

Two R factors in the small- x shockwave formalism

Yuri V. Kovchegov^{1,*}, M. Gabriel Santiago^{2,3,4,5,†} and Huachen Sun^{1,‡}

¹*Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA*

²*Center for Nuclear Fentography, SURA, 12000 Jefferson Avenue Newport News, Virginia 23606, USA*

³*Department of Physics, Old Dominion University, Norfolk, Virginia 23529, USA*

⁴*Jefferson Lab, Newport News, Virginia 23606, USA*

⁵*Department of Physics, Temple University, Philadelphia, Pennsylvania 19122, USA*



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There are two R -factors frequently used in the phenomenology of exclusive processes at small values of the Bjorken x variable. One R -factor takes into account the effects of nonzero longitudinal momentum transfer, which is assumed to be zero in the dipole scattering amplitude. Another R -factor accounts for the real part of the elastic scattering amplitude which is often neglected, with the standard dipole scattering amplitude giving only the imaginary part of the elastic amplitude. In this work we present two new theoretical developments aimed at eliminating the need for the two R -factors. We argue that the R -factors can be replaced by (i) modifying the argument of the dipole scattering amplitude and by (ii) augmenting the initial conditions for its nonlinear small- x evolution. Specifically, we show that to account for the effects of nonzero skewness ξ , one has to replace the rapidity argument $Y = \ln(1/x)$ of the eikonal dipole amplitude N and the odderon dipole amplitude $\mathcal{O}$ by $Y = \ln \min \{1/|x|, 1/|\xi|\}$. The prescription applies to the elastic scattering cross sections, as well as for calculations of the generalized parton distributions and generalized transverse momentum dependent parton distributions at small x and at small but nonzero skewness ξ . We also show that the real part of the scattering amplitude, proportional to $\text{Im } N$, which is intimately connected to the signature factor of the amplitude, can be accounted for by a more careful evaluation of the initial condition for the evolution and by writing the nonlinear evolution equation in an integral form. One can similarly construct $\text{Im } \mathcal{O}$ for the odd-signature odderon amplitude. We hope that future implementation of our prescriptions presented here will eliminate the need for both phenomenological R -factors.

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I. INTRODUCTION

Mapping out the spatial three-dimensional structure of hadrons in terms of their quark and gluon degrees of freedom has been a longstanding problem in quantum chromodynamics (QCD). The three-dimensional hadronic structure is quantified in terms of the generalized parton distributions (GPDs) [1–5], which, unlike the parton distribution functions (PDFs) dependent on the momentum scale Q^2 and the longitudinal momentum fraction x , also depend on the Mandelstam variable t and skewness ξ , quantifying the transverse and longitudinal momentum transfer between the probe and the target hadron. While

in recent decades a lot of progress has been made on understanding GPDs at large and moderate x [6–45], their behavior at small x has been explored to a much lesser degree. (We note the progress made in the recent works [46,47] along with our own [48], which, in an abbreviated form, contains some of the material to be presented below.) The goal of the present work is to understand the small- x asymptotics of spin-independent quark and gluon GPDs with small but nonzero skewness ξ and to improve our understanding of high-energy elastic processes, which can be used to study GPDs.

In high-energy exclusive processes, the hadronic (or nuclear) target acts as a dense color charge medium, being dominated by quasiclassical gluon fields [49–57]. The leading contribution to the scattering amplitude can be described in the small- x /shock wave formalism [49–57], where it is given by Wilson line correlators which encode the eikonal scattering of a quark–antiquark color dipole (and of other color structures, such as quadrupoles [58]) off the hadron’s gluon fields. These Wilson line correlators include information on the three dimensional distribution of gluons in the target, and indeed are related to the GPDs

* Contact author: kovchegov.1@osu.edu

† Contact author: melvin.santiago@temple.edu

‡ Contact author: sun.2885@osu.edu

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which describe the same scattering amplitudes in the formalism of collinear factorization. The GPDs are defined as off-forward matrix elements of bilinear quark and gluon operators, and they encode various properties of the bulk hadron state, including the tomographic distribution of quarks and gluons [59–62], the distribution of the hadron’s mass and angular momentum among its constituents [2,63], as well as various mechanical properties like stresses and pressures within the hadron [7,64–67]. There exists a body of literature containing systematic treatments in the shock wave formalism of various PDFs and transverse momentum dependent parton distribution functions (TMD PDFs or TMDs) at small x in [68–88], which has been recently extended to include GPDs as well as generalized transverse momentum dependent parton distributions (GTMDs) in [47,48,89]. Note that there is recent work on extending the shock wave formalism beyond the usual small- x kinematics for exclusive processes [90].

The calculation of the scattering amplitudes for exclusive processes within the shock wave formalism typically ignores two major features of such processes, namely the nonzero longitudinal momentum transfer (termed skewness in the collinear factorization framework) between the probe and the target hadron and the real part of the scattering amplitude [46,91–100]. These contributions can be restored with significant phenomenological corrections [96,101,102], but a calculation purely within the shock wave formalism has been missing. In this work, we will establish a prescription for including each of these contributions within the small- x /shock wave formalism. Furthermore, we will lay out a framework for calculating the GTMDs and GPDs within the shock wave formalism starting from their operator definitions in factorization theorems. Although we will restrict our calculations to the unpolarized GTMDs/GPDs, our framework allows for the calculation of quantities which are energy suppressed and thus absent in the usual shock wave calculations, including spin-dependent parton distributions. The authors have previously studied the simplification of GTMD/GPD operators at small- x and the inclusion of skewness in the leading logarithmic approximation (LLA) of their small- x evolution in [48]. Here we will give a detailed derivation of the results obtained in [48] and introduce a prescription for systematically including the correction due to the real part of the scattering amplitude into the small- x shock wave formalism.

The paper is structured as follows. In Sec. II A we take the operator definitions of the unpolarized gluon and quark GTMDs and GPDs and convert them to shock wave formalism expressions written in terms of Wilson line correlators. These Wilson line correlators have an unspecified rapidity with respect to the target hadron, which we address in the following Sec. II B. There, we consider the effect of keeping a nonzero longitudinal momentum

transfer in the diagrams which yield the small- x evolution of the dipole scattering amplitude. (To be more specific, we are talking about the “plus” momentum transfer for the mostly x^+ -moving target in the light cone coordinates notation.) We consider general diagrams including non-linear terms where a gluon emission within one dipole splits it into two, generating two independent gluon cascades, and find that the effect of including longitudinal momentum transfer is to set the rapidity Y of the evolved dipole amplitude $N_{10}(Y)$ [along with the odderon dipole amplitude $\mathcal{O}_{10}(Y)$] to $Y = \ln \min \{1/|x|, 1/|\xi|\}$ for x the average parton longitudinal momentum fraction in the GTMD/GPD matrix element and ξ the skewness variable denoting the shift in the parton longitudinal momentum fraction.

After establishing the expressions for the GTMDs/GPDs at small- x and the prescription for including skewness in their small- x evolution, we then derive a prescription for including the real part of the scattering amplitude through the small- x evolution in Sec. III. We find that the imaginary correction to the dipole scattering amplitude (which yields the real part of the exclusive-process scattering amplitude) is obtained by modifying the Glauber-Gribov-Mueller/McLerran-Venugopalan (GGM/MV) [103–106] initial condition/inhomogeneous term in the integral form of the small- x evolution equation [107–110] to¹

$$N_{10}^{(0)}(s) = 1 - \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10}\Lambda} \right) \left[\theta(s - \Lambda^2) + \frac{i}{\pi} \ln \left(\frac{|s - \Lambda^2|}{|s + \Lambda^2|} \right) \right] \right\}. \quad (1)$$

Here $\underline{x}_1$ and $\underline{x}_0$ are the transverse positions of the quark and antiquark in the dipole with $\underline{x}_{10} = \underline{x}_1 - \underline{x}_0$ and $x_{10} = |\underline{x}_{10}|$, Q_{s0} is the initial saturation scale of the hadronic or nuclear target, and Λ is an infrared (IR) cutoff scale. Note that Eq. (1) is written for $x_{10} < 1/\Lambda$ and with $s > 0$ the center of mass energy squared for the scattering of the dipole off the target hadron or nucleus.

We summarize our findings in Sec. V. In this manuscript we use light cone coordinates defined as $u^\mu = (u^+, u^-, \underline{u})$ for an arbitrary 4-vector u^μ with $u^\pm = (u^0 \pm u^3)/\sqrt{2}$ and the scalar product $u \cdot v = u^+ v^- + u^- v^+ - \underline{u} \cdot \underline{v}$. Transverse vectors are denoted by $\underline{u} = (u^1, u^2)$, while the transverse integration measures are given by $d^2 u_\perp$. We also write the magnitude of a transverse vector as $u_T = |\underline{u}|$. For all of the calculations in this manuscript, we take the hadron to move mainly in the x^+ direction.

¹Here everything in the square brackets in the exponent is a modification advocated below, absent in the original GGM/MV expression except for the θ -function, which is often implicitly assumed.

II. UNPOLARIZED GPDS AND GTMDS AT SMALL x

A. Expressions for unpolarized GPDs and GTMDs at small x

We begin with the operator definitions of the gluon and quark GPDs as defined from collinear factorization [1,3–5]. At leading-twist, the gluon GPDs are given by an off-forward matrix element of a nonlocal product of two gluon field strength operators $F_a^{\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g_s f^{abc} A^{b\mu} A^{c\nu}$, such as

$$\int_{-\infty}^{\infty} \frac{dz^-}{2\pi} e^{ix\bar{P}^+z^-} \langle P', S' | F_b^{+i} \left(-\frac{z}{2} \right) U_0^{ba} \left[-\frac{z^-}{2}, \frac{z^-}{2} \right] \times F_a^{+j} \left(\frac{z}{2} \right) | P, S \rangle. \quad (2)$$

One has an analogous definition in the quark case using quark field operators

$$\int_{-\infty}^{\infty} \frac{dz^-}{2\pi} e^{ix\bar{P}^+z^-} \langle P', S' | \bar{\psi} \left(-\frac{z}{2} \right) V_0 \left[-\frac{z^-}{2}, \frac{z^-}{2} \right] \Gamma \psi \left(\frac{z}{2} \right) | P, S \rangle. \quad (3)$$

In these definitions, $z^\mu = (0^+, z^-, \underline{0})$ is the separation vector between the field operators, $i, j = 1, 2$, $\bar{P}^+ \equiv (P'^+ + P^+)/2$ is the average momentum of the hadron, Γ is a Dirac matrix, and $U_0^{ba}[-z^-/2, z^-/2]$ is a light cone Wilson line in the adjoint representation, defined as

$$U_{\underline{x}}[x_f^-, x_i^-] \equiv \mathcal{P} \exp \left[ig_s \int_{x_i^-}^{x_f^-} dx^- A^{a+}(0^+, x^-, \underline{x}) T^a \right]. \quad (4)$$

Here g_s is the strong coupling constant, while the $SU(N_c)$ generator matrices in the adjoint representation are $(T^a)_{bc} = -if_{abc}$ and $\mathcal{P}$ is the path-ordering operator. (N_c is the number of quark colors.) For the quark correlator, we also employ $V_0[-z^-/2, z^-/2]$, a light cone Wilson line in the fundamental representation,

$$V_{\underline{x}}[x_f^-, x_i^-] \equiv \mathcal{P} \exp \left[ig_s \int_{x_i^-}^{x_f^-} dx^- A^{a+}(0^+, x^-, \underline{x}) t^a \right], \quad (5)$$

with generator matrices t^a in the fundamental representation. We denote by x the average fraction of the longitudinal momentum of the hadron which is carried by the quark or gluon.

1. Unpolarized gluon GPD and GTMD at small x

Following the pioneering work [46] (and the more recent [47]), and employing the methods used for PDFs [68,71–83,86,87], the first step in studying the small- x asymptotics of GPDs is to generalize the nonlocal operator to that of GTMDs. This introduces a transverse separation between the two field operators which allows one to convert the operators into a dipole scattering amplitude. Here we focus on the GPDs for unpolarized partons in unpolarized hadrons, with the relevant unpolarized dipole gluon GTMD $F_{1,1}^g$ defined by [111]

$$\begin{aligned} & \frac{1}{2} \sum_S \int \frac{dz^- d^2 z_\perp}{(2\pi)^3} e^{ix\bar{P}^+z^- - i\bar{k} \cdot \underline{z}} \delta^{ij} \langle P', S | \\ & \times 2\text{tr} \left[F^{+i} \left(-\frac{z}{2} \right) V^{[+]} \left[-\frac{z}{2}, \frac{z}{2} \right] F^{+j} \left(\frac{z}{2} \right) V^{[-]} \left[\frac{z}{2}, -\frac{z}{2} \right] \right] | P, S \rangle \\ & = \frac{1}{2} \sum_S \frac{\bar{P}^+}{2M} \bar{u}(P', S) F_{1,1}^g(x, \xi, \underline{\Delta}, \underline{k}) u(P, S), \end{aligned} \quad (6)$$

where now $z^\mu = (0^+, z^-, \underline{z})$, the trace and field strength operators are in the fundamental representation, $u(P, S)$ and $\bar{u}(P', S)$ are on shell spinors for the hadron of mass M , $\Delta = P' - P$ is the momentum transferred to the hadron, x is the average longitudinal momentum fraction of the gluons, and $\xi = -\Delta^+/2\bar{P}^+$. There is an important caveat to our notation: Lorentz invariance requires that the GTMDs only depend on scalar products of the available transverse vectors, namely k_T^2 , Δ_T^2 and $\underline{k} \cdot \underline{\Delta}$. We write the arguments of the GTMDs as the full transverse momenta $\underline{\Delta}$ and $\underline{k}$ only for compactness. Similarly, we suppress the renormalization scale dependence in the arguments of GTMDs. In Eq. (6), the gauge links

$$V^{[\pm]}[x_f, x_i] = V_{\underline{x}_f}[x_f^-, \pm\infty^-] V_{\pm\infty^-}[\underline{x}_f, \underline{x}_i] V_{\underline{x}_i}[\pm\infty^-, x_i^-] \quad (7)$$

are fundamental-representation staples in the positive and negative light cone minus directions with $V_{\pm\infty^-}[\underline{x}_f, \underline{x}_i]$ the transverse gauge links at x^- -infinities, defined by

$$V_{\pm\infty^-}[\underline{x}_f, \underline{x}_i] \equiv \mathcal{P} \exp \left[-ig_s \int_{\underline{x}_i}^{\underline{x}_f} dx^i A^{ai}(0^+, \pm\infty^-, \underline{x}) t^a \right]. \quad (8)$$

This means that we are considering the dipole-type gluon GTMDs, while the Weizsacker-Williams-type gluon GTMDs could also be analyzed and would result in expressions in terms of four Wilson line correlators. We leave such investigations to future work.

From Eq. (6), we read off the unpolarized gluon GTMD

$$F_{1,1}^g(x, \xi, \underline{\Delta}, \underline{k}) = \sqrt{1 - \xi^2} \sum_s \int \frac{dz^- d^2 z_\perp}{(2\pi)^3 2\bar{P}^+} e^{ix\bar{P}^+ z^- - i\bar{k} \cdot \underline{z}} \times \langle P', S | 2\text{tr} \left[F^{+i} \left(-\frac{z}{2} \right) V^{[+]} \left[-\frac{z}{2}, \frac{z}{2} \right] F^{+i} \left(\frac{z}{2} \right) V^{[-]} \left[\frac{z}{2}, -\frac{z}{2} \right] \right] | P, S \rangle. \quad (9)$$

Next, we simplify the expression (9) for the gluon GTMD at small x and small ξ by employing the light cone operator treatment (LCOT) formalism developed in [76–79,112], assuming that one does not choose the $A^+ = 0$ light cone gauge of the hadron so that we can put the transverse links at infinities (8) to one. We want to switch to the shock wave picture, so we need to define the notation for the off-forward “averaging” of operators in the background fields of the target (cf. Ref. [113]). For forward matrix elements, the standard unpolarized color glass condensate (CGC)/saturation average of a two-point operator $\hat{\mathcal{O}}(x_0, x_1)$, depending on positions $x_p^\mu = (0^+, x_p^-, \underline{x}_p)$ with $p = 0, 1$, is defined as [104–106,114,115]

$$\frac{1}{2} \sum_s \langle P, S | \hat{\mathcal{O}}(x_0, x_1) | P, S \rangle = 2P^+ \int d^2 b_\perp db^- \langle \hat{\mathcal{O}}(x_0, x_1) \rangle \quad (10)$$

(and denoted by $\langle \dots \rangle$), where the factor of $2P^+$ results from the normalization of the single-particle states

$$\langle P | P \rangle = 2P^+ (2\pi)^3 \delta(0^+) \delta^2(\underline{0}) = 2P^+ V^- \quad (11)$$

with the (infinite) volume $V^- = \int d^2 x_\perp dx^-$. The impact parameter is defined by

$$b^\mu = \alpha x_1^\mu + (1 - \alpha) x_0^\mu \quad (12)$$

with α an arbitrary real number while $x_{10}^\mu = x_1^\mu - x_0^\mu$, such that

$$x_1^\mu = b^\mu + (1 - \alpha) x_{10}^\mu, \quad x_0^\mu = b^\mu - \alpha x_{10}^\mu. \quad (13)$$

Since

$$F_{1,1}^g(x, \xi, \underline{\Delta}, \underline{k}) = \sqrt{1 - \xi^2} \int \frac{dx_1^- dx_0^- d^2 x_{1\perp} d^2 x_{0\perp}}{2\pi^3} e^{ix\bar{P}^+ x_{10}^- + i\bar{P}^+ \xi(x_1^- + x_0^-) - i\bar{k} \cdot \underline{x}_{10} + i\bar{\Delta} \cdot \underline{b}} \times \langle \text{tr} [V_{\underline{x}_0}[-\infty, x_0^-] F^{+i}(x_0) V_{\underline{x}_0}[x_0^-, \infty] V_{\underline{x}_1}[\infty, x_1^-] F^{+i}(x_1) V_{\underline{x}_1}[x_1^-, -\infty]] \rangle. \quad (16)$$

Expanding Eq. (16) in x and ξ to the lowest nontrivial order (cf. Refs. [74,78,81] for the helicity-dependent PDFs case) by using

$$\begin{aligned} & \int_{-\infty}^{\infty} dx_1^- e^{i(x+\xi)\bar{P}^+ x_1^-} V_{\underline{x}_1}[\infty, x_1^-] F^{+i}(0^+, x_1^-, \underline{x}_1) V_{\underline{x}_1}[x_1^-, -\infty] \\ &= - \int_{-\infty}^{\infty} dx_1^- e^{i(x+\xi)\bar{P}^+ x_1^-} V_{\underline{x}_1}[\infty, x_1^-] [\partial^i A^+ + i(x + \xi)\bar{P}^+ A^i] V_{\underline{x}_1}[x_1^-, -\infty] = \frac{i}{g_s} \partial^i V_{\underline{x}_1} + \mathcal{O}(x) + \mathcal{O}(\xi) \end{aligned} \quad (17)$$

$$\begin{aligned} \hat{\mathcal{O}}(x_0, x_1) &= \hat{\mathcal{O}}(b - \alpha x_{10}, b + (1 - \alpha) x_{10}) \\ &= e^{i\hat{P} \cdot b} \hat{\mathcal{O}}(-\alpha x_{10}, (1 - \alpha) x_{10}) e^{-i\hat{P} \cdot b} \end{aligned} \quad (14)$$

with $\hat{P}$ the momentum operator, the left-hand side of Eq. (10) is, indeed, independent of b .

