

Anisotropic fluctuations of momentum and angular momentum of heavy quarks in the pre-equilibrium stage of pA collisions at the LHC

Gabriele Parisi^{1,2,*}, Vincenzo Greco^{1,2,†} and Marco Ruggieri^{1,3,‡}

¹*Department of Physics and Astronomy, University of Catania, Via S. Sofia 64, I-95125 Catania, Italy*

²*Laboratori Nazionali del Sud, INFN-LNS, Via S. Sofia 62, I-95123 Catania, Italy*

³*INFN-Sezione di Catania, Via S. Sofia 64, I-95123 Catania, Italy*



(Received 3 June 2025; revised 18 September 2025; accepted 7 October 2025; published 23 October 2025)

We simulate the real-time evolution of the SU(3)-glasma generated in the early stages of high-energy proton-nucleus collisions, employing classical lattice gauge theory techniques. Our setup incorporates a realistic modeling of the proton's internal structure and includes longitudinal fluctuations in the initial state, enabling the study of genuinely nonboost-invariant collision dynamics. Focusing on the momentum and angular momentum anisotropies of heavy quarks in the infinite mass limit, we find that the system retains significant anisotropy well beyond the characteristic timescale $\tau = 1/Q_s$, with Q_s being the saturation energy of the glasma, the only scale in our model. This persistence of anisotropy is further confirmed in the more realistic, nonboost-invariant scenario, across a range of fluctuation amplitudes. These findings pave the way for future investigations involving dynamical heavy quarks and more quantitative initializations of the glasma.

DOI: [10.1103/lvv6-xx41](https://doi.org/10.1103/lvv6-xx41)

I. INTRODUCTION

The analysis of a relativistic heavy-ion collision, as performed at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory and at the Large Hadron Collider (LHC) at CERN, provides a unique opportunity to investigate the behavior of QCD matter at extreme temperatures and densities. The study of such events requires different kinds of descriptions of the spacetime evolution of the system, since the matter produced in the collision experiences different stages in its evolution. For instance, over the past few decades many experimental results have indicated the emergence of a new state of matter, referred to as the quark-gluon plasma (QGP), which forms after ~ 1 fm after the collision and in which quarks and gluons behave as a hydrodynamic fluid [1–3].

If we instead consider the early stages of the collision (≤ 1 fm), the classical Yang-Mills (CYM) theory offers a good description of this phase. It can describe well the nonequilibrium evolution of the highly occupied gluonic system, called the glasma, that appears immediately after

the collision. In particular, the glasma simulation with the CYM field plays an important role in understanding the nonequilibrium stage between the moment of the collision and the onset of the hydrodynamic evolution of the QGP. In fact the glasma simulation, in the analysis of experimental data, is widely used to establish the initial conditions for the subsequent hydrodynamic evolution [4,5]. The theoretical background for why the CYM theory offers a good picture of the initial gluonic matter is based on the color glass condensate (CGC) framework, which stands as a valid description of the high-energy nucleus [6–8], see [9–12] for reviews. In such a high-energy nucleus the dominant degrees of freedom are soft gluons emitted from hard partons. In particular, the McLerran-Venugopalan (MV) model in the CGC effective theory describes the soft gluons as the CYM fields and the hard partons as their color sources. Consequently, the glasma generated in the collision of such high-energy nuclei can be described well by the CYM field. The greatest simplicity of the CGC formalism is the fact that the complexity of a nonlinear many-body problem at high energy is reduced into a one-scale problem, with a hard saturation scale Q_s as the only dimensionful relevant scale at which gluon recombination effects start to become as important as the gluon radiation ones. The CYM equations can be studied both analytically (e.g., by employing a small time expansion [13,14]) or numerically [15,16].

However, such a success of the glasma simulations has been largely limited by the boost invariance assumption, which shows good agreement with experimental data only around the midrapidity region. Recently, much attention has

*Contact author: gabriele.parisi@dfa.unict.it

†Contact author: vincenzo.greco@dfa.unict.it

‡Contact author: marco.ruggieri@dfa.unict.it

been paid to the $3 + 1$ -dimensional ($3 + 1D$) glasma simulations, which aim to go beyond the boost invariance assumption, as necessary to understand observables across a broader region of rapidity [17,18]. In fact, on top of the glasma it is possible to add quantum fluctuations [19–32], which are known to trigger plasma instabilities [33] and are helpful to produce entropy during the early stages of high-energy nuclear collisions [23,34–36]. Quantum fluctuations appear when one considers the finite coupling corrections to the glasma solution (which is obtained in the small coupling limit): the spectrum of these fluctuations has been computed within a perturbative calculation in [24] and it has been shown that they affect both the gauge potential and the color-electric field.

The experimental framework we mentioned refers to Pb-Pb (LHC) or Au-Au (RHIC) collisions. Moving on to measurements in smaller collision systems, such as pp and pA [37,38] (in particular those of anisotropies in multi-particle correlation functions), these have shown very similar features as those found in heavy-ion collisions. While calculations within the hydrodynamic framework have been quite successful in describing observables in these small collision systems, alternative explanations relying entirely on intrinsic momentum correlations of the produced particles can also reproduce many features of the experimental data [37,38]. It is likely that both initial intrinsic momentum correlations and a hydrolike anisotropic expansion are necessary to understand the elliptic anisotropic emission in pA [39,40]. However, regardless of the existence of alternative explanations, the applicability of hydrodynamics becomes increasingly doubtful as the system size decreases and gradients increase [41,42]. For this reason, a thorough treatment of the initial stages is even more important for pA collisions. In this work we will simulate a pA collision by solving the classical Yang-Mills equations in SU(3) with MV initial conditions for the initial charge. The original MV hypothesis will be extended in order to account for the quark structure of the proton.

Particularly sensitive probes of the very early stages of the collision are heavy quarks [43]. Due to their short formation time (of the order of $1/2M$, where M is the heavy quark mass), they experience the initial stage of the collision. Moreover, their mass is greater than Λ_{QCD} , hence their initial production can be studied with perturbative methods. By understanding the imprint of the glasma fields on these probes, one can disentangle important information about the structure of initially produced matter, in both proton-nucleus and nucleus-nucleus collisions. There exists a relevant limiting case in which the accumulated momentum of hard probes in glasma can be evaluated only from glasma lattice field correlators [15,44], without explicitly solving the particle equations of motion—this corresponds to infinitely massive heavy quarks.

The paper is structured as follows: in Sec. II we elaborate on the formalism used for solving the CYM equations,

starting from the charge generation, then moving on toward the initialization of the color-magnetic fields and their time evolution. In Sec. III we deal with the Wong equations governing the motion of heavy quarks (HQs), which are charm and beauty (the top quark here is not taken into account, since its huge mass implies decay times much smaller than the QCD timescales, check, e.g., [45] for a study on top quarks in QGP) to find relevant quantities like the momentum shift and the angular momentum shift of infinitely massive HQs in glasma. In Sec. IV we aim to generalize the CYM formalism, including initial state fluctuations, which break boost invariance. After showing the results, we then summarize and conclude.

In this article we use natural units $k_B = c = \hbar = 1$. Moreover, throughout the paper we use both Minkowski and Milne coordinates. The change of coordinates from one system to the other is given by

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2} \log \frac{t+z}{t-z}, \quad (1)$$

which give a definition of proper time and spatial rapidity.

II. THE EVOLVING GLASMA

Several different strategies have been used to understand and describe the earliest phase of relativistic heavy-ion collisions. One which is commonly applied is the color glass condensate effective theory [6–8], which is based on a separation of scales between the high Bjorken- x degrees of freedom, i.e., the valence partons, and the low- x degrees of freedom which they generate, that is soft gluons.

A. Color charges for p and A

In our work, to realize the aforementioned separation, the sources of the initial gluon fields are generated using the MV model, in which the large- x color sources are randomly distributed on an infinitely thin color sheet. The distribution of these charges is a Gaussian, characterized by zero average

$$\langle \rho^a(\mathbf{x}_\perp) \rangle = 0, \quad (2)$$

and variance given by

$$\langle \rho^a(\mathbf{x}_\perp) \rho^b(\mathbf{y}_\perp) \rangle = g^2 \mu^2 \delta^{ab} \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp). \quad (3)$$

Here, g is the coupling constant (we use $g = 2$, corresponding to $\alpha_s = 0.3$), and μ is the so-called MV parameter, describing the number density of the color charges per unit of area in the transverse plane. The symbol $\perp$ here and throughout the paper refers to quantities in the transverse plane. In our implementation, we relax the single-sheet hypothesis of the original MV model by generating a certain number N_s of color sheets stacked on top of one

another [46]. Consequently, for each of the sheets, the numerical implementation of Eq. (3) we will use in this work is

$$\langle \rho_{n,x}^a \rho_{m,y}^b \rangle = g^2 \mu^2 \frac{\delta_{n,m}}{N_s} \delta^{a,b} \frac{\delta_{x,y}}{a_\perp^2}, \quad (4)$$

where $a, b = 1, \dots, N_c^2 - 1$ are SU(3) color indexes, $n, m = 1, \dots, N_s$ are the color sheet indexes, and x, y span each point of a $N_\perp \times N_\perp$ lattice, whose transverse length is $L_\perp$ and lattice spacing is $a_\perp = L_\perp / N_\perp$. Operatively, the conditions (2) and (4) are satisfied by generating random Gaussian numbers with mean zero and standard deviation equal to $\sqrt{(g^2 \mu^2) / (N_s a_\perp^2)}$: for a Gaussian distribution of mean zero and standard deviation σ , one has $\langle x \rangle = 0$ and $\langle x^2 \rangle = \sigma^2 - \langle x \rangle^2 = \sigma^2$, moreover, the appearance of the factor $a_\perp$ comes from the discretization of the Dirac delta. Below we explain how to build up the gluon fields generated by each sheet, and how to combine all these fields to obtain the initial gluon field in the glasma. We will use $N_s = 50$ in Eq. (4) throughout this work: we have seen that there is a slight dependence of the energy density on the number of color sheets, which stops around this value of N_s . For numerical evidence of this fact, check, e.g., the red line in Fig. 4 of [46], in which we see convergence of the extracted value of Q_s at around $N_s \sim 50$. The value $N_s = 50$, therefore, represents a good compromise between the computational cost and the independence of observables from N_s at larger values. Moreover, such a value smears the statistical fluctuations due to the aforementioned random charge extraction.

