

Super Landau model and Howe duality: From supermonopole harmonics to quantum matrix geometry

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Landau models serve as quantum mechanical systems for generating quantum matrix geometries. In this paper, we demonstrate that Howe duality provides the underlying structure of the super Landau model, reflecting a general feature of coset-type Landau models. The (super) Howe duality relates different Landau levels and accounts for the emergence of a dual fuzzy geometry. By employing superspinor derivative operators, the supermonopole harmonics in both integer and half-integer Landau levels are explicitly constructed and the algebraic structure of the super-Hilbert space is revealed. We propose a consistent probabilistic interpretation for these wave functions defined on a supermanifold. Through a level projection method, we derive the matrix coordinates of fuzzy supersphere geometries for arbitrary Landau levels, along with a precise determination of the noncommutative scale factor. It is shown that the theta correspondence of Howe duality induces a geometric transformation between fuzzy objects. Finally, we point out that Howe duality realizes an internal-external space duality and underlies quantum matrix geometries, suggesting that it may play a fundamental role in understanding Matrix model geometries.

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I. INTRODUCTION

Landau models provide the quantum mechanical foundation of the quantum Hall effect (QHE) and a physical realization of noncommutative geometry (NCG). In the context of the QHE, NCG is not merely an underlying geometric framework but the central mechanism governing the dynamics of collective excitations [1–3]. Furthermore, NCG is essential for elucidating the geometric structures of D-branes and matrix theory [4,5]. Specifically, fuzzy geometries are realized as classical solutions of Matrix models [6–14] and play a fundamental role in the recent developments of emergent gravity [15–18]. It is also known that the nonanticommutative structure of fermionic coordinates referred to as nonanticommutative geometry (NACG) is induced in a graviphoton background [19–22]. The frameworks of NCG and NACG are naturally unified within the supersymmetric (SUSY) NCG formalism [23–26]. An archetypical example of such a manifold is the fuzzy supersphere [24,25,27], which realizes covariant bosonic and fermionic matrix coordinates in a graded Lie algebraic fashion. The fuzzy supersphere has been shown to arise as a classical solution of a supermatrix model [28].

More than two decades ago, with a collaborator the author introduced a supersymmetric generalization of the Landau model on a supersphere [29]. He also formulated the SUSY QHE [30] as a natural SUSY counterpart of Haldane’s spherical QHE [31]. Just as the fuzzy sphere geometry [32,33] appears in the lowest Landau level (LLL) [34,35], it was shown that the fuzzy supersphere geometry emerges in the super LLL [29]. The formulation of the SUSY Landau model was extended from a supersphere to a superplane [36–40] and a superhyperboloid [41]. A SUSY Chern-Simons field theory was also introduced as the effective field theory for the SUSY QHE [42]. Interestingly, the SUSY brings a unification of the Laughlin and Moore-Read wave functions in a single super Laughlin wave function [43]. Over the past two decades, the studies of the SUSY QHE have evolved in various ways. The first line of the development concerns the SUSY generalization of the valence bond solid state: The super Laughlin-Haldane wave function is transformed to the SUSY valence bond solid state by translating the external superspace as an internal superspace [44]. The SUSY valence bond solid state exhibits a hidden order [45] leading to the notion of SUSY protected topological phases [46] and SUSY matrix product states [47]. Recent developments along this line can be found in [48,49]. A more direct application of the SUSY QHE is about the analysis of the neutral fermionic excitation in the Moore-Read QH state. Moore-Read state possesses both QH and superconducting natures, and its bulk collective excitations are bosonic magneto-roton excitation and neutral fermion excitation

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regarded as mutual superpartners. Based on the SUSY formalism, the wave function of the neutral fermion excitation was proposed [50] and its validity was numerically confirmed [51]. A nonrelativistic supergeometry for the Moore-Read state has been discussed in [52]. It is also reported that the SUSY QHE is closely related to the M-theoretic origin of topological order [53]. Other significant results regarding the SUSY structure of the QHE can be found in the literature [54–59]. It is also worth mentioning the developments of superqubits [60–62]. Since both the superqubit and the SUSY QHE are founded on the graded Hopf map, their SUSY generalizations share many analogous properties.

In parallel, the NCG associated with Landau models has been extensively investigated over for more than two decades [63–76]. Based on the idea of utilizing the Landau models to generate quantum matrix geometries [9,77], fuzzy matrix geometries of two-, three-, and four-spheres have been revealed in arbitrary LLs [77–80]. This method has been successfully applied to derive new solutions for the Matrix model [9] and to construct deformed matrix geometries [81]. These results indeed demonstrate the effectiveness of the Landau models for generating matrix geometries and strongly suggest that Matrix model geometries are realized in Landau models. It is worthwhile to investigate whether this approach can be equally applicable to the derivation of supermatrix geometries. Introducing NCG in superspace is known to (partially) break the SUSY structure in general [22], however, fuzzy supermanifolds elude this unfavorable property. Importantly, the graded Lie algebraic construction of SUSY NCG preserves the full SUSY of the original supermanifold. For instance, as the supersphere possesses the $\mathcal{N} = 1$ SUSY, the fuzzy supersphere maintains the same $\mathcal{N} = 1$ SUSY. The preservation of (super)symmetry is a remarkable feature of fuzzy spaces. Although fuzzy spaces are regarded as “discretized” geometries, they retain the continuous symmetries of the original manifolds. This stands in stark contrast to lattice models, where continuous spatial symmetries are reduced to discrete lattice symmetries. This unique property is considered to be a key factor behind the recent development of the numerical simulations of 3D Ising criticality using fuzzy-sphere regularization [82,83]. Furthermore, very recent studies [84,85] suggest a close relationship between the fuzzy supersphere and the 3D SUSY Ising theory. The fuzzy supersphere and the fuzzy super Landau model are of increasing importance. However, an entire picture that encompasses higher super LLs has yet to be established.

In this paper, we investigate the super Landau model on a supersphere [29] to explore a comprehensive understanding of the super LL structures and associated matrix geometries. By incorporating higher LL structures, we unveil that the Howe duality constitutes the fundamental algebraic

structure of the system. We show that the supermonopole harmonics form an orthonormal basis and propose their consistent probabilistic interpretation by projecting them onto the body-sphere along with an appropriate definition of inner product. Using the newly defined inner product, the supermatrix coordinates are evaluated for arbitrary Landau levels. This presents the first explicit derivation of the supermatrix geometry from a Landau model. We also discuss the significance of Howe duality in the context of quantum matrix geometry. The theta correspondence is shown to induce a geometric transformation between fuzzy objects in the framework of NCG. Viewing Howe duality as a duality between internal and external spaces, we argue that it is a general feature of coset-type Landau models and may underlie the geometric structure of quantum geometries in Matrix models.

This paper is organized as follows. In Sec. II, we review the basics of the super Landau model. Superspinor derivative operators are constructed in Sec. III. In Sec. IV, we derive the supermonopole harmonics for integer LLs and explore their probabilistic interpretation. This analysis is extended in Sec. V for half-integer LLs, where we redefine the inner product to handle negative-norm states. The super D-matrix is introduced in Sec. VI, where the underlying super Howe duality of the system is revealed. We derive the supermatrix geometry in Sec. VII, providing a geometric realization of the theta correspondence as a transformation of fuzzy manifolds. Section VIII is devoted to a summary and discussions.

II. REVIEW OF THE SUPER LANDAU MODEL

We review the basic structure of the super Landau model on a supersphere based on [29], adding some more useful information. The correspondence between the original Landau model and its SUSY counterpart is summarized in Table I.

TABLE I. The basic mathematical setups for the original Landau model and the super Landau model.

	The original setup	The SUSY setup
Fiber-bundle	Principal fiber bundle	Principal superfiber bundle
Topological map	Hopf map	Graded Hopf map
Symmetry group	$SU(2)$	$UOSP(1 2)$
Gauge group	$U(1)$	$U(1)$
Base manifold	S^2	$S^{2 2}$
Magnetic source	Dirac Monopole	Supermonopole
Landau level eigenstates	Monopole harmonics	Supermonopole harmonics
Noncommutative geometry	Fuzzy sphere	Fuzzy supersphere

A. The graded Hopf map and supersphere

The graded Hopf map [86–88] is defined by the following mapping

$$S^{3|2} \xrightarrow{S^1} S^{2|2}. \quad (2.1)$$

As a coset, $S^{2|2}$ is expressed as

$$S^{2|2} \simeq S^{3|2}/S^1 \simeq UOSP(1|2)/U(1), \quad (2.2)$$

whose bosonic part realizes the original Hopf map, $S^2 \simeq S^3/S^1 \simeq SU(2)/U(1)$. With two complex Grassmann even quantities, u and v , and a complex Grassmann odd quantity η , arbitrary $UOSP(1|2)^1$ group element is represented as [89]

$$g = \begin{pmatrix} u & -v & \eta \\ v^* & u^* & \eta^* \\ -u\eta^* + v^*\eta & u^*\eta + v\eta^* & 1 - \eta^*\eta \end{pmatrix}. \quad (2.3)$$

The condition $\text{sdet}(g) = 1$ and $g^\dagger = g^{-1}$ imposes a constraint on the parameters as

$$u^*u + v^*v - \eta^*\eta = 1, \quad (2.5)$$

which geometrically signifies $UOSP(1|2) \simeq S^{3|2}$. With a three-component spinor ψ (the super Hopf spinor) subject to the normalization condition $\psi^\dagger\psi = \psi^t\psi^* = 1$, the graded Hopf map (2.1) is explicitly given by

$$\psi = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \longrightarrow \frac{x_i}{r} = 2\psi^t l_i \psi^*, \quad \frac{\theta_\alpha}{r} = 2\psi^t l_\alpha \psi^*, \quad (2.6)$$

where r signifies a constant parameter and

$$\psi^\dagger := (u^* \quad v^* \quad -\eta^*). \quad (2.7)$$

¹While the notation $OSP(1|2)$ is frequently used in place of $UOSP(1|2)$ in the literature, we strictly distinguish between these in the present paper for preciseness. The bosonic subgroup of the $UOSP(1|2)$ is the compact $SU(2)$ group, while that of the $OSP(1|2)$ is the noncompact $SU(1,1)$ group.

²See Appendix B for the definition of the superdeterminant and the super adjoint is given by

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}^\dagger := \begin{pmatrix} A^\dagger & -D^\dagger \\ C^\dagger & B^\dagger \end{pmatrix}. \quad (2.4)$$

Note the minus sign in front of η^* .³ The symbols l_i and l_α denote the $UOSP(1|2)$ superspin matrices

$$l_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad l_{\theta_1} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$l_{\theta_2} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}. \quad (2.10)$$

The $uosp(1|2)$ algebra is summarized in Appendix A. The supercoordinates (2.6), the bosonic coordinates x_i and the fermionic coordinates θ_α , automatically satisfy the condition for $S^{2|2}$

$$x_i x_i + C_{\alpha\beta} \theta_\alpha \theta_\beta = (\psi^t \psi^*)^2 = r^2, \quad (2.11)$$

where C denotes the charge conjugation matrix of the $SU(2)$

$$C = i\sigma_2. \quad (2.12)$$

In detail, the supercoordinates (2.6) are given by

$$\frac{x_1}{r} = uv^* + vu^*, \quad \frac{x_2}{r} = -iuv^* + ivu^*, \quad \frac{x_3}{r} = uu^* - vv^*,$$

$$\frac{\theta_1}{r} = v^*\eta + u\eta^*, \quad \frac{\theta_2}{r} = -u^*\eta + v\eta^*. \quad (2.13)$$

Note that the supercoordinates are constructed from the super Hopf spinor, suggesting that the Hopf spinor is more fundamental than the coordinates. The supersphere (2.11) is invariant under the $\mathcal{N} = 1$ SUSY transformation:

$$\delta_\xi x_i = \theta_\alpha (\sigma_i C)_{\alpha\beta} \xi_\beta, \quad \delta_\xi \theta_\alpha = x_i (\sigma_i)_{\beta\alpha} \xi_\beta, \quad (2.14)$$

where ξ_α are Grassmann odd pseudo-Majorana parameters, $\xi_\alpha^* = C_{\alpha\beta} \xi_\beta$.

³Here, $*$ denotes the pseudoconjugation which acts on the Grassmann odd quantities as

$$(\eta_1 \eta_2)^* = \eta_1^* \eta_2^*, \quad (\eta^*)^* = -\eta. \quad (2.8)$$

Then, θ_α (2.6) satisfy

$$\theta_\alpha^* = C_{\alpha\beta} \theta_\beta, \quad (2.9)$$

which implies that θ_α form an $SU(2)$ (pseudo-)Majorana spinor.

B. Super Hopf spinor and the supermonopole gauge field

The original Hopf spinor can be expressed as⁴

$$\phi(y_i) = \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \frac{1}{\sqrt{2r(r+y_3)}} \begin{pmatrix} r+y_3 \\ y_1-iy_2 \end{pmatrix} \cdot e^{i\chi}, \quad (2.19)$$

where y_i denote the ordinary c-number coordinates on S^2 :

$$y_i y_i = r^2. \quad (2.20)$$

The $U(1)$ phase $e^{i\chi}$ signifies the S^1 -fiber on S^2 . The corresponding gauge field is

$$\mathcal{A} = dy_i \mathcal{A}_i = 2ig\phi^\dagger d\phi = -g \frac{1}{r(r+y_3)} \epsilon_{ij3} y_j dy_i. \quad (2.21)$$

We took $e^{i\chi} = 1$ in (2.21). Similarly, the super Hopf spinor can be represented as

$$\begin{aligned} \psi(x_i, \theta_\alpha) &= \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \\ &= \frac{1}{\sqrt{2r^3(r+x_3)}} \begin{pmatrix} (r+x_3) \left(r - \frac{1}{4(r+x_3)} \theta C \theta \right) \\ (x_1 - ix_2) \left(r + \frac{1}{4(r+x_3)} \theta C \theta \right) \\ (x_1 - ix_2) \theta_1 - (r+x_3) \theta_2 \end{pmatrix} \cdot e^{i\chi}. \end{aligned} \quad (2.22)$$

The $U(1)$ phase factor $e^{i\chi}$ denotes the S^1 -fiber on $S^{2|2}$. Note that the x_i are Grassmann even quantities but *not* ordinary c-numbers, as x_i “contain” a Grassmann quantity $\theta C \theta$: indeed from (2.11),

⁴The Hopf spinor regular on the southern hemisphere can be taken as

$$\phi' = \frac{1}{\sqrt{2r(r-y_3)}} \begin{pmatrix} y_1 + iy_2 \\ r - y_3 \end{pmatrix}. \quad (2.15)$$

The corresponding gauge field is given by

$$\mathcal{A}' = dy_i \mathcal{A}'_i = 2ig\phi'^\dagger d\phi' = g \frac{1}{r(r-y_3)} \epsilon_{ij3} y_j dy_i. \quad (2.16)$$

The gauge transformation relating the $\mathcal{A}'$ and $\mathcal{A}$ is

$$\mathcal{A}' = \mathcal{A} - iu^* du, \quad (2.17)$$

where

$$u = e^{-2ig\varphi} \left(\tan(\varphi) := \frac{y_2}{y_1} \right). \quad (2.18)$$

$$x_i = y_i \sqrt{1 - \frac{1}{r^2} \theta C \theta}. \quad (2.23)$$

In the context of supermanifolds, the c -numbers y_i are called the coordinates of the body, whereas the θ_α are referred to as the soul coordinates. When the θ_α are dropped, the bosonic coordinates x_i reduce to the body coordinates y_i . Note that the η (2.22) is an $SU(2)$ singlet

$$\eta = \frac{1}{r} C_{\alpha\beta} \theta_\alpha \psi_\beta = \frac{1}{r} (v\theta_1 - u\theta_2), \quad (2.24)$$

and then

$$\eta^* \eta = -\frac{1}{2r^2} \theta C \theta. \quad (2.25)$$

The super Hopf spinor (2.22) is related to the original Hopf spinor (2.19) as

$$\psi = \left(1 - \frac{1}{4r^2} \theta C \theta \right) \begin{pmatrix} \mu \\ \nu \\ \frac{1}{r} (\nu\theta_1 - \mu\theta_2) \end{pmatrix}. \quad (2.26)$$

The pseudoconjugation of ψ is given by

$$\begin{aligned} \psi^* &= \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} \\ &= \frac{1}{\sqrt{2r^3(r+x_3)}} \begin{pmatrix} (r+x_3) \left(r - \frac{1}{4(r+x_3)} \theta C \theta \right) \\ (x_1 + ix_2) \left(r + \frac{1}{4(r+x_3)} \theta C \theta \right) \\ (r+x_3) \theta_1 + (x_1 + ix_2) \theta_2 \end{pmatrix} \cdot e^{-i\chi}. \end{aligned} \quad (2.27)$$

The supermonopole gauge field is derived as

$$A = 2ig\psi^\dagger d\psi = -2ig\psi^{*\dagger} d\psi^*, \quad (2.28)$$

where g signifies the magnetic charge of the supermonopole, $g = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$ [We took $e^{i\chi} = 1$ in (2.28).] In differential form, (2.28) is given by

$$A = dx_i A_i + d\theta_\alpha A_\alpha, \quad (2.29)$$

where

$$\begin{aligned} A_i &= -g \frac{1}{r+x_3} \epsilon_{ij3} x_j \left(1 + \frac{2r+x_3}{2r^2(r+x_3)} \theta C \theta \right), \\ A_\alpha &= -ig \frac{1}{r^3} (\theta \sigma_i C)_\alpha x_i. \end{aligned} \quad (2.30)$$

Note that the Dirac string singularity appears only in A_i , which is identical to the standard monopole gauge field⁵

$$dx_i A_i = dy_i \mathcal{A}_i. \quad (2.33)$$

Since the Dirac singularity reflects the nontrivial bundle topology, the “twist” of the bosonic part of the superfiber bundle (2.33) is accounted for by that of the original monopole bundle on the body-sphere. The gauge transformation between the supermonopole gauge fields on the norther and southern hemispheres is exactly the same as the original one (2.17):

$$A' = A - iu^* du, \quad (2.34)$$

where u denotes the transition function

$$\begin{aligned} u &:= e^{-2ig\varphi} = \left(\frac{y_1 - iy_2}{\sqrt{r(r - y_3)}} \right)^{2g} \\ &= \left(\frac{x_1 - ix_2}{\sqrt{r(r - x_3)}} \right)^{2g} \left(1 + \frac{g}{r^2 - x_3^2} \theta C \theta \right). \end{aligned} \quad (2.35)$$

We then have

$$\begin{aligned} -iu^* du &= \frac{2g}{r^2 - y_3^2} \epsilon_{ij3} y_j dy_i \\ &= \frac{2g}{r^2 - x_3^2} \epsilon_{ij3} x_j dx_i \left(1 + \frac{1}{r^2 - x_3^2} \theta C \theta \right). \end{aligned} \quad (2.36)$$

Note that the $-iu^* du$ does not include the fermion coordinate 1-form $d\theta$. The twist of the bundle is evaluated by the mapping from the equator to the transition function. Since the transition functions of the super and original Hopf maps coincide (2.35), the twist of the S^1 -fiber of the supermonopole can reasonably be identified with that of the original monopole. The winding number from the S^1 -equator to the transition function $u \in U(1)$ is calculated as

⁵In the southern hemisphere, the regular super Hopf spinor is given by

$$\psi' = \frac{1}{\sqrt{2r^3(r - x_3)}} \begin{pmatrix} (x_i + ix_2) \left(r + \frac{1}{4r(r+x_3)} \theta C \theta \right) \\ (r - x_3) \left(r - \frac{1}{4r(r+x_3)} \theta C \theta \right) \\ (r - x_3) \theta_1 - (x_i + ix_2) \theta_2 \end{pmatrix}. \quad (2.31)$$

The corresponding gauge field $A' = 2ig\psi'^{\dagger} d\psi' = dx_i A'_i + d\theta_\alpha A'_\alpha$ is

$$A'_i = g \frac{1}{r^3} \epsilon_{ij3} x_j \left(1 + \frac{2r - x_3}{2r^2(r - x_3)} \theta C \theta \right), \quad A'_\alpha = -ig \frac{1}{r^3} (\theta \sigma_i C)_\alpha x_i. \quad (2.32)$$

The bosonic part satisfies $dx_i A'_i = dy_i \mathcal{A}'_i$ and the fermionic gauge field A'_α is identical to A_α .

$$i \frac{1}{2\pi} \int_0^{2\pi} u^* \frac{\partial u}{\partial \chi} d\chi = 2g, \quad (2.37)$$

which is equal to the Chern number of the monopole:

$$c_1 = \frac{1}{2\pi} \int_{S^2} f = 2g, \quad (2.38)$$

where

$$f = g \frac{1}{r^3} \epsilon_{ijk} y_k dy_i \wedge dy_j. \quad (2.39)$$

The above discussion implies that the Chern number of the supermonopole is also identified as (2.38). From (2.30), the field strengths of the supermonopole are obtained as

$$F = dA = -\frac{1}{2} dx_j \wedge dx_i F_{ij} - d\theta_\alpha \wedge dx_i F_{i\alpha} - \frac{1}{2} d\theta_\beta \wedge d\theta_\alpha F_{\alpha\beta}, \quad (2.40)$$

where

$$\begin{aligned} F_{ij} &= -F_{ji} = \partial_i A_j - \partial_j A_i = g \frac{1}{r^3} \epsilon_{ijk} x_k \left(1 + \frac{3}{2r^2} \theta C \theta \right), \\ F_{i\alpha} &= -F_{\alpha i} = \partial_i A_\alpha - \partial_\alpha A_i = -ig \frac{1}{r^3} \left(\delta_{ij} - \frac{3}{r^2} x_i x_j \right) (\theta \sigma_j C)_\alpha, \\ F_{\alpha\beta} &= F_{\beta\alpha} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = -2ig \frac{1}{r^3} x_i (\sigma_i C)_{\alpha\beta} \left(1 + \frac{3}{2r^2} \theta C \theta \right). \end{aligned} \quad (2.41)$$

Throughout this paper, Grassmann derivatives such as ∂_α are understood to act from the left.