We are now ready to generalize Eq. (10) to the off-forward case. This yields

$$\begin{aligned} & \frac{1}{2} \sum_s \langle P' = P + \Delta, S | \hat{\mathcal{O}}(-\alpha x_{10}, (1 - \alpha) x_{10}) | P, S \rangle \\ &= 2\bar{P}^+ \int db^- d^2 b_\perp e^{-i\Delta^+ b^- + i\bar{\Delta} \cdot \underline{b}} \langle \hat{\mathcal{O}}(x_0, x_1) \rangle \end{aligned} \quad (15)$$

(see Refs. [71,116–118] along with Appendix A of [113] for the relation between the CGC/saturation averaging and operator matrix elements using Wigner distributions, which results in the expression above). The sign difference in the exponent compared to [116,117] is due to our impact parameter b denoting the position of the dipole with respect to the target nucleon, while in those references the impact parameter denotes the position of the nucleon or a large- x parton in a nucleus.

We note that the definitions (6) and (9) for the gluon GTMD correspond to the choice of $\alpha = 1/2$ in Eq. (15): however, Eqs. (6) and (9) can be rewritten for any other value of α by replacing $z/2 \rightarrow (1 - \alpha)z$ and $-z/2 \rightarrow -\alpha z$ in them, which would result in different GTMD (see Ref. [98] for a related discussion). Employing Eq. (15) with $\alpha = 1/2$, and neglecting the transverse gauge links at $x^- \rightarrow \pm\infty$ [114,119–127], we rewrite Eq. (9) as

along with its Hermitian conjugate with $\xi \rightarrow -\xi$, we obtain (cf. Ref. [46])

$$F_{1,1}^g(|x| \ll 1, |\xi| \ll 1, \underline{\Delta}, \underline{k}) = -\frac{N_c}{8\pi^4\alpha_s} \int d^2x_{1\perp} d^2x_{0\perp} e^{-ik \cdot \underline{x}_{10} + i\Delta \cdot \underline{b}} (\nabla_{\underline{x}_1} \cdot \nabla_{\underline{x}_0}) [N_{10}(Y) - i\mathcal{O}_{10}(Y)], \quad (18)$$

where and we have employed the dipole S -matrix

$$D_{10}(Y) \equiv \frac{1}{N_c} \langle \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] \rangle = 1 - N_{10}(Y) + i\mathcal{O}_{10}(Y) \quad (19)$$

and defined the unpolarized C-even dipole scattering amplitude [107,109]

$$N_{10}(Y) \equiv 1 - \frac{1}{2N_c} \langle \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] + \text{tr}[V_{\underline{x}_0} V_{\underline{x}_1}^\dagger] \rangle(Y) \quad (20)$$

along with the C-odd odderon amplitude [128,129] (see also Ref. [130])

$$\mathcal{O}_{10}(Y) \equiv \frac{1}{2iN_c} \langle \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] - \text{tr}[V_{\underline{x}_0} V_{\underline{x}_1}^\dagger] \rangle(Y) \quad (21)$$

with the infinite fundamental light cone Wilson lines denoted by $V_{\underline{x}} = V_{\underline{x}}[\infty^-, -\infty^-]$ and Y the rapidity of the dipole with respect to the target. Here $\alpha_s = g_s^2/(4\pi)$. We stress again that Eq. (18) is derived here for

$$\underline{b} = \frac{\underline{x}_1 + \underline{x}_0}{2} \quad (22)$$

in it, but applies for any definition of the impact parameter $\underline{b} = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_0$, as long as the definition (6) is modified accordingly. The latter modification would, indeed, change the gluon GTMD $F_{1,1}^g$.

The unpolarized gluon GPD at large Q^2 is given, up to power corrections and $\mathcal{O}(\alpha_s)$ scheme choice corrections, by [131,132]

$$\begin{aligned} H^g(|x| \ll 1, |\xi| \ll 1, t, Q^2) &= \int^{Q^2} d^2k_\perp F_{1,1}^g(|x| \ll 1, |\xi| \ll 1, \underline{\Delta}, \underline{k}) \\ &= -\frac{N_c}{2\pi^2\alpha_s} \int d^2b_\perp e^{i\Delta \cdot \underline{b}} [(\nabla_{\underline{x}_1} \cdot \nabla_{\underline{x}_0}) N_{10}(Y)]_{x_{10}^2=1/Q^2}. \end{aligned} \quad (23)$$

As the odderon contribution to the gluon GTMD (18) is odd under $\underline{k} \rightarrow -\underline{k}$, it does not contribute to the gluon GPD in Eq. (23).

2. Unpolarized quark GPD and GTMD at small x

The unpolarized quark GTMD is defined by [133]

$$\frac{1}{2} \sum_S \int \frac{dz^- d^2z_\perp}{2(2\pi)^3} e^{ix\bar{P}^+ z^- - ik \cdot \underline{z}} \langle P', S | \bar{\psi} \left(-\frac{z}{2} \right) V^{[+]} \left[-\frac{z}{2}, \frac{z}{2} \right] \gamma^+ \psi \left(\frac{z}{2} \right) | P, S \rangle = \frac{1}{4M} \sum_S \bar{u}(P', S) F_{1,1}^q(x, \xi, \underline{\Delta}, \underline{k}) u(P, S). \quad (24)$$

Here, we chose the future-pointing Wilson line staple, as used in semi-inclusive deep inelastic scattering (SIDIS). To simplify the quark GTMD at $|x|, |\xi| \ll 1$, we start with the definition (24), employ Eq. (15) and insert a complete set of states to obtain (while approximating $\sqrt{1-\xi^2} \approx 1$ at $|\xi| \ll 1$)

$$\begin{aligned} F_{1,1}^q(|x| \ll 1, |\xi| \ll 1, \underline{\Delta}, \underline{k}) &= \frac{\bar{P}^+}{(2\pi)^3} \int d^2x_{1\perp} d\bar{x}_1^- d^2x_{0\perp} d\bar{x}_0^- e^{i(x+\xi)\bar{P}^+ x_1^- - i(x-\xi)\bar{P}^+ x_0^- - ik \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} (\gamma^+)_{\alpha\beta} \\ &\quad \times \sum_X \langle \bar{\psi}_\alpha(x_0) V_{\underline{x}_0}[x_0^-, \infty] | X \rangle \langle X | V_{\underline{x}_1}[\infty, x_1^-] \psi_\beta(x_1) \rangle \end{aligned} \quad (25)$$

with the implied summation over the quark spinor indices α, β . We can calculate this matrix element diagrammatically, following the calculation detailed in [79]: we take contractions of the two quark field operators, making use of the antiquark propagator in the background field

$$\overline{\psi}_\alpha^i(x_0) \psi_\beta^j(x_1) = \int d^2 w_\perp \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} e^{ik_1^+ x_1^-} e^{ik_1 \cdot (\underline{w} - \underline{x}_1)} e^{-ik_2^+ x_0^-} e^{ik_2 \cdot (\underline{x}_0 - \underline{w})} \times \left\{ \left[\frac{ik_1}{k_1^2 + i\epsilon} \right] \left[\left(\hat{V}_w^\dagger \right)^{ji} (2\pi) \delta(k_1^- - k_2^-) \right] \left[k_2 (2\pi) \delta(k_2^2) \right] \right\}_{\beta\alpha}. \quad (26)$$

The operator $\hat{V}_w$ is both a Dirac matrix and a color matrix, which we will simplify in the eikonal limit. The quarks are assumed to be massless, for simplicity. Anticipating that in our calculation we will put the k_1 antiquark line on mass shell, we replace the k_1 and k_2 factors with polarization sums $\sum_\sigma v_\sigma(k_{1(2)}) \bar{v}_\sigma(k_{1(2)})$ in terms of $\pm$ -reversed BL spinors [77,112]

$$u_\sigma(p) = \frac{1}{\sqrt{p^-}} [p^- + m\gamma^0 + \gamma^0 \underline{\gamma} \cdot \underline{p}] \rho(\sigma), \quad v_\sigma(p) = \frac{1}{\sqrt{p^-}} [p^- - m\gamma^0 + \gamma^0 \underline{\gamma} \cdot \underline{p}] \rho(-\sigma), \quad (27)$$

with $p^\mu = (\frac{p^2 + m^2}{p^-}, p^-, \underline{p})$ and

$$\rho(+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \rho(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad (28)$$

The action of the operator $\hat{V}_w$ at eikonal order is defined by matrix elements with these spinors as

$$\bar{v}_\sigma(p) (\hat{V}_w) v_{\sigma'}(p') = 2\sqrt{p^- p'^-} \delta_{\sigma\sigma'} V_w, \quad (29)$$

so the eikonal propagator becomes

$$\overline{\psi}_\alpha^i(x_0) \psi_\beta^j(x_1) = \int d^2 w_\perp \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} e^{ik_1^+ x_1^-} e^{ik_1 \cdot (\underline{w} - \underline{x}_1)} e^{-ik_2^+ x_0^-} e^{ik_2 \cdot (\underline{x}_0 - \underline{w})} \times \left[\frac{i(2\pi)^2 2k_1^- \delta(k_1^- - k_2^-) \delta(k_2^2)}{k_1^2 + i\epsilon} \right] \left\{ \sum_\sigma v_\sigma(k_1) \left(V_w^\dagger \right)^{ji} \bar{v}_\sigma(k_2) \right\}_{\beta\alpha}. \quad (30)$$

For the leading diagrams contributing to the GTMD, we allow the light cone minus positions of the quark field operators x_0^- and x_1^- to be either before or after the shock wave (in [79] the quark fields were allowed to be taken inside the shock wave as subeikonal corrections were being included), and we truncate the sum over intermediate states at the leading Fock state, that being an antiquark $|\bar{q}\rangle\langle\bar{q}|$. This gives us three diagrams as depicted in Fig. 1, labeled in the same way as in [79]. In the diagram B we have the case where the antiquark propagator and staple gauge link cross the shock wave on only one side of the final state cut. In the diagram C we have the propagator and gauge link crossing the shock wave on both sides of the final state cut. Finally, in the diagram D neither the propagator nor the gauge links cross the shock wave (a pure noninteraction term).

Using the propagator in Eq. (26), we calculate the diagram B (and its conjugate) by taking $x_0^- > 0$, $x_1^- < 0$ (or $x_0^- < 0$, $x_1^- > 0$ for the conjugate diagram), setting the gauge link along x_1 to unity as it does not cross the shock wave, and inserting the propagator in Eq. (25) to obtain

$$B = \frac{\bar{P}^+}{(2\pi)^3} \int d^2 x_{1\perp} dx_1^- dx_{0\perp} dx_0^- d^2 w_\perp \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} e^{i(x+\xi)\bar{P}^+ x_1^- - i(x-\xi)\bar{P}^+ x_0^-} e^{-ik_2 \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} \times e^{ik_1^+ x_1^-} e^{ik_1 \cdot (\underline{w} - \underline{x}_1)} e^{-ik_2^+ x_0^-} e^{ik_2 \cdot (\underline{x}_0 - \underline{w})} \left[\frac{i(2\pi)^2 2k_1^- \delta(k_1^- - k_2^-) \delta(k_2^2)}{k_1^2 + i\epsilon} \right] \left\langle \left(\sum_\sigma \bar{v}_\sigma(k_2) \gamma^+ v_\sigma(k_1) \right) \text{tr}[V_{\underline{x}_1} V_{\underline{w}}^\dagger] \right\rangle + \text{c.c.} \quad (31)$$

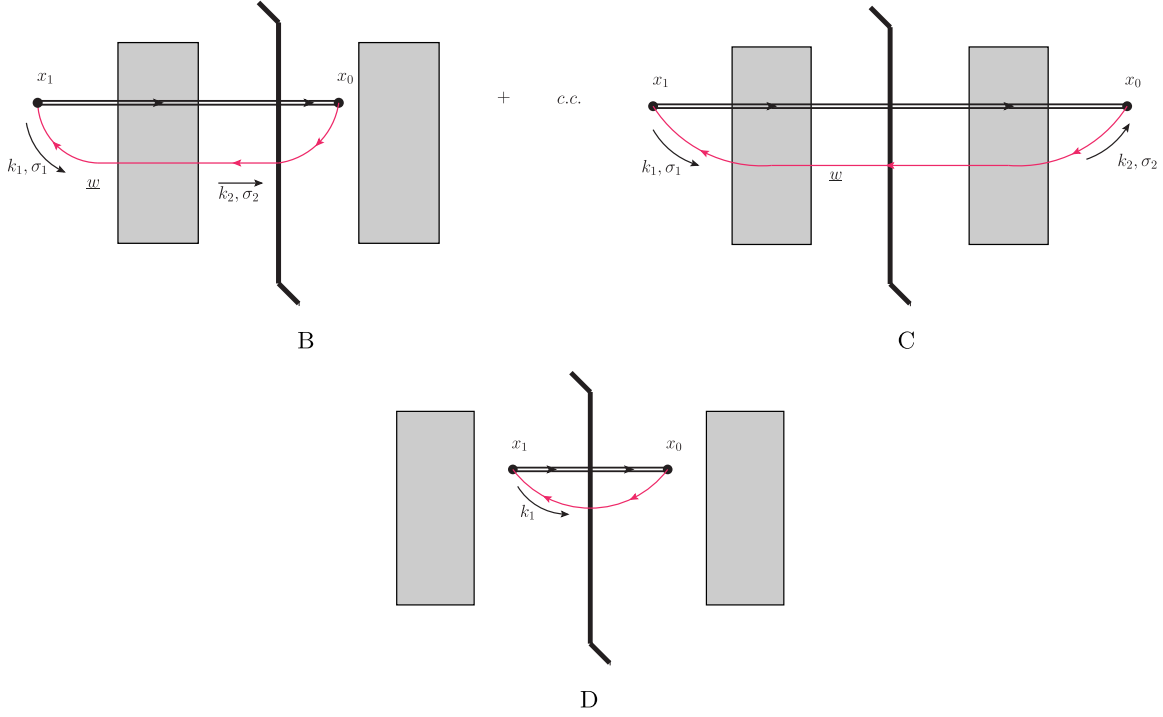


FIG. 1. Eikonal diagrams contributing to the unpolarized quark GTMD at small- x and small- ξ . Double lines represent eikonal, light cone Wilson lines arising from the gauge links in the GTMD staple, and single lines with arrows represent the antiquark propagator from contracting the quark field operators, while the thick vertical line represents the final state cut. Shaded rectangles denote the shock wave.