These steps are shared by the charge generation of both a proton and a heavy ion. Indeed, the original MV model assumed a source charge density that was homogeneous in the transverse plane, which is feasible when dealing with large-A nuclei, of which we can neglect the details at the level of single nucleons. On the other hand, a reasonable description of protons involves a varying nuclear density in the transverse plane, in order to take it into account for the underlying quark structure. The main difference between those two cases lies in the choice of μ : in the A case μ is chosen as a constant equal to $\mu = 0.5$ GeV, which translates into uniform fluctuations of color charge throughout the lattice. The p case is instead more involved. Let us introduce the thickness function of the proton, T_p , as

$$T_p(\mathbf{x}_\perp) = \frac{1}{3} \sum_{i=1}^3 \frac{1}{2\pi B_q} \exp\left(-\frac{(\mathbf{x}_\perp - \mathbf{x}_i)^2}{2B_q}\right), \quad (5)$$

where the $\mathbf{x}_i$ denote the positions of the constituent quarks, which are randomly extracted from the distribution

$$T_{cq}(\mathbf{x}_\perp) = \frac{1}{2\pi B_{cq}} \exp\left(-\frac{\mathbf{x}_\perp^2}{2B_{cq}}\right). \quad (6)$$

Parameters have been chosen as $B_q = 0.3$ GeV⁻² and $B_{cq} = 4$ GeV⁻², i.e., the widths of the two Gaussians differ by a factor of around 4. After doing so, we evaluate the saturation scale Q_s in this model as

$$Q_s^2(x, \mathbf{x}_\perp) = \frac{2\pi^2}{N_c} \alpha_s \cdot xg(x, Q_0^2) \cdot T_p(\mathbf{x}_\perp), \quad (7)$$

from which $\mu(x, \mathbf{x}_\perp)$ of the proton is given by [47]

$$g^2 \mu(x, \mathbf{x}_\perp) = c Q_s(x, \mathbf{x}_\perp), \quad (8)$$

with $c = 1.25$. Note that by virtue of Eq. (7), not only μ but also Q_s depend on the transverse plane coordinates. Also, in principle, xg in (7) should be computed at the scale Q_s by means of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation with a proper initialization. This was done in [47,48], and we reserve this approach for future studies. For the sake of simplicity, in the present study we limit ourselves by assuming that xg is given by the initial condition at $Q_0^2 = 1.51$ GeV², as in [47,48], namely,

$$xg(x, Q_0^2) = A_g x^{-\lambda_g} (1-x)^{f_g}, \quad (9)$$

with $A_g = 2.308$, $\lambda_g = 0.058$, and $f_g = 5.6$. This simplification is partly justified by the fact that the average saturation scale for the proton is of the order of $Q_s \sim 1$ GeV, hence we do not expect the DGLAP evolution of the xg in Eq. (9) to lead to significant changes. For pA collisions at the LHC energy, the relevant values of x are in the range $[10^{-4}, 10^{-3}]$ [4,47]. In our calculations, we will thus fix x in the aforementioned range, then compute xg by virtue of Eq. (9). If not otherwise stated, we will use the corresponding value obtained for $x = 10^{-4}$, that is $xg = 3.94$.

B. Wilson lines and gauge links

Once the charge has been generated, we know that the hard and the soft sectors are coupled via the Yang-Mills equations [49]

$$D_\mu F^{\mu\nu} = J^\nu, \quad (10)$$

where $D_\mu = \partial_\mu - ig[A_\mu, \cdot]$ is the covariant derivative, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$ is the field strength tensor, and J^μ is the color current. Using the light cone coordinates, the conservation of current implies $A^- = 0$. Moreover, by working in the covariant gauge we can also impose $A^i = 0$, where $i = x, y$. The only component left is, therefore, $A^+ \equiv \alpha$, which can be seen to obey the following Poisson equation:

$$\Delta_\perp \alpha(\mathbf{x}_\perp) = -\rho(\mathbf{x}_\perp). \quad (11)$$

In order to solve this equation, we Fourier transform both sides of (11) in order to get

$$\tilde{k}_\perp^2 \tilde{\alpha}_{n,k}^a = \tilde{\rho}_{n,k}^a. \quad (12)$$

The Fourier transformations are easily performed using the fast Fourier transform package in Julia. We get

$$\tilde{\alpha}_{n,k}^a = \frac{\tilde{\rho}_{n,k}^a}{\tilde{k}_\perp^2 + m^2}, \quad (13)$$

where the discretized momentum $\tilde{k}_\perp$ is given by

$$\tilde{k}_\perp^2 = \sum_{i=x,y} \left(\frac{2}{a_\perp} \right)^2 \sin^2 \left(\frac{k_i a_\perp}{2} \right), \quad (14)$$

and we introduce an infrared regulator $m = 0.2 \text{ GeV} = 1 \text{ fm}^{-1}$ acting as a screening mass. By then, Fourier transforming back we get $\alpha(\mathbf{x}_\perp)$ in coordinate space.

Once we obtain a solution for the gauge field in the covariant gauge, the Wilson line for each nucleus is evaluated as

$$V_{\mathbf{x}} = \prod_{n=1}^{N_s} \exp\{ig\alpha_{n,\mathbf{x}}^a t^a\}, \quad (15)$$

and then we obtain the gauge links, U , for each of the two nuclei as

$$E_x = E_y = 0,$$

$$E_\eta = -\frac{i}{4ga_\perp^2} \sum_{i=x,y} [(U_i(\mathbf{x}_\perp) - \mathbb{I})(U_i^{B,\dagger}(\mathbf{x}_\perp) - U_i^{A,\dagger}(\mathbf{x}_\perp)) + (U_i^\dagger(\mathbf{x}_\perp - \hat{i}) - \mathbb{I})(U_i^B(\mathbf{x}_\perp - \hat{i}) - U_i^A(\mathbf{x}_\perp - \hat{i})) - \text{H.c.}]. \quad (18)$$

D. Time evolution

We now discuss the time evolution of the fields whose initialization has been explained in the previous subsection. The electric fields are expressed in terms of link variables as [22]

$$\partial_\tau U_i(\mathbf{x}) = \frac{-iga_\perp}{\tau} E^i(\mathbf{x}) U_i(\mathbf{x}), \quad (19)$$

$$\partial_\tau U_\eta(\mathbf{x}) = -iga_\eta \tau E^\eta(\mathbf{x}) U_\eta(\mathbf{x}). \quad (20)$$

The term a_η denotes the discretization step in the η direction. In order to reduce the discretization error in time, we drive the evolution through a leapfrog algorithm, i.e., by letting the gauge links and the electric fields evolve

$$U_{\mathbf{x},i}^{A,B} = V_{\mathbf{x}}^{A,B} V_{\mathbf{x}+\hat{i}}^{\dagger,A,B}. \quad (16)$$

In the above expressions, t^a are the SU(3) group generators, i.e., the eight Gell-Mann matrices divided by 2. If the point $\mathbf{x}_\perp$ belongs to the edge of the simulation lattice, periodic boundary conditions are implemented.

C. Initialization of the boost-invariant fields

At this point, we would like to derive the gauge link for the combined system of the two nuclei immediately after the collision. The gauge field will be the sum of each nucleus' contribution, but since QCD is a non-Abelian gauge theory, the resulting gauge link will not be the product of the gauge links of each nucleus. The total gauge link, $U_{\mathbf{x},i}$, at the position $\mathbf{x}$ in the transverse plane is determined by solving a set of eight equations, which are

$$\text{Tr}[t_a(U_{\mathbf{x},i}^A + U_{\mathbf{x},i}^B)(\mathbb{I} + U_{\mathbf{x},i}) - \text{H.c.}] = 0, \quad (17)$$

with $a = 1, \dots, N_c^2 - 1$. In the case of $N_c = 2$ there is an exact solution of (17) for $U_{\mathbf{x},i}$ [50]. Instead, for SU(3) there is no exact solution and we have to solve (17) via an iterative method. Using this procedure we calculate the gauge links along the x and y directions, whereas for the longitudinal components we initialize $U_\eta = \mathbb{I}$. We can also evaluate the color-electric fields, i.e., the canonical momenta with respect to the gauge links, at the initial time. The initial conditions on the lattice are implemented as [22]

in different steps alternatively. The time evolution for U_i and U_η is given by [22]

$$U_i(\tau'') = \exp[-2\Delta\tau \cdot iga_\perp E^i(\tau')/\tau'] U_i(\tau), \quad (21)$$

$$U_\eta(\tau'') = \exp[-2\Delta\tau \cdot iga_\eta \tau' E^\eta(\tau')] U_\eta(\tau), \quad (22)$$

where $\tau' = \tau + \Delta\tau/2$ and $\tau'' = \tau + \Delta\tau$. Notice that the exponentiation of the electric field is important, in order to keep the up-to-date gauge links as unitary matrices. In the same fashion, the equations of motion of the electric field are discretized as