C. Super angular momentum operators

The covariant derivatives are introduced as

$$D_i = \partial_i + iA_i, \quad D_\alpha = \partial_\alpha + iA_\alpha, \quad (2.42)$$

and we construct the super angular momentum operators as

$$L_i = \Lambda_i + r^2 B_i, \quad L_\alpha = \Lambda_\alpha + r^2 B_\alpha. \quad (2.43)$$

Here, Λ_i and Λ_α denote the covariant super angular momentum operators

$$\begin{aligned} \Lambda_i &= -i\epsilon_{ijk} x_j D_k + \frac{1}{2} \theta_\alpha (\sigma_i)_{\alpha\beta} D_\beta, \\ \Lambda_\alpha &= \frac{1}{2} x_i (C \sigma_i)_{\alpha\beta} D_\beta - \frac{1}{2} \theta_\beta (\sigma_i)_{\beta\alpha} D_i, \end{aligned} \quad (2.44)$$

and B_i and B_α stand for the supermonopole magnetic field:

$$B_i = g \frac{1}{r^3} x_i, \quad B_\alpha = g \frac{1}{r^3} \theta_\alpha. \quad (2.45)$$

These form a supervector multiplet of the $UOSp(1|2)$ group, implying that B_i and B_α are superpartners.

The supermagnetic field (2.45) is radially distributed in $\mathbb{R}^{3|2}$, which indicates that the supermagnetic monopole is located at the origin. The magnitude of the super magnetic field is

$$B = \sqrt{B_i B_i + C_{\alpha\beta} B_\alpha B_\beta} = \frac{|g|}{r^2}, \quad (2.46)$$

and the magnetic charge of the supermonopole is evaluated from the (super) Gauss law:

$$\frac{\text{sgn}(g)}{2\pi} \int_{S^{2|2}} d\Omega_{2|2} B = \frac{\text{sgn}(g)}{2\pi} \int_{S^2} d\Omega_2 B = 2g, \quad (2.47)$$

where $d\Omega_{2|2}$ denotes the area element of $S^{2|2}$ (see Appendix B)

$$\int_{S^{2|2}} d\Omega_{2|2} = \int_{S^2} d\Omega_2 \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right). \quad (2.48)$$

As shown in (2.47), the integral over the super magnetic field on a supersphere is reduced to that of the magnetic field over the body-sphere, and the supermonopole charge is equal to the Chern number (2.38). This is consistent with the previous observation that the Chern number of the supermonopole is equal to that of the original monopole.

A bit of calculations shows that (2.43) satisfy the $uosp(1|2)$ algebra:

$$\begin{aligned} [L_i, L_j] &= i\epsilon_{ijk} L_k, & [L_i, L_\alpha] &= \frac{1}{2} (\sigma_i)_{\beta\alpha} L_\beta, \\ \{L_\alpha, L_\beta\} &= \frac{1}{2} (C\sigma_i)_{\alpha\beta} L_i. \end{aligned} \quad (2.49)$$

From the first relation, we see that L_i constitute the $su(2)$ algebra and generate $SU(2)$ rotations in the superspace $\mathbb{R}^{3|2}$. The second equation implies that the supercharges L_α transform as an $SU(2)$ spinor and carry angular momentum $1/2$. The last equation shows that L_α are fermionic operators, and the two successive operations of the fermionic generators yield the $SU(2)$ operations. The set of L_i and L_α amounts to the $UOSP(1|2)$ superrotations in $\mathbb{R}^{3|2}$. It is worth mentioning a distinction between the present model and standard SUSY field theories based on the super Poincaré symmetry. While in the latter, the bosonic and fermionic states with different $1/2$ spin are related by the super transformation, in the present model the *orbital* angular momentum, rather than intrinsic spin, is involved: the supercharges L_α relate bosonic and fermionic states whose orbital angular momenta differ by $1/2$. The super charges act as fermionic ladder operators of the angular momentum.

The super angular momentum operators (2.43) are expressed as

$$\begin{aligned} L_x &= L_x^0 + g \frac{x_1}{r + x_3} \left(1 + \frac{1}{2r(r + x_3)} \theta C \theta\right), \\ L_y &= L_y^0 + g \frac{x_2}{r + x_3} \left(1 + \frac{1}{2r(r + x_3)} \theta C \theta\right), \end{aligned} \quad (2.50a)$$

$$L_z = L_z^0 + g, \quad (2.50b)$$

$$\begin{aligned} L_{\theta_1} &= L_{\theta_1}^0 + \frac{g}{2r} \left(\theta_1 + \frac{x_1 + ix_2}{r + x_3} \theta_2\right), \\ L_{\theta_2} &= L_{\theta_2}^0 + \frac{g}{2r} \left(\theta_2 - \frac{x_1 - ix_2}{r + x_3} \theta_1\right), \end{aligned} \quad (2.50c)$$

where L_i^0 and L_α^0 denote the free super angular momentum operators:

$$\begin{aligned} L_i^0 &= -i\epsilon_{ijk} x_j \partial_k + \frac{1}{2} \theta_\alpha (\sigma_i)_{\alpha\beta} \partial_\beta, \\ L_\alpha^0 &= \frac{1}{2} x_i (C\sigma_i)_{\alpha\beta} \partial_\beta - \frac{1}{2} \theta_\beta (\sigma_i)_{\beta\alpha} \partial_i. \end{aligned} \quad (2.51)$$

The super rotations for the supercoordinates are represented as

$$\begin{aligned} [L_i, x_j] &= i\epsilon_{ijk} x_k, & [L_i, \theta_\alpha] &= \frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_\beta, \\ [L_\alpha, x_i] &= -\frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_\beta, & \{L_\alpha, \theta_\beta\} &= \frac{1}{2} (C\sigma_i)_{\alpha\beta} x_i. \end{aligned} \quad (2.52)$$

The last two equations demonstrate SUSY transformations of the supercoordinates. Obviously, the supersphere condition, $x_i x_i + C_{\alpha\beta} \theta_\alpha \theta_\beta = r^2$, is invariant under the $UOSP(1|2)$ SUSY transformation, implying that the supersphere respects the $\mathcal{N} = 1$ SUSY generated by two pseudo-Majorana supercharges, L_{θ_1} and L_{θ_2} .

D. Super Landau Hamiltonian and super Landau levels

Let us consider super quantum mechanics on a supersphere in the supermonopole background:

$$H = \frac{1}{2Mr^2} (\Lambda_i \Lambda_i + C_{\alpha\beta} \Lambda_\alpha \Lambda_\beta). \quad (2.53)$$

Using the orthogonality relation

$$x_i \Lambda_i + C_{\alpha\beta} \theta_\alpha \Lambda_\beta = \Lambda_i x_i + C_{\alpha\beta} \Lambda_\alpha \theta_\beta = 0, \quad (2.54)$$

the Hamiltonian (2.53) can be rewritten as

$$H = \frac{1}{2Mr^2} (L_i L_i + C_{\alpha\beta} L_\alpha L_\beta) - \frac{1}{2Mr^2} g^2, \quad (2.55)$$

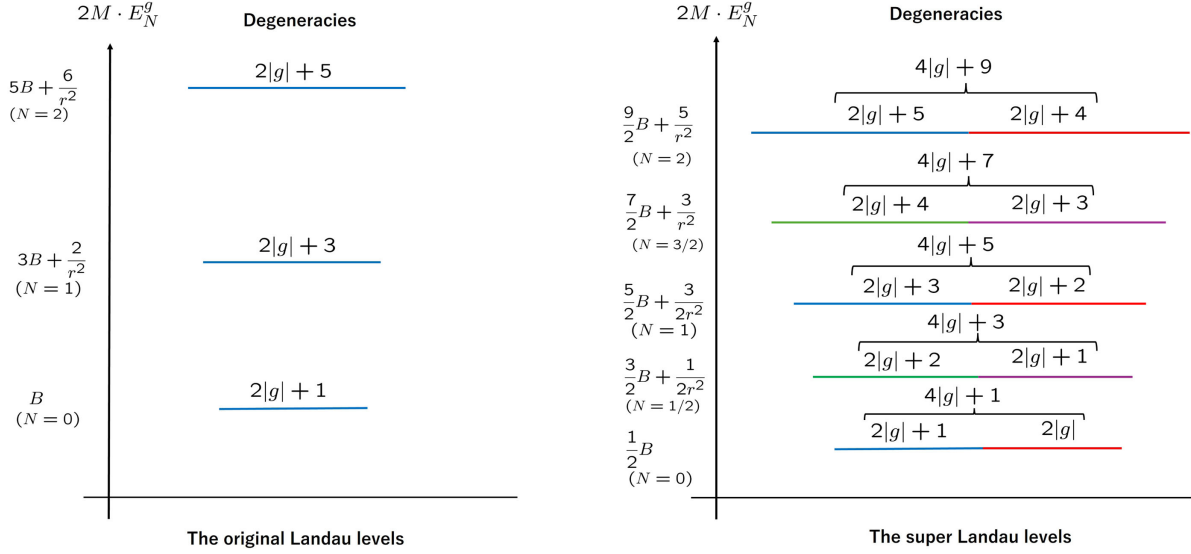


FIG. 1. Left: the original LLs (2.64). Right: the super Landau levels (2.59). Blue/red lines denote bosonic/fermionic states in integer Landau levels, while green/purple lines denote bosonic/fermionic states in half-integer Landau levels. Both of the LLs and the degeneracies are (almost) doubled in the SUSY system.

which is essentially the $UOSP(1|2)$ Casimir operator. Since the SUSY Hamiltonian is invariant under the transformations generated by L_i and L_α , the system possesses $UOSP(1|2)$ superrotational invariance. At the same time, L_i and L_α constitute conserved quantities, as they commute with the Hamiltonian. The standard results from $UOSP(1|2)$ representation theory (see Appendix A) determine the energy eigenvalues and their degeneracies:

$$\text{Casimir eigenvalues: } L_i L_i + C_{\alpha\beta} L_\alpha L_\beta = l \left(l + \frac{1}{2} \right), \quad (2.56a)$$

$$\text{Degeneracies: } 4l + 1. \quad (2.56b)$$

The superspin index l generally takes integer or half-integer values. In particular, for the present system, l is given by

$$l = N + |g| \quad (2.57)$$

where N denotes the super Landau level index

$$N = 0, 1/2, 1, 3/2, 2, \dots \quad (2.58)$$

The Landau levels include both integer and half-integer values due to SUSY. The energy eigenvalues are then given by

$$E_N^g = \frac{1}{2Mr^2} \left(N \left(N + \frac{1}{2} \right) + \left(2N + \frac{1}{2} \right) |g| \right), \quad (2.59)$$

with degeneracies

$$4|g| + 4N + 1 = 4|g| + 1, 4|g| + 3, 4|g| + 5, \dots \quad (2.60)$$

Under the $SU(2)$ group of $UOSP(1|2)$, the supermultiplet in the N th LL decomposes into two $SU(2)$ irreducible representations:

$$4|g| + 4N + 1 = (2|g| + 2N + 1) \oplus (2|g| + 2N), \quad (2.61)$$

where $(2|g| + 2N + 1)$ and $(2|g| + 2N)$ correspond to bosonic and fermionic states, respectively. The LL energies (2.59) can be expressed as the sum of contributions from the (super)spherical harmonics and the planar super Landau levels:

$$E_N^g = E_N^0 + \tilde{E}_N^B \quad (2.62)$$

with

$$E_N^0 = \frac{1}{2Mr^2} N \left(N + \frac{1}{2} \right), \quad (2.63a)$$

$$\tilde{E}_N^B = \left(N + \frac{1}{4} \right) \frac{B}{M}. \quad (2.63b)$$

The formulas (2.59) is quite similar to the LLs on Haldane's sphere [31],⁶ with the exception that the constants differ slightly and the SUSY Landau levels include half-integer values (see Fig. 1 also). In particular, the LLL energy is given by

$$E_{N=0}^g = \tilde{E}_{N=0}^B = \frac{B}{4M}, \quad (2.66)$$

which is half of the original LLL energy due to the cancellation between the bosonic and fermionic degrees of freedom.

III. SUPERSPINOR DERIVATIVE OPERATORS

Spinor derivative operators are quite useful for analyzing the Landau model on a two-sphere [31,73,77,90]. In this section, we introduce superspinor derivative operators that play the role of effective superangular momentum operators and edth operators. We denote the L_i and L_α (2.43) as L_A^g ($A = x_1, x_2, x_3, \theta_1, \theta_2$) with monopole charge g .

A. Super rotations and quantum numbers

Let us first discuss the superrotations of the super-Hopf spinor. Using Eqs. (2.22) and (2.50)

$$L_A^{1/2} \psi^t = \psi^t l_A, \quad (3.1)$$

while, for its complex conjugate (2.27),

$$L_A^{-1/2} \psi^{*t} = \psi^{*t} \tilde{l}_A. \quad (3.2)$$

Here, l_A and $\tilde{l}_A$ respectively denote the superspin matrices (2.10) and their complex representation matrices:

$$\tilde{l}_i = -l_i^*, \quad \tilde{l}_\alpha = C_{\alpha\beta} l_\beta, \quad (3.3)$$

which satisfy

$$[\tilde{l}_i, \tilde{l}_j] = i\epsilon_{ijk} \tilde{l}_k, \quad [\tilde{l}_i, \tilde{l}_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha} \tilde{l}_\beta, \quad \{\tilde{l}_\alpha, \tilde{l}_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta} \tilde{l}_i. \quad (3.4)$$

The super-Hopf spinor and its complex conjugate thus realize a $UOSp(1|2)$ $l = 1/2$ fundamental representation, respectively. These two representation matrices are related as

$$\tilde{l}_A = \mathcal{R}^t l_A \mathcal{R}, \quad (3.5)$$

⁶The original LLs on Haldane's sphere are

$$E_N^g = \frac{1}{2Mr^2} (N(N+1) + (2N+1)|g|), \quad (N=0,1,2,\dots) \quad (2.64)$$

with degeneracies

$$2l+1 = 2|g| + 2N+1 = 2|g| + 1, 2|g| + 3, 2|g| + 5, \dots \quad (2.65)$$

TABLE II. Quantum numbers of the components of the super Hopf spinor and the supercoordinates.

	Superspin l	Spin s	Magnetic quantum number m	Supermonopole charge g
u (u^*)	1/2	1/2	+1/2 (-1/2)	+1/2 (-1/2)
v (v^*)	1/2	1/2	-1/2 (+1/2)	+1/2 (-1/2)
η (η^*)	1/2	0	0 (0)	+1/2 (-1/2)
$x + iy$	1	1	1	0
x_3	1	1	0	0
$x - iy$	1	1	-1	0
θ_1	1	1/2	+1/2	0
θ_2	1	1/2	-1/2	0

where $\mathcal{R}$ signifies the $UOSp(1|2)$ charge conjugation matrix:

$$\mathcal{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (3.6)$$

which satisfies

$$\mathcal{R}^{-1} = \mathcal{R}^t. \quad (3.7)$$

The quantum numbers of the components of the super Hopf spinor (and its conjugate), along with the supercoordinates, are summarized in Table II.

B. Effective super angular momentum operators

An arbitrary LL is specified by the quantum numbers, (N, g) , or equivalently, (l, g) , while $\psi_a = u, v, \eta$ and $\psi_a^* = u^*, v^*, \eta^*$ carry the same superspin index of $l = 1/2$, and $g = 1/2$ and $g = -1/2$, respectively (Table II). Taking into account the matching of the quantum numbers, the N th LL eigenstates are given by (a linear combination of) the following homogeneous polynomials:

$$\Psi^{(n_1, n_2)} := \overbrace{\psi_{a_1} \cdots \psi_{a_{n_1}}}^{n_1} \overbrace{\psi_{b_1}^* \cdots \psi_{b_{n_2}}^*}^{n_2}, \quad (3.8)$$

where n_1 and n_2 are non-negative integers subject to⁷

$$n_1 + n_2 = 2l, \quad n_1 - n_2 = 2g. \quad (3.10)$$

Importantly, the action of L_A^g on these homogeneous functions is effectively represented by that of $L_A^{\pm 1/2}$ on each component⁸:

⁷As we will see in Sec. V, for half-integer LLs, the first relation of (3.10) is modified to

$$n_1 + n_2 = 2l + 1. \quad (3.9)$$

⁸While L_A , ψ_a , and ψ_b^* are assumed to be Grassmann-even, a formula analogous to (3.11) holds for the Grassmann-odd cases.

$$\begin{aligned}
L_A^g \Psi^{(n_1, n_2)} &= \sum_{i=1}^{n_1} \psi_{a_i} \cdots (L_A^{1/2} \psi_{a_i}) \cdots \psi_{a_{n_1}} \psi_{b_1}^* \cdots \psi_{b_{n_2}}^* \\
&+ \sum_{i=1}^{n_2} \psi_{a_1} \cdots \psi_{a_{n_1}} \psi_{b_i}^* \cdots (L_A^{-1/2} \psi_{b_i}^*) \cdots \psi_{b_{n_2}}^*.
\end{aligned} \quad (3.11)$$

Under the constraint (3.10), the action of L_A^g on the homogeneous function reduces to the one-particle action. From the results in Sec. III A, these effective operators are identified as

$$L_A = \psi^t l_A \frac{\partial}{\partial \psi} + \psi^{\dagger} \tilde{l}_A \frac{\partial}{\partial \psi^*}, \quad (3.12)$$

which obviously satisfy the $uosp(1|2)$ algebra. Their explicit forms are given by

$$\begin{aligned}
L_+ &= L_x + iL_y = u \frac{\partial}{\partial v} - v^* \frac{\partial}{\partial u^*}, \\
L_- &= L_x - iL_y = v \frac{\partial}{\partial u} - u^* \frac{\partial}{\partial v^*}, \\
L_z &= \frac{1}{2} \left(u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \right) - \frac{1}{2} \left(u^* \frac{\partial}{\partial u^*} - v^* \frac{\partial}{\partial v^*} \right), \\
L_{\theta_1} &= \frac{1}{2} \left(u \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial v} \right) + \frac{1}{2} \left(v^* \frac{\partial}{\partial \eta^*} - \eta^* \frac{\partial}{\partial u^*} \right), \\
L_{\theta_2} &= \frac{1}{2} \left(v \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial u} \right) - \frac{1}{2} \left(u^* \frac{\partial}{\partial \eta^*} + \eta^* \frac{\partial}{\partial v^*} \right).
\end{aligned} \quad (3.13)$$

When we introduce the complex representation

$$\tilde{\psi} := \mathcal{R}\psi^* = \begin{pmatrix} -v^* \\ u^* \\ -\eta^* \end{pmatrix}, \quad (3.14)$$

L_A (3.12) can also be represented as

$$L_A = \psi^t l_A \frac{\partial}{\partial \psi} + \tilde{\psi}^t \tilde{l}_A \frac{\partial}{\partial \tilde{\psi}}. \quad (3.15)$$

Since these are bilinear differential forms of the super Hopf spinor and its complex conjugate, the degrees n_1 and n_2 are conserved, which is consistent with the fact that g and l are fixed in each LL. In practical calculations associated with the LL eigenstates, the L_A (3.15) are far more efficient than the original coordinate-derivative operators (2.50).

We also introduce an operator L whose eigenvalues signify (the half of) the total degree, $n_1 + n_2$ ⁹:

$$L = \frac{1}{2} \psi^t \frac{\partial}{\partial \psi} + \frac{1}{2} \tilde{\psi}^t \frac{\partial}{\partial \tilde{\psi}} = \frac{1}{2} \psi^t \frac{\partial}{\partial \psi} + \frac{1}{2} \psi^{\dagger} \frac{\partial}{\partial \psi^*}, \quad (3.19)$$

which commutes L_A s:

$$[L, L_A] = 0. \quad (3.20)$$

As demonstrated in Secs. IV B 3 and V A 4, the eigenvalue of L coincides with the superspin index l (2.57) for integer LLs, while it equals $l + 1/2$ for half-integer LLs.

C. Effective edth operators

While the edth operators were originally introduced by Newman and Penrose for the analysis of gravitational radiation [91], these operators were also effective in the analysis of the Landau model [77]. Here, we extend this formalism to the SUSY case.

In contrast to the superangular momentum operators, which do not mix the super-Hopf spinor and its complex representation, the effective edth operators are constructed to interchange these representations:

$$\psi \leftrightarrow \tilde{\psi} = \mathcal{R}\psi^*. \quad (3.21)$$

This interchange is realized by¹⁰

$$R_- = \tilde{\psi}^t \frac{\partial}{\partial \psi} = \psi^{\dagger} \mathcal{R}^t \frac{\partial}{\partial \psi}, \quad R_+ = \psi^t \frac{\partial}{\partial \tilde{\psi}} = \psi^t \mathcal{R} \frac{\partial}{\partial \psi^*} \quad (3.23)$$

or, more explicitly,

⁹In particular in the LLL, L_A reduce to [30]

$$L_A = \psi^t l_A \frac{\partial}{\partial \psi}, \quad (3.16)$$

so the $UOSP(1|2)$ Casimir is expressed as

$$L_i L_i + C_{\alpha\beta} L_{\alpha} L_{\beta} = L \left(L + \frac{1}{2} \right), \quad (3.17)$$

with

$$L = \frac{1}{2} \psi^t \frac{\partial}{\partial \psi}. \quad (3.18)$$

The relation (3.17) does not hold, in general, for L_A (3.15) and L (3.19).

¹⁰ R_- and R_+ act as

$$R_- \psi^t = \tilde{\psi}^t, \quad R_+ \tilde{\psi}^t = \psi^t. \quad (3.22)$$

$$R_- = u^* \frac{\partial}{\partial v} - v^* \frac{\partial}{\partial u} - \eta^* \frac{\partial}{\partial \eta}, \quad R_+ = -u \frac{\partial}{\partial v^*} + v \frac{\partial}{\partial u^*} - \eta \frac{\partial}{\partial \eta^*}. \quad (3.24)$$

These provide a natural SUSY extension of the spinor-derivative operators for Haldane's sphere [73,90].¹¹ Since the super-Hopf spinor and its complex representation carry supermonopole charges of $1/2$ and $-1/2$, respectively, $R_\pm$ shift the supermonopole charge by $\Delta g = \pm 1$. The $R_\pm$ thus act as ladder operators for monopole charge. Note that they do not change the superspin l , as they are bilinear super-spinor differential operators. Since $l = N + |g|$ is fixed, the $R_\pm$ shifts the LL index N by ∓ 1 corresponding to the shift of g by ± 1 . Then, $R_\pm$ can also be viewed as lowering/raising operators for the Landau levels (that simultaneously shift the monopole charge).