Employing the spinor matrix element

$$\bar{v}_\sigma(p)\gamma^+v_{\sigma'}(p') = \frac{1}{2\sqrt{p^-p'^-}}[\delta_{\sigma\sigma'}\underline{p} \cdot \underline{p}' - i\sigma\delta_{\sigma\sigma'}\underline{p} \times \underline{p}'], \quad (32)$$

integrating over $x_0^-, x_1^-, \underline{x}_0, \underline{k}_2$, and k_2^- , defining the longitudinal momentum fraction $z \equiv k_1^-/q^- = 2k_1^-\bar{P}^+/s$ of some projectile's large momentum q^- carried by the antiquark in Fig. 1 with $s \approx 2\bar{P}^+q^-$ the projectile-target center of mass energy squared, and finally relabeling $\underline{w}$ as $\underline{x}_0$ we obtain

$$B = \frac{2N_c s}{(2\pi)^4} \int d^2x_{1\perp} d^2x_{0\perp} \frac{d^2k_{1\perp}}{(2\pi)^2} e^{-i(\underline{k}_1 + \underline{k}) \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} \times \int_{\Lambda^2/s}^1 dz \frac{1}{N_c} \langle \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] \rangle \frac{\underline{k}_1 \cdot (\underline{k} + \frac{1}{2}\Delta)}{[(x + \xi)zs + \underline{k}_1^2 - i\epsilon][(x - \xi)zs + (\underline{k} + \frac{1}{2}\Delta)^2 - i\epsilon]} + \text{c.c.}, \quad (33)$$

with Λ an IR cutoff scale describing the target. The lower integration limit on z stems from the shock wave approximation, which is valid for the x^- -coherence length of the antiquark being much larger than the width of the shock wave, that is, $2k_1^-/\underline{k}_1^2 > 1/\bar{P}^+$, such that $k_1^- > \underline{k}_1^2/(2\bar{P}^+) \geq \Lambda^2/(2\bar{P}^+)$. The upper cutoff on the z -integral, $z < 1$ results from our interpretation here that z is the fraction of some projectile's momentum q^- . Note that, unlike the subeikonal distributions considered, for instance, in [76–82], where the z -integral is logarithmic and both upper and lower regulators are necessary, here the k_1^- integral is finite and does

not require regularization. This allowed the authors of [47] to integrate over k_1^- from 0 to $+\infty$ while assuming that the Wilson line correlator is only mildly dependent on k_1^- , obtaining a zero result for the diagram C.

For the diagram C we have both gauge links crossing the shock wave, so both Wilson lines are kept as interacting. Meanwhile, the gluon exchanges along the antiquark propagator will cancel between the two sides of the final state cut, so we have a free propagator given by setting $(\hat{V}_w^\dagger)^{ji} = \delta^{ji}\gamma^-$. Plugging the noninteracting propagator into Eq. (25) and performing several of the integrals yields

$$C = \frac{2N_c s}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} \frac{d^2 k_{1\perp}}{(2\pi)^2} e^{-i(\underline{k}_1 + \underline{k}) \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} \times \int_{\Lambda^2/s}^1 dz \frac{1}{N_c} \langle \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] \rangle \frac{\underline{k}_1^2}{[(x + \xi)zs + \underline{k}_1^2 - i\epsilon][(x - \xi)zs + \underline{k}_1^2 + i\epsilon]}. \quad (34)$$

Finally, we have the diagram D. Here the antiquark propagator and both gauge links are noninteracting. We can read this contribution off from diagram C by taking the Wilson lines to be identity operators, which gives

$$D = \frac{2N_c s}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} \frac{d^2 k_{1\perp}}{(2\pi)^2} e^{-i(\underline{k}_1 + \underline{k}) \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} \int_{\Lambda^2/s}^1 dz \frac{\underline{k}_1^2}{[(x + \xi)zs + \underline{k}_1^2 - i\epsilon][(x - \xi)zs + \underline{k}_1^2 + i\epsilon]}. \quad (35)$$

Next, we rewrite the quark dipole S -matrix using Eq. (19). We note that the noninteracting terms [arising from 1 in Eq. (19)] in the diagrams B, C, and D add up to yield a nonzero disconnected contribution for an antiquark passing through the hadron without interactions, which is possible in the $x < -|\xi|$ region. Such disconnected terms are subtracted in the TMD operator definition [134], such that the noninteracting terms from the three diagrams fully cancel. Performing a similar subtraction in the GTMD case at hand, we see that the final expression for the quark GTMD at small- x and small- ξ is (cf. Ref. [47]),

$$F_{1,1}^q(|x| \ll 1, |\xi| \ll 1, \underline{\Delta}, \underline{k}) = -\frac{2N_c s}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} \int \frac{d^2 k_{1\perp}}{(2\pi)^2} e^{-i(\underline{k}_1 + \underline{k}) \cdot \underline{x}_{10} + i\Delta \cdot (\underline{x}_1 + \underline{x}_0)/2} \int_{\Lambda^2/s}^1 dz [N_{10}(Y) - i\mathcal{O}_{10}(Y)] \times \left\{ \frac{\underline{k}_1 \cdot (\underline{k} + \frac{1}{2}\underline{\Delta})}{[(x + \xi)zs + \underline{k}_1^2 - i\epsilon][(x - \xi)zs + (\underline{k} + \frac{1}{2}\underline{\Delta})^2 - i\epsilon]} + \frac{\underline{k}_1 \cdot (\underline{k} - \frac{1}{2}\underline{\Delta})}{[(x + \xi)zs + (\underline{k} - \frac{1}{2}\underline{\Delta})^2 + i\epsilon][(x - \xi)zs + \underline{k}_1^2 + i\epsilon]} + \frac{\underline{k}_1^2}{[(x + \xi)zs + \underline{k}_1^2 - i\epsilon][(x - \xi)zs + \underline{k}_1^2 + i\epsilon]} \right\}. \quad (36)$$

Since $N_{10}(Y) = N_{01}(Y)$ and $\mathcal{O}_{10}(Y) = -\mathcal{O}_{01}(Y)$, one can readily see that our expression (36) for quark GTMD satisfies $[F_{1,1}^q(x, \xi, \underline{\Delta}, \underline{k})]^* = F_{1,1}^q(x, -\xi, -\underline{\Delta}, \underline{k})$, in agreement with Eq. (3.18) in [133] following from Hermiticity, as long as $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ are even functions of ξ . (We will see below that this is the case in the LLA.)

The unpolarized quark GPD at $|x|, |\xi| \ll 1$, corresponding to the GTMD in Eq. (36), is given by

$$H^q(|x| \ll 1, |\xi| \ll 1, t, Q^2) = \int^{Q^2} d^2 k_\perp F_{1,1}^q(|x| \ll 1, |\xi| \ll 1, \underline{\Delta}, \underline{k}). \quad (37)$$

While this is not our main goal here, the expression (37) for H^q can be simplified after we substitute $F_{1,1}^q$ into it from Eq. (36). This is done in Appendix A, assuming weak dependence of N_{10} and $\mathcal{O}_{10}$ on z , which is justified in the LLA. The calculation in Appendix A gives

$$H^q(|x| \ll 1, |\xi| \ll 1, t, Q^2) = \frac{2N_c}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} \frac{e^{i\Delta \cdot \underline{x}_1}}{x_{10}^4 \xi^3} \times \left\{ N_{10}(Y) \left[4x\xi + (x^2 - \xi^2) \left[\ln\left(\frac{x - \xi - i\epsilon}{x + \xi - i\epsilon}\right) + \ln\left(\frac{x - \xi + i\epsilon}{x + \xi + i\epsilon}\right) \right] \right] - i\mathcal{O}_{10}(Y)(x^2 - \xi^2) \left[\ln\left(\frac{x - \xi - i\epsilon}{x + \xi - i\epsilon}\right) - \ln\left(\frac{x - \xi + i\epsilon}{x + \xi + i\epsilon}\right) \right] \right\}. \quad (38)$$

We can take $\xi > 0$ without loss of generality due to the symmetry of GPDs in general and of Eq. (38) in particular under $\xi \rightarrow -\xi$. Then we have four regions to consider: the two Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)

[135–137] regions $x > \xi$ with $x > 0$ or $x < -\xi$ with $x < 0$, and the two Efremov-Radyushkin-Brodsky-Lepage (ERBL) [138,139] regions $x < \xi$ with $x > 0$ or $x > -\xi$ with $x < 0$. Simplifying H^q in these regions yields

$$H^q(|x| \ll 1, |\xi| \ll 1, t, Q^2) = \frac{4N_c}{(2\pi)^4} \int d^2x_{1\perp} d^2x_{0\perp} \frac{e^{i\Delta \cdot x_1}}{x_{10}^4} \frac{1}{\xi^3} \times \left\{ N_{10}(Y) \left[2x\xi + (x^2 - \xi^2) \ln \left| \frac{x - \xi}{x + \xi} \right| \right] + \mathcal{O}_{10}(Y) \pi(\xi^2 - x^2) \text{Sign}(\xi) \theta(|\xi| - |x|) \right\}. \quad (39)$$

Note that we have inserted a factor of $\text{Sign}(\xi)$ in the odderon contribution in order to generalize the expression to cover positive and negative ξ values. We see that, within the simplifying assumption that N_{10} and $\mathcal{O}_{10}$ are relatively slowly varying functions of z , the odderon contribution to the unpolarized quark GPD H^q appears to be nonzero, but only within the ERBL region.

Let us summarize our calculations so far. We have obtained expressions (18) and (36) [or (39)] for, respectively, the gluon and quark unpolarized GTMDs at small x and ξ in terms of the impact parameter-dependent dipole scattering amplitudes; these GTMDs give their corresponding GPDs after integrating over the parton transverse momentum [see Eqs. (23) and (37)]. However, the rapidity variable of the dipole amplitudes in these formulas is not yet fixed. Usually, for PDFs and TMDs at small x , one takes $Y = \ln(1/x)$ (within LLA). The question we would like to address next in this work is what Y should one use in Eqs. (18), (23), (36), and (37)? In the next section we will examine the role of ξ in the evolution of the dipole amplitude and establish a prescription for including effects of nonzero skewness by the appropriate choice of rapidity Y in the argument of the dipole amplitudes obtained by solving the full nonlinear evolution equations. We will find that the x^- -lifetime ordering required for resumming the leading logarithms will give us an effective rapidity variable Y for the evolved dipole amplitudes $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ which will depend on the skewness.

B. Dipole amplitudes at nonzero skewness

The dipole amplitudes N_{10} and $\mathcal{O}_{10}$ are constructed out of the light cone Wilson lines and, at first glance, do not seem to account for the longitudinal momentum transfer, which would lead to nonzero skewness. Their impact parameter dependence accounts for the transverse momentum transfer $\underline{\Delta}$ which is Fourier conjugate to the impact parameter of the dipole, as employed in the expressions for the gluon and quark GTMDs derived above and given by Eqs. (18) and (36). Our goal here is to allow for nonzero but small skewness, while still staying at the leading power of energy (eikonal) level. It is, therefore, natural to assume that to keep the leading power of energy, we should still employ the light

cone Wilson line definitions (20) and (21) for the dipole amplitudes, even at nonzero skewness. Below, starting with this eikonal (Wilson-line) assumption, we will derive a prescription for including the longitudinal (plus) momentum transfer into the dipole amplitude N_{10} evolved using the nonlinear Balitsky-Kovchegov (BK) [107–110] and Jalilian-Marian-Iancu-McLerran-Weigert-Leodinov-Kovner (JIMWLK) [115,140–143] equations. Our prescription will also apply to the odderon amplitude $\mathcal{O}_{10}$, whose evolution was derived in [128,129] using the s -channel/shock wave formalism with the result that was equivalent to the odderon evolution from [144] derived in the t -channel formalism [145–147] (see also Refs. [148–154] for the properties of the odderon). Note that in the t -channel Balitsky-Fadin-Kuraev-Lipatov (BFKL) [155,156] formalism and for the Pomeron exchange the case on nonzero longitudinal momentum transfer was considered earlier in [157], resulting in the same prescription for including nonzero longitudinal momentum transfer as we will advocate for below. Therefore, our results here generalize the prescription from [157] to the case of nonlinear small- x evolution and to the case of linear and nonlinear odderon evolution.

1. A simple argument

To begin, let us illustrate our argument by employing a simple example. Consider one step of small- x evolution, which is nonforward in the longitudinal momentum. It is pictured in Fig. 2 using the shock wave formalism: however, our argument can equally be illustrated by using one rung of the ladder in the t -channel formalism. Somewhat unusually for the shock wave approach, we run the evolution from the nucleon target, with the shock wave representing the field of the projectile. For the plus-moving projectile, the logarithm of energy in one step of unpolarized small- x evolution usually arises from the integral over the plus momentum: in the longitudinally nonforward case at hand, the integral is

$$\int_{x\bar{P}^+}^{\bar{P}^+} \frac{d\ell^+}{(\ell^+ - \frac{\Delta^+}{2})(\ell^+ + \frac{\Delta^+}{2})} \Gamma(\ell^+, \Delta^+), \quad (40)$$

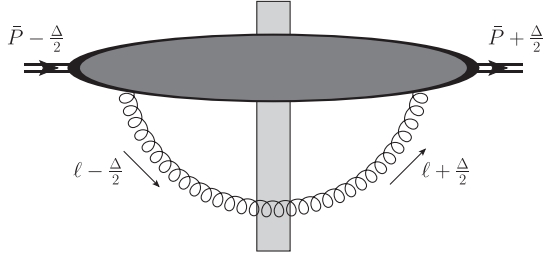


FIG. 2. A diagram illustrating one step of longitudinally nonforward small- x evolution in the target. Evolution correction is generated by the gluon line going through the shock wave (vertical rectangle).

where $\Gamma(\ell^+, \Delta^+)$ represents the “gluon–shock wave vertex,” the factor coming from the gluon interacting with the field of the shock wave dependent on ℓ^+ and Δ^+ , while the factors in the denominator result from the two gluon lines shown in Fig. 2. In the usual forward (in the longitudinal) momentum case one has $\Gamma(\ell^+, \Delta^+ = 0) = 2\ell^+$. In the nonforward case, we can imagine $\Gamma(\ell^+, \Delta^+) = 2\ell^+ + C\Delta^+$ with some constant C , though, in a complete calculation $\Gamma(\ell^+, \Delta^+)$ consists of several linear in ℓ^+ and Δ^+ numerators and denominators, with the number of numerators exceeding that of the denominators by one (while still satisfying $\Gamma(\ell^+, \Delta^+ = 0) = 2\ell^+$). Since the gluons’ plus momenta cannot be larger than those of the incoming and outgoing proton, the upper limit on the ℓ^+ integral is $\bar{P}^+$. Similarly, the lower limit of the ℓ^+ integral is $x\bar{P}^+$, as we are calculating a distribution with the average momentum fraction x ; for simplicity, we assumed that $x > 0$.

Analyzing the expression (40), we see that the ℓ^+ integral is logarithmic only if $\ell^+ \gg |\Delta^+|/2$, or, equivalently, $\ell^+/\bar{P}^+ \gg |\xi|$. In this regime, $\Gamma(\ell^+, \Delta^+) \approx 2\ell^+$ and the integral (40) becomes

$$2 \int_{\max\{x, |\xi|\} \bar{P}^+}^{\bar{P}^+} \frac{d\ell^+}{\ell^+}. \quad (41)$$

This is almost a standard integral in small- x evolution for PDFs and TMDs: the only difference is that now the lower limit has changed from x to $\max\{x, |\xi|\}$. We conclude that in going from small- x evolution for unpolarized PDFs and TMDs to that for unpolarized GPDs and GTMDs, we need to replace $x \rightarrow \max\{x, |\xi|\}$ to account for nonzero skewness (cf. Ref. [157]). Our simple example appears to suggest that the argument of the dipole amplitudes $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ should be modified from $Y = \ln(1/x)$ to $Y = \ln(1/\max\{x, |\xi|\})$ when calculating GPDs and GTMDs: we will show that this is indeed the case in a more detailed analysis below. Another conclusion one can draw from this simple example is that small- x evolution cannot truly be nonforward in the longitudinal

direction: in the region where the momenta of the two gluons, $\ell^+ - \frac{\Delta^+}{2}$ and $\ell^+ + \frac{\Delta^+}{2}$, are truly different, which means that $\ell^+ \lesssim \frac{\Delta^+}{2}$, the integral (40) becomes nonlogarithmic and the evolution stops. Thus, skewness ξ enters the dipole amplitudes only as a cutoff on small- x evolution, but plays no role in the evolution otherwise. Finally, let us point out that the above argument can be generalized to include virtual corrections as well.

2. A general argument

Next, let us construct a more general argument, valid for the nonlinear evolution. We will first consider the s -channel/shock wave formalism’s version of the ladder diagrams contributing to a nonforward elastic process, as shown in Fig. 3, with the evolution running from the projectile now. We take the incoming projectile to have momentum $q^\mu = (-Q^2/(2q^-), q^-, \underline{0})$, where q^- is large. Note that while the projectile is not present in the operator definition of the GTMDs and GPDs, above we have related the unpolarized GTMDs and GPDs to the dipole amplitudes. To calculate a dipole amplitude it is convenient to imagine that the quark and antiquark (giving the light cone Wilson lines) in it were created in a decay of some projectile, like a virtual photon in deep inelastic scattering (DIS). The large q^- momentum of the virtual photon provides a natural upper limit on the k^- integrals of the gluons generated by small- x evolution, $k^- < q^-$. Thus, we take the incoming projectile to produce a color dipole with the quark at position $\underline{x}_1$ and the antiquark at position $\underline{x}_0$. The dipole, in turn, will produce a cascade of gluons through s -channel emissions, yielding a ladder of gluons with the momenta of the gluons corresponding to the i th emission (the “rails” of the ladder attached to the i th “rung”) equal to

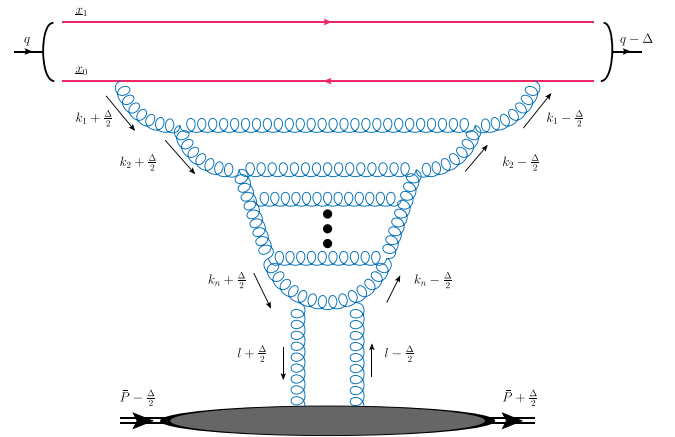


FIG. 3. Example of a ladder diagram which enters the leading logarithmic small- x evolution of a color dipole in nonforward kinematics for an elastic scattering process.

$$k_i^\mu = \left(\frac{(k_i \pm \frac{\Delta}{2})^2}{2(k_i^\pm \pm \frac{\Delta^-}{2})}, k_i^\pm \pm \frac{\Delta^-}{2}, k_i \pm \frac{\Delta}{2} \right). \quad (42)$$

We consider these ladder diagrams as formulated within light cone perturbation theory (LCPT) [158,159], where we have a light cone time x^- ordered picture of the scattering (unlike the light cone time x^+ used in the original works [158,159]). Below, we will also employ the average fractions of the projectile's “minus” light cone momentum $z_i = k_i^-/q^-$ and the fractional minus-momentum transfer $\zeta = \Delta^-/2q^-$.