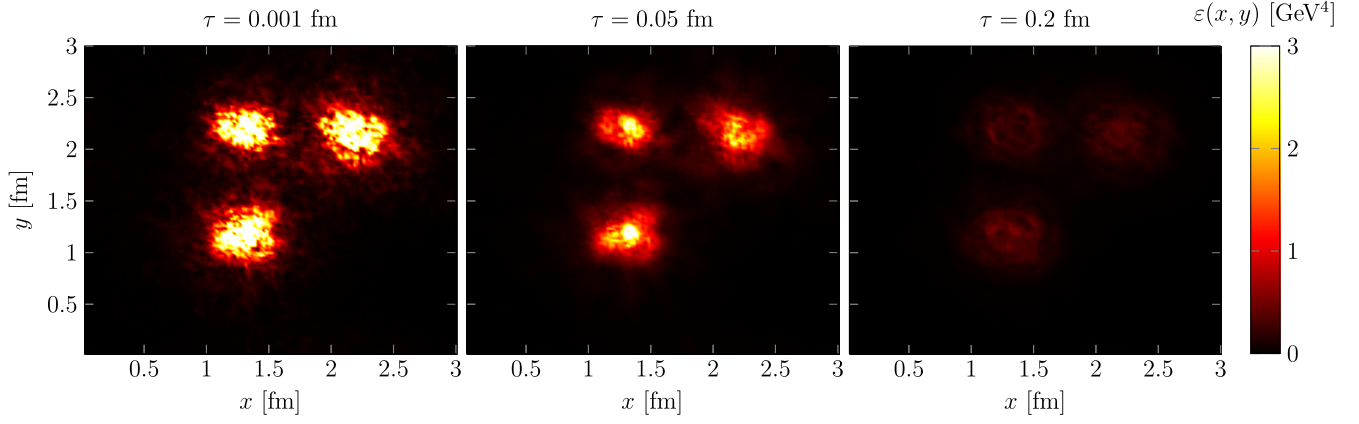


FIG. 1. Energy density (legend on the right side) produced in a pA collision in one event, represented in the transverse plane for three values of τ : the left panel corresponds to $\tau = 0.001$ fm, center panel to $\tau = 0.05$ fm, and right panel to $\tau = 0.2$ fm.

$$\begin{aligned}
 E^i(\tau') &= E^i(\tau - \Delta\tau) + 2\Delta\tau \frac{i}{2ga_\eta^2 a_\perp \tau} [U_{\eta i}(\tau) + U_{-\eta i}(\tau) - (\text{H.c.})] + 2\Delta\tau \frac{i\tau}{2ga_\perp^3} \sum_{j \neq i} [U_{ji}(\tau) + U_{-ji}(\tau) - (\text{H.c.})], \\
 E^\eta(\tau') &= E^\eta(\tau - \Delta\tau) + 2\Delta\tau \frac{i}{2ga_\eta a_\perp^2 \tau} \sum_{j=x,y} [U_{j\eta}(\tau) + U_{-j\eta}(\tau) - (\text{H.c.})],
 \end{aligned} \tag{23}$$

where

$$U_{\mu\nu}(\mathbf{x}) \equiv U_\mu(\mathbf{x}) U_\nu(\mathbf{x} + \hat{\mu}) U_\mu^\dagger(\mathbf{x} + \hat{\nu}) U_\nu^\dagger(\mathbf{x}) \tag{24}$$

are the plaquette variables. Notice that in this way, the gauge links are initialized at time $\tau = 0$ and evolve with integer steps: $\tau = \Delta\tau, 2\Delta\tau, \dots$. Instead, the electric field is initialized at $\tau = \Delta\tau/2$ and proceeds as $\tau = 3\Delta\tau/2, 5\Delta\tau/2, \dots$. The electric fields and gauge fields which we have dealt with so far fit into the definition of the longitudinal/transverse electric and magnetic components of the energy density. In particular, at time τ we have the electric components given by

$$\begin{aligned}
 E_L^2(\tau) &\equiv E^\eta(\tau) \cdot E^\eta(\tau), \\
 E_T^2(\tau) &\equiv \frac{1}{\tau^2} [E^x(\tau) \cdot E^x(\tau) + E^y(\tau) \cdot E^y(\tau)],
 \end{aligned} \tag{25}$$

whereas the magnetic components are given by [22]

$$\begin{aligned}
 B_L^2(\tau) &= \frac{2}{g^2 a_\perp^4} \text{Tr}[\mathbb{I} - U_{xy}(\tau)], \\
 B_T^2(\tau) &= \frac{2}{(ga_\eta a_\perp \tau)^2} \sum_{i=x,y} \text{Tr}[\mathbb{I} - U_{\eta i}(\tau)].
 \end{aligned} \tag{26}$$

The relations in (26) follow easily if one notices that $U_{\mu\nu} \simeq \exp\{-iga_\mu a_\nu F_{\mu\nu}\}$, where a_μ and a_ν denote the lattice spacings in the directions μ and ν (no summation over μ and ν is intended in the exponent).

The above quantities fit into the definition of ε [22],

$$\varepsilon = \langle \text{Tr}[E_L^2 + B_L^2 + E_T^2 + B_T^2] \rangle. \tag{27}$$

The angular brackets $\langle \rangle$ denote the ensemble average, numerically obtained by averaging over the events.

In Fig. 1 we plot the energy density for a pA collision at three different times. In the left panel, which represents the energy density right after the collision, we notice the energy density concentrated over the three lumps standing for the three constituent quarks in the proton. As time passes, we notice that the expansion in the transverse plane is negligible, while the energy density is diluted due to the longitudinal expansion of the system.

In the lower panel, we plot ε for $\tau = 0.15$ fm. We notice that the expansion in the transverse plane is negligible, while the energy density is diluted due to the longitudinal expansion of the system.

For completeness, in Fig. 2 we plot ε versus τ for both pA and AA collisions, averaged over 100 events and over the transverse plane. In order to perform the transverse plane average consistently in the pA case, we have used the thickness function of the proton $T_p(\mathbf{x}_\perp)$ as a weight, i.e., we plot $\langle \langle \text{Tr}[E_L^2 + B_L^2 + E_T^2 + B_T^2] \rangle \rangle$, where $\langle \langle A \rangle \rangle$ for any (transverse space-dependent) quantity A is defined as

$$\langle \langle A \rangle \rangle = \int d^2\mathbf{x}_\perp \langle A(\mathbf{x}_\perp) T_p(\mathbf{x}_\perp) \rangle. \tag{28}$$

In the above definition the function $T_p(\mathbf{x}_\perp)$, defined in Eq. (5), takes into account the initial distribution of the HQs

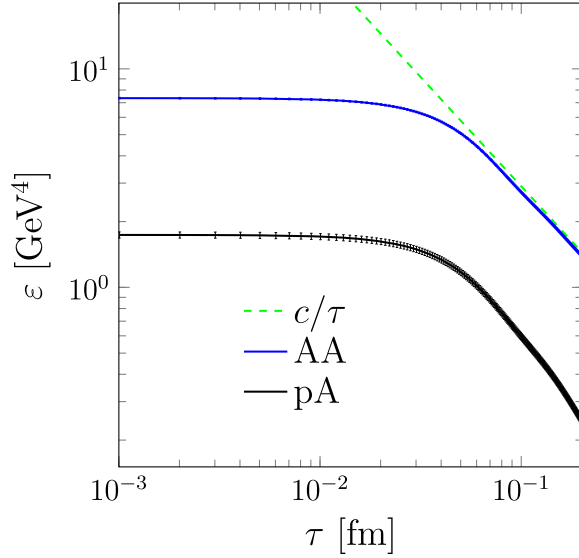


FIG. 2. Energy density ε , versus proper time, averaged over the transverse plane [for the pA case, the average over the transverse plane has been performed using $T_p(\mathbf{x}_\perp)$ as a weight, see the main text]. Error bars are also shown (for the AA case the error is smaller than the line width).

in the configuration space. Moreover, the average on the right-hand side of (28) is over the numerical events.

The results have been obtained for a square grid of transverse size $L = 2$ fm, with 32×32 lattice points. Along with the event averages, here and for all the plots that will follow, we also plot the statistical uncertainty as an error band, obtained as the standard deviation of the mean value: $\sigma_{\text{avg}} = \sigma / \sqrt{N_{\text{events}}}$. We notice that the free-streaming regime, $\varepsilon \sim 1/\tau$, sets in at $\tau \approx 0.1$ fm. For reference, the initial values of ε are, expressed in terms of energy over volume, around 240 GeV/fm³ for pA and 960 GeV/fm³ for AA.

III. QUARKS OF INFINITE MASS IN GLASMA

Next we turn to the diffusion of HQs in the evolving glasma fields. This problem has attracted a lot of interest

recently, see, for example, [16,51–62], as well as references therein. So far, this problem has been solved using a classical approximation in which the equations of motion for the HQs are the Wong equations [63,64]. In this work, we limit ourselves to analyzing the diffusion of heavy quarks in the very-large mass limit, similarly to what has been done in [44].