The commutation relation between R_+ and R_- yields a new operator,

$$R_3 := \frac{1}{2}[R_+, R_-] = \frac{1}{2} \left(\psi^t \frac{\partial}{\partial \psi} - \tilde{\psi}^t \frac{\partial}{\partial \tilde{\psi}} \right) = \frac{1}{2} \left(\psi^t \frac{\partial}{\partial \psi} - \psi^\dagger \frac{\partial}{\partial \psi^*} \right). \quad (3.25)$$

Together with R_3 , R_i satisfy the $SU(2)$ algebra:

$$[R_i, R_j] = i\epsilon_{ijk} R_k, \quad (3.26)$$

where

$$\Phi_{l,m}^g(y_i) = (-1)^{m-g} \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!(l-m)!}{(l+g)!(l-g)!}} \left(\frac{r+y_3}{2r} \right)^{\frac{m+g}{2}} \left(\frac{r-y_3}{2r} \right)^{\frac{m-g}{2}} P_{l-m}^{(m-g, m+g)} \left(\frac{y_3}{r} \right) \left(\frac{y_1 + iy_2}{\sqrt{r^2 - y_3^2}} \right)^{m-g} \quad (g, m = l, l-1, l-2, \dots, -l), \quad (4.2)$$

where $P_{l+m}^{(\alpha, \beta)}$ denote the Jacobi polynomials:

$$P_N^{(\alpha, \beta)}(\cos \vartheta) = \sum_{n=0}^N (-1)^n \binom{N+\alpha}{n+\alpha} \binom{N+\beta}{n} \left(\cos \frac{\vartheta}{2} \right)^{2(N-n)} \left(\sin \frac{\vartheta}{2} \right)^{2n}. \quad (4.3)$$

¹¹To the best of the author's knowledge, the effective edth operators were first employed in [73], where they are realized as right actions of the D-matrix.

¹²We can "interchange" m and g in (4.2) as

$$\begin{aligned} \Phi_{l,m}^g(y_i) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l+g)!(l-g)!}{(l+m)!(l-m)!}} \left(\frac{r+y_3}{2r} \right)^{\frac{g+m}{2}} \left(\frac{r-y_3}{2r} \right)^{\frac{g-m}{2}} P_{l-g}^{(g-m, g+m)} \left(\frac{y_3}{r} \right) \left(\frac{y_1 + iy_2}{\sqrt{r^2 - y_3^2}} \right)^{m-g} \\ &= (-1)^{l-g} \sqrt{\frac{2l+1}{4\pi} \frac{(l+g)!(l-g)!}{(l+m)!(l-m)!}} \sum_{n=\max(0, -m-g)}^{\min(l-g, l-m)} (-1)^n \binom{l+m}{g+m+n} \binom{l-m}{n} \mu^{g+m+n} \nu^{l-m-n} \mu^{*n} \nu^{*l-g-n}. \end{aligned} \quad (4.1)$$

$$R_1 := \frac{1}{2}(R_+ + R_-), \quad R_2 := -i\frac{1}{2}(R_+ - R_-). \quad (3.27)$$

It is straightforward to verify that L_i commute with both L_A and L :

$$[R_i, L_A] = [R_i, L] = 0. \quad (3.28)$$

Although R_i and L_A are commutative, the $SU(2)$ transformations generated by R_i does not bring further degeneracies to Landau levels. Instead, $R_\pm$ act as ladder operators between Landau levels. In other words, L_A represent the symmetry group generators, while R_i are spectrum-generating operators. One may wonder whether there exist fermionic counterparts for R_i , analogous to L_α for L_i . We will revisit this question in Sec. VI along with the discussion of the super Howe duality.

IV. SUPERMONOPOLE HARMONICS IN THE INTEGER LANDAU LEVELS

In this section, we explicitly construct the supermonopole harmonics by using the superspinor derivative operators. We also discuss the probabilistic interpretation of these states.

A. The original monopole harmonics

We begin with a brief review of the original monopole harmonics. In the Dirac gauge, the monopole harmonics are given by [92]¹²:

Here, we parametrized y_i as

$$y_1 = r \sin \vartheta \cos \varphi, \quad y_2 = r \sin \vartheta \sin \varphi, \quad y_3 = r \cos \vartheta. \quad (4.4)$$

The monopole harmonics satisfy

$$\Phi_{l,m}^g(\vartheta, \varphi)^* = (-1)^{g-m} \Phi_{l,-m}^{-g}(\vartheta, \varphi) \quad (4.5)$$

and

$$|\Phi_{l,m}^g(y_i)| = |\Phi_{l,m}^g(y_3)| = |\Phi_{l,-m}^g(-y_3)|. \quad (4.6)$$

Their orthonormality relation on S^2 is expressed as

$$\int_{S^2} d\Omega_2 \Phi_{l',m'}^g(y_i)^* \Phi_{l,m}^g(y_i) = \delta_{l'l} \delta_{m'm} \quad (d\Omega_2 = \sin \vartheta d\vartheta d\varphi). \quad (4.7)$$

It is useful to express the monopole harmonics in terms of the Hopf spinor components (2.19):

$$\begin{aligned} \Phi_{l,m}^g(y_i) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!(l-m)!}{(l+g)!(l-g)!}} \sum_{n=\max(0,m-g)}^{\min(l-g,l+m)} (-1)^n \\ &\times \binom{l+g}{g-m+n} \binom{l-g}{n} \mu^{l+m-n} \nu^{g-m+n} \mu^{*l-g-n} \nu^{*n}. \end{aligned} \quad (4.8)$$

Low-dimensional examples are the components of the Hopf spinor:

$$\begin{aligned} \Phi_{1/2,1/2}^{1/2}(y_i) &= \frac{1}{\sqrt{2\pi}} \mu, \quad \Phi_{1/2,-1/2}^{1/2}(y_i) = \frac{1}{\sqrt{2\pi}} \nu, \\ \Phi_{1/2,1/2}^{-1/2}(y_i) &= -\frac{1}{\sqrt{2\pi}} \nu^*, \quad \Phi_{1/2,-1/2}^{-1/2}(y_i) = \frac{1}{\sqrt{2\pi}} \mu^*, \end{aligned} \quad (4.9)$$

and the vector coordinates:

$$\Phi_{1,\pm 1}^0(y_i) = \mp \sqrt{\frac{3}{8\pi}} \frac{1}{r} (y_1 \pm i y_2), \quad \Phi_{1,0}^0(y_i) = \sqrt{\frac{3}{4\pi}} \frac{1}{r} y_3. \quad (4.10)$$

B. Supermonopole harmonics

In the integer LL ($N = l - |g|$), there are $(4l+1)$ degenerate eigenstates consisting of

$$\Psi_{l,m}^g(x_i, \theta_\alpha) = \{Y_{l,m_1}^g(x_i, \theta_\alpha), \mathcal{Y}_{l,m_2}^g(x_i, \theta_\alpha)\}, \quad (4.11)$$

where

$$\begin{aligned} g &= \overbrace{l, l-1, l-2, \dots, -l}^{2l+1}, & m_1 &= \overbrace{l, l-1, l-2, \dots, -l}^{2l+1}, \\ m_2 &= \overbrace{l-1/2, l-3/2, l-5/2, \dots, -l+1/2}^{2l}. \end{aligned} \quad (4.12)$$

Here, $Y_{l,m_1}^g(x_i, \theta_\alpha)$ and $\mathcal{Y}_{l,m_2}^g(x_i, \theta_\alpha)$ represent the bosonic and fermionic sectors of the supermonopole harmonics, respectively.

1. Bosonic monopole harmonics

It is straightforward to extend the discussion in Sec. IV A to derive the bosonic sector of the supermonopole harmonics. The highest-weight state of the supermultiplet ($m = l$) may be given by

$$u^{l+g} v^{*l-g}. \quad (4.13)$$

We can easily confirm that (4.13) is indeed annihilated by the fermionic raising operator L_{θ_1} (3.13). By repeatedly applying the lowering operator L_- to (4.13), we can derive the remaining monopole harmonics of the bosonic sector. These are identical in form to the standard monopole harmonics, provided that the original Hopf spinor is replaced by the bosonic part of the super-Hopf spinor:

$$\begin{aligned} Y_{l,m}^g(x_i, \theta_\alpha) &= \sqrt{\frac{1}{4\pi} \frac{(l+m)!(l-m)!}{(l+g)!(l-g)!}} \sum_{n=\max(0,m-g)}^{\min(l-g,l+m)} (-1)^n \\ &\times \binom{l+g}{g-m+n} \binom{l-g}{n} u^{l+m-n} v^{g-m+n} u^{*l-g-n} v^{*n} \\ &(g, m = l, l-1, l-2, \dots, -l). \end{aligned} \quad (4.14)$$

Here, the normalization factor is chosen to match the integral over the supersphere (4.29a). Since $Y_{l,m}^g$ are homogeneous functions of the bosonic components of the super Hopf spinor, (4.14) can be factorized into a body part (involving y_i) and a soul part (involving θ_α). The supermonopole harmonics are then related to the original monopole harmonics as

$$Y_{l,m}^g(x_i, \theta_\alpha) = \frac{1}{\sqrt{2l+1}} \left(1 - \frac{l}{2r^2} \theta C \theta \right) \cdot \Phi_{l,m}^g(y_i). \quad (4.15)$$

From this equation, the properties of the bosonic supermonopole harmonics are readily deduced from those of the original monopole harmonics. For instance, the pseudo-complex conjugation is

$$Y_{l,m}^g(x_i, \theta_\alpha)^* = (-1)^{g-m} Y_{l,-m}^{-g}(x_i, \theta_\alpha). \quad (4.16)$$

2. Fermionic monopole harmonics

The fermionic monopole harmonics can be derived by applying the fermionic lowering operator L_{θ_2} to the bosonic monopole harmonics. According to (A9), a fermionic state is obtained from the bosonic state through the formula $|l, m\rangle_F = -\frac{2}{\sqrt{l+m+\frac{1}{2}}} L_{\theta_2} |l, m+\frac{1}{2}\rangle_B$. Substituting $|l, m\rangle_B = Y_{l,m}^g$ (4.15) and $L_{\theta_2} = \frac{1}{\sqrt{l+m+\frac{1}{2}}} (\eta \frac{\partial}{\partial u} + \eta^* \frac{\partial}{\partial v^*})$ (3.13) into this formula, we obtain

$$\mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \sqrt{\frac{1}{4\pi} \frac{(l+m-\frac{1}{2})!(l-m-\frac{1}{2})!}{(l+g)!(l-g)!}} \sum_{n=(0, m-g+\frac{1}{2})}^{\min(l-g, l+m+\frac{1}{2})} (-1)^n \binom{l+g}{g-m+n-\frac{1}{2}} \binom{l-g}{n} \cdot u^{l+m-n-\frac{1}{2}} v^{g-m+n-\frac{1}{2}} u^{*l-g-n} v^{*n-1} \left(\left(l-n+m+\frac{1}{2} \right) v^* \eta + n u \eta^* \right). \quad (4.17)$$

With the bosonic monopole harmonics, (4.17) takes a remarkably compact form

$$\mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \sqrt{l+g} Y_{l-\frac{1}{2},m}^{g-\frac{1}{2}}(x_i, \theta_\alpha) \cdot \eta - \sqrt{l-g} Y_{l-\frac{1}{2},m}^{g+\frac{1}{2}}(x_i, \theta_\alpha) \cdot \eta^* \quad (m = l-1/2, l-1, l-3/2, \dots, -l+1/2, g = l, l-1, l-2, \dots, -l), \quad (4.18)$$

which, in terms of the original monopole harmonics, is expressed as¹³

$$\mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \sqrt{\frac{l+g}{2l}} \Phi_{l-\frac{1}{2},m}^{g-\frac{1}{2}}(y_i) \cdot \eta - \sqrt{\frac{l-g}{2l}} \Phi_{l-\frac{1}{2},m}^{g+\frac{1}{2}}(y_i) \cdot \eta^*. \quad (4.20)$$

This concise expression is particularly efficient in deriving the supermatrix geometry in Sec. VII. The pseudocomplex conjugation is given by

$$\mathcal{Y}_{l,m}^g(x_i, \theta_\alpha)^* = (-1)^{g-m+\frac{1}{2}} \mathcal{Y}_{l,-m}^{-g}(x_i, \theta_\alpha), \quad (4.21)$$

In terms of the body and soul coordinates, (4.20) can also be represented as

$$\mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \mathcal{V}_{l,m-\frac{1}{2}}^g(y_i) \frac{\theta_1}{r} - \mathcal{U}_{l,m+\frac{1}{2}}^g(y_i) \frac{\theta_2}{r}, \quad (4.22)$$

where

$$\begin{aligned} \mathcal{U}_{l,m+\frac{1}{2}}^g(y_i) &:= \frac{g}{l} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \Phi_{l,m+\frac{1}{2}}^g(y_i) \\ &+ \frac{1}{l} \sqrt{\frac{(l+g)(l-g)(l-m-\frac{1}{2})}{2l-1}} \Phi_{l-1,m+\frac{1}{2}}^g(y_i), \end{aligned} \quad (4.23a)$$

¹³With the Clebsch-Gordan coefficients, (4.20) can be written as

$$\begin{aligned} \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) &= \langle l-\frac{1}{2}, g-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | l, g \rangle \Phi_{l-\frac{1}{2},m}^{g-\frac{1}{2}}(y_i) \cdot \eta \\ &- \langle l-\frac{1}{2}, g+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | l, g \rangle \Phi_{l-\frac{1}{2},m}^{g+\frac{1}{2}}(y_i) \cdot \eta^*. \end{aligned} \quad (4.19)$$

$$\begin{aligned} \mathcal{V}_{l,m-\frac{1}{2}}^g(y_i) &:= \frac{g}{l} \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \Phi_{l,m-\frac{1}{2}}^g(y_i) \\ &- \frac{1}{l} \sqrt{\frac{(l+g)(l-g)(l+m-\frac{1}{2})}{2l-1}} \Phi_{l-1,m-\frac{1}{2}}^g(y_i). \end{aligned} \quad (4.23b)$$

In the derivation of (4.22), we used (D9). The pseudoconjugations of $\mathcal{U}_{l,m+\frac{1}{2}}^g$ and $\mathcal{V}_{l,m-\frac{1}{2}}^g$ are respectively given by

$$\begin{aligned} \mathcal{U}_{l,m+\frac{1}{2}}^g(y_i)^* &= (-1)^{g-m+\frac{1}{2}} \mathcal{V}_{l,-m-\frac{1}{2}}^{-g}(y_i), \\ \mathcal{V}_{l,m-\frac{1}{2}}^g(y_i)^* &= -(-1)^{g-m+\frac{1}{2}} \mathcal{U}_{l,-m+\frac{1}{2}}^{-g}(y_i). \end{aligned} \quad (4.24)$$

3. Action of the operators

It is straightforward to show that the superspinor derivative operators act on the supermonopole harmonics as

$$R_\pm Y_{l,m}^g = \sqrt{(l \mp g)(l \pm g + 1)} Y_{l,m}^{g \pm 1}, \quad R_z Y_{l,m}^g = g Y_{l,m}^g, \quad (4.25a)$$

$$R_\pm \mathcal{Y}_{l,m}^g = \sqrt{(l \mp g)(l \pm g + 1)} \mathcal{Y}_{l,m}^{g \pm 1}, \quad R_z \mathcal{Y}_{l,m}^g = g \mathcal{Y}_{l,m}^g, \quad (4.25b)$$

$$L_\pm Y_{l,m}^g = \sqrt{(l \mp m)(l \pm m + 1)} Y_{l,m \pm 1}^g, \quad L_z Y_{l,m}^g = m Y_{l,m}^g, \quad (4.26a)$$

$$L_{\pm} \mathcal{Y}_{l,m}^g = \sqrt{\left(l \mp m - \frac{1}{2}\right) \left(l \pm m + \frac{1}{2}\right)} \mathcal{Y}_{l,m \pm 1}^g, \\ L_z \mathcal{Y}_{l,m}^g = m \mathcal{Y}_{l,m}^g, \quad (4.26b)$$

$$L_{\theta_1} Y_{l,m}^g = \frac{1}{2} \sqrt{l-m} \mathcal{Y}_{l,m+\frac{1}{2}}^g, \quad L_{\theta_2} Y_{l,m}^g = -\frac{1}{2} \sqrt{l+m} \mathcal{Y}_{l,m-\frac{1}{2}}^g, \quad (4.26c)$$

$$L_{\theta_1} \mathcal{Y}_{l,m}^g = \frac{1}{2} \sqrt{l+m} + \frac{1}{2} Y_{l,m+\frac{1}{2}}^g, \\ L_{\theta_2} \mathcal{Y}_{l,m}^g = \frac{1}{2} \sqrt{l-m} + \frac{1}{2} Y_{l,m-\frac{1}{2}}^g, \quad (4.26d)$$

and

$$L Y_{l,m}^g = l Y_{l,m}^g, \quad L \mathcal{Y}_{l,m}^g = l \mathcal{Y}_{l,m}^g. \quad (4.27)$$

Equation (4.26) coincides with (A9), indicating that the $\{Y_{l,m_1}^g, \mathcal{Y}_{l,m_2}^g\}$ form a $UOSP(1|2)$ irreducible representation.

C. Probabilistic interpretation

As the supermonopole harmonics are defined on a supermanifold, they depend on Grassmann variables, so their physical interpretation is far from clear. Here, we propose a probabilistic interpretation by projecting them onto the underlying body-sphere.

Using the Berezin integral on a supersphere (2.48), we can demonstrate that the supermonopole harmonics form an orthonormal basis¹⁴:

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l',m'}^g(x_i, \theta_\alpha) Y_{l,m}^g(x_i, \theta_\alpha) = \delta_{l'l} \delta_{m'm}, \quad (4.29a)$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l',m'}^g(x_i, \theta_\alpha) \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \delta_{l'l} \delta_{m'm}, \quad (4.29b)$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l',m'}^g(x_i, \theta_\alpha) Y_{l,m}^g(x_i, \theta_\alpha) = 0. \quad (4.29c)$$

Since the Grassmann quantities are the primary obstacles for the probabilistic interpretation, we project them out by mapping the absolute square of the super states onto the body-sphere:

$$\rho(y_i) := \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \Psi(x_i, \theta_\alpha) \Psi^*(x_i, \theta_\alpha). \quad (4.30)$$

Here, $\int d\theta_1 d\theta_2 (1 - \frac{1}{2} \theta C \theta)$ denotes the Grassmann part of the integral over a supersphere (2.48). With this definition, the probability densities for $Y_{l,m}^g$ are obtained as

$$\rho_{l,m}^g(y_i) = \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) Y_{l,m}^g(x_i, \theta_\alpha) Y_{l,m}^g(x_i, \theta_\alpha) \\ = |\Phi_{l,m}^g(y_i)|^2, \quad (4.31)$$

which are nothing but the probability densities for the original monopole harmonics. Meanwhile, the probability densities for $\mathcal{Y}_{l,m}^g$ are given by

$$\varrho_{l,m}^g(y_i) = \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) \\ = |\mathcal{U}_{l,m+\frac{1}{2}}^g(y_i)|^2 + |\mathcal{V}_{l,m-\frac{1}{2}}^g(y_i)|^2, \quad (4.32)$$

which represent the sum of the absolute squares of the amplitudes, $\mathcal{U}_{l,m+\frac{1}{2}}^g$ and $\mathcal{V}_{l,m-\frac{1}{2}}^g$, and are thereby manifestly positive semidefinite. From (4.20), we have another expression of (4.32)

$$\varrho_{l,m}^g(y_i) = \frac{l+g}{2l} \rho_{l-\frac{1}{2},m}^{g-\frac{1}{2}}(y_i) + \frac{l-g}{2l} \rho_{l-\frac{1}{2},m}^{g+\frac{1}{2}}(y_i). \quad (4.33)$$

Let us recall that $|\Phi_{l,m}^g(y_i)|$ depends only on y_3 (4.6), so the probability densities also depend solely on y_3 and are axially symmetric about the y_3 -axis. Furthermore, from (4.6), the probability densities satisfy the following property:

$$\rho_{l,-m}^g(y_3) = \rho_{l,m}^g(-y_3), \quad \varrho_{l,-m}^g(y_3) = \varrho_{l,m}^g(-y_3). \quad (4.34)$$

In addition from $|\Phi_{l,-m}^g(y_i)| = |\Phi_{l,m}^g(y_i)|$, it follows that

$$\rho_{l,-m}^{-g}(y_3) = \rho_{l,m}^g(y_3), \quad \varrho_{l,-m}^{-g}(y_3) = \varrho_{l,m}^g(y_3). \quad (4.35)$$

¹⁴For the integrals of Grassmann numbers,

$$\int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) Y_{l',m'}^g(x_i, \theta_\alpha) Y_{l,m}^g(x_i, \theta_\alpha) = \Phi_{l',m'}^g(y_i) \Phi_{l,m}^g(y_i), \\ \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Y}_{l',m'}^g(x_i, \theta_\alpha) \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = \frac{l+g}{2l} \Phi_{l'-\frac{1}{2},m'}^{g-\frac{1}{2}}(y_i) \Phi_{l-\frac{1}{2},m}^{g-\frac{1}{2}}(y_i) + \frac{l-g}{2l} \Phi_{l'-\frac{1}{2},m'}^{g+\frac{1}{2}}(y_i) \Phi_{l-\frac{1}{2},m}^{g+\frac{1}{2}}(y_i), \\ \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Y}_{l',m'}^g(x_i, \theta_\alpha) Y_{l,m}^g(x_i, \theta_\alpha) = 0. \quad (4.28)$$

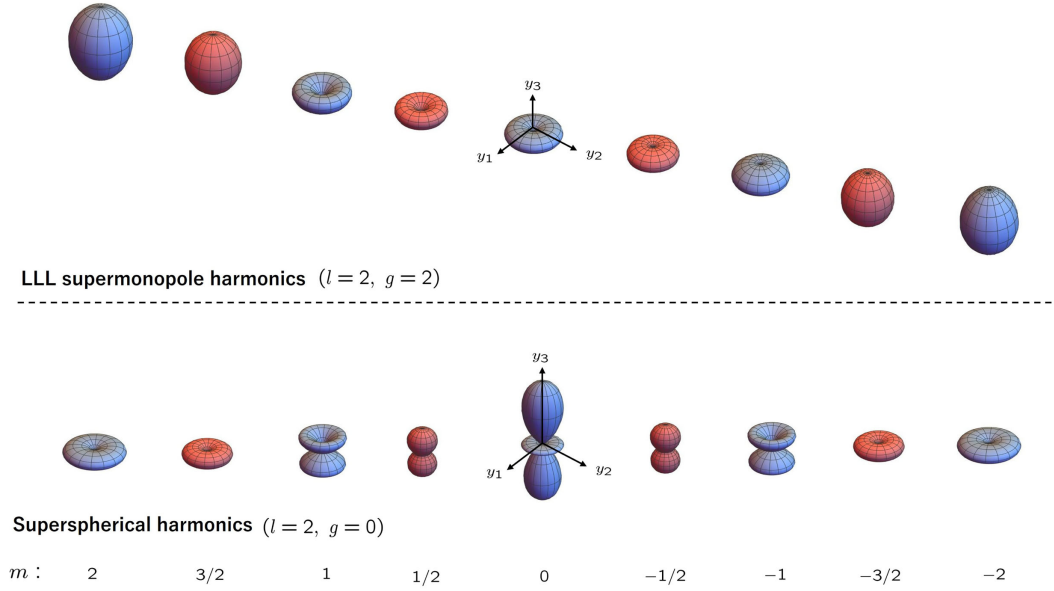


FIG. 2. The probability densities for the LLL supermonopole harmonics and the superspherical harmonics.

