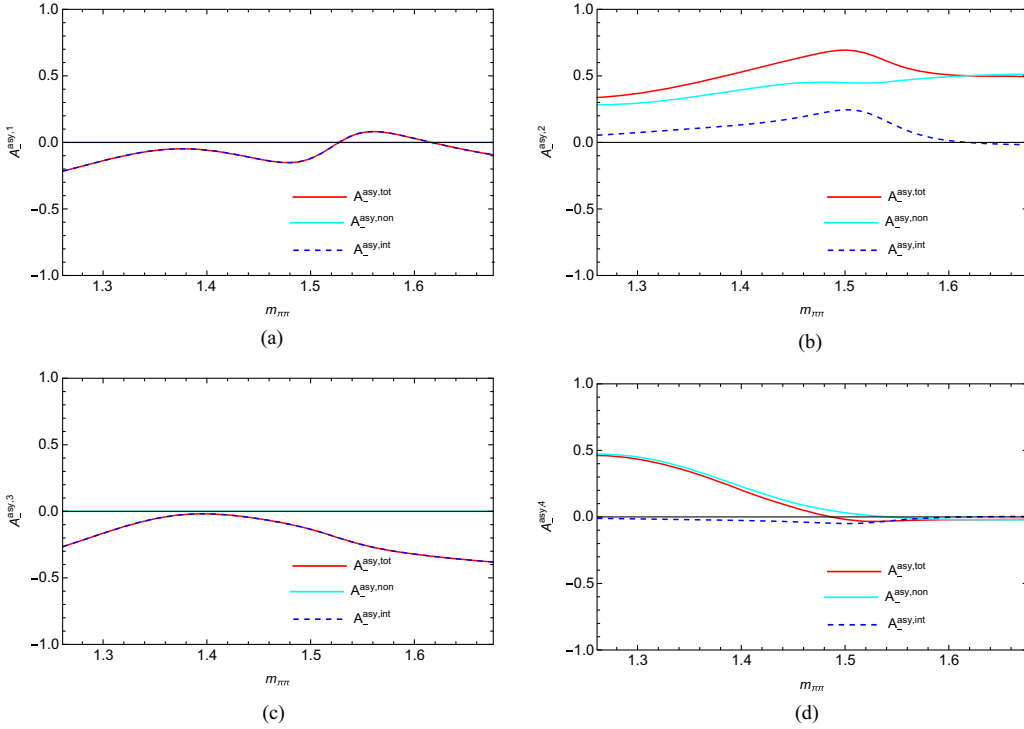
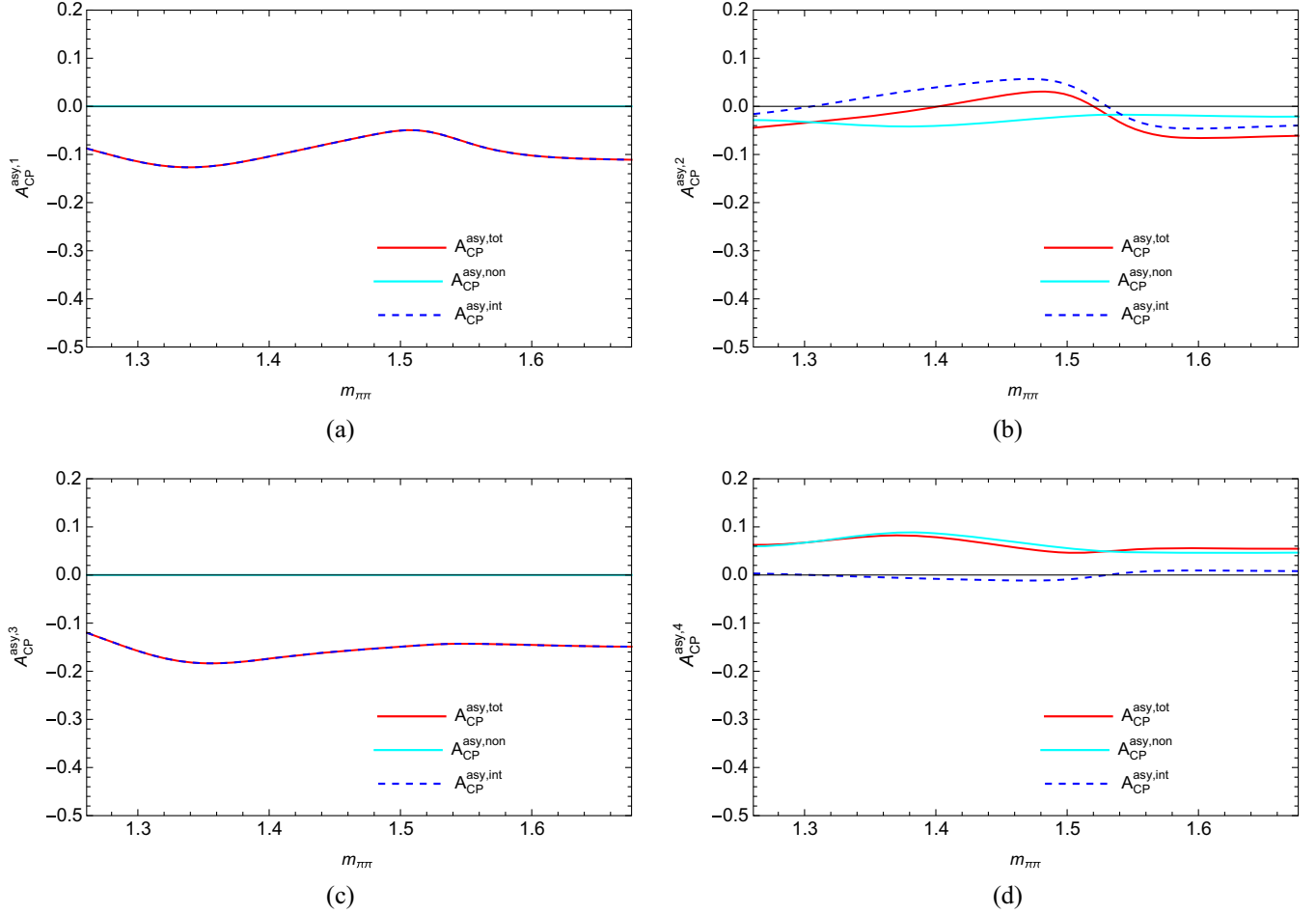