A. Momentum broadening

One of the quantities we want to study is the momentum broadening, δp_i^2 , of the HQs. It has already been studied, for instance, in [16,51–62]. We analyze this quantity in the present study, bringing the novelty of introducing longitudinal fluctuations. As a benchmark, we firstly show the results obtained without fluctuations.

This quantity is defined as

$$\delta p_i^2(\tau) \equiv p_i^2(\tau) - p_i^2(\tau_{\text{form}}), \quad i = x, y, z, \quad (29)$$

where τ_{form} is the formation time of the HQ and $p^i(\tau_{\text{form}})$ is the i th component of the momentum at the formation time. In order to compute the momentum broadening, we need to specify the initial distribution of the HQs in the configuration space. To this end, we assume that HQs are produced near the hot spots of the energy density (see Fig. 1). The idea behind this claim is that heavy quarks can only be produced in the regions with the highest energy density. This is justified since HQs are produced by hard scatterings, them being quite massive, so their formation rate must be higher in correspondence to a large number of binary collisions within the two colliding nuclei (here, p and A). Hence, it is reasonable to distribute the HQs according to the $T_p(\mathbf{x}_\perp)$ in Eq. (5). Following the same arguments presented in [15], from the Wong equations we can write the momentum broadening of each heavy quark, in the $M \rightarrow \infty$ limit, as

$$\delta p_L^2(\tau) = g^2 \int_0^\tau d\tau' \int_0^\tau d\tau'' \text{Tr}[E_z(\tau')E_z(\tau'')], \quad (30)$$

$$\delta p_T^2(\tau) = g^2 \int_0^\tau d\tau' \int_0^\tau d\tau'' \frac{1}{\tau'\tau''} \text{Tr}[E_x(\tau')E_x(\tau'') + E_y(\tau')E_y(\tau'')], \quad (31)$$

where we took the formation time of the infinitely massive HQs equal to zero, since $\tau_{\text{form}} = O(1/M)$ and $M \rightarrow \infty$. The integrands on the right-hand side of Eqs. (30) and (31) also depend on the transverse plane coordinates. Integrating over the whole transverse plane and ensemble-averaging Eqs. (30) and (31) results in

$$\langle\langle \delta p_L^2(\tau) \rangle\rangle = g^2 \int_0^\tau d\tau' \int_0^\tau d\tau'' \int d^2\mathbf{x}_\perp \langle \text{Tr}[E_z(\tau')E_z(\tau'')] T_p(\mathbf{x}_\perp) \rangle, \quad (32)$$

$$\langle\langle \delta p_T^2(\tau) \rangle\rangle = g^2 \int_0^\tau d\tau' \int_0^\tau d\tau'' \int d^2\mathbf{x}_\perp \frac{1}{\tau'\tau''} \langle \text{Tr}[E_x(\tau')E_x(\tau'') + E_y(\tau')E_y(\tau'')] T_p(\mathbf{x}_\perp) \rangle. \quad (33)$$

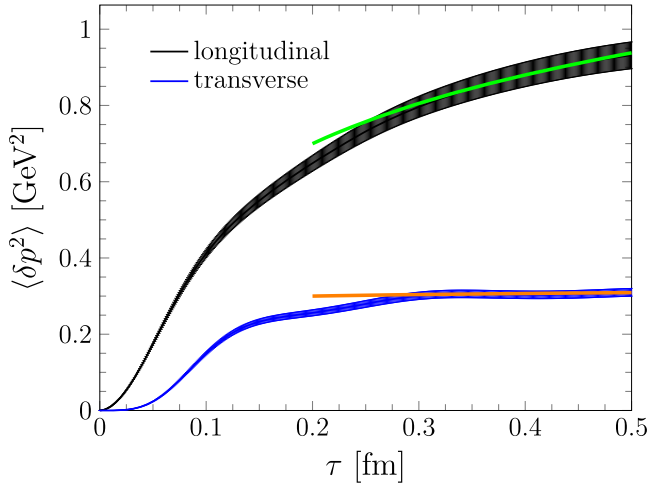


FIG. 3. Longitudinal momentum broadening $\langle \delta p_L^2(\tau) \rangle$ (black) and transverse momentum broadening $\langle \delta p_T^2(\tau) \rangle$ (blue) for a pA collision, along with the corresponding error bars. The green line corresponds to the function $a + b \log(\tau/\bar{\tau})$, where we put $\bar{\tau} = 0.4$ fm, $a = 0.88$ GeV², and $b = 0.26$ GeV². Similarly, the orange solid line represents the function $c + d \log(\tau/\bar{\tau})$, where we put $\bar{\tau} = 0.2$ fm, $c = 0.30$ GeV², and $d = 0.01$ GeV².

In writing Eqs. (32) and (33) we took advantage of the fact that the HQs are infinitely massive and, therefore, their distribution in the configuration space does not change in time.

Equations (32) and (33) are formally similar to what we would obtain in the case of a standard Langevin equation for a purely diffusive motion: $dp^i/dt = \xi^i$, where ξ^i denotes the random force. In fact, in this case we would obtain [56]

$$\langle \delta p_i^2 \rangle = \int_0^t dt_1 \int_0^t dt_2 \langle \xi_i(t_1) \xi_i(t_2) \rangle. \quad (34)$$

We can indeed notice the similarity among (34) and Eqs. (32) and (33) after realizing that the role of the random force in the context of HQs is taken by the force exerted by the color-electric field. Within Langevin dynamics, one can in principle include “memory effects,” which here are not considered, in order to account for coherent field-induced dynamics [65]. Moreover, one can also include radiation effects, but these have been shown to be basically irrelevant within the dynamics of heavy quarks in glasma [56]. We also notice that in the infinite mass limit, Eqs. (32) and (33) are exact, in the sense that the color-magnetic field gives no contribution in this limit, as its effect is proportional to the velocity of the quarks, which vanishes for $M \rightarrow \infty$. For $i = x, y$ the resulting equation for the momenta shift can be evaluated directly from the E_x and E_y fields. On the other hand, in principle we need to explicitly compute the E_z component of the color-electric field from the (τ, η) components. This can be easily achieved, as under the

change of coordinates from $x^{\mu'} = (\tau, \eta)$ to $x^\mu = (t, z)$ in Eq. (1) we have

$$E_z = F_{0z} = \frac{\partial x^{\mu'}}{\partial t} \frac{\partial x^{\nu'}}{\partial z} F_{\mu'\nu'} = F_{\tau\eta} = E_\eta. \quad (35)$$

Consequently, the knowledge of the η component of the field, which we easily extract from the Yang-Mills equations, is enough to compute the momentum spreading along the z direction.

In Fig. 3 we show that $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$ versus proper time calculations correspond to an average of over 100 events, for a transverse plane with 32×32 points and transverse size $L = 2$ fm. The evolution is shown up to $\tau = 0.5$ fm. We notice that the transverse momentum shift is substantially lower than the longitudinal one, and this is a consequence of the predominance of the longitudinal color-electric fields in the first instants after the collision. The anisotropy of the evolving glasma, which reflects onto the difference between the longitudinal and the transverse field correlators, is transmitted to the momentum broadenings. We find that, as a result of the diffusion of the HQs in the evolving glasma fields, both $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$ grow with proper time. After a short transient lasting up to $\tau \approx 0.3$ fm, the growth of $\langle \delta p_T^2(\tau) \rangle$ drastically slows down, while that of $\langle \delta p_L^2(\tau) \rangle$ remains substantial for the whole time range considered here. The values obtained can be compared, for example, with those found in [15] for AA collisions—our momenta shifts are smaller since we are dealing with pA collisions, in which the charge density (and hence the energy density) is smaller.

The late-time behavior of $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$ can be interpreted by assuming a δ -like correlator of the color-electric fields. For instance, let us assume that for E_z we have

$$\langle E_z(\tau') E_z(\tau'') \rangle = \frac{\langle E_z^2(\tau') \rangle}{Q_s} \delta(\tau' - \tau''), \quad (36)$$

where the overall $1/Q_s$ on the right-hand side of the above equation is added in order to balance the units of the δ function. The particular choice of Q_s in the equation is due to the fact that Q_s is the only energy scale in the MV model. Using the ansatz (36) in Eq. (32) we easily get

$$\langle \delta p_L^2(\tau) \rangle_{M \rightarrow \infty} \approx g^2 Q_s^{-1} \int_{\bar{\tau}}^{\tau} d\tau' \langle E_z^2(\tau') \rangle, \quad (37)$$

where we cut the time integral down to $\bar{\tau}$, which represents a value of the proper time of the order of $1/Q_s$. We introduce it since we are interested in the late-time behavior only. Indeed it is large enough that it allows us to use $\langle E_z^2(\tau') \rangle \propto Q_s^4/(Q_s \tau')$, in agreement with the late-time behavior of the energy density, see Fig. 2. We have also inserted the needed powers of Q_s in order to get the correct

dimension of energy to the fourth power.¹ We thus get

$$\langle \delta p_L^2(\tau) \rangle_{M \rightarrow \infty} \propto g^2 Q_s^2 \log(\tau/\bar{\tau}), \quad \text{for } \tau/\bar{\tau} > 1. \quad (38)$$

The late-time behavior (38) of $\langle \delta p_L^2(\tau) \rangle$ [and similar calculations hold for $\langle \delta p_T^2(\tau) \rangle$] is in fair agreement with the results of the full numerical simulations, see the orange and green lines in Fig. 3. Here, by “late time” we refer to timescales which are at the upper limit of the validity of the glasma framework, i.e., around $\tau \sim 0.4$ fm. As a final comment on the results shown in Fig. 3, we notice that we measure some tiny fluctuations of both $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$, similar to what was observed in [15]. The amplitude of these fluctuations is, however, very small in comparison with the bulk value of these quantities, thus they are practically irrelevant.