Namely, the probability densities are invariant under the simultaneous reversal of the signs of m and g . Physically, this implies that the probability density of the supermonopole harmonics with a given magnetic quantum number m in an antimonopole background is identical to that with the opposite quantum number $-m$ in a monopole background. For a better understanding, we depict the probability densities for the $l = 2$ LLL supermonopole harmonics ($g = l$) and the $l = 2$ superspherical harmonics ($g = 0$) in Fig. 2.¹⁵

V. SUPERMONOPOLE HARMONICS IN THE HALF-INTEGER LANDAU LEVELS

In this section, we derive the supermonopole harmonics in the half-integer LLs and discuss their probabilistic interpretation.

A. Supermonopole harmonics

We denote $(4l + 1)$ -fold degenerate supermonopole harmonics in half-integer LLs as

$$\Psi_{l,m}^g(x_i, \theta_\alpha) = \begin{pmatrix} Z_{l,m_1}^g(x_i, \theta_\alpha) \\ Z_{l,m_2}^g(x_i, \theta_\alpha) \end{pmatrix},$$

where Z_{l,m_1}^g and Z_{l,m_2}^g represent the bosonic and fermionic sectors, respectively. The ranges of the quantum numbers are given by

¹⁵See Appendix C for more about the LLL eigenstates and the superspherical harmonics.

$$\begin{aligned} g &= \overbrace{l - 1/2, l - 3/2, l - 5/2, \dots, -l + 1/2}^{2l}, \\ m_1 &= \overbrace{l, l - 1, l - 2, \dots, -l}^{2l+1}, \\ m_2 &= \overbrace{l - 1/2, l - 3/2, l - 5/2, \dots, -l + 1/2}^{2l}. \end{aligned} \quad (5.1)$$

Since the LL index $N (= l - |g|)$ is a half-integer, the difference between g and l must also be a half-integer. Consequently, the monopole charge g takes $2l$ distinct values, in contrast to the $2l + 1$ values in the integer LL case.

1. Highest weight states of the half-integer Landau levels

Let us first construct the highest weight states for the $1/2$ th Landau level from those of the $N = 0$ and $N = 1$ LLs with the same superspin $l (= g > 0)$,

$$Y_{g,g}^g(x_i, \theta_\alpha) = \frac{1}{\sqrt{4\pi}} u^{2g}, \quad Y_{g,g}^{g-1}(x_i, \theta_\alpha) = -\sqrt{\frac{2g}{4\pi}} u^{2g-1} v^*. \quad (5.2)$$

We then introduce the following Grassmann odd state:

$$Z_{l',m'}^g \propto Y_{g,g}^g(x_i, \theta_\alpha) \eta^* + \alpha Y_{g,g}^{g-1}(x_i, \theta_\alpha) \eta, \quad (5.3)$$

where α is a constant to be determined. Let us recall that η and η^* are $SU(2)$ singlets, while $Y_{g,g}^g$ and $Y_{g,g}^{g-1}$ are $SU(2)$ highest weight states with spin $z_s = g$; thereby, $Z_{l',m'}^g$ is also an $SU(2)$ highest weight state (satisfying $L_+ Z_{l',m'}^g = 0$)

with spin $s = g$. By considering the matching of the quantum numbers between the left- and right-hand sides of (5.3), we can identify the quantum numbers of the $Z_{l',m'}^{g'}$ as¹⁶

$$g' = g - \frac{1}{2}, \quad m' = g. \quad (5.4)$$

The coefficient α is determined by requiring that $Z_{l',m'}^{g'}$ be the highest weight state of a supermultiplet:

$$L_{\theta_1} Z_{g,g}^{g-1/2} = 0. \quad (5.5)$$

This yields $\alpha = \frac{1}{\sqrt{2g}}$, and so

$$\begin{aligned} Z_{l',g}^{g-1/2} &\propto Y_{g,g}^g(x_i, \theta_\alpha) \eta^* + \frac{1}{\sqrt{2g}} Y_{g,g}^{g-1}(x, \theta_\alpha) \eta \\ &= \frac{1}{\sqrt{4\pi}} u^{2g} \eta^* - \frac{1}{\sqrt{4\pi}} u^{2g-1} v^* \eta. \end{aligned} \quad (5.6)$$

Now $Z_{l',g}^{g-1/2}$ is promoted to the highest weight state of the $UOSP(1|2)$ supermultiplet with superspin

$$l' = g. \quad (5.7)$$

The LL to which (5.6) belongs is then determined as

$$N = l' - g' = 1/2. \quad (5.8)$$

Other eigenstates in the $1/2$ th LL can be obtained by applying L_{θ_2} (and L_-) to (5.6).

Furthermore, by acting R_- on $Z_{g,g}^{g-1/2}$, we can construct other highest weight states with the same superspin $l = g$ but with different monopole charges:

$$Z_{g,g}^{g-n-1/2} \propto R_-^n Z_{g,g}^{g-1/2}. \quad (5.9)$$

Since L s and R s commute, $Z_{g,g}^{g-n-1/2}$ obviously satisfies the highest weight state conditions, $L_{\theta_1} Z_{g,g}^{g-n-1/2} = 0$. Therefore, $Z_{g,g}^{g-n-1/2}$ realize the highest weight states of the following half-integer LLs:

$$N = g - (g - n - 1/2) = n + 1/2. \quad (5.10)$$

As before, by applying L_{θ_2} (and L_-) to (5.9), we can explicitly derive the degenerate eigenstates for the half-integer LLs (5.10).

The half-integer LL eigenstates are represented by homogeneous functions of degree $2l + 1$ (in contrast to degree $2l$ for integer LLs) with $\max(|g|) = l - 1/2$ and $\max(|m|) = l$.

2. Bosonic monopole harmonics

Following the above procedure, we obtain the (normalized) bosonic monopole harmonics as the following Grassmann *odd* homogeneous polynomials:

$$\begin{aligned} Z_{l,m}^g(x_i, \theta_\alpha) &= \sqrt{l - g + \frac{1}{2}} Y_{l,m}^{g-\frac{1}{2}}(x_i, \theta_\alpha) \cdot \eta + \sqrt{l + g + \frac{1}{2}} Y_{l,m}^{g+\frac{1}{2}}(x_i, \theta_\alpha) \cdot \eta^* \\ &= \sqrt{\frac{l - g + \frac{1}{2}}{2l + 1}} \Phi_{l,m}^{g-\frac{1}{2}}(y_i) \cdot \eta + \sqrt{\frac{l + g + \frac{1}{2}}{2l + 1}} \Phi_{l,m}^{g+\frac{1}{2}}(y_i) \cdot \eta^* \\ &\quad \times \left(m = l, l - 1, l - 2, \dots, -l, g = l - \frac{1}{2}, l - \frac{3}{2}, \dots, -\left(l - \frac{1}{2}\right) \right). \end{aligned} \quad (5.11)$$

By employing θ_1 and θ_2 instead of η and η^* , one can rewrite (5.11) as

$$\begin{aligned} Z_{l,m}^g(x_i, \theta_\alpha) &= \frac{1}{l + \frac{1}{2}} \left(\sqrt{\frac{(l + g + \frac{1}{2})(l - g + \frac{1}{2})(l - m + 1)}{2(l + 1)}} \Phi_{l+\frac{1}{2},m-\frac{1}{2}}^g(y_i) + g \sqrt{\frac{l + m}{2l}} \Phi_{l-\frac{1}{2},m-\frac{1}{2}}^g(y_i) \right) \frac{\theta_1}{r} \\ &\quad - \frac{1}{l + \frac{1}{2}} \left(\sqrt{\frac{(l + g + \frac{1}{2})(l - g + \frac{1}{2})(l + m + 1)}{2(l + 1)}} \Phi_{l+\frac{1}{2},m+\frac{1}{2}}^g(y_i) - g \sqrt{\frac{l - m}{2l}} \Phi_{l-\frac{1}{2},m+\frac{1}{2}}^g(y_i) \right) \frac{\theta_2}{r}, \end{aligned} \quad (5.12)$$

¹⁶ η and η^* carry the supermonopole charge $\pm 1/2$, respectively.

where (D9) and (4.9) were used. The complex conjugation is given by

$$Z_{l,m}^g(x_i, \theta_\alpha)^* = (-1)^{g-m-\frac{1}{2}} Z_{l,-m}^{-g}(x_i, \theta_\alpha). \quad (5.13)$$

3. Fermionic monopole harmonics

As in the integer LL case, the fermionic monopole harmonics can be derived using the formula $Z_{l,m}^g = -\frac{2}{\sqrt{l+m+\frac{1}{2}}} L_{\theta_2} Z_{l,m+\frac{1}{2}}^g$:

$$\begin{aligned} Z_{l,m}^g(x_i, \theta_\alpha) &= \overbrace{(u^* u + v^* v - (2l+1)\eta^* \eta)}^{=1-2l\eta^* \eta} Y_{l-\frac{1}{2},m}^g(x_i, \theta_\alpha) \\ &\left(m = l - \frac{1}{2}, l - \frac{3}{2}, \dots, -\left(l - \frac{1}{2}\right), \right. \\ &\left. g = l - \frac{1}{2}, l - \frac{3}{2}, \dots, -\left(l - \frac{1}{2}\right)\right), \end{aligned} \quad (5.14)$$

which are Grassmann *even* polynomial functions. In terms of θ_1 and θ_2 , (5.14) is represented as

$$Z_{l,m}^g(x_i, \theta_\alpha) = \frac{1}{\sqrt{2l}} \left(1 + \frac{2l+1}{4r^2} \theta C \theta\right) \Phi_{l-\frac{1}{2},m}^g(y_i). \quad (5.15)$$

The complex conjugation is

$$Z_{l,m}^g(x_i, \theta_\alpha)^* = (-1)^{g-m} Z_{l,-m}^{-g}(x_i, \theta_\alpha). \quad (5.16)$$

Note that for the half-integer LL eigenstates, the fermion parity and the Grassmann parity are opposite: $Z_{l,m}^g$ are Grassmann odd, while $Z_{l,m}^g$ are Grassmann even.

Just as the monopole harmonics (with a given g) form a complete set for functions on a sphere [92], the supermonopole harmonics realize a complete basis on a supersphere. Any function on a supersphere can be expanded in terms of the fermion coordinates and represented as the sum of the four terms

$$f(x_i, \theta_\alpha) = f_0(y_i) + f_1(y_i)\theta_1 + f_2(y_i)\theta_2 + f_3(y_i)\theta_1\theta_2. \quad (5.17)$$

The supermonopole harmonics $Y_{l,m}^g$ and $Z_{l,m}^g$ can span the first and last terms, while $Y_{l,m}^g$ and $Z_{l,m}^g$ the second and third terms. Thus, the supermonopole harmonics (with a given g) realize a complete set on the supersphere.

4. Action of the operators

The superspinor derivative operators act as

$$R_\pm Z_{l,m}^g = \sqrt{\left(l \mp g - \frac{1}{2}\right)\left(l \pm g + \frac{1}{2}\right)} Z_{l,m}^{g\pm 1}, \quad R_z Z_{l,m}^g = g Z_{l,m}^g, \quad (5.18a)$$

$$R_\pm Z_{l,m}^g = \sqrt{\left(l \mp g - \frac{1}{2}\right)\left(l \pm g + \frac{1}{2}\right)} Z_{l,m}^{g\pm 1}, \quad R_z Z_{l,m}^g = g Z_{l,m}^g, \quad (5.18b)$$

$$L_\pm Z_{l,m}^g = \sqrt{(l \mp m)(l \pm m + 1)} Z_{l,m\pm 1}^g, \quad L_z Z_{l,m}^g = m Z_{l,m}^g, \quad (5.19a)$$

$$L_\pm Z_{l,m}^g = \sqrt{\left(l \mp m - \frac{1}{2}\right)\left(l \pm m + \frac{1}{2}\right)} Z_{l,m\pm 1}^g, \quad L_z Z_{l,m}^g = m Z_{l,m}^g, \quad (5.19b)$$

$$L_{\theta_1} Z_{l,m}^g = \frac{1}{2} \sqrt{l-m} Z_{l,m+\frac{1}{2}}^g, \quad L_{\theta_2} Z_{l,m}^g = -\frac{1}{2} \sqrt{l+m} Z_{l,m-\frac{1}{2}}^g, \quad (5.19c)$$

$$\begin{aligned} L_{\theta_1} Z_{l,m}^g &= \frac{1}{2} \sqrt{l+m+\frac{1}{2}} Z_{l,m+\frac{1}{2}}^g, \\ L_{\theta_2} Z_{l,m}^g &= \frac{1}{2} \sqrt{l-m+\frac{1}{2}} Z_{l,m-\frac{1}{2}}^g, \end{aligned} \quad (5.19d)$$

and

$$L Z_{l,m}^g = \left(l + \frac{1}{2}\right) Z_{l,m}^g, \quad L Z_{l,m}^g = \left(l + \frac{1}{2}\right) Z_{l,m}^g. \quad (5.20)$$

The relations (5.19) are equal to those for the integer LLs (4.26), implying that the $\{Z_{l,m_1}^g, Z_{l,m_2}^g\}$ form another $UOSP(1|2)$ multiplet. The relations (5.18) and (5.20) are distinct from the integer LL results, (4.25) and (4.27). These differences are consistent with the previous observations: (5.18) implies $\max(|g|) = l - 1/2$, and (5.20) implies that the half-integer supermonopole harmonics have degree $2l + 1$.

B. Ghosts and a new definition of the inner product

The integrals of the supermonopole harmonics over a supersphere are evaluated as

$$\int_{S^{2|2}} d\Omega_{2|2} Z_{l',m'}^g(x_i, \theta_\alpha)^* Z_{l,m}^g(x_i, \theta_\alpha) = \delta_{l'l} \delta_{m'm}, \quad (5.21a)$$

$$\int_{S^{2|2}} d\Omega_{2|2} Z_{l',m'}^g(x_i, \theta_\alpha)^* Z_{l,m}^g(x_i, \theta_\alpha) = -\delta_{l'l} \delta_{m'm}, \quad (5.21b)$$

$$\int_{S^{2|2}} d\Omega_{2|2} Z_{l',m'}^g(x_i, \theta_\alpha)^* Z_{l,m}^g(x_i, \theta_\alpha) = 0, \quad (5.21c)$$

and

$$\begin{aligned}
& \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l,m'}^g(x_i, \theta_\alpha)^* Y_{l,m}^g(x_i, \theta_\alpha) \\
&= \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l,m'}^g(x_i, \theta_\alpha)^* \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) \\
&= \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l,m'}^g(x_i, \theta_\alpha)^* Y_{l,m}^g(x_i, \theta_\alpha) \\
&= \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l,m'}^g(x_i, \theta_\alpha)^* \mathcal{Y}_{l,m}^g(x_i, \theta_\alpha) = 0. \quad (5.22)
\end{aligned}$$

The supermonopole harmonics thus form an orthonormal basis, except for the minus sign on the right-hand side of (5.21b). This implies that the fermionic monopole harmonics $\mathcal{Z}_{l,m}^g$ are negative norm states, i.e., ghosts. It is known that representations of the $UOSp(1|2)$ group are pseudo-orthogonal [93,94] and super Landau models contain ghost states [36,37,39]. To handle these ghosts, there are basically three approaches. The first is simply to discard the half-integer Landau levels as unphysical and decouple the physical sector from ghosts. As the whole Hilbert space is spanned by the integer and the half-integer Landau level eigenstates, almost half of the Hilbert space has to be abandoned in this approach. The second is to utilize only the bosonic monopole harmonics in the half-integer LLs, but discarding the fermionic sectors containing ghosts. In this remedy, the SUSY structure in the half-integer LLs has to be abandoned, leaving only the $SU(2)$ symmetry. The last one is to modify the definition of the inner product to retain the ghost states as physical states [36]. All of the fermionic monopole harmonics in the half-integer Landau levels are ghosts, but by just flipping the sign of their norms simultaneously, we can transform them positive norm states without ruining the SUSY structure. This is equivalent to modifying the metric of the super Hilbert space [39]. In this paper, we adopt the last. We add a minus sign for the definition of bra of $\mathcal{Z}$:

$$\langle \mathcal{Z}_{l,m}^g | := -\mathcal{Z}_{l,m}^g, \quad (5.23)$$

while the bras for other supermonopole harmonics are as usual. We can represent this new definition of the bras by introducing the Scasimir

$$S = -2C_{\alpha\beta} L_\alpha L_\beta + \frac{1}{4}, \quad (5.24)$$

whose eigenvalues are $\pm(l + \frac{1}{4})$ for bosonic and fermionic eigenstates, respectively, so the eigenvalues correspond to the fermion parity (see Appendix A 2 for details). We then define the bras of the supermonopole harmonics in a unified fashion:

$$\langle \Psi | := (\text{sgn}(S)^{2N} \Psi(x_i, \theta_\alpha))^*. \quad (5.25)$$

Here, N denotes the LL of $\Psi(x_i, \theta_\alpha)$.¹⁷ For instance, if $\Psi = \mathcal{Z}$, where N is a half-integer and $\text{sgn}(S) = -1$, then (5.25) reduces to (5.23). Accordingly, the inner product is modified as

$$\langle \Psi | \Psi' \rangle = \int_{S^{2|2}} d\Omega_{2|2} (\text{sgn}(S)^{2N} \Psi(x_i, \theta_\alpha))^* \Psi'(x_i, \theta_\alpha), \quad (5.27)$$

and matrix element of an operator O is given by

$$\langle \Psi | O | \Psi' \rangle = \int_{S^{2|2}} d\Omega_{2|2} (\text{sgn}(S)^{2N} \Psi(x_i, \theta_\alpha))^* O \Psi'(x_i, \theta_\alpha). \quad (5.28)$$

The correct matrix elements of L_α (A15) can be reproduced from (5.19c) under the new formula (5.28), which supports the validity of this definition. Henceforth, we use these new definitions when evaluating the inner products and matrix elements. For instance, the probability densities are calculated as

$$\begin{aligned}
\rho_{l,m}^g(y_i) &= \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Z}_{l,m}^g(x_i, \theta_\alpha)^* \mathcal{Z}_{l,m}^g(x_i, \theta_\alpha) \\
&= \frac{l+g+\frac{1}{2}}{2l+1} |\Phi_{l,m}^{g+\frac{1}{2}}(y_i)|^2 + \frac{l-g+\frac{1}{2}}{2l+1} |\Phi_{l,m}^{g-\frac{1}{2}}(y_i)|^2, \quad (5.29a)
\end{aligned}$$

$$\begin{aligned}
\varrho_{l,m}^g(y_i) &= - \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Z}_{l,m}^g(x_i, \theta_\alpha)^* \mathcal{Z}_{l,m}^g(x_i, \theta_\alpha) \\
&= |\Phi_{l-\frac{1}{2},m}^g(y_i)|^2. \quad (5.29b)
\end{aligned}$$

VI. SUPER D-MATRIX AND HOWE DUALITY

In this section, we introduce the super D-matrix of the $UOSp(1|2)$ group and discuss a systematic realization of the supermonopole harmonics from the perspective of super Howe duality.