FIG. 7. Results of  $\mathcal{A}_-^{\text{asy,int}}$ ,  $\mathcal{A}_-^{\text{asy,non}}$ , and  $\mathcal{A}_-^{\text{asy,tot}}$  with the contribution from  $f_0(1500)$  in S2.

together with the parity properties of the polynomials. For the odd- $l$ , the interference and noninterference contributions are proportional to the  $P_{1,3}$  and  $P_{0,2,4}$  terms, respectively. After the phase-space integration, the  $P_{0,2,4}$  terms

vanish. For even- $l$ , the interference and noninterference parts are proportional to the  $P_2$  and  $P_{0,2,4}$  terms, respectively. In this case, the phase-space integrations of  $P_{0,2,4}$  are nonzero, so both contributions remain and their relative


 FIG. 8. Results of  $\mathcal{A}_{CP}^{asy,int}$ ,  $\mathcal{A}_{CP}^{asy,non}$ , and  $\mathcal{A}_{CP}^{asy,tot}$  with the contribution from  $f_0(1500)$  in S2.

magnitudes depend on the specific theoretical predictions. Consequently, the choices of the  $l = 1$  and 3 are well suited to isolating interference contributions, whereas  $l = 2$  and 4, particularly  $l = 4$ , are more suitable for studying the noninterference effects.<sup>1</sup>

#### IV. CONCLUSIONS

$CP$  violation in multibody decays of heavy mesons contains rich dynamical information associated with both resonance structures and their interferences. In the

<sup>1</sup>It should be noted that, although the weight  $w_{\pm}^4$  in Eq. (14) contains only the noninterference term  $\frac{18}{35}|\mathcal{M}_{\pm}^{f_2}|^2$ , the observable  $\mathcal{A}_{\pm}^{asy,l=4}$  still receives an interference contribution. This is because  $\text{sgn}[P_4(\cos\theta)]$ , unlike  $P_4(\cos\theta)$  itself, is not orthogonal to the other Legendre polynomials. Being an even function of  $\cos\theta$ , it has nonvanishing overlaps with  $P_0$  and  $P_2$  so that the numerator of  $\mathcal{A}_{\pm}^{asy,4}$  also picks up  $w_{\pm}^0$  and  $w_{\pm}^2$ . The residual interference contribution at  $l = 4$  in Figs. 6 and 7 therefore stems from the interference term  $2\text{Re}[\mathcal{M}_{\pm}^{f_0}\mathcal{M}_{\pm}^{f_2*}]$  contained in  $w_{\pm}^2$ , while the odd- $l$  interference terms do not contribute by parity.