We are interested in resumming the leading-logarithmic, high energy contribution from the gluon cascade, so we must impose the following conditions (cf. Refs. [160,161]). First of all, the light cone “minus” momenta have to be ordered,

$$q^- \gg k_1^- \pm \frac{\Delta^-}{2} \gg k_2^- \pm \frac{\Delta^-}{2} \gg \dots \gg k_n^- \pm \frac{\Delta^-}{2}. \quad (43)$$

Note also that $k_i^\pm \pm \frac{\Delta^\pm}{2} > 0$ by the LCPT rules: the particles are moving forward in the light cone “minus” direction and must thus have positive light cone “minus” momentum. On either side of the shock wave, the contribution of the i th gluon emission must dominate the light cone energy denominator for the intermediate state immediately following that emission. This leads to the lifetime ordering condition

$$\begin{aligned} \frac{Q^2}{2q^-}, |\Delta^+| &\ll \frac{(k_1 \pm \frac{\Delta}{2})^2}{2(k_1^\pm \pm \frac{\Delta^-}{2})} \ll \frac{(k_2 \pm \frac{\Delta}{2})^2}{2(k_2^\pm \pm \frac{\Delta^-}{2})} \ll \dots \\ &\ll \frac{(k_{n-1} \pm \frac{\Delta}{2})^2}{2(k_{n-1}^\pm \pm \frac{\Delta^-}{2})} \ll \frac{(k_n \pm \frac{\Delta}{2})^2}{2(k_n^\pm \pm \frac{\Delta^-}{2})}. \end{aligned} \quad (44)$$

Finally, as is usual for unpolarized small- x evolution, we take all the transverse momenta to be of the same order (and comparable to Q),

$$Q \sim \left| k_1 \pm \frac{\Delta}{2} \right| \sim \left| k_2 \pm \frac{\Delta}{2} \right| \sim \dots \sim \left| k_n \pm \frac{\Delta}{2} \right|. \quad (45)$$

Consider the case where some k_i^- momenta are comparable to or smaller in magnitude than $|\Delta^-|$. Then, either $k_i^- + \Delta^-/2$ or $k_i^- - \Delta^-/2$ are negative, and we would have violated the LCPT rule about the minus momentum flowing in the forward (right-ward) direction being positive. Hence, we can rule out this contribution and conclude that $k_i^- \gg \Delta^-$ for all i .

Because $\Delta^-/q^- \sim 1/s$ is an energy suppressed quantity, with $s \approx 2q^- \bar{P}^+$, we can safely neglect it relative to z_i near the top of the cascade and satisfy our orderings. However, as the light cone momenta k_i^- decrease through the cascade and approach the minimum value, eventually one has the situation where $k_i^\pm \pm \Delta^\pm/2$ become too small and there are

no further leading logarithmic emissions allowed by LCPT. We see, again, that one needs to have $k_i^- \gg \Delta^-$ for all i . At the same time, the absolute minimum for the light cone “minus” momentum fractions of the gluons z_i is set by the shock wave as Λ^2/s (with Λ the IR cutoff). One can see this by requiring that the x^- -lifetime of a gluon is longer than the proton shock wave, $2k^-/k_\perp^2 \gtrsim 1/\bar{P}^+$, such that $z = k^-/q^- \gtrsim k_\perp^2/s \geq \Lambda^2/s$. Gluon emissions with lifetimes shorter than this constraint would not generate energy logarithms and would thus be beyond the LLA. From the above discussion, we see that in the case where $\Lambda^2/s < |\zeta|$, one must terminate the cascade when the gluon light cone “minus” momentum fractions reach $z_n \approx |\zeta|$. On the other hand, if $\Lambda^2/s > |\zeta|$ then one can take the momenta down to the usual shock wave cutoff $z > \Lambda^2/s$ and have the usual small- x evolution logarithms. Thus, we find that the proper ordering condition for the light cone momentum fractions is the cascade is

$$z_1 \gg z_2 \gg \dots \gg z_{n-1} \gg z_n \gg \max \left\{ \frac{\Lambda^2}{s}, |\zeta| \right\}. \quad (46)$$

The diagram containing n gluon emissions in the cascade (and thus giving n steps of evolution) will give us a leading logarithmic contribution given by

$$\alpha_s^n \int_{z_{\min}}^1 \frac{dz_1}{z_1} \int_{z_{\min}}^{z_1} \frac{dz_2}{z_2} \dots \int_{z_{\min}}^{z_{n-1}} \frac{dz_n}{z_n} = \frac{1}{n!} \alpha_s^n \ln^n \left(\frac{1}{z_{\min}} \right) \quad (47)$$

with $z_{\min} = \max \{ \Lambda^2/s, |\zeta| \}$. The situation here is very similar to the example considered in Sec. II B 1.

In this work, we take the total momentum transfer $|t| \approx \Delta_\perp^2$ to be small compared to the hard scales in the process, following the typical kinematics for collinear factorization of exclusive processes where t is of the same order as the IR scales of the target: $|t| \sim \Lambda^2 \ll Q^2$. In addition, one can argue by invoking the uncertainty principle that $\Delta_\perp \sim 1/R$ with R the radius of the target nucleon: this makes $|t| \approx \Delta_\perp^2$ similarly small. Thus, the shock wave cutoff Λ^2/s coming from the forward limit will generically be of the same order as ζ , since $|\zeta| \sim \Delta_\perp^2/s \approx |t|/s \sim \Lambda^2/s$. Thus, we can neglect the difference between $|\zeta|$ and Λ^2/s in $\ln(1/z_{\min})$, putting $z_{\min} = \max \{ \Lambda^2/s, |\zeta| \} \approx \Lambda^2/s$ with the logarithmic accuracy, and thus ignoring the effect of nonzero ζ on small- x evolution.

We have analyzed the effect of nonzero minus longitudinal momentum transfer in the case of ladder diagrams, but even linear BFKL evolution requires going beyond these ladder diagrams within the modern dipole framework [162–164]. It appears that, in order to study nonzero plus momentum transfer, one has to generalize the above discussion to generic gluon or dipole cascade diagrams. Furthermore, we wish to consider nonlinear

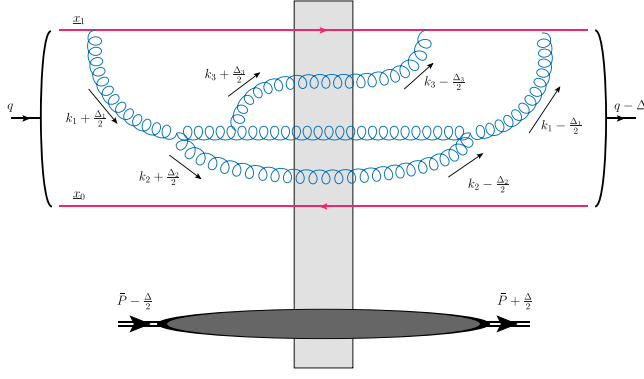


FIG. 4. A nonladder gluon cascade in the nonforward elastic scattering case. The rectangle represents the shock wave, comprising subsequent small- x evolution and interaction of the resulting cascade with the target.

evolution [107–110,115,140–143] where successive emissions of gluons generate more and more dipoles which can each have their own cascade and final interaction with the target. This is illustrated in Fig. 4: we see that the original quark antiquark dipole emits a gluon labeled by the momentum k_1 . Then, subsequent emissions of gluons labeled by k_2 and k_3 can generate a cascade within either the dipole formed by the quark and the gluon with momentum k_1 or the dipole formed by the antiquark and the gluon with momentum k_1 . All of the gluons in this diagram cross the shock wave denoted by the rectangle, and thus can form dipoles which will generate various further dipole cascades which will then interact with the target. Because each emission/absorption can be nonforward, it must be labeled by a separate momentum transfer Δ_i : each Δ_i is related to an independent loop momentum variable and can thus vary significantly, as it is integrated over. Each gluon line has a momentum labeled as

$$k_i^\mu = \left(\frac{(k_i \pm \frac{\Delta_i}{2})^2}{2(k_i^\pm \pm \frac{\Delta_i^\pm}{2})}, k_i^\pm \pm \frac{\Delta_i^\pm}{2}, k_i \pm \frac{\Delta_i}{2} \right) \quad (48)$$

with the plus signs to the left of the shock wave, and the minus signs to its right. We again require that $k_i^\pm \pm \Delta_i^\pm/2 > 0$ in order to have valid LCPT diagrams. However, given the independent momentum transfers Δ_i , we might have complications with our resummation when the longitudinal momentum transfer on a given line gets very large or very small. In the case that we have a very large Δ_i^- , the diagrams violate the $k_i^\pm \pm \Delta_i^\pm/2 > 0$ condition and can be discarded. On the other hand, in the case where we have a vanishingly small Δ_i^- , such that $\Delta_i^- \ll \Delta^-$ with Δ^- the net minus momentum transfer, further emissions from that gluon line may include contributions where $k_{j>i}^-$ is very small, $\Delta_i^- \ll k_{j>i}^- \ll \Delta^-$. Such emissions may still be outside the shock wave if $2(k_i^\pm \pm \Delta_i^\pm/2)/(k_i \pm \frac{\Delta_i}{2})^2 \gg 1/\bar{P}^+$, but

may be inside the effective shock wave if $2(k_i^\pm \pm \Delta_i^\pm/2)/(k_i \pm \frac{\Delta_i}{2})^2 \ll \Delta^-/\Lambda^2 = (\zeta/P^+)(s/\Lambda^2)$ with $\zeta = \Delta^-/2q^-$ again. This is possible if $\zeta > \Lambda^2/s$. While such emissions lead to longitudinal logarithms, these logarithms would yield $\ln(\zeta s/\Lambda^2) \sim \ln(|t|/\Lambda^2)$, which will not be logarithms of energy and thus will fall outside of the LLA we are working with: again, we discard such contributions. Note that contributions proportional to $\alpha_s \ln(t/\Lambda^2)$ would be negligible in the typical kinematics we are considering where $t \sim \Lambda_{\text{QCD}}^2$. We can thus generalize the ordering conditions from the ladder case (44) and (45) to the nonlinear evolution case as

$$\begin{aligned} \frac{Q^2}{2q^-}, |\Delta^+| &\ll \frac{(k_1 \pm \Delta_1/2)^2}{2(k_1^\pm \pm \Delta_1^\pm/2)} \ll \frac{(k_2 \pm \Delta_2/2)^2}{2(k_2^\pm \pm \Delta_2^\pm/2)} \ll \dots \\ &\ll \frac{(k_{n-1} \pm \Delta_{n-1}/2)^2}{2(k_{n-1}^\pm \pm \Delta_{n-1}^\pm/2)} \ll \frac{(k_n \pm \Delta_n/2)^2}{2(k_n^\pm \pm \Delta_n^\pm/2)} \ll \bar{P}^+ \end{aligned} \quad (49)$$

and

$$Q \sim \left| k_1 \pm \frac{\Delta_1}{2} \right| \sim \left| k_2 \pm \frac{\Delta_2}{2} \right| \sim \dots \sim \left| k_n \pm \frac{\Delta_n}{2} \right|. \quad (50)$$

These lifetime ordering conditions give us the set of diagrams which contribute to the eikonal, LLA nonlinear evolution with nonzero longitudinal momentum transfer. Note that the numbering of the momenta reflects the ordering in x^- light cone time over which the gluons are emitted, such that the gluon with the shortest lifetime (largest $\frac{(k_n \pm \Delta_n/2)^2}{2(k_n^\pm \pm \Delta_n^\pm/2)}$) is the last emission among all of the cascades at each step in the evolution.

We have analyzed evolution which is logarithmic in the “minus” light cone momenta of the emitted gluons. However, the variables x and ξ in the GTMDs/GPDs are “plus” momentum fraction variables. Therefore, to incorporate the effects of nonzero ξ we need to track the plus momenta, as we did in Sec. II B 1. Note that there is no plus momenta in LCPT ordered along the x^- -axis which we employ here (e.g., in Fig. 4). Instead of the plus momenta, this version of LCPT has energy denominators, which are related to the gluon x^- -light cone lifetimes, and are employed in the lifetime ordering condition (49). We will now show how the lifetime ordering in the gluon cascade yields a dependence of the dipole amplitude on the skewness variable ξ in the LLA small- x evolution. Returning to the energy denominators, the leading logarithms come from the contribution where the last emitted gluon has the largest light cone energy, dominating the energy denominator. This implies that the light cone energy (the plus momentum) of each newly emitted gluon should be larger than that for all the preexisting gluons (and quarks), and also larger than the plus momentum of the particles in the initial state. On the

left side of the shock wave this latter requirement leads to

$$\frac{(\underline{k}_i + \underline{\Delta}_i/2)^2}{2(k_i^- + \Delta_i^-/2)} \gg |q^+|. \quad (51)$$

This is already satisfied by the lifetime ordering (49) we established above, which we see by noting that $q^+ = -Q^2/(2q^-)$. On the right side of the shock wave we similarly require that the gluon light cone energies are larger than the energy of the outgoing projectile particle [which, when we use the GPD in a scattering process, will be a photon in deeply virtual Compton scattering (DVCS) or a vector meson in deeply virtual meson production (DVMP)]. This gives

$$\frac{(\underline{k}_i - \underline{\Delta}_i/2)^2}{2(k_i^- - \Delta_i^-/2)} \gg q^+, |\Delta^+|. \quad (52)$$

This inequality is included in Eq. (49) as well. In order to satisfy the inequality (52) with $|\Delta^+|$ on the right we must have

$$\frac{(\underline{k}_i - \underline{\Delta}_i/2)^2}{2(k_i^- - \Delta_i^-/2)} \gg 2|\xi|\bar{P}^+. \quad (53)$$

Far from the bottom of the cascade we have $k_i^- = z_i q^- \gg \Delta_i^-/2$, so we can neglect Δ_i^- to find

$$\frac{(\underline{k}_i - \underline{\Delta}_i/2)^2}{z_i s} \gg 2|\xi|. \quad (54)$$

The left hand side of Eq. (54) is proportional to the inverse of the x^- -lifetime of the i th gluon, with the lifetime decreasing as we go down the cascade. The strongest constraint is then at the top of the cascade for the first emitted gluon, $i = 1$, where the gluon lives the longest in x^- : putting $i = 1$ in Eq. (54) yields

$$\frac{(\underline{k}_1 - \underline{\Delta}_1/2)^2}{z_1 s} \gg 2|\xi|, \quad (55)$$

where the initial dipole size $x_{10} \sim 1/Q$ sets the typical momentum for the first emitted gluon as $|\underline{k}_1 - \underline{\Delta}_1/2| \sim 1/x_{10} \sim Q$. We can then rewrite this inequality as

$$z_1 \ll \frac{Q^2}{2|\xi|s}. \quad (56)$$

Finally, remembering that $z_1 \leq 1$, because it is a momentum fraction, and, moreover, $z_1 \ll 1$ in the LLA, we write this as

$$z_1 \ll \min \left\{ 1, \frac{x}{2|\xi|} \right\}, \quad (57)$$

where we have defined $x = Q^2/s$ from the longitudinal momentum fraction in the forward limit.

To justify this value of the average parton momentum fraction x , let us go back to the definitions of GTMDs in Eqs. (6) and (24). There, from the Fourier exponent, we see that the x^- -lifetime of the first gluon or (anti)quark is

$$\Delta x^- \lesssim \frac{1}{x\bar{P}^+}. \quad (58)$$

Noticing that, on the other hand, the x^- -lifetime can be written as

$$\Delta x^- \approx \frac{2k^-}{k_1^2} \approx \frac{2k^-}{Q^2} \quad (59)$$

with the LLA accuracy employed here (with k the momentum of the first gluon for the gluon distribution or the momentum of the antiquark for the quark distribution). Substituting Eq. (59) into Eq. (58), we arrive at

$$k^- \lesssim \frac{Q^2}{2x\bar{P}^+} \quad (60)$$

as the upper cutoff on the k^- momentum components resulting from the definitions of GTMDs (and GPDs). On the other hand, the q^- momentum was introduced at the beginning of this subsection as the upper cutoff on the k^- momenta. Identifying the right-hand side of Eq. (60) with q^- , and remembering that $s = 2\bar{P}^+q^-$, we arrive at $x = Q^2/s$, as desired.