B. Angular momentum broadening

The momentum anisotropy is also transmitted to other observables of the HQs. In particular, one can look at the angular momentum anisotropy parameter, Δ_2 , defined as [59]

$$\Delta_2 \equiv \frac{\langle L_x^2 - L_z^2 \rangle}{\langle L_x^2 + L_z^2 \rangle}, \quad (39)$$

where L_i , with $i = x, y, z$, denotes the i th component of the orbital angular momentum of the HQ. Δ_2 , therefore, measures the anisotropy of the fluctuations of the orbital angular momentum of the HQs (the fluctuations of the HQ spin are suppressed by a power of M [59]). The notation makes clear that this quantity is basically a second moment of the HQ angular momentum distribution. It is also worth noting that Δ_2 receives contributions from both the anisotropy of the color fields and from the geometry of the system [59].

Such quantity can be rewritten in terms of the averages of the squared components of the momentum of the HQs. This can be done following the same steps given in [59] and assuming that momenta and positions of the HQs are not correlated with each other—this holds since we are working in the static limit for HQs. We have

$$\begin{aligned} \langle L_x^2 \rangle &= \langle (yp_z - zp_y)^2 \rangle \simeq \langle y^2 \rangle \langle p_z^2 \rangle + \langle z^2 \rangle \langle p_y^2 \rangle, \\ \langle L_z^2 \rangle &= \langle (xp_y - yp_x)^2 \rangle \simeq \langle x^2 \rangle \langle p_y^2 \rangle + \langle y^2 \rangle \langle p_x^2 \rangle. \end{aligned} \quad (40)$$

The above expressions allow us to write

¹For this argument we do not need to be precise and fix the overall coefficient, as we are only interested in extracting the late-time behavior of the momentum broadening.

$$\Delta_2 = \frac{[\langle z^2 \rangle - \langle x^2 \rangle - \langle y^2 \rangle] \langle p_T^2 \rangle / 2 + \langle y^2 \rangle \langle p_z^2 \rangle}{[\langle z^2 \rangle + \langle x^2 \rangle + \langle y^2 \rangle] \langle p_T^2 \rangle / 2 + \langle y^2 \rangle \langle p_z^2 \rangle}. \quad (41)$$

This expression can be further specialized to the case of a pA collision as follows. Firstly, we estimate the mean squared value of the longitudinal $\langle z^2 \rangle$ and of the transverse coordinates $\langle x^2 \rangle$ and $\langle y^2 \rangle$. As far as the extension along the beam axis z is concerned, we will assume that the HQs are generated uniformly along the whole η interval. This means that $\langle z^2 \rangle$ will be given by

$$\langle z^2 \rangle = \frac{\int_{-L_\eta/2}^{L_\eta/2} d\eta (\tau \sinh \eta)^2}{\int_{-L_\eta/2}^{L_\eta/2} d\eta} = \frac{1}{2} \tau^2 \left(\frac{\sinh L_\eta}{L_\eta} - 1 \right). \quad (42)$$

We fix the η extension as $L_\eta = 2$, corresponding to the rapidity range $\eta \in [-1, 1]$. For what concerns the transverse coordinates, firstly we assume that the geometrical distribution of the HQs in the transverse plane follows the same profile as the initial energy density, which in turn resembles that of the color charges of the colliding proton. This is a fair approximation considering that most HQs will be produced by hard QCD scatterings, where the density of the color charges is larger, as well as the fact that the transverse expansion is not so important in the pre-equilibrium stage, see, for example, Fig. 1. Within these reasonable approximations, we can estimate $\langle \mathbf{x}_\perp^2 \rangle = \langle x^2 + y^2 \rangle$. Firstly, we fix the location of the three constituent quarks, $\mathbf{x}_i$, and we average over the transverse coordinates using the thickness function $T_p(\mathbf{x}_\perp)$ in (5), getting

$$\langle \mathbf{x}_\perp^2 \rangle_{\mathbf{x}_i} = \frac{\int d^2 \mathbf{x}_\perp \mathbf{x}_\perp^2 T_p(\mathbf{x}_\perp)}{\int d^2 \mathbf{x}_\perp T_p(\mathbf{x}_\perp)} = 2B_q + \frac{1}{3} \sum_{i=1}^3 \mathbf{x}_i^2. \quad (43)$$

Then, an average over the locations of the constituent quarks, $\mathbf{x}_i$, is needed. We perform this using the distribution $T_{cq}(\mathbf{x}_i)$ in (6), resulting in

$$\langle \mathbf{x}_\perp^2 \rangle = \frac{\int d^2 \mathbf{x}_i \langle \mathbf{x}_\perp^2 \rangle_{\mathbf{x}_i} T_{cq}(\mathbf{x}_i)}{\int d^2 \mathbf{x}_i T_{cq}(\mathbf{x}_i)} = 2B_q + 2B_{cq}. \quad (44)$$

Hence,

$$\langle x^2 \rangle = \langle y^2 \rangle = \frac{1}{2} \langle \mathbf{x}_\perp^2 \rangle = B_q + B_{cq}. \quad (45)$$

Using the results of (42) and (45) in Eq. (41), along with the momenta shifts we previously got from (32) and (33), we can compute Δ_2 for a pA collision.

We have already mentioned that a nonzero Δ_2 may not only be due to the anisotropy of the fields (which is reflected on to a anisotropic momentum distribution) but entails a geometric contribution as well. The latter is related to the shape of the fireball and could be nonzero also in case

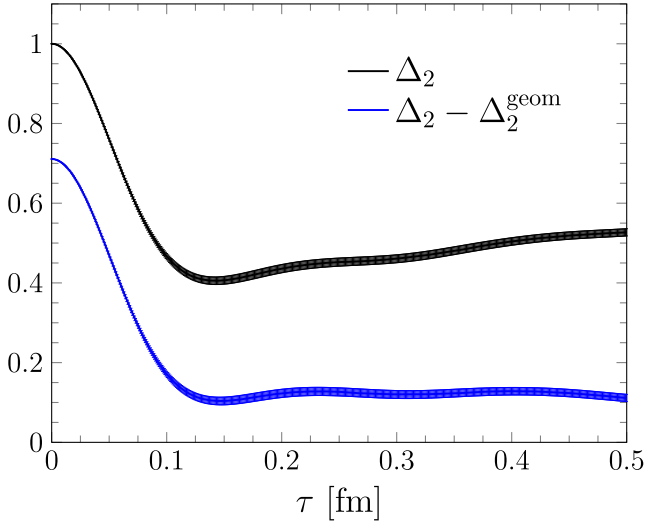


FIG. 4. Δ_2 (black) and $\Delta_2 - \Delta_2^{\text{geom}}$ (blue) versus proper time for pA collisions, along with the corresponding error bars.

momenta are isotropic. In order to extract the geometric contribution to Δ_2 , we start with Eq. (41), in which we evaluate $\langle p_z^2 \rangle$ as follows. Since we have

$$p_z = p_T \sinh y \simeq p_T \sinh \eta, \quad (46)$$

by assuming once again that the HQs are uniformly spread in η we have

$$\langle p_z^2 \rangle = \frac{\int_{-L_\eta/2}^{L_\eta/2} d\eta \langle p_T^2 \rangle (\sinh \eta)^2}{\int_{-L_\eta/2}^{L_\eta/2} d\eta} = \frac{1}{2} \langle p_T^2 \rangle \left(\frac{\sinh L_\eta}{L_\eta} - 1 \right). \quad (47)$$

This allows us to define the geometric component of Δ_2 , we call it Δ_2^{geom} , namely,

$$\begin{aligned} \Delta_2^{\text{geom}} &= \frac{\langle z^2 \rangle + \langle x^2 \rangle - \langle y^2 \rangle + \langle y^2 \rangle (\sinh L_\eta / L_\eta - 1)}{\langle z^2 \rangle + \langle x^2 \rangle + \langle y^2 \rangle + \langle y^2 \rangle (\sinh L_\eta / L_\eta - 1)} \\ &= \frac{[\frac{1}{2} \tau^2 + (B_q + B_{cq})] (\sinh L_\eta / L_\eta - 1)}{[\frac{1}{2} \tau^2 + (B_q + B_{cq})] (\sinh L_\eta / L_\eta - 1) + 2(B_q + B_{cq})}. \end{aligned} \quad (48)$$

In Fig. 4 we show our results for both Δ_2 and $\Delta_2 - \Delta_2^{\text{geom}}$. The calculations correspond to a pA collision for a 32×32 lattice and an average over 100 events. From Eq. (48) we get $\Delta_2^{\text{geom}}(\tau = 0) = (\sinh L_\eta / L_\eta - 1) / (\sinh L_\eta / L_\eta + 1) \simeq 0.289$ and, therefore, we have a geometric correction already at initial time. Notice also that $\Delta_2^{\text{geom}}(\tau = 0)$ is independent with respect to the widths of the proton charge, namely, B_q and B_{cq} . For times up to $\tau \sim 0.15$ fm, $\Delta_2^{\text{geom}}(\tau)$ varies with time but quite slowly

[see the τ^2 term in (48)], hence $\Delta_2^{\text{geom}}(\tau) \simeq \Delta_2^{\text{geom}}(\tau = 0)$, and the two curves in Fig. 4 differ basically by a constant. During this time we observe most of the damping for both curves. For even larger times we see that Δ_2 (black curve) shows an increasing behavior, but a look at $\Delta_2 - \Delta_2^{\text{geom}}$ on the same timescales shows that the increase in angular momentum anisotropy is likely due to an expansion of the system, mostly along the longitudinal direction. Indeed, by subtracting the geometric component we see that the pure anisotropy of the glasma fields (blue curve) remains constant and small over time.