A. D-matrix and Howe duality

The Euler angle representation of Wigner's D-matrix is given by

$$D_s(\alpha, \beta, \gamma) = e^{-i\alpha S_z^{(s)}} e^{-i\beta S_y^{(s)}} e^{-i\gamma S_z^{(s)}}, \quad (6.1)$$

where $S_i^{(s)}$ denote the $SU(2)$ spin matrices with spin s , and thereby D_s denotes a $(2s+1) \times (2s+1)$ matrix. It is

¹⁷With the Scasimir, N can be represented as

$$N = |S| - |g| - \frac{1}{4}. \quad (5.26)$$

known that the monopole harmonics are realized as the matrix elements of the D-matrix¹⁸: [92,95,96]

$$\Phi_{s,m}^g(y_i) = \sqrt{\frac{2l+1}{4\pi}} D_s(\varphi, -\vartheta, -\varphi)_{g,m}, \quad (6.4)$$

where ϑ and φ signify the angle coordinates in $\mathbb{R}^3$ (4.4). The $\mathcal{D}_s$ consists of $(2s+1)$ rows comprising the LLL and higher LL eigenstates: its g th row realizes the $N = s - |g|$ th LL eigenstates as components [9,77]. (This is a consequence of the Peter-Weyl theorem.) The left and right actions of the D-matrix independently induce transformations between rows and columns, respectively [73,77,90]. In other words, for a fixed g , the $(2s+1)$ elements in a given row form an irreducible representation under the right $SU(2)$ action, while for a fixed m , the $(2s+1)$ elements in a given column constitute another irreducible representation under the left $SU(2)$ action. This interchangeability manifests the $(SU(2), SU(2))$ self-dual Howe duality [97]¹⁹ of the D-matrix, which realizes a duality between its rows and columns, or equivalently, between the quantum numbers g and m . The Hilbert space $\mathcal{H}$ of the Landau model then admits a multiplicity-free decomposition:

$$\mathcal{H} \cong \bigoplus_{s=0}^{\infty} (V_{SU(2)_L}^{(s)} \otimes V_{SU(2)_R}^{(s)}), \quad (6.5)$$

where $V_{SU(2)_{L/R}}^{(s)}$ denotes the vector space on which each $SU(2)$ group with a common spin s acts.²⁰ The one-to-one mapping between $V_{SU(2)_L}^{(s)}$ and $V_{SU(2)_R}^{(s)}$ is known as the theta correspondence. It should be emphasized that the existence of the Howe duality is only realized through the inclusion of higher Landau levels.

¹⁸With Wigner's small D-matrix

$$d_{s,g,m}(\beta) = (e^{i\beta S_y^{(s)}})_{g,m}, \quad (6.2)$$

the monopole harmonics (6.4) are represented as

$$\Phi_{s,m}^g(y_i) = \sqrt{\frac{2l+1}{4\pi}} d_{s,g,m}(\vartheta) e^{i(m-g)\varphi}. \quad (6.3)$$

¹⁹The $(SU(2), SU(2))$ duality is a particular realization of Howe duality in the Landau model. Howe duality, however, is a mathematical structure that also arises in certain other quantum mechanical systems. See [98,99] as pedagogical accounts of Howe duality for physicists.

²⁰Physically, this duality corresponds to a transformation between monopole harmonics and spin-coherent states [100].

B. Super D-matrix and supermonopole harmonics

An arbitrary group element of $UOSp(1|2)$ is given by [89]

$$g = F \cdot B, \quad (6.6)$$

where F and B are defined as

$$\begin{aligned} F(\eta, \eta^*) &:= e^{2\eta L_{\theta_1} + 2\eta^* L_{\theta_2}} \\ &= \mathbf{1} + 2\eta L_{\theta_1} + 2\eta^* L_{\theta_2} - 2\eta^* \eta (L_{\theta_1} L_{\theta_2} - L_{\theta_2} L_{\theta_1}), \end{aligned} \quad (6.7a)$$

$$B(\chi, \vartheta, \varphi) := e^{-i\chi L_z} e^{-i\vartheta L_y} e^{-i\varphi L_z}. \quad (6.7b)$$

Here, L_i and L_α are the $UOSp(1|2)$ representation matrices rather than differential operators. For an irreducible representation with superspin l , the super D-matrix is expressed as

$$\mathcal{D}_l(\alpha, \beta, \gamma, \eta, \eta^*) = F_l(\eta, \eta^*) \cdot B_l(\alpha, \beta, \gamma), \quad (6.8)$$

where

$$F_l(\eta, \eta^*) = \begin{pmatrix} (1 + l\eta^* \eta) \mathbf{1}_{2l+1} & \eta \tau_{\theta_1}^{(l)} + \eta^* \tau_{\theta_2}^{(l)} \\ \eta \tau_{\theta_2}^{(l)\dagger} - \eta^* \tau_{\theta_1}^{(l)\dagger} & (1 - (l + \frac{1}{2})\eta^* \eta) \mathbf{1}_{2l} \end{pmatrix}, \quad (6.9)$$

and

$$B_l(\alpha, \beta, \gamma) = \begin{pmatrix} D_l(\alpha, \beta, \gamma) & 0 \\ 0 & D_{l-\frac{1}{2}}(\alpha, \beta, \gamma) \end{pmatrix}. \quad (6.10)$$

The block diagonal components of B_l are the standard D-matrices (6.1). See Appendix A for the explicit forms of $\tau_\alpha^{(l)}$ and $F_l(\eta, \eta^*)$. In the following, we take the Dirac gauge in which (6.10) is given by

$$B_l(\varphi, \vartheta, -\varphi) = e^{-i\varphi L_z^{(l)}} e^{-i\vartheta L_y^{(l)}} e^{i\varphi L_z^{(l)}} = e^{i\vartheta (L_x^{(l)} \sin \varphi - L_y^{(l)} \cos \varphi)}, \quad (6.11)$$

where $L_{i=x,y,x}^{(l)}$ represent the bosonic generators of the $uosp(1|2)$ matrices (A15). For low-dimensional representations, (6.8) are given by

$$\mathcal{D}_{1/2} = \begin{pmatrix} u & -v & \eta \\ v^* & u^* & \eta^* \\ -u\eta^* + v^*\eta & u^*\eta + v\eta^* & 1 - \eta^*\eta \end{pmatrix}, \quad (6.12a)$$

$$\mathcal{D}_1 = \begin{pmatrix} u^2 & -\sqrt{2}uv & v^2 & \sqrt{2}u\eta & -\sqrt{2}v\eta \\ \sqrt{2}uv^* & u^*u - v^*v & -\sqrt{2}u^*v & u\eta^* + v^*\eta & u^*\eta - v\eta^* \\ v^{*2} & \sqrt{2}u^*v^* & u^{*2} & \sqrt{2}v^*\eta^* & \sqrt{2}u^*\eta^* \\ -\sqrt{2}(u^2\eta^* - uv^*\eta) & (u^*u - v^*v)\eta + 2uv\eta^* & -\sqrt{2}(v^2\eta^* + u^*v\eta) & (1 - 2\eta^*\eta)u & -(1 - 2\eta^*\eta)v \\ \sqrt{2}(v^{*2}\eta - uv^*\eta^*) & -(u^*u - v^*v)\eta^* + 2u^*v^*\eta & \sqrt{2}(u^{*2}\eta + u^*v\eta^*) & (1 - 2\eta^*\eta)v^* & (1 - 2\eta^*\eta)u^* \end{pmatrix}. \quad (6.12b)$$

Matrix $\mathcal{D}_{1/2}$ is exactly equal to (2.3). The first two rows of $\mathcal{D}_{1/2}$ denote the $g = 1/2$ and $g = -1/2$ LLL eigenstates, (C2a) and (C4a). The last row stands for the $l = 1/2$ superspherical harmonics (C16). Similarly, the first and third rows of $\mathcal{D}_1$ represent the $g = 1$ and $g = -1$ LLL eigenstates, (C2b) and (C4b). The second row is the $l = 1$ superspherical harmonics (C12b). The last two rows respectively correspond to the $g = 1/2$ and $g = -1/2$ lowest half-integer LL eigenstates, (C20a) and (C20b). In general, the supermonopole harmonics are realized as the matrix elements of the super D-matrix:

$$\Psi_{l,m}^g(x_i, \theta_\alpha) = \frac{1}{\sqrt{4\pi}} \mathcal{D}_l(\varphi, \vartheta, -\varphi, \eta, \eta^*)_{g,m}. \quad (\text{up to sign}) \quad (6.13)$$

As illustrated in Fig. 3, the integer LL supermonopole harmonics with $l = N + |g|$ appear as the rows of the upper $(2l+1) \times (4l+1)$ rectangular matrix of $\mathcal{D}_l$, while the half-integer LL supermonopole harmonics with $l = N + |g|$ appear as the rows of the lower $(2l) \times (4l+1)$ rectangular matrix of $\mathcal{D}_l$.

C. Super Howe duality

As in the case of the standard D-matrix, the super D-matrix undergoes two independent transformations, namely the left and right $UOSp(1|2)$ superrotations. This suggests that the $(4l+1)$ matrix elements in a given column (with varying g) constitute a left $UOSp(1|2)$ multiplet, and those in a given row (with varying m) form a right $UOSp(1|2)$ multiplet. While the $SU(2)$ ladder operators $R_\pm$ (3.24) partially realize the left $UOSp(1|2)$ transformations, they only connect the components within each of the two $SU(2)$ sectors in a column, so they fail to relate integer and half-integer LLs. To bridge this gap, we construct the fermionic ladder operators that shift the monopole charge g by $\pm 1/2$.

Let us recall the lowest-dimensional super D-matrix (6.12), in which the fundamental representations for $g = 0$ and $g = \pm 1/2$ are realized as its rows. We then construct the fermionic ladder operators interchanging these states:

$$R_\eta := \frac{1}{2} \Phi^t \mathcal{R} \frac{\partial}{\partial \psi^*} = \frac{1}{2} \left((-u\eta^* + v^*\eta) \frac{\partial}{\partial v^*} - (u^*\eta + v\eta^*) \frac{\partial}{\partial u^*} - (1 - \eta^*\eta) \frac{\partial}{\partial \eta^*} \right), \quad (6.14a)$$

$$R_{\eta^*} := -\frac{1}{2} \Phi^t \frac{\partial}{\partial \psi} = \frac{1}{2} \left((-u\eta^* + v^*\eta) \frac{\partial}{\partial u} - (u^*\eta + v\eta^*) \frac{\partial}{\partial v} + (1 - \eta^*\eta) \frac{\partial}{\partial \eta} \right), \quad (6.14b)$$

where Φ denotes the $l = 1/2$ superspherical harmonics:

$$\Phi := \begin{pmatrix} u\eta^* - v^*\eta \\ u^*\eta + v\eta^* \\ -(1 - \eta^*\eta) \end{pmatrix}. \quad (6.15)$$

Notably, $R_{\alpha=\eta,\eta^*}$ are quadratic-order polynomial differential operators unlike the other superspinor bilinear derivative operators. It is straightforward to verify that R_α and R_i satisfy the $uosp(1|2)$ algebra:

$$\begin{aligned} [R_i, R_j] &= i\epsilon_{ijk} R_k, & [R_i, R_\alpha] &= \frac{1}{2} (\sigma_i)_{\beta\alpha} R_\beta, \\ \{R_\alpha, R_\beta\} &= \frac{1}{2} (C\sigma_i)_{\alpha\beta} R_i \quad (i, j, k = 1, 2, 3, \alpha, \beta = \eta, \eta^*). \end{aligned} \quad (6.16)$$

Furthermore, the generators L_A and R_A are shown to be supercommutative:

$$[L_i, R_j] = [L_i, R_\alpha] = [L_\alpha, R_\alpha] = \{L_\alpha, R_\beta\} = 0, \quad (6.17)$$

where we used the Grassmann properties of η and η^* along with (2.5). They act as the fermionic ladder operators for the monopole charge:

$$R_\eta Y_{l,m}^g = \frac{1}{2} \sqrt{l-g} Z_{l,m}^{g+\frac{1}{2}}, \quad R_{\eta^*} Y_{l,m}^g = -\frac{1}{2} \sqrt{l+g} Z_{l,m}^{g-\frac{1}{2}}, \quad (6.18a)$$

$$R_\eta Z_{l,m}^g = \frac{1}{2} \sqrt{l+g+\frac{1}{2}} Y_{l,m}^{g+\frac{1}{2}}, \quad R_{\eta^*} Z_{l,m}^g = \frac{1}{2} \sqrt{l-g+\frac{1}{2}} Y_{l,m}^{g-\frac{1}{2}}, \quad (6.18b)$$

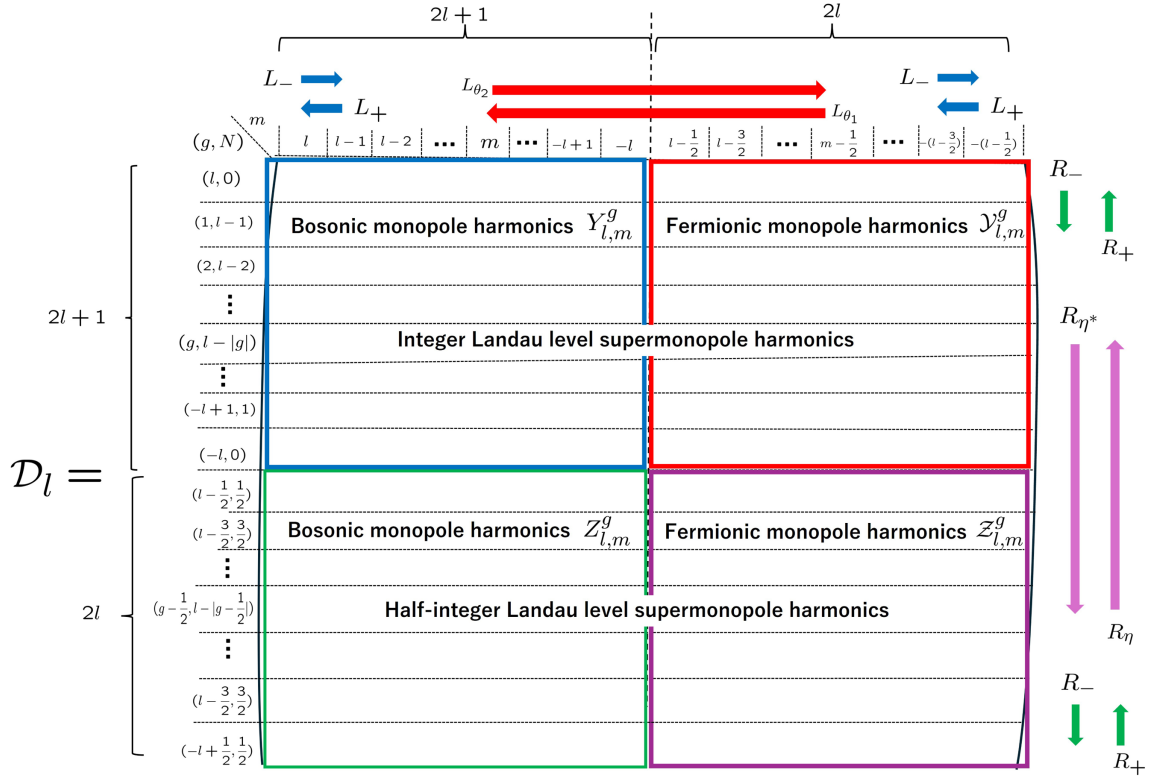


FIG. 3. Structure of the super D -matrix ($l = N + |g|$). The rows in the blue (red) rectangles correspond to bosonic (fermionic) monopole harmonics in the integer Landau levels, while those in the green (purple) rectangles realize the bosonic (fermionic) monopole harmonics in the half-integer Landau levels. The colored arrows indicate the actions of the ladder operators of the two independent $uosp(1|2)$ algebras.

and

$$R_{\eta} \mathcal{Y}_{l,m}^g = -\frac{1}{2} \sqrt{l-g} \mathcal{Z}_{l,m}^{g+\frac{1}{2}}, \quad R_{\eta^*} \mathcal{Y}_{l,m}^g = \frac{1}{2} \sqrt{l+g} \mathcal{Z}_{l,m}^{g-\frac{1}{2}}, \quad (6.19a)$$

$$R_{\eta} \mathcal{Z}_{l,m}^g = -\frac{1}{2} \sqrt{l+g+\frac{1}{2}} \mathcal{Y}_{l,m}^{g+\frac{1}{2}}, \quad R_{\eta^*} \mathcal{Z}_{l,m}^g = -\frac{1}{2} \sqrt{l-g+\frac{1}{2}} \mathcal{Y}_{l,m}^{g-\frac{1}{2}}. \quad (6.19b)$$

The R_{α} operators thus shift the supermonopole charge by $\pm 1/2$, and they interchange the integer and half-integer Landau level eigenstates.²¹ We identified two independent $uosp(1|2)$ algebras acting on the rows and columns indices, with R_A and L_A serving as their respective ladder operators (see Fig. 3). The original ($SU(2), SU(2)$) Howe

duality is now extended to the $(UOSp(1|2), UOSp(1|2))$ duality for the super D -matrix.

The super Hilbert space is then decomposed as

$$\mathcal{H} \cong \bigoplus_{l=0}^{\infty} (V_{UOSp(1|2)_L}^{(l)} \otimes V_{UOSp(1|2)_R}^{(l)}), \quad (6.20)$$

where $V_{UOSp(1|2)_{L/R}}^{(l)}$ denotes the vector space on which each $UOSp(1|2)$ group with a common superspin l acts. More specifically, $V_{UOSp(1|2)_L}^{(l)}$ and $V_{UOSp(1|2)_R}^{(l)}$ represent the space of supermonopole harmonics generated by L_A and the $UOSp(1|2)$ irreducible representation space generated by R_A , respectively. The theta correspondence realizes a one-to-one map between $V_{UOSp(1|2)_L}^{(l)}$ and $V_{UOSp(1|2)_R}^{(l)}$ in the present SUSY system.

VII. SUPER QUANTUM MATRIX GEOMETRY

There are basically three methods for deducing NCG from Landau models. The former two, summarized in Appendix E, are intuitive but restricted to the LLL. On the other hand, the third approach, known as the level projection method, is rigorous for deriving precise matrix

²¹Careful readers may wonder about the consistency of the degrees in (6.18b) and (6.19b). The left-hand sides involve the quadratic-order polynomial differential operators acting on degree- $(2l+1)$ variables, which seemingly suggests a total degree $2l+2$. However, the nilpotency of the Grassmann variables reduces the actual degree to $2l$, which indeed matches the degrees of Y_l and $\mathcal{Y}_l$ on the right-hand sides.

coordinates and applicable not only to the LLL but also to higher LLs [9,77].

A. Fuzzy sphere geometry of the body-sphere

As the body-part of the present SUSY model realizes Haldane's sphere, its effective geometry is naturally given by a fuzzy sphere geometry. Within each LL, the matrix coordinates of the body-sphere are evaluated as

$$(Y_i)_{m'm} = \langle \Phi_{sm'}^g | y_i | \Phi_{sm}^g \rangle = \int_{S^2} d\Omega_2 \Phi_{sm'}^g * y_i \Phi_{sm}^g. \quad (7.1)$$

From (D11), the matrix elements are readily obtained as [77]

$$\begin{aligned} \langle \Phi_{sm'}^g | y_1 \pm iy_2 | \Phi_{sm}^g \rangle &= r \frac{g}{s(s+1)} \sqrt{s(s+1) \mp m(m \pm 1)} \\ &\quad \times \delta_{m', m \pm 1}, \\ \langle \Phi_{sm'}^g | y_3 | \Phi_{sm}^g \rangle &= r \frac{g}{s(s+1)} m \delta_{m', m}, \end{aligned} \quad (7.2)$$

which implies

$$Y_i = \alpha S_i^{(s)}, \quad (7.3)$$

with the NC scale factor

$$\alpha = r \frac{g}{s(s+1)}. \quad (7.4)$$

Note that the $SU(2)$ Casimir eigenvalue $s(s+1)$ appears as the denominator of α . The matrix coordinates (7.3) constitute the NCG of the fuzzy sphere:

$$\begin{aligned} [Y_i, Y_j] &= i\alpha \epsilon_{ijk} Y_k, \\ Y_i Y_i &= \alpha^2 s(s+1) \mathbf{1}_{2s+1} = r^2 \frac{g^2}{s(s+1)} \mathbf{1}_{2s+1}. \end{aligned} \quad (7.5)$$

B. Fuzzy supersphere geometry

We extend the level projection method in Sec. VII A to derive the supermatrix geometry.