Taking inequality (57) into account, we find that the logarithmic integral over z_1 corresponding to the first step of evolution becomes

$$\int_{\frac{1}{sx_{10}^2}}^{\min\{1, \frac{x}{2|\xi|}\}} \frac{dz_1}{z_1} \approx \int_x^{\min\{1, \frac{x}{2|\xi|}\}} \frac{dz_1}{z_1} \approx \ln \left(\min \left\{ \frac{1}{x}, \frac{1}{|\xi|} \right\} \right). \quad (61)$$

Note that we have taken the lower limit of the z_1 integral to be $1/sx_{10}^2$ for simplicity. Recall that we established that this lower bound is equal to $\Lambda^2/s \approx \zeta$ for kinematics with $|t| \approx \Delta_\perp^2 < Q^2$. Our choice of $1/(sx_{10}^2) \approx Q^2/s \approx x$ for the lower limit of the z_1 integration is different from Λ^2/s : however, the difference by an additive term $\sim \alpha_s \ln(Q^2/\Lambda^2)$ is outside of our LLA precision. We similarly neglected a factor of 2 under the logarithm in Eq. (61) within our precision.

We have established the upper bound on the z_1 integral corresponding to the first step of evolution. The momentum fractions ordering (46) enforces this upper bound on all further successive emissions. We have therefore confirmed the prescription from the example in Sec. II B 1 that in order to include nonzero skewness, one has to replace

$$Y = \ln \frac{1}{x} \rightarrow Y = \ln \left(\min \left\{ \frac{1}{x}, \frac{1}{|\xi|} \right\} \right) \quad (62)$$

in the arguments of the unpolarized dipole amplitudes. We can then write the gluon GPD at small- x and small- ξ (assuming $|x| \gg |\xi|$ or $|x| \ll |\xi|$) as

$$H^g(|x| \ll 1, |\xi| \ll 1, t, Q^2) = -\frac{N_c}{2\pi^2\alpha_s} \int d^2b_\perp e^{i\Delta \cdot b} \left[(\nabla_{\underline{x}_1} \cdot \nabla_{\underline{x}_0}) N_{10} \left(Y = \ln \left(\min \left\{ \frac{1}{|x|}, \frac{1}{|\xi|} \right\} \right) \right) \right]_{x_{10}=1/Q^2}. \quad (63)$$

Similarly, one can specify the rapidity in the dipole amplitudes for the quark GPD in Eq. (37) and for the gluon and quark GTMDs in Eqs. (18) and (36), respectively, to be

$$Y = \ln \left(\min \left\{ \frac{1}{|x|}, \frac{1}{|\xi|} \right\} \right). \quad (64)$$

Note that we have replaced $x \rightarrow |x|$, as x ranges from -1 to 1 in GPDs and GTMDs. Allowing for positive or negative x and using symmetry to work with $|\xi|$ instead of ξ , we see that the dipole amplitude rapidity is determined by the intrinsic averaged parton momentum fraction x in the DGLAP region $|x| > |\xi|$, while it is determined by the longitudinal momentum transfer in the ERBL region $|x| < |\xi|$.

Let us restate the main result of this section: the effect of skewness on the leading logarithmic evolution of the unpolarized GPDs and GTMDs at low x and low ξ is to set the effective rapidity of the unpolarized dipole amplitudes $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ which enter these distributions in Eqs. (18), (23), (36), and (37) to equal the logarithm of the minimum of $1/|x|$ and $1/|\xi|$, as shown in Eq. (64).

III. THE REAL PART OF THE SCATTERING AMPLITUDE R -FACTOR

A. Signature factor and the dipole scattering amplitude

The developments in the small- x /saturation physics over the last decades [49–56] using the s -channel/shock wave approach [107–110, 115, 141–143, 162–166] have largely neglected the signature factor in the Regge theory terminology, with the exception of Ref. [167]. In Regge theory one distinguishes the contributions dependent on either of these two configurations (see, e.g., Ref. [168])

$$M^\pm(s) = \frac{1}{2} [M(s) \pm M(-s)], \quad (65)$$

which are symmetric and antisymmetric under the $s \leftrightarrow u \approx -s$ interchange. Here $M(s)$ is the scattering amplitude averaged over all initial- and final-state helicities (since we are interested in unpolarized scattering). We also assume some projection onto a definite color state. The plus sign in

Eq. (65) corresponds to the positive signature, while the minus sign corresponds to the negative signature.

At the eikonal order, the scattering amplitude is proportional to the center-of-mass energy squared s , that is, $M(s) \sim s$ at the Born level, without small- x evolution. It is customary to work with the rescaled amplitude [55]

$$A(s) = \frac{M(s)}{2|s|}. \quad (66)$$

We see that

$$M^\pm(s) = |s| [A(s) \pm A(-s)]. \quad (67)$$

Consider scattering of a dipole on some target at the eikonal level. For the rescaled amplitude $A(s)$ we write

$$iA^{q\bar{q}A}(s) = \frac{1}{N_c} \langle T \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] \rangle(s) - 1, \quad (68)$$

where the letter A in the superscript denotes the target, which may be a nucleon or a nucleus. This amplitude corresponds to the incoming quark at $\underline{x}_1$ and an incoming antiquark at $\underline{x}_0$: see the left panel of Fig. 5 for a diagrammatic representation. The $s \leftrightarrow u$ interchange corresponds to swapping the initial and final state particles in the projectile: hence, $A(-s)$ corresponds to the incoming quark at $\underline{x}_0$ and an incoming antiquark at $\underline{x}_1$:

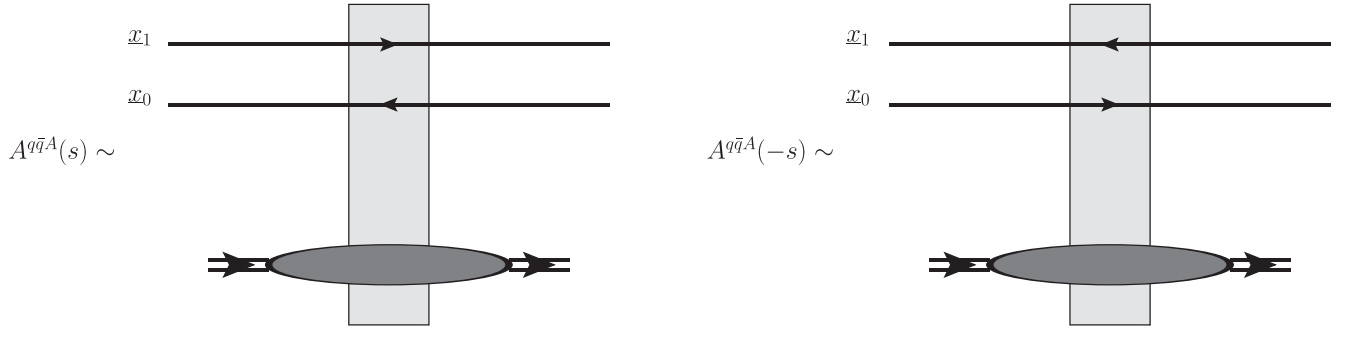
$$iA^{q\bar{q}A}(-s) = \frac{1}{N_c} \langle T \text{tr}[V_{\underline{x}_0} V_{\underline{x}_1}^\dagger] \rangle(s) - 1. \quad (69)$$

This is illustrated in the right panel of Fig. 5. We conclude that

$$iM_{q\bar{q}A}^\pm(s) = \frac{|s|}{N_c} [\langle T \text{tr}[V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] - N_c \rangle(s) \pm \langle T \text{tr}[V_{\underline{x}_0} V_{\underline{x}_1}^\dagger] - N_c \rangle(s)]. \quad (70)$$

B. Signature factor in the nonlinear small- x evolution

Consider the dipole S -matrix from Eq. (19), now with the time-ordering sign T ,

FIG. 5. Dipole-target scattering diagrams related by the $s \leftrightarrow u \approx -s$ interchange.

$$D_{10}(Y) = \frac{1}{N_c} \langle \text{Tr} [V_{\underline{x}_1} V_{\underline{x}_0}^\dagger] \rangle(Y). \quad (71)$$

At large N_c , it obeys the standard BK evolution equation [107–110] (in the LLA)

$$\partial_Y D_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [D_{12}(Y) D_{20}(Y) - D_{10}(Y)]. \quad (72)$$

Interchanging 1 and 0 we write

$$\partial_Y D_{01}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [D_{02}(Y) D_{21}(Y) - D_{01}(Y)]. \quad (73)$$

Defining positive- and negative-signature objects (see, e.g., Ref. [130])

$$S_{10}(Y) \equiv \frac{1}{2} [D_{10}(Y) + D_{01}(Y)], \quad \mathcal{O}_{10}(Y) \equiv \frac{1}{2i} [D_{10}(Y) - D_{01}(Y)], \quad (74)$$

we see that the half-sum of Eqs. (72) and (73) yields

$$\partial_Y S_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \left[\frac{D_{12}(Y) D_{20}(Y) + D_{02}(Y) D_{21}(Y)}{2} - S_{10}(Y) \right]. \quad (75)$$

Since

$$D_{12} D_{20} + D_{02} D_{21} = \frac{1}{2} (D_{12} + D_{21})(D_{20} + D_{02}) + \frac{1}{2} (D_{12} - D_{21})(D_{20} - D_{02}) = 2S_{12} S_{20} - 2\mathcal{O}_{12} \mathcal{O}_{20}, \quad (76)$$

we have (cf. Refs. [128,129])

$$\partial_Y S_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [S_{12}(Y) S_{20}(Y) - S_{10}(Y) - \mathcal{O}_{12}(Y) \mathcal{O}_{20}(Y)]. \quad (77)$$

If we drop the odderon amplitude, we will be back to the BK equation for the positive-signature symmetric S -matrix $S_{10}(Y)$. Finally, defining

$$N_{10}(Y) = 1 - S_{10}(Y), \quad (78)$$

we rewrite Eq. (77) as

$$\partial_Y N_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [N_{12}(Y) + N_{20}(Y) - N_{10}(Y) - N_{12}(Y)N_{20}(Y) + \mathcal{O}_{12}(Y)\mathcal{O}_{20}(Y)]. \quad (79)$$

The difference of Eqs. (72) and (73), divided by $2i$, gives

$$\partial_Y \mathcal{O}_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \left[\frac{D_{12}(Y)D_{20}(Y) - D_{02}(Y)D_{21}(Y)}{2i} - \mathcal{O}_{10}(Y) \right], \quad (80)$$

or, equivalently,

$$\partial_Y \mathcal{O}_{10}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [S_{12}(Y)\mathcal{O}_{20}(Y) + \mathcal{O}_{12}(Y)S_{20}(Y) - \mathcal{O}_{10}(Y)] \quad (81)$$

This is the odderon evolution equation, in the presence of saturation effects [128,129].

From Eqs. (79) and (81) we conclude that the evolution of amplitudes of different signature mixes, but in a way that preserves the signature, in the sense that $(-1)(-1) = +1$ and $+1(-1) = -1$. Note that $N_{10}(Y)$ has a positive signature, while $\mathcal{O}_{10}(Y)$ has a negative signature. For the future use, let us rewrite these equations in the integral form as (cf. Refs. [76–78,80,81,86,160,169–171])

$$\begin{aligned} N_{10}(zs) &= N_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_0^z \frac{dz'}{z'} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [N_{12}(z's) + N_{20}(z's) - N_{10}(z's) - N_{12}(z's)N_{20}(z's) \\ &\quad + \mathcal{O}_{12}(z's)\mathcal{O}_{20}(z's)] \end{aligned} \quad (82)$$

and

$$\mathcal{O}_{10}(zs) = \mathcal{O}_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_0^z \frac{dz'}{z'} \int d^2 x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [S_{12}(z's)\mathcal{O}_{20}(z's) + \mathcal{O}_{12}(z's)S_{20}(z's) - \mathcal{O}_{10}(z's)]. \quad (83)$$

We have switched from rapidity $Y = \ln(zsx_{10}^2) \approx \ln(zs/\Lambda^2)$ as the energy-dependent argument to the center-of-mass energy squared zs . Here, $N_{10}^{(0)}$ and $\mathcal{O}_{10}^{(0)}$ are the initial conditions for these two dipole amplitudes: if they are energy-independent, then a simple differentiation of Eqs. (82) and (83) with respect to $\ln z \sim Y$ reduces them to Eqs. (79) and (81), respectively, above. However, such a simplification is not possible if $N_{10}^{(0)}$ and $\mathcal{O}_{10}^{(0)}$ have a nontrivial energy dependence. In this way, Eqs. (82) and (83) are more general than Eqs. (79) and (81): they allow for energy-dependent inhomogeneous terms/initial conditions (cf. Refs. [76–82,86,160,172] for subeikonal small- x evolution).

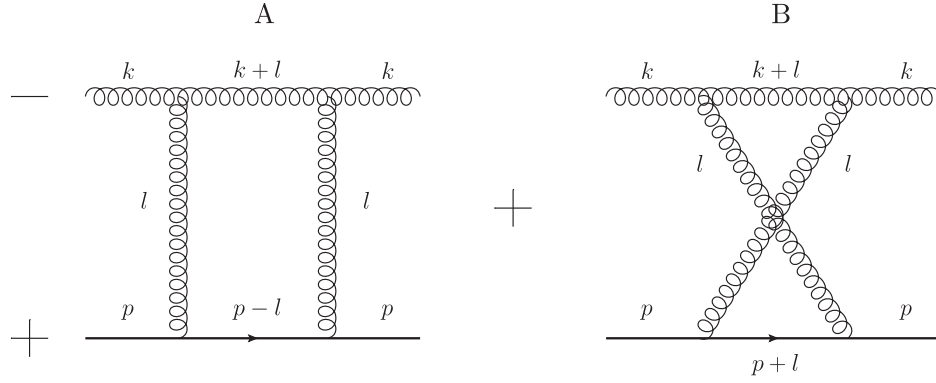
Note that the lower limit of the z' integrals in Eqs. (82) and (83) is 0 and not Λ^2/s coming from the region of validity of the shockwave picture [76–78,80,81,86,160,169–171]) (with Λ the IR cutoff). We are, thus, extending the integration a little bit outside of the allowed region. Below we will argue

how one can modify the initial conditions $N_{10}^{(0)}$ and $\mathcal{O}_{10}^{(0)}$ to make the z' integrals convergent near $z' = 0$ by effectively cutting them off at $z' \approx \Lambda^2/s$ while generating the proper signature factors.

C. Initial conditions in the nonlinear small- x evolution for positive and negative signature factor amplitudes

To fully connect the previous two subsections, consider a gluon from the minus-moving cascade generated by nonlinear small- x evolution of, say, a dipole, scattering on a single quark in the target. The scattering is considered forward, for simplicity, and is depicted in Fig. 6. If the exchange is color-singlet, we have two diagrams at the eikonal order, as shown in Fig. 6.

Starting with the diagram A (on the left) in Fig. 6 we concentrate on the denominators of the propagators. Working in Feynman gauge, we write

FIG. 6. High-energy scattering of a gluon generated by small- x s -channel evolution on a quark in the target.

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(k+l)^2 + i\epsilon} \frac{1}{(l^2 + i\epsilon)^2} \frac{1}{(p-l)^2 + i\epsilon} &= \int \frac{d^4 l}{(2\pi)^4} \frac{1}{2(k^- + l^-)(\frac{k_\perp^2}{2k^-} + l^+) - (\underline{k} + \underline{l})^2 + i\epsilon} \frac{1}{(2l^+ l^- - \underline{l}^2 + i\epsilon)^2} \\ &\times \frac{1}{-2(p^+ - l^+)l^- - \underline{l}^2 + i\epsilon} \approx \frac{i}{4p^+ k^-} \int \frac{d^2 l_\perp}{(2\pi)^2} \frac{1}{(l_\perp^2)^2} \int_{-k^-}^0 \frac{dl^-}{2\pi} \frac{1}{l^- + \frac{l_\perp^2}{2p^+} - i\epsilon}. \end{aligned} \quad (84)$$

We work in the frame where the (massless) quark carries momentum $p^\mu = (p^+, 0^-, \underline{0})$ and assume that the incoming gluon is on mass shell, $k^2 = 0$. Moreover, $p^+ > 0$ and $k^- > 0$. In arriving at the final result in Eq. (84), we have also assumed that $k^- \gg l^-$, which is justified by the logarithmic integral over l^- we have obtained. Our final conclusion below can be shown to apply if we relax this assumption as well.