IV. INCLUSION OF FIELD FLUCTUATIONS

As already explained in Sec. I, in relativistic nuclear collisions one clearly does not have exact boost invariance. In particular, the glasma is the solution of the Yang-Mills equations only in the case of weak coupling g . For realistic values of g it is likely that quantum fluctuations develop, with a strength that increases with g . For instance, in [24] it has been shown that increasing g leads to a larger contribution of fluctuations to the longitudinal pressures. Since a finite value of g is relevant for realistic collisions, it is meaningful to repeat the analysis presented in the previous section to the case in which we add fluctuations on top of the glasma. We construct the fluctuations by adding terms to the boost-invariant electric fields at initial time. These additions satisfy the Gauss law

$$D_i E_i + D_\eta E_\eta = 0. \quad (49)$$

Firstly, we evaluate random configurations $\xi_i(\mathbf{x}_\perp)$ ($i = x, y$) such that

$$\langle \xi_i(\mathbf{x}_\perp) \xi_j(\mathbf{y}_\perp) \rangle = \delta_{ij} \delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp), \quad (50)$$

which are operatively generated by sampling random Gaussian numbers with zero average and standard deviation $1/a_\perp$. After that, the additional terms to the glasma color-electric fields are evaluated as [21,22]

$$\delta E^i(\mathbf{x}_\perp, \eta) = -\partial_\eta F(\eta) \xi_i(\mathbf{x}_\perp), \quad (51)$$

$$\delta E^\eta(\mathbf{x}_\perp, \eta) = F(\eta) \sum_{i=x,y} D_i \xi_i(\mathbf{x}_\perp). \quad (52)$$

The information about the η distribution of the fluctuations is encoded in the function $F(\eta)$. By construction, these fluctuations satisfy the Gauss law constraint (49). The above relations are discretized on the lattice as [22]

$$\delta E^i(\mathbf{x}_\perp, \eta) = a_\eta^{-1} [F(\eta - a_\eta) - F(\eta)] \xi_i(\mathbf{x}_\perp), \quad (53)$$

$$\delta E^\eta(\mathbf{x}_\perp, \eta) = -a_\perp^{-1} F(\eta) \sum_{i=x,y} [\Xi^i(\mathbf{x}_\perp) - \xi^i(\mathbf{x}_\perp)], \quad (54)$$

where

$$\Xi^i(\mathbf{x}_\perp) = U_i^\dagger(\mathbf{x}_\perp - \hat{i}) \xi_i(\mathbf{x}_\perp - \hat{i}) U_i(\mathbf{x}_\perp - \hat{i}). \quad (55)$$

In the above equations, $a_\eta = L_\eta/N_\eta$ is the lattice spacing along the η direction. The term $\Xi^i(\mathbf{x}_\perp) - \xi^i(\mathbf{x}_\perp)$ is the covariant derivative of ξ^i , expressed in terms of the gauge links. Moreover, also in the longitudinal direction, periodic boundary conditions have been implemented for points at the edge of the η domain. As far as the choice of the rapidity-dependent term $F(\eta)$ is concerned, one may consider different approaches. For instance, in [21] the authors implemented a model in which $F(\eta)$ is a Gaussian random number with zero mean and standard deviation equal to 1. In this work, we instead follow a simpler model of fluctuations, similar to the one introduced in [22], in which $F(\eta)$ takes the form

$$F(\eta) = \frac{\Delta}{N_\perp} \sum_{n \in \mathcal{I}} \frac{1}{|\mathcal{I}|} \cos\left(\frac{2\pi n \eta}{L_\eta}\right), \quad (56)$$

where $\mathcal{I} \in \mathbb{N}$ denotes a set of positive integers with cardinality $|\mathcal{I}|$. The higher the order of the harmonics that contribute to $F(\eta)$ in (56), the higher the energy carried by the transverse fields of the fluctuations, see the η derivative in Eq. (51). Notice the presence of a free parameter, Δ , which encodes the strength of the fluctuating fields.

A. Momentum broadening fluctuations

In this subsection we present our results on the impact of the initial fluctuations on the momentum broadening of the HQs in the pre-equilibrium stage of pA collisions. The fluctuations are added on top of the boost-invariant glasma at $\tau \equiv \tau_0 = 0.05$ fm. We checked that the results are not significantly affected by changing τ_0 in the range [0.01, 0.05] fm. The plots which follow have been obtained in a $32 \times 32 \times 32$ lattice, as an average over 30 events, for different values of Δ .

In Fig. 5 we show, for the set of harmonics $\mathcal{I} = \{1, \dots, 10\}$ and several values of Δ , the ratio of longitudinal pressure over transverse pressure, P_L/P_T , versus proper time. These quantities are defined as [22]

$$P_L = \langle \text{Tr}[E_T^2 + B_T^2 - E_L^2 - B_L^2] \rangle, \quad (57)$$

$$P_T = \langle \text{Tr}[E_L^2 + B_L^2] \rangle, \quad (58)$$

where the ensemble average is performed as in (28). The main purpose of the results shown in Fig. 5 is to highlight

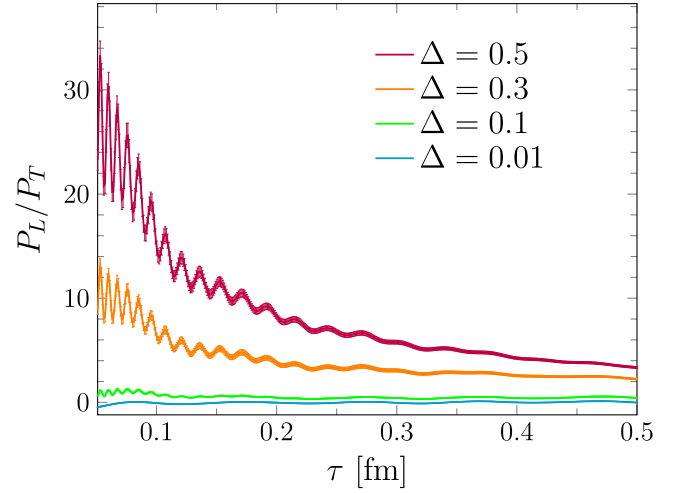


FIG. 5. The ratio of the longitudinal pressure over the transverse pressure P_L/P_T versus proper time for different values of Δ . Error bars are also shown.

the net effect of the fluctuations on the bulk gluon fields. Indeed, for the highest values of Δ we find a nontrivial behavior of P_L/P_T , which is quite different from the one we find for smaller values of Δ as well as for $\Delta = 0$ (the result for $\Delta = 0$ is not shown, since we checked that it is indistinguishable from the $\Delta = 0.01$ result). In particular, for large Δ , P_L/P_T stands significantly higher than the close-to-zero values obtained for small Δ . We remark that the largest value of Δ used in Fig. 5, that is $\Delta = 0.5$, is probably too large, as it corresponds to having a system whose energy is carried half by glasma and half by fluctuations. Nevertheless, it is shown to illustrate that fluctuations are actually introduced in the bulk and substantially modify it.

Next, we turn to the momentum broadening. In Fig. 6 we plot $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$ versus proper time. We show results for the same set of harmonics $\mathcal{I} = \{1, \dots, 10\}$ and for the same parameters Δ we have already considered in Fig. 5. We have split the datasets into two different plots, just for the sake of clarity. In addition to the averaging over the transverse plane [as in (28)], we average on the spacetime rapidity of the HQs, restricting ourselves to the rapidity range $[-1, 1]$.

We find no net effect of the rapidity fluctuations on the momenta shifts, since each result lies within the error bars of all the others, including the highest value $\Delta = 0.5$. Our conclusion is that the fluctuations do not substantially affect the HQ momenta shifts in the static limit. Our interpretation is that momentum broadening is related, in the static limit, to the time correlator of the color-electric fields, see Eqs. (32) and (33), and these correlators are not very much affected by the longitudinal fluctuations, which add a nontrivial dependence of the fields on η and mostly modify correlations along the η direction [66].