1. Fuzzy supersphere geometry in the integer Landau levels

Using the integer LL eigenstates (4.11); $|\Psi_{lm}^g\rangle = (|Y_{lm_1}^g\rangle |\mathcal{Y}_{lm_2}^g\rangle)^t$, we evaluate the matrix elements of the supersphere coordinates as

$$(X_i)_{m'm} = \langle \Psi_{lm'}^g | x_i | \Psi_{lm}^g \rangle = \int_{S^{2|2}} d\Omega_{2|2} \Psi_{lm'}^g * x_i \Psi_{lm}^g, \quad (7.6a)$$

$$(\Theta_\alpha)_{m'm} = \langle \Psi_{lm'}^g | \theta_\alpha | \Psi_{lm}^g \rangle = \int_{S^{2|2}} d\Omega_{2|2} \Psi_{lm'}^{(g)} * \theta_\alpha \Psi_{lm}^{(g)}, \quad (7.6b)$$

where $\int_{S^{2|2}} d\Omega_{2|2}$ is defined by (2.48). From the $SU(2)$ selection rules, it is easy to see that they take the following

block-structured forms:

$$\begin{aligned} X_i &= \begin{pmatrix} \langle Y | x_i | Y \rangle & 0 \\ 0 & \langle \mathcal{Y} | x_i | \mathcal{Y} \rangle \end{pmatrix}, \\ \Theta_\alpha &= \begin{pmatrix} 0_{2l+1} & \langle Y | \theta_\alpha | \mathcal{Y} \rangle \\ \langle \mathcal{Y} | \theta_\alpha | Y \rangle & 0_{2l} \end{pmatrix}. \end{aligned} \quad (7.7)$$

The remaining matrix elements can be evaluated using either the standard monopole harmonics formulas, (D9) and (D11), or the newly derived supermonopole harmonics formula (D17). Here we adopt the latter. By applying (D17a) and (D17b) together with (C12b), we readily obtain

$$\begin{aligned} \langle Y_{l,m_1}^g | x_1 \pm ix_2 | Y_{l,m_1}^g \rangle &= r \frac{g}{l(l+\frac{1}{2})} \sqrt{l(l+1) \mp m_1(m_1 \pm 1)} \delta_{m_1', m_1 \pm 1}, \\ \langle Y_{l,m_1}^g | x_3 | Y_{l,m_1}^g \rangle &= r \frac{g}{l(l+\frac{1}{2})} m_1 \delta_{m_1', m_1}, \end{aligned} \quad (7.8)$$

and, from (D17c) and (D17d),

$$\begin{aligned} \langle \mathcal{Y}_{l,m_2}^g | x_1 \pm ix_2 | \mathcal{Y}_{l,m_2}^g \rangle &= r \frac{g}{l(l+\frac{1}{2})} \sqrt{\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right) \mp m_2'(m_2 \pm 1)} \delta_{m_2', m_2 \pm 1}, \\ \langle \mathcal{Y}_{l,m_2}^g | x_3 | \mathcal{Y}_{l,m_2}^g \rangle &= r \frac{g}{l(l+\frac{1}{2})} m_2 \delta_{m_2', m_2}. \end{aligned} \quad (7.9)$$

Similarly, using (D17e) and (D17f) with (C12b), the fermionic components are evaluated as

$$\begin{aligned} \langle Y_{l,m_1}^g | \theta_{1,2} | \mathcal{Y}_{l,m_2}^g \rangle &= r \frac{g}{2l(l+\frac{1}{2})} \sqrt{l \pm m_2 + \frac{1}{2}} \delta_{m_1', m_2 \pm \frac{1}{2}}, \\ \langle \mathcal{Y}_{l,m_2}^g | \theta_{1,2} | Y_{l,m_1}^g \rangle &= \pm r \frac{g}{2l(l+\frac{1}{2})} \sqrt{l \mp m_1} \delta_{m_2', m_1 \pm \frac{1}{2}}. \end{aligned} \quad (7.10)$$

The supermatrix coordinates are then identified as

$$X_i = \alpha L_i^{(l)} \quad \Theta_\alpha = \alpha L_\alpha^{(l)}, \quad (7.11)$$

with the NC scale factor

$$\alpha = r \frac{g}{l(l+\frac{1}{2})}, \quad (7.12)$$

where $L_i^{(l)}$ and $L_\alpha^{(l)}$ are the $uosp(1|2)$ matrices with superspin l (A15). These are $(4l+1) \times (4l+1)$ matrices, reflecting the total degeneracy of the given Landau level. Analogous to the $SU(2)$ case (7.4), the $UOSp(1|2)$ Casimir eigenvalue $l(l+1/2)$ appears as the denominator of the NC scale factor (7.12). In the LLL ($l = |g|$), (7.12) reduces to

$$\alpha \rightarrow \frac{r}{g + \frac{1}{2}\text{sgn}(g)}, \quad (7.13)$$

which (in the limit $|g| \rightarrow \infty$) coincides with the NC scale factor obtained in Appendix E. It is also noted that, for a fixed l , α (7.12) is proportional to g , which will be important in Sec. VII D. With the identification (7.11), one may find that the fuzzy supersphere geometry is realized in an arbitrary super Landau level:

$$\begin{aligned} [X_i, X_j] &= i\alpha\epsilon_{ijk}X_k, & [X_i, \Theta_\alpha] &= \frac{\alpha}{2}(\sigma_i)_{\beta\alpha}\Theta_\beta, \\ \{\Theta_\alpha, \Theta_\beta\} &= \frac{\alpha}{2}(C\sigma_i)_{\alpha\beta}X_i, \end{aligned} \quad (7.14)$$

The first relation signifies the NC algebra of the bosonic coordinates, while the last denotes the NAC geometry of the fermionic sector. The second relation implies that these bosonic and fermionic coordinates are entangled, forming a closed SUSY NC algebra as a whole. The supermatrix coordinates satisfy

$$X_i X_i + C_{\alpha\beta}\Theta_\alpha\Theta_\beta = \alpha^2 l \left(l + \frac{1}{2} \right) \mathbf{1}_{4l+1} = r^2 \frac{g^2}{l(l + \frac{1}{2})} \mathbf{1}_{4l+1}. \quad (7.15)$$

The right-hand side (the $UOSP(1|2)$ Casimir) is interpreted as the square of the radius of the fuzzy supersphere:

$$R_F := r \frac{|g|}{\sqrt{l(l + \frac{1}{2})}}. \quad (7.16)$$

With higher LLs included, there are two independent parameters: g and l , or equivalently, g and $N(=l - |g|)$. In either of the limits $|g| \rightarrow \infty$ or $N \rightarrow \infty$, the NC scale factor (7.12) vanishes, indicating that the fuzzy supersphere reduces to a continuous supersphere. However, the NC scale factor decreases more rapidly in the $N \rightarrow \infty$ limit than in the $|g| \rightarrow \infty$ limit. Consequently, the radius R_F (7.16) also vanishes as $N \rightarrow \infty$, whereas it converges to the finite classical radius r of the classical sphere as $|g| \rightarrow \infty$.

2. Fuzzy supersphere geometry in the half-integer Landau levels

In a manner analogous to the analysis of the integer LLs, we deduce the supermatrix geometry from the half-integer LL eigenstates (5.1); $|\Psi_{lm}^g\rangle = (|Z_{lm_1}^g\rangle|Z_{lm_2}^g\rangle)^t$. From the $SU(2)$ selection rules, we find the matrix coordinates again take the following form:

$$\begin{aligned} X_i &= \begin{pmatrix} \langle Z|x_i|Z\rangle & 0 \\ 0 & \langle Z|x_i|Z\rangle \end{pmatrix}, \\ \Theta_\alpha &= \begin{pmatrix} 0_{2l+1} & \langle Z|\theta_\alpha|Z\rangle \\ \langle Z|\theta_\alpha|Z\rangle & 0_{2l} \end{pmatrix}. \end{aligned} \quad (7.17)$$

Let us recall that the inner products (5.28) associated with $\langle Z|$ are given by

$$\begin{aligned} \langle Z|x_i|Z\rangle &= - \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}^* x_i Z, \\ \langle Z|\theta_\alpha|Z\rangle &= - \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}^* \theta_\alpha Z. \end{aligned} \quad (7.18)$$

By proceeding in a similar manner to Sec. VII B 1, we derive the matrix coordinates as

$$X_i = \alpha L_i^{(l)}, \quad \Theta_\alpha = \alpha L_\alpha^{(l)}. \quad (7.19)$$

These are identical to those obtained for the integer LLs (7.11). We have thus confirmed that the fuzzy supersphere geometries emerge also in the half-integer Landau levels under the new definition of the inner products for the ghosts.

C. Implications of the SUSY NC algebra

The matrix coordinates X_i and Θ_α act also as the generators of $UOSP(1|2)$ transformations. The SUSY NC algebra (7.14) represents the superrotational covariance of the fuzzy supersphere, serving as the NC realization of the covariance of the classical supersphere (2.52). The fuzzy supersphere condition (7.15) remains manifestly invariant under the SUSY transformations generated by Θ_1 and Θ_2 , indicating that the $\mathcal{N} = 1$ SUSY remains exact after fuzzification. More specifically, under the $\mathcal{N} = 1$ SUSY transformation identical to that of the classical supersphere (2.14)

$$\delta_\xi X_i = \Theta_\alpha (\sigma_i C)_{\alpha\beta} \xi_\beta, \quad \delta_\xi \Theta_\alpha = X_i (\sigma_i)_{\beta\alpha} \xi_\beta, \quad (7.20)$$

the fuzzy supersphere condition (2.11) is invariant, and the SUSY NC algebra (7.14) is also covariantly preserved. Thus, the NCG does not break the SUSY structure of the original classical system. This exact preservation of $\mathcal{N} = 1$ SUSY is a key feature of the fuzzy geometries constructed from super Lie algebras, as mentioned in the Introduction.

In terms of the supermonopole field strengths (2.41), the SUSY NC algebra (7.14) can be expressed as

$$\begin{aligned} [X_i, X_j] &= i \frac{r^4}{l(l + \frac{1}{2})} F_{ij} + \dots, \\ [X_i, \Theta_\alpha] &= i \frac{r^4}{2l(l + \frac{1}{2})} C_{\alpha\beta} F_{i\beta} + \dots, \\ \{\Theta_\alpha, \Theta_\beta\} &= i \frac{r^4}{4l(l + \frac{1}{2})} C_{\alpha\gamma} C_{\beta\delta} F_{\gamma\delta} + \dots, \end{aligned} \quad (7.21)$$

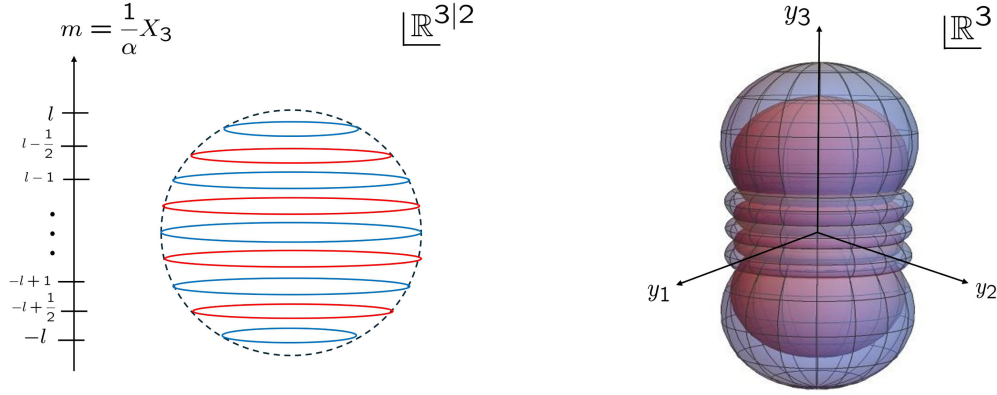


FIG. 4. Left: the fuzzy supersphere. The blue and red latitudes respectively correspond to the bosonic and fermionic degrees of freedom. Right: the probability densities of the $l = 2$ LLL eigenstates from Fig. 2 [also see (C8)]. The elements (= latitudes) in the left figure correspond to the supermonopole harmonics in the right figure.

where $\dots$ denotes higher-order corrections. Equations (7.21) clearly show that the existence of SUSY field strengths is crucial for the realization of the SUSY noncommutative geometry.

Since the three bosonic matrix coordinates are equivalent under the $SU(2)$ symmetry, we may, without loss of generality, diagonalize the third component of X_i (A15):

$$X_3 = \alpha \begin{pmatrix} S_z^{(l)} & 0 \\ 0 & S_z^{(l-1/2)} \end{pmatrix} = \alpha \cdot \text{diag}(l, l-1, l-2, \dots, -l+1/2, l-3/2, \dots, -l+1/2). \quad (7.22)$$

The bosonic degrees of freedom are represented by the eigenvalues of the upper-left block $S_z^{(l)}$, while the fermionic degrees of freedom correspond to those of the lower-right block $S_z^{(l-1/2)}$. The bosonic and fermionic components constitute the quantum elements of the fuzzy supersphere (see Fig. 4). The radius of the latitude corresponding to the quantum element at $X_3 = \alpha \cdot m$ ($m = l, l-1/2, l-1, \dots, -l$) is given by

$$R_m = \sqrt{R_F^2 - X_3^2} = r \sqrt{\frac{1}{l(l+\frac{1}{2})} - \left(\frac{m}{l(l+\frac{1}{2})}\right)^2} \cdot |g|. \quad (7.23)$$

In other words, the supermonopole harmonics are eigenstates of $|\alpha| \sqrt{L_1 L_1 + L_2 L_2 + C_{\alpha\beta} L_\alpha L_\beta}$ with the eigenvalues (7.23).

D. Fuzzy geometry transformation via the super Howe duality

Here, we fix $l = N + |g|$ and vary g from l to $-l$ under this constraint. The fuzzy supersphere matrix coordinates are given by the $(4l+1) \times (4l+1)$ $uosp(1|2)$ matrices (7.11). Such fuzzy supermatrix geometries correspond to the columns of the super D-matrix $\mathcal{D}_l$, since each row of $\mathcal{D}_l$

describes the $(4l+1)$ component basis states of the super Landau level (Fig. 3). More specifically, the rows of $\mathcal{D}_l$, labeled by g , represent fuzzy supersphere geometries of the same $(4l+1) \times (4l+1)$ matrix size, but with distinct NC scale factors $\alpha = g/(l(l+1/2))$, or equivalently, distinct radii (7.16). Then, as g varies from l to $-l$, we have $(4l+1)$ fuzzy supersphere geometries of different radii. When g is regarded as “time,” this represents an evolution of the fuzzy supersphere of $(4l+1)$ latitudes as shown in Fig. 5.

Let us consider the evolution of the quantum state with fixed m . This quantum state corresponds to the latitude with radius R_m (7.23) which depends linearly on $|g|$. As g decreases from l to $-l$ (with $\Delta g = 1/2$), the R_m sweeps out a (fuzzy) cone. In the super D matrix picture, this procedure realizes a sequence of transitions between the columns of the integer and half-integer Landau levels. The resulting fuzzy cone is then composed of $(4l+1)$ latitudes corresponding to the integer and half-integer LL eigenstates connected by the supercharges R_α . We refer to this conic structure as the fuzzy supercone (of finite length). The slope of the “surface” of the fuzzy supercone is given by²²

$$c := \frac{l}{r} \frac{\partial R_m}{\partial |g|} = \sqrt{\frac{l}{l+\frac{1}{2}} - \left(\frac{m}{l+\frac{1}{2}}\right)^2} < 1. \quad (7.24)$$

As $|m|$ decreases, c increases and reaches the maximum value $c = 1/\sqrt{1+1/(2l)}$ at $m = 0$. With fixed l , m varies from l to $-l$, so there are $(4l+1)$ fuzzy supercones with distinct slopes (see Fig. 6). Regarding m as “time,” we see the evolution of the fuzzy supercone in Fig. 6.

The roles of g and m are interchanged between the fuzzy supersphere and the fuzzy supercone. Furthermore, the evolution of the fuzzy supercone (Fig. 5) is uniquely

²²We assume the physical time to be $\frac{r}{l} \cdot g$. This definition accounts for the range of g (from $-l$ to l) and the required dimension of length.

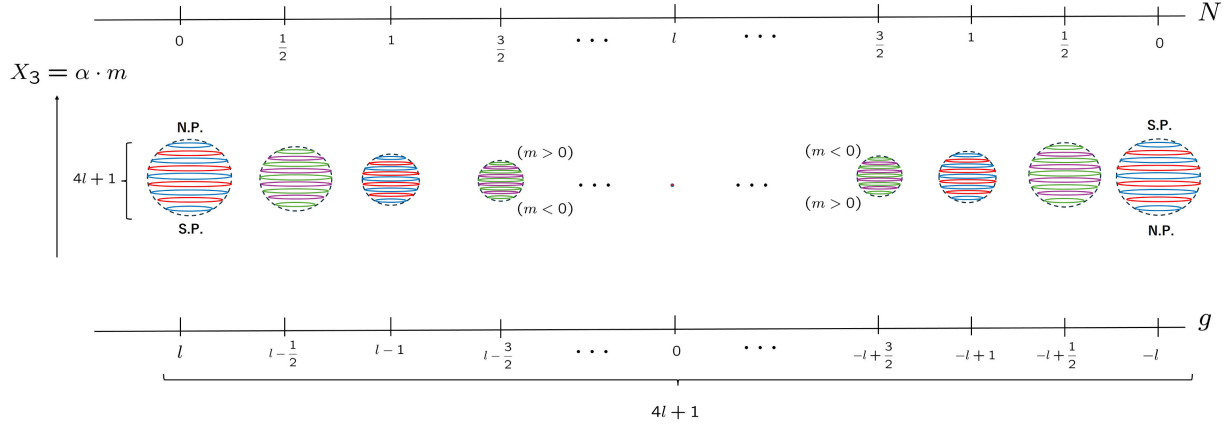


FIG. 5. Evolution of the fuzzy supersphere geometry. As the supermonopole charge g varies, the fuzzy superspheres associated with integer LLs (constituted from blue and red latitudes) and half-integer LLs (constituted from green and purple latitudes) appear alternately. The radius of the fuzzy supersphere shrinks as $|g|$ decreases, and upon crossing $g = 0$, the northern ($m > 0$) and southern ($m < 0$) hemispheres are interchanged.

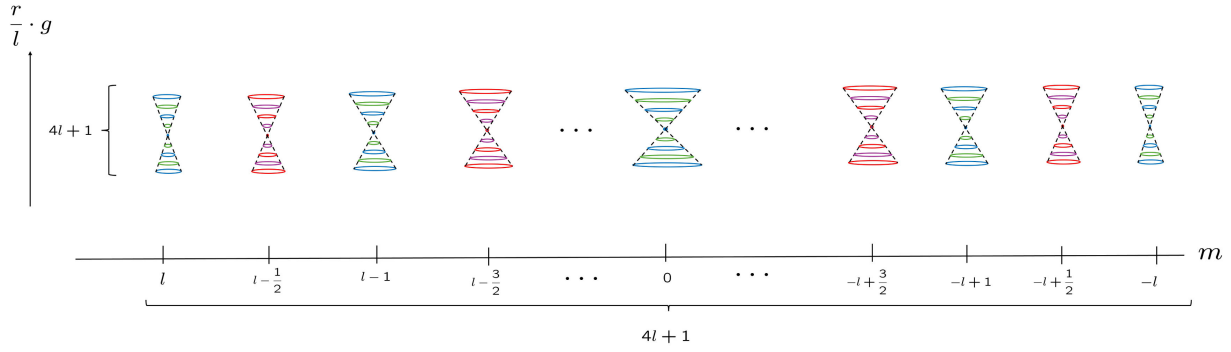


FIG. 6. Evolution of the fuzzy supercone. The slope of the fuzzy super cones decreases, as m varies from l to 0 , and subsequently increases as m moves from 0 to $-l$.

determined by that of the fuzzy supersphere (Fig. 6), and vice versa. Let us recall that the Howe duality represents the duality between the row (g) and column (m). Hence, the present geometric correspondence demonstrates a geometric realization of the theta correspondence. Some of the readers may be curious about the appearance of the fuzzy supercone instead of the fuzzy supersphere. Since the present Howe duality is self-dual, $(UOSp(1|2), UOSp(1|2))$, the interchange of m and g should lead to dual fuzzy superspheres instead of fuzzy supercones. In this perspective, the finite-length fuzzy supercone is considered to be a “disguise” of the dual fuzzy supersphere.²³ In the language of the orbit method, each fuzzy supersphere indexed by g corresponds to a quantized coadjoint orbit generated by one supergroup

²³The physical origin of the conical geometry is, however, still unclear. The light cone describes the geometry of massless particle trajectories, and it is also known that massless particle is closely related to charge-monopole system [101,102]. There might be some deeper reason behind the appearance of the conical geometry.

$UOSp(1|2)$. When we consider a family of such quantized orbits labeled by g from l to $-l$, another family of quantized orbits emerges along the “stacking direction” of the original orbits, which manifests the Howe duality in the framework of NCG. More specifically, the stacked fuzzy superspheres yield a new nested fuzzy geometry, which can also be viewed as a family of the fuzzy supercones (see Fig. 7). These two superficially different descriptions are in fact equivalent and represent the same nested object. This provides an intuitive understanding of the Howe duality in the context of NCG.

E. Howe duality as an underlying structure

Finally, we discuss whether Howe duality appears in other Landau models and their associated matrix geometries.²⁴ Let us recall the discussions in Sec. VIC, where

²⁴In [103,104], the authors introduced a doubled space containing the Hilbert space of the Landau model, which naturally gives rise to a Howe duality. The relation between their Howe duality and the present one remains an open problem.

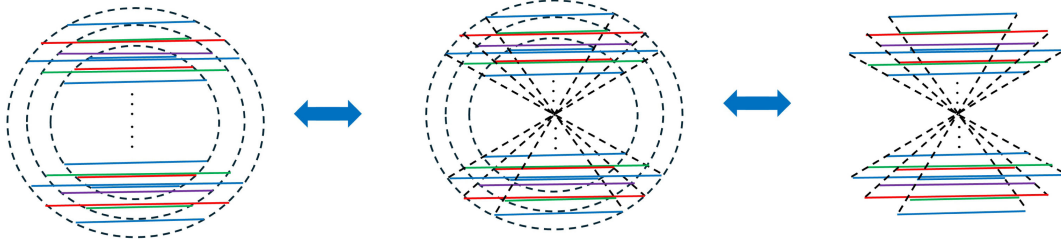


FIG. 7. An intuitive picture of the emergence of fuzzy supercones. The left denotes the stacked fuzzy superspheres with their latitudes. From this nested structure, fuzzy supercone geometries naturally emerge as shown in the middle. The right depicts the dual description as a family of the fuzzy supercones.

L_A , which change the external index m , act as the symmetry generators of the external space, while R_A that change the internal index g , act as symmetry generators of the internal space. Since the $UOSp(1|2)_L$ and $UOSp(1|2)_R$ form a dual pair, the Howe duality can be interpreted as a duality between the internal and external spaces:

$$\text{Internal (SUSY) space} \xleftrightarrow{\text{(Super)Howe duality}} \text{External (SUSY) space} \quad (7.25)$$

In this interpretation, the theta correspondence realizes a bijection between the internal and external spaces (6.20). Specifically, in the present system, the $\mathcal{N} = 1$ SUSY relating the bosonic and fermionic monopole harmonics in the external space *necessarily* induces the corresponding $\mathcal{N} = 1$ structure in the internal space, leading to the appearance of half-integer Landau levels as superpartners of the integer ones. The internal space structure is thus uniquely determined by that of the external space and vice versa.