Diagram B in Fig. 6 is only different from the diagram A by the quark line propagator's denominator. Similar to the above, it gives

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(k+l)^2 + i\epsilon} \frac{1}{(l^2 + i\epsilon)^2} \frac{1}{(p+l)^2 + i\epsilon} &= \int \frac{d^4 l}{(2\pi)^4} \frac{1}{2(k^- + l^-)(\frac{k_\perp^2}{2k^-} + l^+) - (\underline{k} + \underline{l})^2 + i\epsilon} \frac{1}{(2l^+ l^- - \underline{l}^2 + i\epsilon)^2} \\ &\times \frac{1}{2(p^+ + l^+)l^- - \underline{l}^2 + i\epsilon} \approx -\frac{i}{4p^+ k^-} \int \frac{d^2 l_\perp}{(2\pi)^2} \frac{1}{(l_\perp^2)^2} \int_{-k^-}^0 \frac{dl^-}{2\pi} \frac{1}{l^- - \frac{l_\perp^2}{2p^+} + i\epsilon}. \end{aligned} \quad (85)$$

In the case of an infinite x^- light cone Wilson line instead of the gluon projectile in Fig. 6 the $(k+l)^2 + i\epsilon$ denominator will be effectively replaced by $l^+ + i\epsilon$: the rest of the calculation will proceed similarly, although the l^- integral would have to be regulated in the ultraviolet (UV), perhaps by tilting the Wilson line away from the light cone.

We conclude that

$$A + B \sim \int_{-k^-}^0 dl^- \left[\frac{1}{l^- + \frac{l_\perp^2}{2p^+} - i\epsilon} - \frac{1}{l^- - \frac{l_\perp^2}{2p^+} + i\epsilon} \right] = \ln \left(\frac{l_\perp^2 - i\epsilon}{-2p^+ k^- + l_\perp^2 - i\epsilon} \right) - \ln \left(\frac{l_\perp^2 - i\epsilon}{2p^+ k^- + l_\perp^2 - i\epsilon} \right). \quad (86)$$

Note that in the case when the projectile is a dipole, and the two t -channel gluons in Fig. 6 can couple to different lines in a dipole, there will be no difference between diagrams A and B, such that only one diagram will contribute the result in Eq. (86).

Now, in the usual eikonal calculation, see, e.g., pp. 134–135 of [55], we assume that $zs = 2p^+ k^- \gg l_\perp^2$ and simplify Eq. (86) as

$$A + B \sim -\ln \frac{zs}{l_\perp^2} + i\pi + \ln \frac{zs}{l_\perp^2} = i\pi. \quad (87)$$

This is the origin of the energy-independent initial condition for the small- x evolution equation (79).

However, if we remember that the gluon k was generated by the small- x evolution cascade, we would realize that we need to integrate $A + B$ over k^- with the logarithmic integration measure,

$$\int_0^{p_2^-} \frac{dk^-}{k^-} (A + B), \quad (88)$$

as in Eqs. (82) and (83), since $z = k^-/p_2^-$. Furthermore, if we employ the simplification (87) in Eq. (88), we will not generate any additional factors of $i\pi$. Doing so neglects the real part of the scattering amplitude $M_{q\bar{q}A}^+$ which results from summing such powers of $i\pi$. Therefore, to keep those $i\pi$'s, it appears necessary to employ the full exact expression (86) in Eq. (88).

To see what this gives us, we can consider n steps of evolution, which yield

$$\int_0^{p_2^-} \frac{dk_1^-}{k_1^-} \int_0^{k_1^-} \frac{dk_2^-}{k_2^-} \dots \int_0^{k_{n-1}^-} \frac{dk_n^-}{k_n^-} \left[\ln \left(\frac{l_\perp^2 - i\epsilon}{-2p^+ k_n^- + l_\perp^2 - i\epsilon} \right) - \ln \left(\frac{l_\perp^2 - i\epsilon}{2p^+ k_n^- + l_\perp^2 - i\epsilon} \right) \right]. \quad (89)$$

Note that we allow the evolution integrals to start at zero as the lower limit: the integrals are still convergent.

The two terms in Eq. (89) are related by

$$\textcircled{1} = \textcircled{2}(p^+ \rightarrow -p^+), \quad (90)$$

or, equivalently, by

$$\textcircled{1} = \textcircled{2}(s \rightarrow -s). \quad (91)$$

Calculating the second term first, we get

$$\begin{aligned} \textcircled{2} &= \int_0^{p_2^-} \frac{dk_1^-}{k_1^-} \int_0^{k_1^-} \frac{dk_2^-}{k_2^-} \dots \int_0^{k_{n-1}^-} \frac{dk_n^-}{k_n^-} \ln \left(\frac{l_\perp^2 - i\epsilon}{2p^+ k_n^- + l_\perp^2 - i\epsilon} \right) \approx \int_{l_\perp^2/(2p^+)}^{p_2^-} \frac{dk_1^-}{k_1^-} \int_{l_\perp^2/(2p^+)}^{k_1^-} \frac{dk_2^-}{k_2^-} \dots \int_{l_\perp^2/(2p^+)}^{k_{n-1}^-} \frac{dk_n^-}{k_n^-} \ln \left(\frac{l_\perp^2}{2p^+ k_n^- - i\epsilon} \right) \\ &= -\frac{1}{(n+1)!} \ln^{n+1} \left(\frac{s - i\epsilon}{l_\perp^2} \right). \end{aligned} \quad (92)$$

Using Eq. (91), we arrive at

$$\textcircled{1} - \textcircled{2} = -\frac{1}{(n+1)!} \left[\ln^{n+1} \left(\frac{-s - i\epsilon}{l_\perp^2} \right) - \ln^{n+1} \left(\frac{s - i\epsilon}{l_\perp^2} \right) \right]. \quad (93)$$

For $n = 0$ we readily reproduce Eq. (87). Note also that Eq. (93), after being multiplied by $\text{Sign}(s)$ resulting from $s/|s|$ one gets from Eq. (68), becomes an even function of s . Hence, as a two-gluon t -channel exchange contribution to D_{10} , it survives in the positive-signature combination $M_{q\bar{q}A}^+$ and disappears in the negative-signature $M_{q\bar{q}A}^-$,

$$\begin{aligned} iM_{q\bar{q}A}^\pm(s) &= |s|[(D_{10}(s) - 1) \pm (D_{10}(-s) - 1)] \\ &= |s|[(D_{10}(s) - 1) \pm (D_{01}(s) - 1)], \end{aligned} \quad (94)$$

as it should.

Equation (93) illustrates the terms we are trying to resum by including the signature factor. While at LLA one sums up leading logarithms of energy, that is, powers of $\alpha_s \ln s$, we see that some logarithms of energy come in as

$\ln(-s - i\epsilon) = \ln s - i\pi$. The resummation parameter for such logarithms is $\alpha_s \ln(-s - i\epsilon) = \alpha_s \ln s - \alpha_s i\pi$. We see that in addition to resumming the usual LLA parameter $\alpha_s \ln s$, keeping the signature factor effectively resums powers of $-\alpha_s i\pi$. While such powers of $-\alpha_s i\pi$, strictly speaking, constitute nonlogarithmic power-of- α_s higher-order corrections, their resummation is well-defined, as it allows one to reconstruct the real part of the LLA scattering amplitude $M(s)$, just like it would have been obtained using dispersion relations (see Ref. [55] and references therein).

Let us note that in the case of linear (BFKL) evolution, the terms in Eq. (93) can be summed over all integer $n \geq 0$, after being multiplied by the BFKL intercept $\alpha_p - 1$ to the n th power (taken in the Mellin space of the γ variable, where $\alpha_p - 1 = (\alpha_s N_c / \pi) \chi(\gamma)$ in leading-order (LO) BFKL evolution, with $\chi(\gamma)$ the eigenvalue of the LO BFKL kernel [155,156]). This gives

$$\sum_{n=0}^{\infty} \frac{(\alpha_p - 1)^n}{(n+1)!} \left[\ln^{n+1} \left(\frac{-s - i\epsilon}{l_{\perp}^2} \right) - \ln^{n+1} \left(\frac{s - i\epsilon}{l_{\perp}^2} \right) \right] \sim \left(\frac{-s - i\epsilon}{l_{\perp}^2} \right)^{\alpha_p - 1} - \left(\frac{s - i\epsilon}{l_{\perp}^2} \right)^{\alpha_p - 1}, \quad (95)$$

a factor associated with the positive-signature exchange [173,174]. The signature factor, in general, is defined by [173,174]

$$\xi_p(j) = -\frac{e^{-i\pi j} + p}{\sin \pi j}, \quad (96)$$

where j is the intercept (angular momentum in Regge theory) or the Mellin moment-space variable conjugate to x : $j = \alpha_p$ in Eq. (95). Here, $p = +1$ corresponds to the positive signature and $p = -1$ corresponds to the negative signature. Our goal is to find a way to include the effect of this signature factor $\xi_p(j)$ into the nonlinear evolution of Eqs. (82) and (83).

What we have just shown is that by keeping naively energy-suppressed terms in the initial condition (86) for the small- x evolution [and not simplifying $A + B$ down to the standard Eq. (87)] allows one to restore the signature factor in the BFKL evolution sourced by such an initial condition. The generalization to the nonlinear case can now be done by analogy via modification of the initial conditions.

Since we are not keeping track of corrections other than $i\pi$, we can replace $l_{\perp}^2 \rightarrow \Lambda^2$ with Λ our IR cutoff. Based on the above, by comparing Eq. (86) to its standard approximation (87), we see that to include the signature factor into the linear BFKL evolution in the dipole/shockwave formalism, the prescription is to multiply the initial condition/inhomogeneous term by

$$\frac{\text{Sign}(zs)}{i\pi} \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-zs + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{zs + \Lambda^2 - i\epsilon} \right) \right] = \theta(|zs| - \Lambda^2) + i \frac{\text{Sign}(zs)}{\pi} \ln \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) \quad (97)$$

and use the resulting $N_{10}^{(0)}$ in the linearized version of Eq. (82). The overall factor of $\text{Sign}(zs)$ on the left-hand side of Eq. (97) comes from the fact that the eikonal two-gluon exchange amplitude (or, to that matter, any eikonal amplitude) is proportional to s , while to make it energy-independent we divide it by $|s|$, per (68). The factor in Eq. (97) is equal to 1 for $|zs| \gg \Lambda^2$, thus restoring the original initial conditions in this high-energy limit. As we have seen above, not assuming that $|zs| \gg \Lambda^2$ allows one to keep the effects of the signature factor in small- x evolution. Note also that the form of the expression on the right-hand side of Eq. (97) clarifies the physical meaning of our factor on the left of the same equation: its real part is simply a θ -function imposing the cutoff $z > \Lambda^2/s$ on the integral

over z , as is done in the standard shockwave calculations. The imaginary part of the factor is strictly speaking subeikonal: however, after (multiple) z -integral(s) in the small- x evolution (written in an integral form), this part ceases to be subeikonal: rather, it becomes eikonal (power of energy suppression disappears) and generates the factors of $i\pi$, which, when assembled together, form the signature factor for the linear BFKL evolution. This justifies us keeping such subeikonal terms in our approximation for the initial conditions.

In the case of the nonlinear (BK/JIMWLK) evolution, the GGM/MV initial condition [103–106] for the dipole amplitude

$$N_{10}^{(0)}(zs) = 1 - \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \right\} \quad (98)$$

should be replaced by

$$N_{10}^{(0)}(zs) = 1 - \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \frac{\text{Sign}(zs)}{i\pi} \times \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-zs + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{zs + \Lambda^2 - i\epsilon} \right) \right] \right\} \quad (99)$$

to be used as the inhomogeneous term in Eq. (82).

To justify the result (99), we note that here the two-gluon exchange cross section of a dipole scattering on a single nucleon in the nucleus (multiplied by appropriate factors) simply exponentiates [55]. Our above calculation of the two-gluon exchange showed that this cross section needs to be multiplied by the factor in Eq. (97). However, the factor of $\text{Sign}(zs)$ did not arise from the diagrammatic calculation; rather, it appeared due to the definition (68). To show that this factor exponentiates as well, consider two consecutive scatterings of a gluon in a target nucleus, as depicted in Fig. 7. The incoming gluon has a large p^- momentum, while the two scatterings are separated by the distance Δx^- on the light cone.

Repeating the standard calculation [55] yields for the longitudinal Fourier transform of the internal gluon

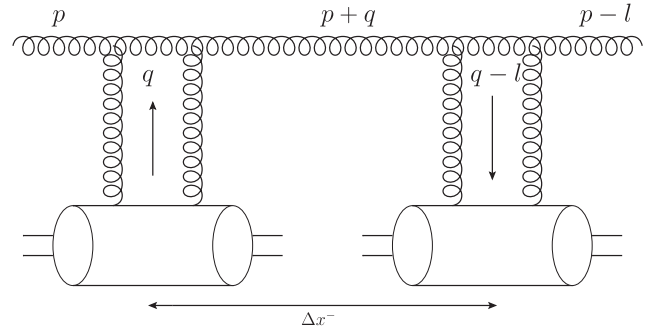


FIG. 7. Double scattering of a gluon in a target nucleus.

propagator's denominator

$$\int_{-\infty}^{\infty} \frac{dq^+}{2\pi} \frac{e^{-iq^+ \Delta x^-}}{2p^- q^+ + i\epsilon} = -\frac{i}{2|p^-|} [\theta(p^-) \theta(\Delta x^-) + \theta(-p^-) \theta(-\Delta x^-)]. \quad (100)$$

We see that one obtains the absolute value of the gluon's minus momentum, $|p^-|$, in the denominator. This results in dividing the amplitude for the second scattering by $|s|$, giving us a factor of $\text{Sign}(zs)$ for each rescattering. We conclude that indeed this $\text{Sign}(zs)$ factor exponentiates as well, justifying Eq. (99). Defining rapidity $Y = \ln(zsx_{10}^2)$ and employing the substitution in Eq. (97) we reduce Eq. (99) to Eq. (1) in the physical regime of $zs > 0$.

For the odderon amplitude in the scattering on a large nucleus, the initial condition including saturation effects reads [130]

$$\mathcal{O}_{10}^{(0)}(zs) = o_{10}^{(0)} \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \right\}, \quad (101)$$

where $o_{10}^{(0)}$ is the lowest-order odderon amplitude due to an exchange of three gluons in the d^{abc} color configuration. To include the signature factor effects, we have to perform an analysis similar to the one above, but only for the d^{abc} three-gluon exchange. This analysis is detailed in Appendix B below. There, we obtain

$$\begin{aligned} \mathcal{O}_{10}^{(0)}(zs) &= o_{10}^{(0)} \frac{\text{Sign}(zs)}{(i\pi)^2} \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-zs + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{zs + \Lambda^2 - i\epsilon} \right) \right]^2 \\ &\quad \times \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \left[\theta(|zs| - \Lambda^2) + \frac{i}{\pi} \text{Sign}(zs) \ln \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) \right] \right\} \\ &= o_{10}^{(0)} \text{Sign}(zs) \left[\theta(|zs| - \Lambda^2) + \frac{2i}{\pi} \theta(|zs| - \Lambda^2) \text{Sign}(zs) \ln \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) - \frac{1}{\pi^2} \ln^2 \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) \right] \\ &\quad \times \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \left[\theta(|zs| - \Lambda^2) + \frac{i}{\pi} \text{Sign}(zs) \ln \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) \right] \right\}. \end{aligned} \quad (102)$$

Let us point out that the real part of the scattering amplitude M (or A), via Eq. (68), corresponds to the (negative) imaginary part of the dipole amplitude N . The imaginary part of the odderon amplitude $\mathcal{O}$ corresponds to the imaginary part of the scattering amplitudes M and A .

D. Summary

Let us summarize the results of this section. In order to include the real part of the scattering amplitude M , or, equivalently, the imaginary part of the positive-signature dipole amplitude $N_{10}(s)$, one has to solve the nonlinear evolution equation [107–110] written in an integral form,

$$N_{10}(zs) = N_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_0^z \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [N_{12}(z's) + N_{20}(z's) - N_{10}(z's) - N_{12}(z's) N_{20}(z's)], \quad (103)$$

with the initial condition/inhomogeneous term given by

$$\begin{aligned} N_{10}^{(0)}(zs) &= 1 - \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \frac{\text{Sign}(zs)}{i\pi} \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-zs + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{zs + \Lambda^2 - i\epsilon} \right) \right] \right\} \\ &= 1 - \exp \left\{ -\frac{1}{4} x_{10}^2 Q_{s0}^2 \ln \left(\frac{1}{x_{10} \Lambda} \right) \left[\theta(|zs| - \Lambda^2) + i \frac{\text{Sign}(zs)}{\pi} \ln \left(\frac{|zs - \Lambda^2|}{|zs + \Lambda^2|} \right) \right] \right\}. \end{aligned} \quad (104)$$

This prescription should replace the phenomenological R-factor used in the literature [96,101,102]. It is valid in the case when the odderon contribution is negligible: indeed the odderon amplitude is suppressed by one power of α_s in

the small- x power counting compared to the C-even (Pomeron) amplitude [128,129,144].

To also include the odderon effects and allow for the imaginary part to the negative-signature odderon

amplitude $\mathcal{O}_{10}(s)$ one needs to solve the coupled integral equations [128,129] (see also Refs. [144,148–154])

$$N_{10}(zs) = N_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_0^z \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [N_{12}(z's) + N_{20}(z's) - N_{10}(z's) - N_{12}(z's)N_{20}(z's) + \mathcal{O}_{12}(z's)\mathcal{O}_{20}(z's)], \quad (105a)$$

$$\mathcal{O}_{10}(zs) = \mathcal{O}_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_0^z \frac{dz'}{z'} \int d^2x_2 \frac{x_{10}^2}{x_{21}^2 x_{20}^2} [S_{12}(z's)\mathcal{O}_{20}(z's) + \mathcal{O}_{12}(z's)S_{20}(z's) - \mathcal{O}_{10}(z's)], \quad (105b)$$

with the initial condition/inhomogeneous term $N_{10}^{(0)}$ given by Eq. (104) and $\mathcal{O}_{10}^{(0)}$ given by Eq. (102) and $S_{10}(s) = 1 - N_{10}(s)$. Note that the inhomogeneous terms (104) and (102) make sure that the z' integrals in Eqs. (103) and (105) are convergent, and are effectively cut off by $z' > \Lambda^2/s$, in agreement with the conventional calculations where Λ^2/s is chosen to be the lower limit of the z' integral.