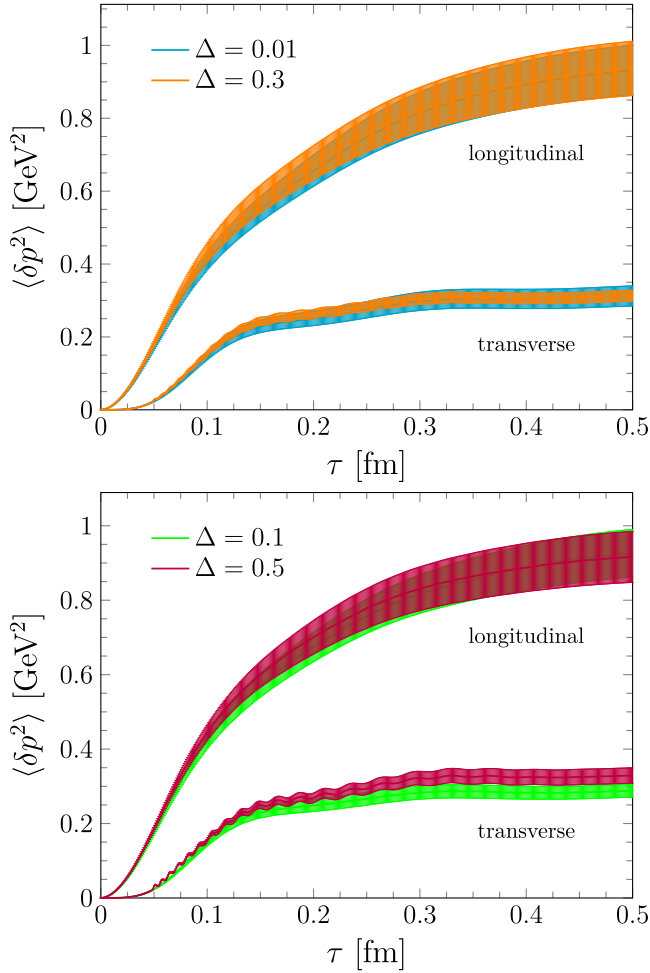


FIG. 6. The longitudinal and transverse momentum shifts, $\langle \delta p_L^2(\tau) \rangle$ and $\langle \delta p_T^2(\tau) \rangle$, versus proper time for different values of Δ . We also show the corresponding error bars. We used $\mathcal{I} = \{1, \dots, 10\}$ in Eq. (56). The four curves have been split into two plots for the sake of clarity.

V. CONCLUSIONS

We studied the diffusion of heavy quarks in the early stage of high-energy proton-nucleus collisions. Our initial condition is based on the glasma picture and includes event-by-event fluctuations of the color charges in the proton, as well as fluctuations that break the boost invariance. We performed 3 + 1D real-time statistical simulations of the SU(3) Yang-Mills fields produced immediately after the collision. Both the modeling of the proton and the longitudinal fluctuations for an expanding geometry in SU(3) have not been considered before in the literature in the context of heavy quarks. For simplicity, we limited ourselves to studying the large-mass limit of the heavy quarks. In this limit, the solution of the Wong equations for the momentum diffusion amounts to computing the time correlator of the color-electric fields. As a model of initial state field fluctuations, we adopted a simple superposition

of several harmonics [22], with a free parameter, Δ , that measures the strength of the fluctuating fields, see Eq. (56). The fact that the strength of the fluctuating fields scales as $\Delta/|\mathcal{I}|$ in Eq. (56) allows us to keep the initial energy density carried by the fluctuations unaffected by the inclusion of more harmonics.

We found that momentum broadening both in the longitudinal direction and in the transverse plane, $\langle \delta p_L^2 \rangle$ and $\langle \delta p_T^2 \rangle$, respectively, are unaffected (within error bars) by the inclusion of η -dependent fluctuations. Our interpretation of this result is that the fluctuations do not alter in a substantial way the time correlations of the color-electric fields, which are directly related to momentum broadening. Hence, our conclusion is that the initial state fluctuations, at least in the form introduced within our study, might be not sufficient to achieve isotropization of heavy quark momenta within the early stage. This occurs despite the fact that for the (illustrative) case of intense fluctuations considered in our work, the quantity P_L/P_T , which is a measure of the amount of anisotropy of the bulk gluon fields, may be an order of magnitude larger than 1.

Additionally, we computed the anisotropy of the angular momentum fluctuations, which we quantified by the coefficient Δ_2 introduced in [59]. In [59] the Δ_2 has been studied for a static geometry, within the gauge group SU(2) and without initial state fluctuations, hence the present work improves these several aspects with respect to [59]. We found that Δ_2 remains considerably large during the whole early stage. In the time range where the picture based on an evolving glasma is phenomenologically relevant, that is for τ in the range [0.2, 0.4] fm, Δ_2 remains in the range [0.4, 0.6], signaling a substantial amount of anisotropic angular momentum fluctuations. For larger times we can grasp a linear trend for Δ_2 , however, such behavior not only may not be accessible phenomenologically, but it is also likely due to the expansion of the system, rather than a genuine feature of the glasma fields.

The work we presented here can be improved in several ways. Certainly, the most important improvement would be the inclusion of dynamical heavy quarks, hence going beyond the large-mass limit. That should be accompanied by a proper initialization of the heavy quarks in momentum space, as already done in [67]. Moreover, it would be interesting to match a quark-gluon plasma stage after the glasma phase, the former being modeled, e.g., via relativistic transport theory. This would allow a more direct comparison with experimental observables. These improvements will be presented in the near future.

ACKNOWLEDGMENTS

M. R. acknowledges Francesco Acerbi and John Petrucci for inspiration. The authors acknowledge numerous discussions with Dana Avramescu, Bjorn Schenke, and Raju Venugopalan. This work has been partly funded by the European Union—Next Generation EU, through the

research Grant No. P2022Z4P4B “SOPHYA—Sustainable Optimised PHYsics Algorithms: Fundamental physics to build an advanced society,” under the program PRIN 2022 PNRR of the Italian Ministero dell’Università e Ricerca (MUR).

DATA AVAILABILITY

The data that support the findings of this article are openly available in [68].

-
- [1] U. W. Heinz and M. Jacob, Evidence for a new state of matter: An Assessment of the results from the CERN lead beam program, [arXiv:nucl-th/0002042](#).
 - [2] U. Heinz and R. Snellings, Collective flow and viscosity in relativistic heavy-ion collisions, *Annu. Rev. Nucl. Part. Sci.* **63**, 123 (2013).
 - [3] J. W. Harris and B. Muller, The search for the quark—gluon plasma, *Annu. Rev. Nucl. Part. Sci.* **46**, 71 (1996).
 - [4] B. Schenke, P. Tribedy, and R. Venugopalan, Event-by-event gluon multiplicity, energy density, and eccentricities in ultrarelativistic heavy-ion collisions, *Phys. Rev. C* **86**, 034908 (2012).
 - [5] B. Schenke, P. Tribedy, and R. Venugopalan, Fluctuating Glasma initial conditions and flow in heavy ion collisions, *Phys. Rev. Lett.* **108**, 252301 (2012).
 - [6] L. D. McLerran and R. Venugopalan, Computing quark and gluon distribution functions for very large nuclei, *Phys. Rev. D* **49**, 2233 (1994).
 - [7] L. D. McLerran and R. Venugopalan, Gluon distribution functions for very large nuclei at small transverse momentum, *Phys. Rev. D* **49**, 3352 (1994).
 - [8] L. D. McLerran and R. Venugopalan, Green’s functions in the color field of a large nucleus, *Phys. Rev. D* **50**, 2225 (1994).
 - [9] F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, The color glass condensate, *Annu. Rev. Nucl. Part. Sci.* **60**, 463 (2010).
 - [10] E. Iancu and R. Venugopalan, The color glass condensate and high-energy scattering in QCD, in *Quark-Gluon Plasma 4*, edited by R. C. Hwa and X.-N. Wang (World Scientific, Singapore, 2003), pp. 249–363.
 - [11] L. McLerran, A brief introduction to the color glass condensate and the glasma, in *Proceedings of the 38th International Symposium on Multiparticle Dynamics* (2009), pp. 3–18, <https://doi.org/10.48550/arXiv.0812.4989>.
 - [12] F. Gelis, Color glass condensate and glasma, *Int. J. Mod. Phys. A* **28**, 1330001 (2013).
 - [13] M. E. Carrington, A. Czajka, and S. Mrowczynski, The energy-momentum tensor at the earliest stage of relativistic heavy-ion collisions, *Eur. Phys. J. A* **58**, 5 (2022).
 - [14] M. E. Carrington, A. Czajka, and S. Mrówczyński, Physical characteristics of glasma from the earliest stage of relativistic heavy ion collisions, *Phys. Rev. C* **106**, 034904 (2022).
 - [15] D. Avramescu, V. Băran, V. Greco, A. Ipp, D. I. Müller, and M. Ruggieri, Simulating jets and heavy quarks in the glasma using the colored particle-in-cell method, *Phys. Rev. D* **107**, 114021 (2023).
 - [16] D. Avramescu, V. Greco, T. Lappi, H. Mäntysaari, and D. Müller, The impact of glasma on heavy flavor azimuthal correlations and spectra, *Phys. Rev. D* **111**, 074036 (2025).
 - [17] L. Pang, Q. Wang, and X.-N. Wang, Effect of longitudinal fluctuation in event-by-event (3 + 1)D hydrodynamics, *Nucl. Phys.* **A904–905**, 811c (2013).
 - [18] L.-G. Pang, G.-Y. Qin, V. Roy, X.-N. Wang, and G.-L. Ma, Longitudinal decorrelation of anisotropic flows in heavy-ion collisions at the CERN Large Hadron Collider, *Phys. Rev. C* **91**, 044904 (2015).
 - [19] K. Fukushima, Turbulent pattern formation and diffusion in the early-time dynamics in relativistic heavy-ion collisions, *Phys. Rev. C* **89**, 024907 (2014).
 - [20] P. Romatschke and R. Venugopalan, Collective non-Abelian instabilities in a melting color glass condensate, *Phys. Rev. Lett.* **96**, 062302 (2006).
 - [21] P. Romatschke and R. Venugopalan, The unstable glasma, *Phys. Rev. D* **74**, 045011 (2006).
 - [22] K. Fukushima and F. Gelis, The evolving glasma, *Nucl. Phys.* **A874**, 108 (2012).
 - [23] H. Iida, T. Kunihiro, A. Ohnishi, and T. T. Takahashi, Time evolution of gluon coherent state and its von Neumann entropy in heavy-ion collisions, [arXiv:1410.7309](#).
 - [24] T. Epelbaum and F. Gelis, Pressure isotropization in high energy heavy ion collisions, *Phys. Rev. Lett.* **111**, 232301 (2013).
 - [25] T. Epelbaum and F. Gelis, Fluctuations of the initial color fields in high energy heavy ion collisions, *Phys. Rev. D* **88**, 085015 (2013).
 - [26] R. Ryblewski and W. Florkowski, Equilibration of anisotropic quark-gluon plasma produced by decays of color flux tubes, *Phys. Rev. D* **88**, 034028 (2013).
 - [27] M. Ruggieri, A. Puglisi, L. Oliva, S. Plumari, F. Scardina, and V. Greco, Modelling early stages of relativistic heavy ion collisions: Coupling relativistic transport theory to decaying color-electric flux tubes, *Phys. Rev. C* **92**, 064904 (2015).
 - [28] N. Tanji and K. Itakura, Schwinger mechanism enhanced by the Nielsen–Olesen instability, *Phys. Lett. B* **713**, 117 (2012).
 - [29] J. Berges and S. Schlichting, The nonlinear glasma, *Phys. Rev. D* **87**, 014026 (2013).
 - [30] J. Berges, K. Boguslavski, S. Schlichting, and R. Venugopalan, Universal attractor in a highly occupied non-Abelian plasma, *Phys. Rev. D* **89**, 114007 (2014).
 - [31] J. Berges, K. Boguslavski, S. Schlichting, and R. Venugopalan, Basin of attraction for turbulent thermalization