It should be emphasized that this internal-external space correspondence is not a special feature in the present SUSY system, but rather a generic feature of systems with Howe duality. In a general prescription for constructing fuzzy coset manifolds G/H from Landau models, the gauge groups of the Landau models are identified with the stabilizer groups H [9]. While H originally represents a redundancy in the external symmetry G to define the coset manifold, it is reinterpreted in the Landau model as a redundancy of the internal space. In other words, the redundant external symmetry is equivalent to the internal gauge symmetry in the framework of coset-type Landau models, indicating the internal-external space correspondence. The D-matrix provides a concrete nonlinear realization of the coset, as demonstrated in the previous studies of the Landau models on complex (projective) manifolds [65,66,73,74,105] and on higher dimensional spheres [75,77–80]. Since Howe duality originates from the commutativity of the left- and right-actions of the D-matrix, it arises naturally in coset-type Landau models and their associated matrix geometries. Thus, Howe duality can be regarded as an underlying structure of these matrix

geometries. It has been shown that the analogous structure between internal and external spaces gives rise to a dimensional hierarchy [68,71]. Such a duality is closely related to the intrinsic structure of Matrix models, as their internal and external spaces are described by matrices of the same size arising from a common algebra [13,106]. Since the internal and external spaces share a common origin, a large external space necessarily implies an extended internal space. This particular feature of the internal space gives rise to a higher-spin structure in Matrix models [15–17,107,108].

In the language of the orbit method, a coadjoint orbit generated by a group action corresponds to an irreducible representation of that group, while the theta correspondence determines a relation between two representation spaces. Therefore, it is natural to expect a unique correspondence between the two families of such orbits. Since the fuzzy geometries are represented as quantized coadjoint orbits, the theta correspondence between fuzzy objects can be regarded as a quantized version of the relation between two orbit families, as exemplified by the geometric transformation in Sec. VII D. This observation also indicates the underlying role of Howe duality in fuzzy geometries.

VIII. SUMMARY AND DISCUSSIONS

We have performed a comprehensive analysis of the super-Landau model on a supersphere, encompassing both integer and half-integer higher LLs. By employing super-spinor derivative operators, we derived explicit forms of the supermonopole harmonics. In the half-integer LLs, the fermionic monopole harmonics are negative-norm states. A consistent prescription for resolving this probability problem is proposed using a modified inner product and a projection onto the body-sphere. We revealed that the $(UOSp(1|2), UOSp(1|2))$ self-dual super Howe duality constitutes the underlying structure of the entire super Hilbert space based on the super D-matrix formalism. We then explicitly derived the matrix elements for the supersphere-coordinates in arbitrary LLs, showing that the fuzzy supersphere geometries arise in all LLs. We finally presented a geometric interpretation of the super Howe duality

as the theta correspondence between the fuzzy super-spheres and fuzzy supercones.

We have also discussed that the Howe duality realizes an internal-external space duality. This observation leads to two intriguing possibilities. The first concerns a new interpretation of quantum numbers associated with external-space symmetries. Since the external space and internal spaces are essentially equivalent, the external quantum numbers may be interpreted as internal ones, for instance a possible topological interpretation of the magnetic quantum number. This issue will be discussed elsewhere. The second pertains to the structure of Matrix model geometries. As discussed in the main text, the internal and external spaces in Matrix models share a common matrix origin, while Howe duality naturally realizes this equivalence as an internal-external space duality. Furthermore, the previous studies, together with the orbit method, also suggest that Howe duality serves as an underlying structure in fuzzy geometries. Needless to mention, for Matrix models, matrices are of prime importance, while we have observed that matrix geometries are obtained from quantum states or irreducible representations. This representation-theoretic point of view may offer new insights into a fundamental understanding Matrix model geometries.

Furthermore, though we highlighted Howe duality in the Landau models, the concept is not restricted to such models. Howe duality is a central concept in representation theory, providing a unified perspective that links irreducible representations of two distinct groups as manifestations of a single underlying representation. Since fuzzy manifolds are constructed from irreducible representations (and their algebraic structures), the theta correspondence naturally relates two geometrically distinct fuzzy manifolds as distinct realizations of the same fuzzy object, namely a family of quantized coadjoint orbits. Howe duality encompasses cases where one member of the two groups is compact and the other is noncompact, with different dimensions. In such cases, we may have phenomenologically plausible matrix geometries that feature a compact internal space and a noncompact external space of different dimensions. The theta correspondence then accounts for geometric transformations between fuzzy Riemannian (positive-definite) and pseudo-Riemannian (indefinite) spaces. Additionally, the see-saw structure of Howe duality allows a single Hilbert space to accommodate multiple dual pairs. This suggests that a fuzzy space possesses multiple equivalent descriptions in terms of pairs of fuzzy spaces. Another remarkable feature of Howe duality is that irreducible representations of a larger group, which are difficult to construct directly, can be derived from those of a simpler group. In the context of NCG, this approach may be applied to obtain an intricate fuzzy manifold from a simpler one.

The application of supermonopole harmonics also offers an interesting direction to be pursued. Since monopole harmonics are mathematically equivalent to spin-weighted

spherical harmonics or Wigner's D-matrices, the super-monopole harmonics will possess an equally broad utility. One promising application is about the quantum mechanics of massless superparticle. As spin-weighted spherical harmonics describe massless particle states, their SUSY counterpart naturally describes massless superparticle states, which will be important in studies of supergravity and supertwistor theory.

As mentioned in the Introduction, recent studies indicate the existence of a SUSY structure in the Moore-Read state. Specifically, in the zero-momentum limit, spin-3/2 gravitino-mode excitations are expected to appear along with the graviton-mode. NC algebra is crucial for describing the magneto-roton along with its low energy limit, the recently observed chiral graviton [109,110]. We then naturally anticipate that the SUSY NCG may provide a possible geometric framework for the chiral gravitino-mode. It is rather stimulating to speculate how such a sophisticated mathematics of SUSY NCG could manifest itself in real physics.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE $uosp(1|2)$ ALGEBRA AND RELEVANT MATHEMATICS

1. The $uosp(1|2)$ algebra and irreducible representation

The $uosp(1|2)$ algebra consists of three bosonic generators L_i and two fermionic generators L_α , which satisfy

$$\begin{aligned} [L_i, L_j] &= i\epsilon_{ijk}L_k, & [L_i, L_\alpha] &= \frac{1}{2}(\sigma_i)_{\beta\alpha}L_\beta, \\ \{L_\alpha, L_\beta\} &= \frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i, \end{aligned} \quad (A1)$$

where $C := i\sigma_2$ denotes the $SU(2)$ charge conjugation matrix. The first equation denotes the $su(2)$ algebra, and the second relation shows the $SU(2)$ spinor property of the fermionic generators. The last equation implies that two successive applications of the fermionic operators generate the bosonic angular momentum operations:

$$L_{\theta_1}^2 = \frac{1}{4}L_+, \quad L_{\theta_2}^2 = -\frac{1}{4}L_- \quad (L_\pm := L_x \pm iL_y), \quad (A2)$$

and

$$\{L_{\theta_1}, L_{\theta_2}\} = -\frac{1}{2}L_z. \quad (\text{A3})$$

The $UOSP(1|2)$ Casimir is given by

$$\mathcal{C} = L_i L_i + C_{\alpha\beta} L_\alpha L_\beta. \quad (\text{A4})$$

Irreducible representations of the $UOSP(1|2)$ are indexed by the following superspin index

$$l = 0, 1/2, 1, 3/2, \dots, \quad (\text{A5})$$

and the corresponding Casimir eigenvalue and the dimension of the representation are respectively given by

$$l(l+1/2), \quad (\text{A6a})$$

$$4l+1. \quad (\text{A6b})$$

Under the $SU(2)$ subgroup of $UOSP(1|2)$, the irreducible representation decomposes into

$$4l+1 = (2l+1) \oplus (2l). \quad (\text{A7})$$

We refer to the $(2l+1)$ -dimensional and $(2l)$ -dimensional representations as the bosonic and fermionic states, denoted by $|l, m_1\rangle_B$ and $|l, m_2\rangle_F$, respectively. The magnetic quantum numbers range as $m_1 = l, l-1, \dots, -l$ and $m_2 = l-1/2, l-3/2, \dots, -l+1/2$. These states are assumed to satisfy

$$\mathcal{C}|l, m\rangle_{B/F} = l\left(l + \frac{1}{2}\right)|l, m\rangle_{B/F}. \quad (\text{A8})$$

The action of the $UOSP(1|2)$ operators on these states is given by [89,94,111]

$$\begin{aligned} L_\pm |l, m\rangle_B &= \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle_B, & L_3 |l, m\rangle_B &= m |l, m\rangle_B, \\ L_\pm |l, m\rangle_F &= \sqrt{\left(l \mp m - \frac{1}{2}\right)\left(l \pm m + \frac{1}{2}\right)} |l, m \pm 1\rangle_F, & L_3 |l, m\rangle_F &= m |l, m\rangle_F, \\ L_{\theta_1} |l, m\rangle_B &= \frac{1}{2} \sqrt{l-m} \left|l, m + \frac{1}{2}\right\rangle_F, & L_{\theta_2} |l, m\rangle_B &= -\frac{1}{2} \sqrt{l+m} \left|l, m - \frac{1}{2}\right\rangle_F, \\ L_{\theta_1} |l, m\rangle_F &= \frac{1}{2} \sqrt{l+m+\frac{1}{2}} \left|l, m + \frac{1}{2}\right\rangle_B, & L_{\theta_2} |l, m\rangle_F &= \frac{1}{2} \sqrt{l-m+\frac{1}{2}} \left|l, m - \frac{1}{2}\right\rangle_B. \end{aligned} \quad (\text{A9})$$

2. Scasimir and fermion parity

The Scasimir [112] is introduced as

$$S = -2C_{\alpha\beta} L_\alpha L_\beta + \frac{1}{4}. \quad (\text{A10})$$

This operator is a “square root” of the Casimir operator $\mathcal{C}$ (A4) in the following sense:

$$S^2 = \mathcal{C} + \frac{1}{16}. \quad (\text{A11})$$

Note that the Scasimir plays the role of a “center”; it commutes with the bosonic generators but anticommutes with the fermionic generators:

$$[S, L_i] = \{S, L_\alpha\} = 0. \quad (\text{A12})$$

The bosonic and fermionic states are the eigenstates of S with eigenvalues of the same magnitude but opposite signs:

$$S|l, m\rangle_B = \left(l + \frac{1}{4}\right)|l, m\rangle_B, \quad S|l, m\rangle_F = -\left(l + \frac{1}{4}\right)|l, m\rangle_F. \quad (\text{A13})$$

The sign of S indicates the fermion parity. The fermion number F and the fermion parity $(-1)^F$ are then respectively identified as

$$F = \frac{1}{2}(1 - \text{sgn}(S)), \quad (-1)^F = \text{sgn}(S). \quad (\text{A14})$$

3. Supermatrix realization

The $uosp(1|2)$ matrices for superspin l are expressed as

$$L_i^{(l)} = \begin{pmatrix} S_i^{(l)} & 0 \\ 0 & S_i^{(l-\frac{1}{2})} \end{pmatrix}, \quad L_\alpha^{(l)} = \frac{1}{2} \begin{pmatrix} 0_{2l+1} & \tau_\alpha^{(l)} \\ C_{\alpha\beta} \tau_\beta^{(l)\dagger} & 0_{2l} \end{pmatrix}. \quad (\text{A15})$$

Here, $S_i^{(s)}$ denote the $SU(2)$ spin matrices of spin s :

$$S_i^{(s)} S_i^{(s)} = s(s+1) \mathbf{1}_{2s+1}. \quad (\text{A16})$$

The $(2l+1) \times (2l)$ rectangular matrices $\tau_\alpha^{(l)}$ in $L_\alpha^{(l)}$ (A15) are defined as

$$\tau_{\theta_1}^{(l)} := \begin{pmatrix} \sqrt{2l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2l-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2l-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_{\theta_2}^{(l)} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2l-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2l} \end{pmatrix}. \quad (\text{A17})$$

The bosonic generators L_i in (A15) are the direct sum of $SU(2)$ spin generators whose spins differ by $1/2$. In the matrix representation, the Scasimir is expressed as

$$S^{(l)} = \left(l + \frac{1}{4}\right) \begin{pmatrix} 1_{2l+1} & 0 \\ 0 & -1_{2l} \end{pmatrix}. \quad (\text{A18})$$

The transformation matrix F_l (6.9) is given by

$$F_l(\eta, \eta^*) = e^{2\eta L_{\theta_1}^{(l)} + 2\eta^* L_{\theta_2}^{(l)}} = \begin{pmatrix} 1+l\eta^*\eta & 0 & 0 & 0 & 0 & 0 & \sqrt{2l}\eta & 0 & 0 & 0 & 0 \\ 0 & 1+l\eta^*\eta & 0 & 0 & 0 & 0 & \eta^* & \sqrt{2l-1}\eta & 0 & 0 & 0 \\ 0 & 0 & 1+l\eta^*\eta & 0 & 0 & 0 & 0 & \sqrt{2}\eta^* & \sqrt{2l-2}\eta & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \sqrt{3}\eta^* & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1+l\eta^*\eta & 0 & 0 & 0 & 0 & \ddots & \eta \\ 0 & 0 & 0 & 0 & 0 & 1+l\eta^*\eta & 0 & 0 & 0 & 0 & \sqrt{2l}\eta^* \\ -\sqrt{2l}\eta^* & \eta & 0 & 0 & 0 & 0 & 1-(l+\frac{1}{2})\eta^*\eta & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2l-1}\eta^* & \sqrt{2}\eta & 0 & 0 & 0 & 0 & 1-(l+\frac{1}{2})\eta^*\eta & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2l-2}\eta^* & \sqrt{3}\eta & 0 & 0 & 0 & 0 & 1-(l+\frac{1}{2})\eta^*\eta & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\eta^* & \sqrt{2l}\eta & 0 & 0 & 0 & 0 & 1-(l+\frac{1}{2})\eta^*\eta \end{pmatrix}. \quad (\text{A19})$$

APPENDIX B: BEREZINIAN AND THE AREA ELEMENT OF $S^{2|2}$

The Berezinian integral on $S^{2|2}$ is derived as follows. We consider the coordinate transformation from the super-Euclidean coordinates in $\mathbb{R}^{3|2}$ to the superspherical coordinates:

$$\begin{aligned} x_1 &= r \left(1 - \frac{1}{2r^2} \theta C \theta\right) \cos(\varphi) \sin(\vartheta), \\ x_2 &= r \left(1 - \frac{1}{2r^2} \theta C \theta\right) \sin(\varphi) \sin(\vartheta), \\ x_3 &= r \left(1 - \frac{1}{2r^2} \theta C \theta\right) \cos(\vartheta), \quad \theta'_1 = \theta_1, \quad \theta'_2 = \theta_2. \end{aligned} \quad (\text{B1})$$

Let us recall that $x_i = (1 - \frac{1}{2r^2}\theta C\theta)y_i$, in which y_i represent the standard Cartesian coordinates in $\mathbb{R}^3$ (4.4). The superdeterminant (Berezinian) associated with the coordinates transformation is given by

$$\text{Ber} = \text{sdet} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \frac{\det(A - C \frac{1}{B} D)}{\det(B)}. \quad (\text{B2})$$

In the present case (B1), the blocks of the super-Jacobian matrix take the form of

$$A = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{1}{2r^2}\theta C\theta\right) \frac{\partial y}{\partial r} & \left(1 - \frac{1}{2r^2}\theta C\theta\right) \frac{\partial y}{\partial \vartheta} & \left(1 - \frac{1}{2r^2}\theta C\theta\right) \frac{\partial y}{\partial \varphi} \end{pmatrix}, \quad (\text{B3})$$

and

$$\begin{aligned} C_{i\alpha} &= \frac{\partial x_i}{\partial \theta_\alpha} = -\frac{1}{r^2} x_i C_{\alpha\beta} \theta_\beta, & D_{ai} &= \frac{\partial \theta_\alpha}{\partial x_i} = 0, \\ B_{\alpha\beta} &= \frac{\partial \theta'_\alpha}{\partial \theta_\beta} = \delta_{\alpha\beta}. \end{aligned} \quad (\text{B4})$$

Since $D = 0$ and $\det(B) = 1$, the Berezinian simplifies to $\det(A)$, which is evaluated as

$$\begin{aligned} \text{Ber} = \det(A) &= \left(1 + \frac{1}{2r^2}\theta C\theta\right) \left(1 - \frac{1}{2r^2}\theta C\theta\right)^2 \\ &\times \det \begin{pmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \left(1 - \frac{1}{2r^2}\theta C\theta\right) r^2 \sin \vartheta. \end{aligned} \quad (\text{B5})$$

Consequently, the Berezin integral on $S^{2|2}$ is related to the integral on S^2 as

$$\begin{aligned} \int_{S^{2|2}} d\Omega_{2|2} &= \int_{S^2} d\vartheta d\varphi d\theta_1 d\theta_2 \text{Ber} \\ &= \int_{S^2} \overbrace{d\vartheta d\varphi \sin \vartheta}^{=d\Omega_2} \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2}\theta C\theta\right). \end{aligned} \quad (\text{B6})$$

For an arbitrary function on $\mathbb{R}^{3|2}$

$$f(x_i, \theta_\alpha) = f_0(y_i) + f_1(y_i)\theta_1 + f_2(y_i)\theta_2 + f_3(y_i)\theta_1\theta_2, \quad (\text{B7})$$

its integral over $S^{2|2}$ yields

$$\int_{S^{2|2}} d\Omega_{2|2} f(x_i, \theta_\alpha) = \int_{S^2} d\Omega_2 f_0(y_i) - \int_{S^2} d\Omega_2 f_3(y_i). \quad (\text{B8})$$

In particular, the integral of unity over $S^{2|2}$ is identical to that over S^2 :

$$\int_{S^{2|2}} d\Omega_{2|2} 1 = \int_{S^2} d\Omega_2 1. \quad (\text{B9})$$

APPENDIX C: SPECIAL LL EIGENSTATES

We examine two special limits of g , the LLL limit $|g| = \max(|g|) = l$, and the superspherical harmonics limit, $g = \min(|g|) = 0$. In these limits, the right-hand sides of (4.23) reduce to the first and second terms, respectively. We also discuss the lowest half-integer LL eigenstates.

1. Lowest Landau level eigenstates ($|g| = \max |g| = l$)

In the LLL ($N = 0$), the energy eigenvalues are given by (2.66). For $g > 0$, from (4.14) and (4.18), the LLL eigenstates are obtained as

$$Y_{g,m}^g(x_i, \theta_\alpha) = \sqrt{\frac{1}{4\pi} \frac{(2g)!}{(g+m)!(g-m)!}} u^{g+m} v^{g-m}, \quad (\text{C1a})$$

$$\begin{aligned} \mathcal{Y}_{g,m}^g(x_i, \theta_\alpha) &= \sqrt{2g} Y_{g-\frac{1}{2},m}^{g-\frac{1}{2}}, \\ \eta &= \sqrt{\frac{1}{4\pi} \frac{(2g)!}{(g+m-\frac{1}{2})!(g-m-\frac{1}{2})!}} u^{g+m-\frac{1}{2}} v^{g-m-\frac{1}{2}} \eta. \end{aligned} \quad (\text{C1b})$$

These are holomorphic functions of the super Hopf spinor components [29]. For example,

$$\begin{aligned} (l, g) = (1/2, 1/2): & Y_{1/2,1/2}^{1/2} = \frac{1}{\sqrt{4\pi}} u, \\ Y_{1/2,-1/2}^{1/2} &= \frac{1}{\sqrt{4\pi}} v, & \mathcal{Y}_{1/2,0}^{1/2} &= \frac{1}{\sqrt{4\pi}} \eta, \end{aligned} \quad (\text{C2a})$$

$$\begin{aligned} (l, g) = (1, 1): & Y_{1,1}^1 = \frac{1}{\sqrt{4\pi}} u^2, \\ Y_{1,0}^1(x_i, \theta_\alpha) &= \frac{1}{\sqrt{2\pi}} uv, & Y_{1,-1}^1 &= \frac{1}{\sqrt{4\pi}} v^2, \\ \mathcal{Y}_{1,1/2}^1 &= \frac{1}{\sqrt{2\pi}} u\eta, & \mathcal{Y}_{1,-1/2}^1 &= \frac{1}{\sqrt{2\pi}} v\eta. \end{aligned} \quad (\text{C2b})$$

Equation (C2a) corresponds to the components of the super Hopf spinor themselves, whereas (C2b) appears in Sec. VI B. Meanwhile for $g < 0$, the LLL eigenstates are

$$Y_{|g|,m}^{-|g|}(x_i, \theta_\alpha) = (-1)^{|g|+m} \sqrt{\frac{1}{4\pi} \frac{(2|g|)!}{(|g|+m)!(|g|-m)!}} u^{*|g|-m} v^{*|g|+m},$$

$$\mathcal{Y}_{|g|,m}^{-|g|}(x_i, \theta_\alpha) = -\sqrt{2|g|} Y_{|g|-\frac{1}{2},m}^{-|g|+\frac{1}{2}} \cdot \eta^* = (-1)^{|g|+m+\frac{1}{2}} \sqrt{\frac{1}{4\pi} \frac{(2|g|)!}{(|g|+m-\frac{1}{2})!(|g|-m-\frac{1}{2})!}} u^{*|g|-m-\frac{1}{2}} v^{*|g|+m-\frac{1}{2}} \eta^*, \quad (\text{C3})$$

which are antiholomorphic functions of the super Hopf spinor components. For instance,

$$(l, g) = (1/2, -1/2): Y_{1/2,1/2}^{-1/2} = -\frac{1}{\sqrt{4\pi}} v^*, \quad Y_{1/2,-1/2}^{-1/2} = \frac{1}{\sqrt{4\pi}} u^*, \quad \mathcal{Y}_{1/2,0}^{-1/2} = -\frac{1}{\sqrt{4\pi}} \eta^*, \quad (\text{C4a})$$

$$(l, g) = (1, -1): Y_{1,1}^{-1} = \frac{1}{\sqrt{4\pi}} v^{*2}, \quad Y_{1,0}^{-1} = -\frac{1}{\sqrt{2\pi}} u^* v^*, \quad Y_{1,-1}^{-1} = \frac{1}{\sqrt{4\pi}} v^{*2}, \quad \mathcal{Y}_{1,1/2}^{-1} = \frac{1}{\sqrt{2\pi}} v^* \eta^*, \quad \mathcal{Y}_{1,-1/2}^{-1} = -\frac{1}{\sqrt{2\pi}} u^* \eta^*. \quad (\text{C4b})$$

Equation (C4a) represents the pseudocomplex representation (3.14). The LLL eigenstates for $\pm g$ are simply related under the exchange:

$$u \leftrightarrow -v^*, \quad v \leftrightarrow u^*, \quad \eta \leftrightarrow -\eta^*. \quad (\text{C5})$$

The probability densities are

$$\rho_{|g|,m}^g(y_3) = |\Phi_{|g|,m}^g(y_3)|^2 = \frac{(2|g|+1)!}{4\pi(|g|+m)!(|g|-m)!} \left(\frac{r+y_3}{2r}\right)^{|g|+\text{sgn}(g)m} \left(\frac{r-y_3}{2r}\right)^{|g|-\text{sgn}(g)m}, \quad (\text{C6a})$$

$$\varrho_{|g|,m}^g(y_3) = \frac{1}{2(2|g|+1)} ((2|g|-2m+1)\rho_{|g|,m-\frac{1}{2}}^g(y_3) + (2|g|+2m+1)\rho_{|g|,m+\frac{1}{2}}^g(y_3)), \quad (\text{C6b})$$

where (C6b) can also be expressed in a concise form

$$\varrho_{|g|,m}^g(y_3) = \rho_{|g|-\frac{1}{2},m}^{g-\frac{1}{2}\text{sgn}(g)}(y_3). \quad (\text{C7})$$

The probability densities of the LLL fermionic harmonics thus coincide exactly with those of the original LLL monopole harmonics, provided the magnetic charge is shifted by $-\frac{1}{2}\text{sgn}(g)$. The extreme values (maxima) of $\rho_{|g|,m}^g(y_3)$ and $\varrho_{|g|,m}^g(y_3)$ occur at

$$\frac{y_3}{r} = \frac{m}{|g|}, \quad \frac{m}{|g|-\frac{1}{2}}, \quad (\text{C8})$$

which are distributed along the y_3 -axis with constant intervals of $\Delta y_3 = \frac{r}{|g|}$ and $\frac{r}{|g|-1/2}$, respectively.