It is instructive to study the high-energy asymptotics of the dipole amplitudes N_{10} and $\mathcal{O}_{10}$ with the initial conditions allowing for nonzero imaginary parts for these dipole amplitudes, like those in Eqs. (102) and (104). The fixed points of Eq. (105) are obtained by requiring that the integrands on their right-hand sides are zero for constant N_{10} and $\mathcal{O}_{10}$. Just like in the case of purely real N_{10} we get the fixed points at $N = 0, \mathcal{O} = 0$ and $N = 1, \mathcal{O} = 0$.² The former is unstable, since, given a small fluctuation about $N = 0$ the linear (BFKL) part of the evolution would drive the system away from the fixed point. The fixed point at $N = 1, \mathcal{O} = 0$ is stable and represents the high-energy asymptotics of the dipole amplitudes N and $\mathcal{O}$. This means that

$$\lim_{s \rightarrow \infty} \text{Im}[N_{10}(s)] = 0, \quad \lim_{s \rightarrow \infty} \text{Im}[\mathcal{O}_{10}(s)] = 0. \quad (106)$$

The imaginary parts of N_{10} and $\mathcal{O}_{10}$ vanish in the black disk limit. Given that

$$\int d^2b_{\perp} \mathcal{O}_{10}(s) = 0, \quad (107)$$

we see that the total and elastic dipole-target cross sections are

$$\begin{aligned} \sigma_{\text{tot}}^{q\bar{q}A} &= 2 \int d^2b_{\perp} \text{Re}[N_{10}(s) - i\mathcal{O}_{10}(s)] \\ &= 2 \int d^2b_{\perp} \text{Re}[N_{10}(s)], \\ \sigma_{el}^{q\bar{q}A} &= \int d^2b_{\perp} |N_{10}(s) - i\mathcal{O}_{10}(s)|^2 \end{aligned} \quad (108)$$

²There are also fixed points at $N = 1/2$ and $\mathcal{O} = \pm i/2$. However, these fixed points (along with any other constant and nonzero high-energy asymptotic expression for the odderon) violate Eq. (107) below. Therefore, $\mathcal{O} = \pm i/2$ cannot be the high-energy asymptotics of the odderon: this invalidates the fixed points at $N = 1/2$ and $\mathcal{O} = \pm i/2$.

[for a dipole with the quark fixed at transverse position x_1 and the antiquark fixed at x_0 , as in Eq. (71)]. We conclude that the $N = 1, \mathcal{O} = 0$ fixed point still implies that

$$\lim_{s \rightarrow \infty} \frac{\sigma_{el}^{q\bar{q}A}}{\sigma_{\text{tot}}^{q\bar{q}A}} = \frac{1}{2}, \quad (109)$$

in agreement with conventional wisdom.

We also would like to note that in the linear (BFKL) evolution regime, where Eq. (95) applies, our prescription above can be approximated by the relation

$$\text{Im}[N_{10}(s)] \approx -\frac{\pi}{2} \frac{d}{d \ln s} \text{Re}[N_{10}(s)], \quad (110)$$

valid at the first nontrivial order in α_s and often used in phenomenology [175]. From our results presented here, it also follows that the relation (110) is consistent with the $N_{10}(s) \rightarrow 1$ high-energy asymptotics of the dipole amplitude. However, we cannot verify the validity of Eq. (110) in the intermediate energy region between the BFKL evolution and the black-disk limit without fully solving the above evolution equation(s).

IV. RECONCILING THE TWO PARTS OF THE PAPER

A careful reader probably noticed that the definitions of dipole amplitudes differ in our Secs. II and III. In the former, the dipole amplitudes are defined in Eqs. (20) and (21), which contain no time ordering operator. Since the operator in the angle brackets of Eq. (20) is Hermitian, while the operator in the angle brackets of Eq. (21) is anti-Hermitian and divided by i , both amplitudes $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ are manifestly real.

In Sec. III, we study an elastic scattering amplitude, and our definitions of the dipole amplitudes imply a time-ordering operator T ; namely

$$N_{10}^T(Y) \equiv 1 - \frac{1}{2N_c} \langle \text{Tr}[V_{x_1} V_{x_0}^\dagger] + \text{Tr}[V_{x_0} V_{x_1}^\dagger] \rangle(Y), \quad (111a)$$

$$\mathcal{O}_{10}^T(Y) \equiv \frac{1}{2iN_c} \langle \text{Tr}[V_{x_1} V_{x_0}^\dagger] - \text{Tr}[V_{x_0} V_{x_1}^\dagger] \rangle(Y). \quad (111b)$$

To accentuate the difference, we have added the superscript T to the amplitudes. Due to the time-ordering signs, the operators defining $N_{10}^T(Y)$ and $\mathcal{O}_{10}^T(Y)$ are no longer Hermitian and anti-Hermitian, respectively, and, therefore, these amplitudes are not necessarily real. This allowed us to calculate the imaginary parts of both $N_{10}^T(Y)$ and $\mathcal{O}_{10}^T(Y)$ in Sec. III.

This difference between $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ on one side and $N_{10}^T(Y)$ and $\mathcal{O}_{10}^T(Y)$ on the other appears to violate the equivalence between the two objects observed in [176]. While the calculation in [176] was eikonal, it included next-to-leading order (NLO) small- x evolution, demonstrating that small- x evolution at both leading order and at NLO leaves both types of eikonal amplitudes identical,

$$N_{10}(Y) = N_{10}^T(Y) \quad \text{and} \quad \mathcal{O}_{10}(Y) = \mathcal{O}_{10}^T(Y). \quad (112)$$

However, the calculation in [176] did not include the signature effects: we believe this is why we do not see the equalities (112) in our present analysis. Rather, we conjecture that Eqs. (112) should be replaced by

$$N_{10}(Y) = \text{Re}[N_{10}^T(Y)] \quad \text{and} \quad \mathcal{O}_{10}(Y) = \text{Re}[\mathcal{O}_{10}^T(Y)]. \quad (113)$$

Another important consequence of the difference between $N_{10}(Y)$ and $N_{10}^T(Y)$ concerns Eq. (23). Since $N_{10}(Y)$ is real, so is the gluon GPD H^g given by Eq. (23). (For the unpolarized target, the dipole amplitude $N_{10}(Y)$ is symmetric under $\underline{b} \rightarrow -\underline{b}$, such that the Fourier transform in Eq. (23) does not generate an imaginary part of H^g .) This is in accordance with the known fact that while GTMDs may have an imaginary part, the GPDs, as defined above, without the time-ordering sign, are real (see Ref. [9] and references therein). At the same time, since GPDs are defined for elastic scattering amplitudes, the time-ordering sign is more natural for them (and follows from factorization theorems). Repeating the analysis of Sec. II with time-ordering inserted in the definition of gluon GPD will result in Eq. (23) with the dipole amplitude $N_{10}^T(Y)$ instead of $N_{10}(Y)$. Since, as we have shown in Sec. III, $N_{10}^T(Y)$ has an imaginary part, it appears that so will the GPD H^g . (It is possible that a similar concern would apply to the quark GPD H^q : however, our derivation of H^q at small- x in Sec. II assumed no time-ordering sign. Hence, to verify this concern the calculation would have to be redone assuming time-ordering in the definition of H^q , which is beyond the scope of this paper.) The possibility of a GPD with the time-ordering sign being different from the one without one seems to be in violation of the conclusion reached in [177] (see also Refs. [178–181]). Since the difference between $N_{10}(Y)$ and $N_{10}^T(Y)$ is beyond the LLA and resides in the factors of $i\pi$ which are suppressed by a logarithm of energy (they come in multiplied by α_s without a $\ln s$ enhancement), the difference in the H^g with and without time ordering is also beyond the LLA approximation. However, the

difference may still be nonzero, suppressed by only one power of a logarithm of energy [see, e.g., Eq. (110)]. While it is possible that our derivation of Eq. (23) given above only applies in the LLA, it appears we have only neglected subeikonal (proportional to x) terms in this derivation, and, therefore, it should contain all the eikonal terms, at LLA and beyond. We note that in [182], the signature factor in the expression for the elastic cross section results from the conformal moment of the coefficient function: this appears to be different from our case, where the signature factor is a part of the GPD itself. We leave the resolution of this puzzle for future work.

V. CONCLUSIONS

In this work, we have considered two physical phenomena which are often modeled by R -factors in small- x phenomenology. We have studied the effects of longitudinal momentum transfer and the imaginary correction to the dipole amplitude at high energy. In the process, we derived expressions for the unpolarized quark and gluon GTMDs in the small- x / shock wave formalism, obtaining results which agree with prior calculations in the literature when projected down to the GPD limit. The results of that calculation are given in Eqs. (18), (23), (36), and (37). In the quark GPD case, we found a novel appearance of the spin-independent odderon exchange amplitude, confined to only a portion of the GPD phase space in (x, ξ) —the ERBL region. Both quark and gluon unpolarized GTMDs appear to depend on the odderon amplitude as well.

The operator calculations at small x yield the dipole amplitudes entering the expressions for quark and gluon GTMDs/GPDs without a specified rapidity with respect to the hadron. The first part of the above analysis concentrated on determining the value of rapidity Y in the arguments of the dipole amplitudes $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ entering the expressions for these distributions. We addressed the relation between the rapidity of the dipole amplitudes and the partonic longitudinal momentum fraction variables x and ξ by including skewness in the diagrams for LLA small- x evolution. Restricting our analysis to the kinematics $|x| \ll |\xi| \ll 1$ or $|\xi| \ll |x| \ll 1$, we found that the result of including longitudinal momentum transfer in the shock wave formalism is to set the rapidity Y in the argument of the dipole amplitudes resulting from the solution of the full nonlinear evolution equations (BK or JIMWLK equations) at LLA equal to $Y = \ln \min \{1/|x|, 1/|\xi|\}$ [cf. Eq. (64)].

In the second half of the paper, we studied the contribution to the imaginary part of the dipole amplitude $N_{10}(Y)$, corresponding to the real part of the conventional elastic scattering amplitude M . By keeping the naively subeikonal terms in the logarithmic piece of the initial conditions, we modified the initial conditions for the small- x evolution of the dipole to encode both the real and the leading imaginary parts. Explicitly, in Eq. (104) we give the modified initial condition for the GGM/MV model,

allowing one to obtain the dominant imaginary contribution to the dipole amplitude by solving the nonlinear evolution equation (103) written in an integral form. The generalization including the odderon amplitude $\mathcal{O}_{10}(Y)$ into evolution is given by Eq. (105) with the initial conditions given in Eqs. (102) and (104).

We have thus constructed two specific prescriptions describing how both the R -factors can be systematically accounted for in small- x calculations: our prescriptions are based on theoretical considerations and do not require modeling associated with the R -factors. Going forward, we hope that both prescriptions can be successfully implemented in phenomenology, including the analysis of the data to be generated at the Electron Ion Collider [183–187].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SIMPLIFIED EXPRESSION FOR THE QUARK GPD H^q

To simplify Eq. (37) for the quark GPD H^q at small x and ξ , let us examine the $\underline{k}$ and $\underline{k}_1$ integrals. The last term in the curly brackets of Eq. (36) does not depend on $\underline{k}$, so the $\underline{k}$ -integral, which can be extended to infinity at large Q^2 , yields a factor of $\delta^{(2)}(\underline{x}_{10})$ which sets the dipole amplitudes contribution equal to zero (both N_{10} and $\mathcal{O}_{10}$ go to zero when $x_{10} \rightarrow 0$), making that term also zero. We are left with the contributions coming from the first and second terms in the curly brackets of Eq. (36), where we need to consider integrals of the form

$$\int d^2 k_{\perp} d^2 k_{1\perp} e^{-i(\underline{k}+\underline{k}_1)\cdot\underline{x}_{10}} [N_{10}(Y) - i\mathcal{O}_{10}(Y)] \left\{ \frac{\underline{k}_1 \cdot (\underline{k} + \frac{1}{2}\underline{\Delta})}{[(x+\xi)zs + \underline{k}_1^2 - i\epsilon][(x-\xi)zs + (\underline{k} + \frac{1}{2}\underline{\Delta})^2 - i\epsilon]} + \frac{\underline{k}_1 \cdot (\underline{k} - \frac{1}{2}\underline{\Delta})}{[(x+\xi)zs + (\underline{k} - \frac{1}{2}\underline{\Delta})^2 + i\epsilon][(x-\xi)zs + \underline{k}_1^2 + i\epsilon]} \right\}. \quad (\text{A1})$$

We can replace $\underline{k} \rightarrow -\underline{k}$, $\underline{k}_1 \rightarrow -\underline{k}_1$, and $\underline{x}_1 \leftrightarrow \underline{x}_0$ in the second term in curly brackets of Eq. (A1), making use of $N_{10} = N_{01}$, and $\mathcal{O}_{10} = -\mathcal{O}_{01}$ to obtain

$$\int d^2 k_{\perp} d^2 k_{1\perp} e^{-i(\underline{k}+\underline{k}_1)\cdot\underline{x}_{10}} \left\{ \frac{[N_{10}(Y) - i\mathcal{O}_{10}(Y)]\underline{k}_1 \cdot (\underline{k} + \frac{1}{2}\underline{\Delta})}{[(x+\xi)zs + \underline{k}_1^2 - i\epsilon][(x-\xi)zs + (\underline{k} + \frac{1}{2}\underline{\Delta})^2 - i\epsilon]} + \frac{[N_{10}(Y) + i\mathcal{O}_{10}(Y)]\underline{k}_1 \cdot (\underline{k} + \frac{1}{2}\underline{\Delta})}{[(x+\xi)zs + (\underline{k} + \frac{1}{2}\underline{\Delta})^2 + i\epsilon][(x-\xi)zs + \underline{k}_1^2 + i\epsilon]} \right\}. \quad (\text{A2})$$

Next, we perform the integrals over $\underline{k}_1$ and $\underline{k}$, obtaining

$$\begin{aligned} & - (2\pi)^2 e^{i\Delta\cdot\underline{x}_{10}/2} \left\{ N_{10}(Y) \left[\sqrt{(x+\xi)zs - i\epsilon} \sqrt{(x-\xi)zs - i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs - i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs - i\epsilon} \right) \right. \right. \\ & \quad \left. \left. + \sqrt{(x+\xi)zs + i\epsilon} \sqrt{(x-\xi)zs + i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs + i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs + i\epsilon} \right) \right] \right. \\ & \quad \left. - i\mathcal{O}_{10}(Y) \left[\sqrt{(x+\xi)zs - i\epsilon} \sqrt{(x-\xi)zs - i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs - i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs - i\epsilon} \right) \right. \right. \\ & \quad \left. \left. - \sqrt{(x+\xi)zs + i\epsilon} \sqrt{(x-\xi)zs + i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs + i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs + i\epsilon} \right) \right] \right\}. \quad (\text{A3}) \end{aligned}$$

As $zs > 0$, we can see that for $x > |\xi|$ the term proportional to $\mathcal{O}_{10}(Y)$ cancels as $\epsilon \rightarrow 0$. In the rest of the (x, ξ) phase space we have no clear cancellation, and thus we have the curious result that there may be an odderon contribution to the quark GPD in a limited part of its (x, ξ) phase space.