- and the range of validity of classical-statistical simulations, *J. High Energy Phys.* **05** (2014) 054.
- [32] J. Berges, K. Boguslavski, S. Schlichting, and R. Venugopalan, Turbulent thermalization process in heavy-ion collisions at ultrarelativistic energies, *Phys. Rev. D* **89**, 074011 (2014).
- [33] S. Bazak and S. Mrowczynski, Stability of initial glasma fields, *Phys. Rev. C* **109**, 024903 (2024).
- [34] H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, and T. T. Takahashi, Entropy production from chaoticity in Yang-Mills field theory with use of the Husimi function, *Phys. Rev. D* **94**, 091502 (2016).
- [35] H. Tsukiji, T. Kunihiro, A. Ohnishi, and T. T. Takahashi, Entropy production and isotropization in Yang-Mills theory using a quantum distribution function, *Prog. Theor. Exp. Phys.* **2018**, 013D02 (2018).
- [36] H. Matsuda, T. Kunihiro, A. Ohnishi, and T. T. Takahashi, Entropy production in a longitudinally expanding Yang-Mills field with use of the Husimi function: semiclassical approximation, *Prog. Theor. Exp. Phys.* **2022**, 073D02 (2022).
- [37] K. Dusling, W. Li, and B. Schenke, Novel collective phenomena in high-energy proton-proton and proton-nucleus collisions, *Int. J. Mod. Phys. E* **25**, 1630002 (2016).
- [38] M. Greif, C. Greiner, B. Schenke, S. Schlichting, and Z. Xu, Importance of initial and final state effects for azimuthal correlations in p + Pb collisions, *Nucl. Phys. A* **982**, 491 (2019).
- [39] M. Greif, C. Greiner, B. Schenke, S. Schlichting, and Z. Xu, Importance of initial and final state effects for azimuthal correlations in p + Pb collisions, *Phys. Rev. D* **96**, 091504 (2017).
- [40] V. Nugara, V. Greco, and S. Plumari, Far-from-equilibrium attractors with full relativistic Boltzmann approach in 3 + 1D: Moments of distribution function and anisotropic flows v_n , *Eur. Phys. J. C* **85**, 311 (2025).
- [41] A. Kurkela, U. A. Wiedemann, and B. Wu, Opacity dependence of elliptic flow in kinetic theory, *Eur. Phys. J. C* **79**, 759 (2019).
- [42] A. Kurkela, U. A. Wiedemann, and B. Wu, Flow in AA and pA as an interplay of fluid-like and non-fluid like excitations, *Eur. Phys. J. C* **79**, 965 (2019).
- [43] X. Dong and V. Greco, Heavy quark production and properties of quark-gluon plasma, *Prog. Part. Nucl. Phys.* **104**, 97 (2019).
- [44] K. Boguslavski, A. Kurkela, T. Lappi, and J. Peuron, Heavy quark diffusion in an overoccupied gluon plasma, *J. High Energy Phys.* **09** (2020) 077.
- [45] L. Apolinário, J. G. Milhano, G. P. Salam, and C. A. Salgado, Probing the time structure of the quark-gluon plasma with top quarks, *Phys. Rev. Lett.* **120**, 232301 (2018).
- [46] T. Lappi, Wilson line correlator in the MV model: Relating the glasma to deep inelastic scattering, *Eur. Phys. J. C* **55**, 285 (2008).
- [47] B. Schenke, C. Shen, and P. Tribedy, Running the gamut of high energy nuclear collisions, *Phys. Rev. C* **102**, 044905 (2020).
- [48] A. H. Rezaeian, M. Siddikov, M. Van de Klundert, and R. Venugopalan, Analysis of combined HERA data in the impact-parameter dependent saturation model, *Phys. Rev. D* **87**, 034002 (2013).
- [49] C.-N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, *Phys. Rev.* **96**, 191 (1954).
- [50] A. Krasnitz and R. Venugopalan, Nonperturbative computation of gluon minijet production in nuclear collisions at very high-energies, *Nucl. Phys. B* **557**, 237 (1999).
- [51] M. Ruggieri and S. K. Das, Diffusion of charm and beauty in the Glasma, *EPJ Web Conf.* **192**, 00017 (2018).
- [52] M. Ruggieri and S. K. Das, Cathode tube effect: Heavy quarks probing the glasma in p -Pb collisions, *Phys. Rev. D* **98**, 094024 (2018).
- [53] Y. Sun, G. Coci, S. K. Das, S. Plumari, M. Ruggieri, and V. Greco, Impact of Glasma on heavy quark observables in nucleus-nucleus collisions at LHC, *Phys. Lett. B* **798**, 134933 (2019).
- [54] J. H. Liu, S. Plumari, S. K. Das, V. Greco, and M. Ruggieri, Diffusion of heavy quarks in the early stage of high-energy nuclear collisions at energies available at the BNL relativistic Heavy Ion Collider and at the CERN Large Hadron Collider, *Phys. Rev. C* **102**, 044902 (2020).
- [55] M. Y. Jamal, S. K. Das, and M. Ruggieri, Energy loss versus energy gain of heavy quarks in a hot medium, *Phys. Rev. D* **103**, 054030 (2021).
- [56] J.-H. Liu, S. K. Das, V. Greco, and M. Ruggieri, Ballistic diffusion of heavy quarks in the early stage of relativistic heavy ion collisions at RHIC and the LHC, *Phys. Rev. D* **103**, 034029 (2021).
- [57] Y. Sun, G. Coci, S. K. Das, S. Plumari, M. Ruggieri, and V. Greco, Impact of glasma on heavy quark R_{AA} and v_2 in nucleus-nucleus collisions at LHC, *Nucl. Phys. A* **1005**, 121913 (2021).
- [58] P. Khowal, S. K. Das, L. Oliva, and M. Ruggieri, Heavy quarks in the early stage of high energy nuclear collisions at RHIC and LHC: Brownian motion versus diffusion in the evolving glasma, *Eur. Phys. J. Plus* **137**, 307 (2022).
- [59] Pooja, S. K. Das, V. Greco, and M. Ruggieri, Anisotropic fluctuations of angular momentum of heavy quarks in the glasma, *Eur. Phys. J. Plus* **138**, 313 (2023).
- [60] Pooja, M. Y. Jamal, P. P. Bhaduri, M. Ruggieri, and S. K. Das, $c\bar{c}$ and $b\bar{b}$ suppression in glasma, *Phys. Rev. D* **110**, 094018 (2024).
- [61] Pooja, S. K. Das, and M. Ruggieri, Diffusion of heavy quarks in glasma and a plasma of gluons, *Springer Proc. Phys.* **304**, 1225 (2024).
- [62] D. Avramescu, V. Greco, T. Lappi, H. Mäntysaari, and D. Müller, Heavy flavor angular correlations as a direct probe of the glasma, *Phys. Rev. Lett.* **134**, 172301 (2025).
- [63] S. K. Wong, Field and particle equations for the classical Yang-Mills field and particles with isotopic spin, *Nuovo Cimento A* **65**, 689 (1970).
- [64] U. W. Heinz, Quark-Gluon transport theory. Part 1. The classical theory, *Ann. Phys. (N.Y.)* **161**, 48 (1985).

- [65] M. Ruggieri, Pooja, J. Prakash, and S. K. Das, Memory effects on energy loss and diffusion of heavy quarks in the quark-gluon plasma, *Phys. Rev. D* **106**, 034032 (2022).
- [66] M. Ruggieri, L. Oliva, G. X. Peng, and V. Greco, Evolution of pressures and correlations in the glasma produced in high energy nuclear collisions, *Phys. Rev. D* **97**, 076004 (2018).
- [67] L. Oliva, G. Parisi, and M. Ruggieri, Melting of $c\bar{c}$ and $b\bar{b}$ pairs in the pre-equilibrium stage of proton-nucleus collisions at the Large Hadron Collider, *Phys. Rev. D* **112**, 014008 (2025).
- [68] <https://drive.google.com/drive/folders/1CfGO-DNBwoLdzLiN8Tc-C9rHMMOIM5wi?usp=sharing>.