conventional treatment, however, integrations over the full angular phase space remove the contributions from higher-order terms in the partial-wave expansion of the total decay amplitude squared, thereby reducing or even eliminating the sensitivity to interference effects among different intermediate states. To address these limitations within the partial-wave analysis framework, we introduce a class of angular observables constructed with sign functions of Legendre polynomials, together with the corresponding  $CP$  asymmetries, denoted by  $\mathcal{A}_{\pm}^{asy,l}$  and  $\mathcal{A}_{CP}^{asy,l}$ , respectively. As an illustrative example,  $B^{\pm} \rightarrow \pi^{\pm}\pi^{+}\pi^{-}$  decays are considered. Depending on whether the  $\pi\pi$   $S$ -wave resonances influence the study region, we adopt two research schemes, and the comparison shows that including  $f_0(1500)$  provides a more satisfactory description. The theoretical results for  $\mathcal{A}_{\pm}^{asy,l}$  and  $\mathcal{A}_{CP}^{asy,l}$  are obtained with the fitted parameters, as shown in Figs. 4 and 5. We further quantify the relative impacts of the interference and non-interference components with the aids of  $\mathcal{A}_{\pm}^{asy,int/non}$  and  $\mathcal{A}_{CP}^{asy,int/non}$ , whose results are presented in Figs. 6–8. Through comparison, it is found that for  $l = 1$  and 3,  $\mathcal{A}_{\pm}^{asy,l}$  and  $\mathcal{A}_{CP}^{asy,l}$  are

dominated by interference contributions, whereas the situation is opposite for  $l = 2$  and 4. Such behavior originates from the angular structure of the contributions. The interference (noninterference) contributions are proportional to  $P_{1,3}$  and ( $P_{0,2,4}$ ) for odd  $l$  and to  $P_2$  ( $P_{0,2,4}$ ) for even  $l$ . Since the  $P_{0,2,4}$  terms vanish under phase-space integration only in the odd- $l$  case, odd- $l$  observables isolate the interference contribution, whereas even- $l$  ones retain significant sensitivity to noninterference terms. Their relative magnitudes depend on specific theoretical predictions. These findings can be expected to generalize to other heavy-hadron decay processes.