We note that, as follows from the z -integral in Eq. (36), the typical z -value is of the order-1, such that it does not significantly affect the value of rapidity Y in the argument of the dipole amplitude. (As $\ln(zs) \approx \ln(s)$ for $z = \mathcal{O}(1)$, the rapidity Y , which usually depends on $\ln(zs)$, can be thought of as dependent only on $\ln(s)$ instead.) This allows

us to neglect the dependence of $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ on z , as a constant under the logarithm in the LLA approximation. In this simplified case where $N_{10}(Y)$ and $\mathcal{O}_{10}(Y)$ are independent of z , we can substitute Eq. (A3) into Eqs. (36) and (37) and allow the lower limit of z integration to go to zero, obtaining

$$\begin{aligned}
 & H^q(|x| \ll 1, |\xi| \ll 1, t, Q^2) \\
 &= \frac{2N_c s}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} e^{i\Delta \cdot x_1} \int_0^1 dz \left\{ N_{10}(Y) \left[\sqrt{(x+\xi)zs - i\epsilon} \sqrt{(x-\xi)zs - i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs - i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs - i\epsilon} \right) \right. \right. \\
 &+ \left. \left. \sqrt{(x+\xi)zs + i\epsilon} \sqrt{(x-\xi)zs + i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs + i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs + i\epsilon} \right) \right] \right. \\
 &- i\mathcal{O}_{10}(Y) \left[\sqrt{(x+\xi)zs - i\epsilon} \sqrt{(x-\xi)zs - i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs - i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs - i\epsilon} \right) \right. \\
 &- \left. \left. \sqrt{(x+\xi)zs + i\epsilon} \sqrt{(x-\xi)zs + i\epsilon} K_1 \left(x_{10} \sqrt{(x+\xi)zs + i\epsilon} \right) K_1 \left(x_{10} \sqrt{(x-\xi)zs + i\epsilon} \right) \right] \right\}. \tag{A4}
 \end{aligned}$$

Following [47], we change the integration variable from z to zs , and, assuming that for very large s the upper limit of the resulting integral can be replaced by infinity, we write

$$\begin{aligned}
 & H^q(|x| \ll 1, |\xi| \ll 1, t, Q^2) \\
 &= \frac{2N_c}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} e^{i\Delta \cdot x_1} \int_0^\infty d(zs) zs \left\{ N_{10}(Y) \left[\sqrt{x+\xi-i\epsilon} \sqrt{x-\xi-i\epsilon} K_1 \left(x_{10} \sqrt{zs} \sqrt{x+\xi-i\epsilon} \right) K_1 \left(x_{10} \sqrt{zs} \sqrt{x-\xi-i\epsilon} \right) \right. \right. \\
 &+ \left. \left. \sqrt{x+\xi+i\epsilon} \sqrt{x-\xi+i\epsilon} K_1 \left(x_{10} \sqrt{zs} \sqrt{x+\xi+i\epsilon} \right) K_1 \left(x_{10} \sqrt{zs} \sqrt{x-\xi+i\epsilon} \right) \right] \right. \\
 &- i\mathcal{O}_{10}(Y) \left[\sqrt{x+\xi-i\epsilon} \sqrt{x-\xi-i\epsilon} K_1 \left(x_{10} \sqrt{zs} \sqrt{x+\xi-i\epsilon} \right) K_1 \left(x_{10} \sqrt{zs} \sqrt{x-\xi-i\epsilon} \right) \right. \\
 &- \left. \left. \sqrt{x+\xi+i\epsilon} \sqrt{x-\xi+i\epsilon} K_1 \left(x_{10} \sqrt{zs} \sqrt{x+\xi+i\epsilon} \right) K_1 \left(x_{10} \sqrt{zs} \sqrt{x-\xi+i\epsilon} \right) \right] \right\} \\
 &= \frac{N_c}{(2\pi)^4} \int d^2 x_{1\perp} d^2 x_{0\perp} \frac{e^{i\Delta \cdot x_1}}{x_{10}^4 \xi^3} \\
 &\times \left\{ N_{10}(Y) \left[(x+\xi-i\epsilon)^2 - (x-\xi-i\epsilon)^2 + 2(x+\xi-i\epsilon)(x-\xi-i\epsilon) \ln \left(\frac{x-\xi-i\epsilon}{x+\xi-i\epsilon} \right) \right. \right. \\
 &+ \left. \left. (x+\xi+i\epsilon)^2 - (x-\xi+i\epsilon)^2 + 2(x+\xi+i\epsilon)(x-\xi+i\epsilon) \ln \left(\frac{x-\xi+i\epsilon}{x+\xi+i\epsilon} \right) \right] \right. \\
 &- i\mathcal{O}_{10}(Y) \left[(x+\xi-i\epsilon)^2 - (x-\xi-i\epsilon)^2 + 2(x+\xi-i\epsilon)(x-\xi-i\epsilon) \ln \left(\frac{x-\xi-i\epsilon}{x+\xi-i\epsilon} \right) \right. \\
 &- \left. \left. \left[(x+\xi+i\epsilon)^2 - (x-\xi+i\epsilon)^2 + 2(x+\xi+i\epsilon)(x-\xi+i\epsilon) \ln \left(\frac{x-\xi+i\epsilon}{x+\xi+i\epsilon} \right) \right] \right] \right\}. \tag{A5}
 \end{aligned}$$

Outside of the logarithms we can safely take $\epsilon \rightarrow 0$ regardless of the values of x and ξ . This leads to Eq. (38) in the main text.

APPENDIX B: IMAGINARY CORRECTIONS TO THE BORN-LEVEL ODDERON AMPLITUDE

The initial condition for the dipole amplitude and odderon amplitude (before including multiple rescatterings) is given by the lowest-order two-gluon and three-gluon exchange diagrams. Considering the scattering amplitude of a quark-quark scattering with t -channel gluon exchanges, we see that a convenient way to obtain these amplitudes is by using dispersion relations. For instance, instead of calculating directly the two-gluon exchange diagrams, we would only need to calculate the imaginary part of these diagrams given by the cut diagram. The cut diagram would only consist of two one-gluon exchange diagrams (see Fig. 9 below). In this way we can build up the amplitudes iteratively.

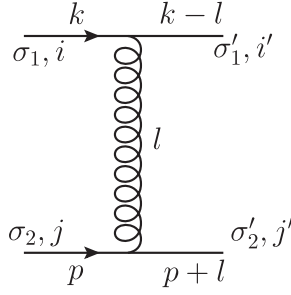


FIG. 8. One-gluon exchange diagrams for a quark scattering on a quark at high energy.

The high-energy amplitude for quark-quark scattering via one t -channel gluon exchange (shown in Fig. 8) is, in the eikonal approximation,

$$M_{1g}(s, t) = -8\pi\alpha_s \frac{s}{l_\perp^2} (t^a)_{i'i} (t^a)_{j'j}, \quad (\text{B1})$$

where $s = 2p^+k^-$ is the center-of-mass energy. The imaginary part of the quark-quark scattering amplitude via two-gluon exchange is given by the cut diagram shown in Fig. 9, which can be obtained using Cutkosky rules and the one-gluon amplitude in Eq. (8):

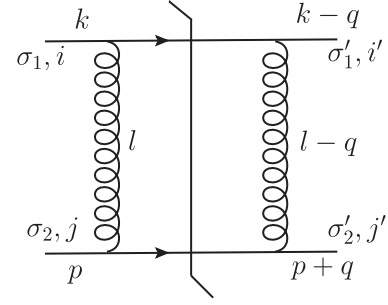


FIG. 9. Imaginary part of a two-gluon exchange scattering amplitude for a quark scattering on another quark.

$$\begin{aligned} 2 \text{Im} M_{2g}(s, t = -\underline{q}^2) \\ = \int \frac{d^4 l}{(2\pi)^4} M_{1g}(s, -l^2) M_{1g}^*(s, -(l-q)^2) 2\pi\delta((k-l)^2) \\ \times 2\pi\delta((p+l)^2). \end{aligned} \quad (\text{B2})$$

This gives

$$\text{Im} M_{2g}(s, t = -\underline{q}^2) = 4\alpha_s^2 s (t^a t^b)_{i'i} (t^a t^b)_{j'j} \int \frac{d^2 l_\perp}{l^2 (l-q)^2}. \quad (\text{B3})$$

Since $\text{Im} M_{2g} \propto s$, we shall use the double-subtracted dispersion relation to obtain (the leading part of) M_{2g} from its imaginary part [55]:

$$\begin{aligned} M_{2g}(s, t = -\underline{q}^2) &= M_{2g}(0, t = -\underline{q}^2) + s \partial_s M_{2g}(0, t = -\underline{q}^2) \\ &+ \frac{1}{\pi} \left\{ s^2 \int_{\Lambda^2}^\infty ds' \frac{\text{Im} M_{2g}(s', t)}{(s' - i\epsilon)^2 (s' - s - i\epsilon)} + u^2 \int_{\Lambda^2}^\infty du' \frac{\text{Im} M_{2g}(u', t)}{(u' - i\epsilon)^2 (u' - u - i\epsilon)} \right\} \\ &= -\frac{4\alpha_s^2}{\pi} s (t^a t^b)_{i'i} (t^a t^b)_{j'j} \int \frac{d^2 l_\perp}{l^2 (l-q)^2} \left[\ln \left(\frac{-s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) - \ln \left(\frac{s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) \right]. \end{aligned} \quad (\text{B4})$$

In the last step, we have neglected $M_{2g}(0, t = -\underline{q}^2) + s \partial_s M_{2g}(0, t = -\underline{q}^2)$ and assumed that the color factors in the two terms are the same, in anticipation of calculating the odderon contribution. The structure of Eq. (B4) agrees with that in Eq. (86). We see that the two-gluon exchange scattering amplitude aside from the color factor is [see Eq. (3.49) in [55]]

$$M_{2g}(s, t = -l^2) = -\frac{4\alpha_s^2 s}{\pi} \int \frac{d^2 k_\perp}{k^2 (l-k)^2} \left[\ln \left(\frac{-s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) - \ln \left(\frac{s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) \right]. \quad (\text{B5})$$

The odderon exchange at the lowest 3-gluon order is illustrated in Fig. 10. Since odderon is defined to be the antisymmetric part of the dipole S -matrix, the color factor of every possible configuration of gluon exchanges takes the form $\text{tr}[t^a t^b t^c] \text{tr}[t^a \{t^b, t^c\}] = \frac{1}{8} (d^{abc})^2$. Therefore, in calculating the odderon amplitude we can drop the color factors for simplicity as they are all the same.

Similar to how we obtain the 2-gluon exchange amplitude, our method is to reconstruct the odderon amplitude from its imaginary part. The imaginary part is given by all the possible



FIG. 10. Diagrammatic representation of the odderon exchange at the lowest 3-gluon order.

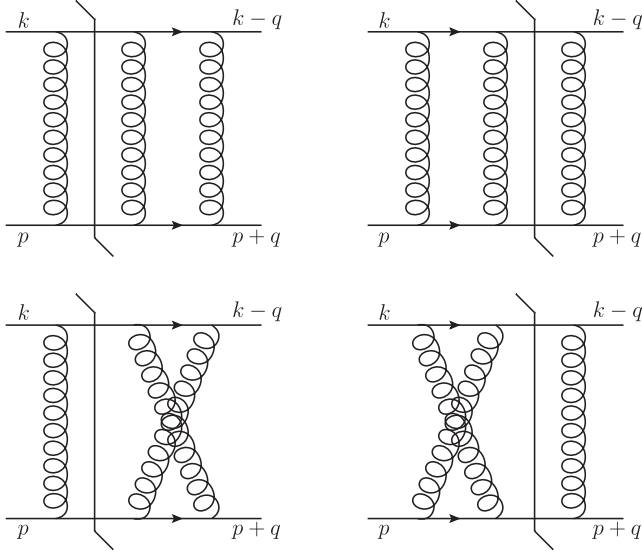


FIG. 11. Cut diagrams of the odderon exchange at the lowest 3-gluon order.

cuts between the t -channel gluon exchanges, as shown in Fig. 11 (see more on this below). Gluing the 1-gluon

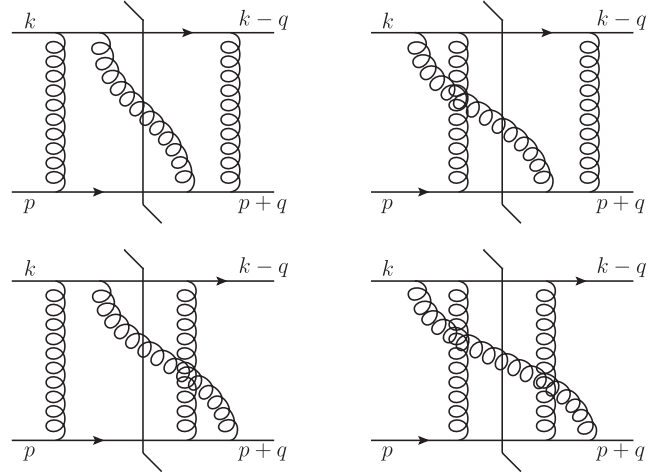


FIG. 12. Diagrams with the cuts that go through one of the gluons. These diagrams give an overall subeikonal contribution due to the gluons being in the d^{abc} color state.

and 2-gluon exchange amplitudes together we get the Im part of the 3-gluon exchange contribution (with the color factor removed)

$$\begin{aligned} \text{Im } M_{3g}(s, t = -\underline{q}^2) &= \frac{2\alpha_s^3 s}{\pi^2} \int \frac{d^2 k_\perp d^2 l_\perp}{\underline{k}^2 (\underline{l} - \underline{k})^2 (\underline{l} - \underline{q})^2} \text{Sign}(s) \left[\ln \left(\frac{-s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) - \ln \left(\frac{s + \Lambda^2 - i\epsilon}{\Lambda^2 - i\epsilon} \right) + \text{c.c.} \right] \theta(|s| - \Lambda^2) \\ &= \frac{4\alpha_s^3 s}{\pi^2} \int \frac{d^2 k_\perp d^2 l_\perp}{\underline{k}^2 (\underline{l} - \underline{k})^2 (\underline{l} - \underline{q})^2} \text{Sign}(s) \ln \left(\frac{|s - \Lambda^2|}{|s + \Lambda^2|} \right) \theta(|s| - \Lambda^2). \end{aligned} \quad (\text{B6})$$

The u -channel contribution is obtained by reversing the order of the vertices attaching the gluons to the projectile and then by applying crossing symmetry to change the quark into an antiquark in the projectile in Fig. 11. This procedure would then require us to change the sign of all three quark-gluon vertices while keeping the d^{abc} color factor the same. Hence, $\text{Sign}(s)$ is added to account for the

additional sign change when we do the $s \rightarrow u \approx -s$ replacement to obtain the u -channel contribution to (the imaginary part of the) odderon amplitude. The theta-function describes the threshold above which the scattering is possible (in both the s - and u -channels), which is taken to be the same Λ^2 for simplicity.

Squaring Eq. (97) and putting $z = 1$ for simplicity we get

$$\begin{aligned} -\frac{1}{\pi^2} \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-s + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{s + \Lambda^2 - i\epsilon} \right) \right]^2 &= \theta(|s| - \Lambda^2) + 2i\theta(|s| - \Lambda^2) \frac{\text{Sign}(s)}{\pi} \ln \left(\frac{|s - \Lambda^2|}{|s + \Lambda^2|} \right) \\ &\quad - \frac{1}{\pi^2} \ln^2 \left(\frac{|s - \Lambda^2|}{|s + \Lambda^2|} \right). \end{aligned} \quad (\text{B7})$$

From Eq. (B7) we see that the imaginary part in Eq. (B6) may arise from the following full expression for M_{3g} ,

$$M_{3g}(s, t = -\underline{q}^2) = \frac{4\alpha_s^3 s}{\pi^2} \int \frac{d^2 k_\perp d^2 l_\perp}{\underline{k}^2 (\underline{l} - \underline{k})^2 (\underline{l} - \underline{q})^2} \left(-\frac{1}{2\pi} \right) \left[\ln \left(\frac{\Lambda^2 - i\epsilon}{-s + \Lambda^2 - i\epsilon} \right) - \ln \left(\frac{\Lambda^2 - i\epsilon}{s + \Lambda^2 - i\epsilon} \right) \right]^2. \quad (\text{B8})$$

It appears impossible to reconstruct the full $\text{Re } M_{3g}$ using dispersion relations, as they generate only the subeikonal contribution to $\text{Re } M_{3g}$. Employing the function (B7) in the derivation of the single-subtracted dispersion relation, one can see that the semicircle at infinity would contain the eikonal contribution to $\text{Re } M_{3g}$ and cannot be discarded, making it impossible to construct $\text{Re } M_{3g}$ from $\text{Im } M_{3g}$ only. For the double-subtracted dispersion relation, like that used in (B4), the integral again gives the subeikonal contribution to $\text{Re } M_{3g}$ only, with the needed eikonal contribution likely residing in the $s\partial_s M_{3g}(0, t)$ term. Hence, while we could not reconstruct the full $\text{Re } M_{3g}$ using dispersion relations, we have found the analytic function (B7) whose imaginary part is the same (up to a multiplicative factor) as that in Eq. (B6) and which contains $\text{Re } M_{3g}$ as well. The latter happens to be eikonal, as expected for the odderon. Moreover, the amplitude in Eq. (B8) has a negative signature. Therefore, Eq. (B8) is our well-motivated conjecture for the full odderon amplitude at the 3-gluon exchange level: it contains the correction we have included in Eq. (102) in the main text.

We also notice that some of the 3-gluon exchange diagrams may contain a cut going through a gluon line. For instance, for the diagrams in Fig. 10, when all three gluons connect to either the quark or the antiquark line in the dipole, it is possible to place a cut going through the gluon line with the color index b in the figure. This is

depicted in Fig. 12. A straightforward calculation of these diagrams, which we will not present here, leads to the following contribution to the imaginary part of the odderon amplitude coming from such a cut,

$$\text{Im } M_{3g}(s, t = -\underline{q}^2) \supset \theta(|s| - \Lambda^2) \text{Sign}(s) \ln\left(\frac{|s| - \Lambda^2}{\Lambda^2}\right). \quad (\text{B9})$$

This contribution is missing an overall factor of s and is, therefore, subeikonal. (While our calculation gave us terms with either $\ln((|s| - \Lambda^2)/\Lambda^2)$ or $\ln((|s| + \Lambda^2)/\Lambda^2)$, we have replaced the latter logarithm with the former in Eq. (B9) within the present subeikonal accuracy.) The amplitude which leads to the imaginary part in Eq. (B9) can be easily identified as proportional to

$$M_{3g}(s, t = -\underline{q}^2) \supset \ln^2\left(\frac{\Lambda^2 - i\epsilon}{-s + \Lambda^2 - i\epsilon}\right) - \ln^2\left(\frac{\Lambda^2 - i\epsilon}{s + \Lambda^2 - i\epsilon}\right). \quad (\text{B10})$$

This contribution is also subeikonal; it can be neglected at high energies compared to that in Eq. (B8) above, as a subeikonal correction to the (real part of the) odderon scattering amplitude.

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