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## DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Evidence for the  $2\pi$  decay of the  $K_2^0$  meson, *Phys. Rev. Lett.* **13**, 138 (1964).
  - [2] M. Kobayashi and T. Maskawa,  $CP$  violation in the renormalizable theory of weak interaction, *Prog. Theor. Phys.* **49**, 652 (1973).
  - [3] A. Alavi-Harati *et al.* (KTeV Collaboration), Observation of direct  $CP$  violation in  $K_{S,L} \rightarrow \pi\pi$  decays, *Phys. Rev. Lett.* **83**, 22 (1999).
  - [4] B. Aubert *et al.* (BABAR Collaboration), Observation of  $CP$  violation in the  $B^0$  meson system, *Phys. Rev. Lett.* **87**, 091801 (2001).
  - [5] K. Abe *et al.* (Belle Collaboration), Observation of large  $CP$  violation in the neutral  $B$  meson system, *Phys. Rev. Lett.* **87**, 091802 (2001).
  - [6] A. Poluektov *et al.* (Belle Collaboration), Evidence for direct  $CP$  violation in the decay  $B^\pm \rightarrow D^{(*)}K^\pm$ ,  $D \rightarrow K_S^0\pi^+\pi^-$  and measurement of the CKM phase  $\phi_3$ , *Phys. Rev. D* **81**, 112002 (2010).
  - [7] P. del Amo Sanchez *et al.* (BABAR Collaboration), Measurement of  $CP$  observables in  $B^\pm \rightarrow D_{CP}K^\pm$  decays and constraints on the CKM angle  $\gamma$ , *Phys. Rev. D* **82**, 072004 (2010).
  - [8] R. Aaij *et al.* (LHCb Collaboration), First observation of  $CP$  violation in the decays of  $B_s^0$  mesons, *Phys. Rev. Lett.* **110**, 221601 (2013).
  - [9] R. Aaij *et al.* (LHCb Collaboration), Observation of  $CP$  violation in charm decays, *Phys. Rev. Lett.* **122**, 211803 (2019).
  - [10] R. Aaij *et al.* (LHCb Collaboration), Observation of charge-parity symmetry breaking in baryon decays, *Nature (London)* **643**, 1223 (2025).
  - [11] J. F. Donoghue, B. R. Holstein, and G. Valencia, Survey of present and future tests of  $CP$  violation, *Int. J. Mod. Phys. A* **02**, 319 (1987).
  - [12] G. Valencia, Angular correlations in the decay  $B \rightarrow VV$  and  $CP$  violation, *Phys. Rev. D* **39**, 3339 (1989).
  - [13] B. Kayser, Kinematically nontrivial  $CP$  violation in beauty decay, *Nucl. Phys. B, Proc. Suppl.* **13**, 487 (1990).
  - [14] W. Bensalem and D. London,  $T$  odd triple product correlations in hadronic  $b$  decays, *Phys. Rev. D* **64**, 116003 (2001).
  - [15] J.-P. Wang, Q. Qin, and F.-S. Yu, Complementary  $CP$  violation induced by T-odd and T-even correlations, *Phys. Rev. D* **111**, L111301 (2025).
  - [16] Z.-H. Zhang and J.-J. Qi, Analysis of angular distribution asymmetries and the associated  $CP$  asymmetries in three-body decays of bottom baryons, *Eur. Phys. J. C* **83**, 133 (2023).
  - [17] R. Hu and Z.-H. Zhang, Data-based analysis of the forward-backward asymmetry in  $B^\pm \rightarrow K^\pm K^\mp K^\pm$ , *Phys. Rev. D* **105**, 093007 (2022).
  - [18] Z.-H. Zhang and X.-H. Guo, A novel strategy for searching for  $CP$  violations in the baryon sector, *J. High Energy Phys.* **07** (2021) 177.
  - [19] J.-J. Qi, J.-Y. Yang, and Z.-H. Zhang,  $CP$  asymmetries corresponding to the imaginary parts of the interference terms in cascade decays of heavy hadrons, *Phys. Rev. D* **110**, L111301 (2024).
  - [20] J.-J. Qi, Z.-Y. Wang, Z.-H. Zhang, and X.-H. Guo, Normalization of partial wave  $CP$  asymmetries in three-body decays of heavy hadrons, *Phys. Rev. D* **113**, L111301 (2026).
  - [21] R. Aaij *et al.* (LHCb Collaboration), Amplitude analysis of the  $B^+ \rightarrow \pi^+\pi^+\pi^-$  decay, *Phys. Rev. D* **101**, 012006 (2020).
  - [22] R. Aaij *et al.* (LHCb Collaboration), Observation of several sources of  $CP$  violation in  $B^+ \rightarrow \pi^+\pi^+\pi^-$  decays, *Phys. Rev. Lett.* **124**, 031801 (2020).
  - [23] Y.-R. Wei and Z.-H. Zhang, Forward-backward asymmetry induced  $CP$  asymmetry of  $B^\pm \rightarrow \pi^\pm\pi^\pm\pi^\mp$ , *Phys. Rev. D* **106**, 113002 (2022).
  - [24] H.-Y. Cheng,  $CP$  violation in the interference between  $\rho(770)^0$  and S-wave in  $B^\pm \rightarrow \pi^\pm\pi^\pm\pi^\mp$  decays and its implication for  $CP$  asymmetry in  $B^\pm \rightarrow \rho^0\pi^\pm$ , *Phys. Rev. D* **106**, 113004 (2022).
  - [25] H.-Y. Cheng and K.-C. Yang, Charmless hadronic B decays into a tensor meson, *Phys. Rev. D* **83**, 034001 (2011).

- [26] Q. Chang, L. Yang, Z.-T. Zou, and Y. Li, Study of the  $B^+ \rightarrow \pi^+(\pi^+\pi^-)$  decay in PQCD approach, *Eur. Phys. J. C* **84**, 753 (2024).
- [27] Y. Li, A.-J. Ma, Z. Rui, W.-F. Wang, and Z.-J. Xiao, Quasi-two-body decays  $B_{(s)} \rightarrow Pf_2(1270) \rightarrow P\pi\pi$  in the perturbative QCD approach, *Phys. Rev. D* **98**, 056019 (2018).
- [28] Z.-T. Zou, X. Yu, and C.-D. Lu, Nonleptonic two-body charmless B decays involving a tensor meson in the perturbative QCD approach, *Phys. Rev. D* **86**, 094015 (2012).
- [29] B. Aubert *et al.* (BABAR Collaboration), An amplitude analysis of the decay  $B^\pm \rightarrow \pi^\pm\pi^\pm\pi^\mp$ , *Phys. Rev. D* **72**, 052002 (2005).
- [30] B. Aubert *et al.* (BABAR Collaboration), Dalitz plot analysis of  $B^\pm \rightarrow \pi^\pm\pi^\pm\pi^\mp$  decays, *Phys. Rev. D* **79**, 072006 (2009).
- [31] S. Navas *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).