2. Superspherical harmonics ($g = \min(|g|) = 0$)

The superspherical harmonics are the eigenstates of

$$H = \frac{1}{2Mr^2} (L_i^0 L_i^0 + C_{\alpha\beta} L_\alpha^0 L_\beta^0), \quad (\text{C9})$$

with the eigenvalues (2.63a). From (4.15) and (4.22), the superspherical harmonics ($l = 0, 1, 2, \dots$) are obtained as²⁵

$$Y_{l,m}^0(x_i) = \frac{1}{\sqrt{2l+1}} \Phi_{l,m}^0(x_i) = \frac{1}{\sqrt{2l+1}} \left(1 - \frac{l}{2} \theta C \theta\right) \Phi_{l,m}^0(y_i), \quad (\text{C11a})$$

²⁵With the Clebsch-Gordan coefficients, (C11b) can be written as

$$\mathcal{Y}_{l,m}^0(x_i, \theta_\alpha) = -\left\langle \frac{1}{2}, \frac{1}{2}; l-1, m-\frac{1}{2} \middle| l-\frac{1}{2}, m \right\rangle \Phi_{l-1, m-\frac{1}{2}}^0(x_i) \frac{\theta_1}{r} - \left\langle \frac{1}{2}, -\frac{1}{2}; l-1, m+\frac{1}{2} \middle| l-\frac{1}{2}, m \right\rangle \Phi_{l-1, m-\frac{1}{2}}^0(x_i) \frac{\theta_2}{r}. \quad (\text{C10})$$

$$\mathcal{Y}_{l,m}^0(x_i, \theta_a) = -\sqrt{\frac{l+m-\frac{1}{2}}{2l-1}} \Phi_{l-1, m-\frac{1}{2}}^0(x_i) \frac{\theta_1}{r} - \sqrt{\frac{l-m-\frac{1}{2}}{2l-1}} \Phi_{l-1, m+\frac{1}{2}}^0(x_i) \frac{\theta_2}{r}, \quad (\text{C11b})$$

where $\Phi_{l,m}^0$ denote the ordinary spherical harmonics. (Since the arguments x_i are not c-numbers, $\Phi_{l,m}^0(x_i)$ should be understood as a formal expression.) The bosonic spherical harmonics depend only on the bosonic coordinates x_i . Low-dimensional examples are

$$l=0: Y_{0,0}^0 = \frac{1}{\sqrt{4\pi}}, \quad (\text{C12a})$$

$$l=1: Y_{1,1}^0 = -\frac{1}{\sqrt{8\pi}} \frac{x_1 + ix_2}{r}, \quad Y_{1,0}^0 = \frac{1}{\sqrt{4\pi}} \frac{x_3}{r}, \quad Y_{1,-1}^0 = \frac{1}{\sqrt{8\pi}} \frac{x_1 - ix_2}{r}, \quad \mathcal{Y}_{1,1/2}^0 = -\frac{1}{\sqrt{4\pi}} \frac{\theta_1}{r}, \quad \mathcal{Y}_{1,-1/2}^0 = -\frac{1}{\sqrt{4\pi}} \frac{\theta_2}{r}, \quad (\text{C12b})$$

$$\begin{aligned} l=2: Y_{2,2}^0 &= \sqrt{\frac{3}{32\pi}} \left(\frac{x_1 + ix_2}{r} \right)^2, & Y_{2,1}^0 &= -\sqrt{\frac{3}{8\pi}} \frac{(x_1 + ix_2)x_3}{r^2}, & Y_{2,0}^0 &= \frac{1}{\sqrt{16\pi}} \frac{2x_3^2 - x_1^2 - x_2^2}{r^2}, \\ Y_{2,-1}^0 &= \sqrt{\frac{3}{8\pi}} \frac{(x_1 - ix_2)x_3}{r^2}, & Y_{2,-2}^0 &= \sqrt{\frac{3}{32\pi}} \left(\frac{x_1 - ix_2}{r} \right)^2, \\ \mathcal{Y}_{2,3/2}^0 &= \sqrt{\frac{3}{8\pi}} \frac{(x_1 + ix_2)\theta_1}{r^2}, & \mathcal{Y}_{2,1/2}^0 &= -\frac{1}{\sqrt{2\pi}} \frac{x_3\theta_1}{r^2} + \frac{1}{\sqrt{8\pi}} \frac{(x_1 + ix_2)\theta_2}{r^2}, \\ \mathcal{Y}_{2,-1/2}^0 &= -\frac{1}{\sqrt{8\pi}} \frac{(x_1 - ix_2)\theta_1}{r^2} - \frac{1}{\sqrt{2\pi}} \frac{x_3\theta_2}{r^2}, & \mathcal{Y}_{2,-3/2}^0 &= -\sqrt{\frac{3}{8\pi}} \frac{(x_1 - ix_2)\theta_2}{r^2}. \end{aligned} \quad (\text{C12c})$$

As expected, the components of the $l=1$ supermultiplet (C12b) are given by the supervector coordinates. The probability densities for (C11) are respectively given by

$$\rho_{l,m}^0(y_3) = |\Phi_{l,m}^0(y_i)|^2, \quad (\text{C13a})$$

$$\begin{aligned} q_{l,m}^0(y_3) &= \frac{1}{2(2l-1)} ((2l+2m-1)\rho_{l-1, m-\frac{1}{2}}^0(y_3) \\ &+ (2l-2m-1)\rho_{l-1, m+\frac{1}{2}}^0(y_3)). \end{aligned} \quad (\text{C13b})$$

Equation (C13b) indicates that the probability densities of the fermionic spherical harmonics are composed of the original ones with the angular momentum shifted by -1 . (C13b) can also be expressed as

$$q_{l,m}^0(y_i) = \frac{1}{2} \left(\rho_{l-\frac{1}{2}, m}^{\frac{1}{2}} + \rho_{l-\frac{1}{2}, m}^{\frac{3}{2}} \right). \quad (\text{C14})$$

The probability densities of superspherical harmonics exhibit reflection symmetry with respect to the $y_1 y_2$ plane: $\rho_{l,m}^0(y_3) = \rho_{l,m}^0(-y_3)$ and $q_{l,m}^0(y_3) = q_{l,m}^0(-y_3)$.

For half-integer l s ($l = 1/2, 3/2, \dots$), the superspherical harmonics are given by

$$\begin{aligned} Z_{l,m}^0(x_i, \theta_a) &= \sqrt{\frac{l-m+1}{2(l+1)}} \Phi_{l+\frac{1}{2}, m-\frac{1}{2}}^0(x_i) \frac{\theta_1}{r} \\ &- \sqrt{\frac{l+m+1}{2(l+1)}} \Phi_{l+\frac{1}{2}, m+\frac{1}{2}}^0(x_i) \frac{\theta_2}{r}, \end{aligned} \quad (\text{C15a})$$

$$\begin{aligned} \mathcal{Z}_{l,m}^0(x_i, \theta_i) &= \frac{1}{\sqrt{2l}} (1 + l\theta C\theta) \Phi_{l-\frac{1}{2}, m}^0(x_i) \\ &= \frac{1}{\sqrt{2l}} \left(1 + \frac{1}{2} \left(l + \frac{1}{2} \right) \theta C\theta \right) \Phi_{l-\frac{1}{2}, m}^0(y_i). \end{aligned} \quad (\text{C15b})$$

For instance,

$$\begin{aligned} l=1/2: \quad Z_{1/2, 1/2}^0 &= \frac{1}{\sqrt{4\pi}r^2} (x_3\theta_1 + (x_1 + ix_2)\theta_2), \\ Z_{1/2, -1/2}^0 &= \frac{1}{\sqrt{4\pi}r^2} ((x_1 - ix_2)\theta_1 - x_3\theta_2), \\ \mathcal{Z}_{1/2, 0}^0 &= \frac{1}{\sqrt{4\pi}} \left(1 + \frac{1}{2r^2} \theta C\theta \right). \end{aligned} \quad (\text{C16})$$

The probability densities for (C15) are

$$\begin{aligned} \rho_{l,m}^0(y_3) &:= \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C\theta \right) Z_{l,m}^{0*} Z_{l,m}^0 \\ &= \frac{l-m+1}{2(l+1)} |\Phi_{l+\frac{1}{2}, m-\frac{1}{2}}^0(y_3)|^2 + \frac{l+m+1}{2(l+1)} |\Phi_{l+\frac{1}{2}, m+\frac{1}{2}}^0(y_3)|^2, \end{aligned} \quad (\text{C17a})$$

$$\begin{aligned} \varrho_{l,m}^0(y_3) &:= - \int d\theta_1 d\theta_2 \left(1 - \frac{1}{2} \theta C \theta\right) \mathcal{Z}_{l,m}^{0*} \mathcal{Z}_{l,m}^0 \\ &= |\Phi_{l-\frac{1}{2},m}^0(y_3)|^2. \end{aligned} \quad (\text{C17b})$$

To the best of the author's knowledge, the superspherical harmonics were first constructed in Ref. [93]. While the specific forms may appear to differ, the present

formulas (C11) and (C15) are mathematically equivalent to those in [93].

3. Lowest half-integer Landau level eigenstates ($N=1/2$)

From the general formulas (5.11) and (5.14), we readily obtain the eigenstates for the lowest half-integer LL ($N=1/2$). In the case of a positive monopole charge $g > 0$, the eigenstates are given by

$$\begin{aligned} Z_{g+\frac{1}{2},m}^g &= \sqrt{\frac{2g+1}{4\pi} \frac{(2g+1)!}{(g+\frac{1}{2}+m)!(g+\frac{1}{2}-m)!}} u^{g+\frac{1}{2}+m} v^{g+\frac{1}{2}-m} \eta^* + \sqrt{\frac{1}{4\pi} \frac{(g+\frac{1}{2}+m)!(g+\frac{1}{2}-m)!}{(2g)!}} \\ &\cdot \left(\binom{2g}{g-m-\frac{1}{2}} u^{g+\frac{1}{2}+m} v^{g-\frac{1}{2}-m} u^* - \binom{2g}{g-m+\frac{1}{2}} u^{g-\frac{1}{2}+m} v^{g+\frac{1}{2}-m} v^* \right) \eta, \end{aligned} \quad (\text{C18a})$$

$$\mathcal{Z}_{g+\frac{1}{2},m}^g = \sqrt{\frac{1}{4\pi} \frac{(2g)!}{(g+m)!(g-m)!}} (1 - (2g+1)\eta^* \eta) u^{g+m} v^{g-m}. \quad (\text{C18b})$$

Similarly, for a negative monopole charge $g = -|g| < 0$, we obtain

$$\begin{aligned} Z_{|g|+\frac{1}{2},m}^{-|g|} &= (-1)^{|g|+\frac{1}{2}+m} \sqrt{\frac{2|g|+1}{4\pi} \frac{(2|g|+1)!}{(|g|+\frac{1}{2}+m)!(|g|+\frac{1}{2}-m)!}} u^{*|g|+\frac{1}{2}-m} v^{*|g|+\frac{1}{2}+m} \eta + (-1)^{|g|+\frac{1}{2}+m} \sqrt{\frac{1}{4\pi} \frac{(|g|+\frac{1}{2}+m)!(|g|+\frac{1}{2}-m)!}{(2|g|)!}} \\ &\cdot \left(\binom{2|g|}{|g|+m+\frac{1}{2}} u^{*|g|-\frac{1}{2}-m} v^{*|g|+\frac{1}{2}+m} v - \binom{2|g|}{|g|+m-\frac{1}{2}} u^{*|g|+\frac{1}{2}-m} v^{*|g|-\frac{1}{2}+m} u \right) \eta^*, \end{aligned} \quad (\text{C19a})$$

$$\mathcal{Z}_{|g|+\frac{1}{2},m}^{-|g|} = (-1)^{|g|+\frac{1}{2}+m} \sqrt{\frac{1}{4\pi} \frac{(2|g|)!}{(|g|+m)!(|g|-m)!}} (1 - (2|g|+1)\eta^* \eta) u^{*|g|-m} v^{*|g|+m}. \quad (\text{C19b})$$

Specifically, for $g = \pm 1/2$,

$$\begin{aligned} Z_{1,1}^{\frac{1}{2}} &= \frac{1}{\sqrt{2\pi}} (u^2 \eta^* - uv^* \eta), & Z_{1,0}^{\frac{1}{2}} &= \frac{1}{\sqrt{4\pi}} (2uv\eta^* + (uu^* - vv^*)\eta), & Z_{1,-1}^{\frac{1}{2}} &= \frac{1}{\sqrt{2\pi}} (u^* v \eta + v^2 \eta^*), \\ \mathcal{Z}_{1,\frac{1}{2}}^{\frac{1}{2}} &= \frac{1}{\sqrt{4\pi}} (1 - 2\eta^* \eta) u, & \mathcal{Z}_{1,\frac{1}{2}}^{\frac{1}{2}} &= \frac{1}{\sqrt{4\pi}} (1 - 2\eta^* \eta) v, \end{aligned} \quad (\text{C20a})$$

$$\begin{aligned} Z_{1,1}^{-\frac{1}{2}} &= \frac{1}{\sqrt{2\pi}} (v^{*2} \eta - uv^* \eta^*), & Z_{1,0}^{-\frac{1}{2}} &= -\frac{1}{\sqrt{4\pi}} (2u^* v^* \eta - (uu^* - vv^*) \eta^*), & Z_{1,-1}^{-\frac{1}{2}} &= \frac{1}{\sqrt{2\pi}} (u^* v \eta^* + u^{*2} \eta), \\ \mathcal{Z}_{1,\frac{1}{2}}^{-\frac{1}{2}} &= -\frac{1}{\sqrt{4\pi}} (1 - 2\eta^* \eta) v^*, & \mathcal{Z}_{1,\frac{1}{2}}^{-\frac{1}{2}} &= \frac{1}{\sqrt{4\pi}} (1 - 2\eta^* \eta) u^*. \end{aligned} \quad (\text{C20b})$$

APPENDIX D: USEFUL FORMULAS

Here, we summarize the formulas for the monopole harmonics and supermonopole harmonics used in the main text. The products of the monopole harmonics are

determined by the $SU(2)$ Clebsch-Gordan decomposition rule:

$$s \otimes s' = \bigoplus_{S=|s-s'|}^{s+s'} S \quad (\Delta s = \Delta s' = \Delta S = 1), \quad (\text{D1})$$

or

$$(2s+1)(2s'+1) = \sum_{S=|s-s'|}^{s+s'} (2S+1). \quad (\text{D2})$$

Similarly, the products of the supermonopole harmonics are determined by the $UOSP(1|2)$ Clebsch-Gordan decomposition [89,94,111,113]:

$$l \otimes l' = \bigoplus_{L=|l-l'|}^{l+l'} L \quad (\Delta l = \Delta l' = \Delta L = 1/2), \quad (\text{D3})$$

or

$$(4l+1)(4l'+1) = \sum_{L=|l-l'|}^{l+l'} (4L+1). \quad (\text{D4})$$

1. For the monopole harmonics

The properties of monopole harmonics are discussed in [114].

a. General formulas

The product of two Wigner D -functions is expanded as

$$D_{l,m_1,m_2} \cdot D_{l',m'_1,m'_2} = \sum_{L,M_1,M_2} \langle l, m_1; l', m'_1 | L, M_1 \rangle D_{L,M_1,M_2} \langle L, M_2 | l, m_2; l', m'_2 \rangle. \quad (\text{D5})$$

In terms of the monopole harmonics (6.4), (D5) becomes

$$\Phi_{l,m}^g \cdot \Phi_{l',m'}^{g'} = \sum_{L,G,M} \sqrt{\frac{1}{4\pi} \frac{(2l+1)(2l'+1)}{2L+1}} \langle l, g; l', g' | L, G \rangle \Phi_{L,M}^G \langle L, M | l, m; l', m' \rangle. \quad (\text{D6})$$

The triple integral over S^2 is then given by

$$\int_{S^2} d\Omega_2 \Phi_{L,M}^G \cdot \Phi_{l,m}^g \cdot \Phi_{l',m'}^{g'} = \sqrt{\frac{1}{4\pi} \frac{(2l+1)(2l'+1)}{2L+1}} \langle l, g; l', g' | L, G \rangle \langle L, M | l, m; l', m' \rangle, \quad (\text{D7})$$

which can be expressed symmetrically using $3j$ -symbols:

$$\int_{S^2} d\Omega_2 \Phi_{l_1,m_1}^{g_1} \cdot \Phi_{l_2,m_2}^{g_2} \cdot \Phi_{l_3,m_3}^{g_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \quad (\text{D8})$$

b. Low-dimensional examples

From (D6), we have

$$\Phi_{1/2,\pm 1/2}^{1/2} \Phi_{l,m}^g = \sqrt{\frac{(l+g+1)(l \pm m+1)}{4\pi(l+1)(2l+1)}} \Phi_{l+\frac{1}{2},m \pm \frac{1}{2}}^{g+\frac{1}{2}} \pm \sqrt{\frac{(l-g)(l \mp m)}{4\pi l(2l+1)}} \Phi_{l-\frac{1}{2},m \pm \frac{1}{2}}^{g-\frac{1}{2}}, \quad (\text{D9a})$$

$$\Phi_{1/2,\pm 1/2}^{-1/2} \Phi_{l,m}^g = \sqrt{\frac{(l-g+1)(l \pm m+1)}{4\pi(l+1)(2l+1)}} \Phi_{l+\frac{1}{2},m \pm \frac{1}{2}}^{g-\frac{1}{2}} \mp \sqrt{\frac{(l+g)(l \mp m)}{4\pi l(2l+1)}} \Phi_{l-\frac{1}{2},m \pm \frac{1}{2}}^{g+\frac{1}{2}}, \quad (\text{D9b})$$

and

$$\begin{aligned} \Phi_{1,\pm 1}^0 \Phi_{lm}^g &= \mp \sqrt{\frac{3}{8\pi l(l+1)}} \sqrt{(l \pm m+1)(l \mp m)} \Phi_{l,m \pm 1}^g \\ &+ \sqrt{\frac{3}{8\pi l+1}} \sqrt{\frac{((l+1)^2 - g^2)(l \pm m+2)(l \pm m+1)}{(2l+3)(2l+1)}} \Phi_{l+1,m \pm 1}^g - \sqrt{\frac{3}{8\pi l}} \sqrt{\frac{(l^2 - g^2)(l \mp m)(l \mp m-1)}{(2l+1)(2l-1)}} \Phi_{l-1,m \pm 1}^g, \\ \Phi_{1,0}^0 \Phi_{lm}^g &= \sqrt{\frac{3}{4\pi l(l+1)}} m \Phi_{l,m}^g + \sqrt{\frac{3}{4\pi l+1}} \sqrt{\frac{((l+1)^2 - g^2)((l+1)^2 - m^2)}{(2l+3)(2l+1)}} \Phi_{l+1,m}^g \\ &+ \sqrt{\frac{3}{4\pi l}} \sqrt{\frac{(l^2 - g^2)(l^2 - m^2)}{(2l+1)(2l-1)}} \Phi_{l-1,m}^g. \end{aligned} \quad (\text{D10})$$

Hence,

$$\int_{S^2} d\Omega_2 \Phi_{l,m'}^g \cdot \Phi_{1,\pm 1}^0 \cdot \Phi_{l,m}^g = \mp \sqrt{\frac{3}{8\pi}} \frac{g}{l(l+1)} \sqrt{(l \pm m + 1)(l \mp m)} \delta_{m',m \pm 1}, \quad (\text{D11a})$$

$$\int_{S^2} d\Omega_2 \Phi_{l,m'}^g \cdot \Phi_{1,0}^0 \cdot \Phi_{l,m}^g = \sqrt{\frac{3}{4\pi}} \frac{g}{l(l+1)} m \delta_{m',m}. \quad (\text{D11b})$$

These formulas are used in Sec. VII A.

2. For the supermonopole harmonics

We extend the formulas of the monopole harmonics to the supermonopole harmonics.

a. General formulas

The products of the supermonopole harmonics are expanded as

$$Y_{l,m}^g \cdot Y_{l',m'}^{g'} = \frac{1}{\sqrt{4\pi}} \left(1 + (L - l - l') \frac{1}{2} \theta C \theta \right) \sum_{L,G,M} \langle l, g; l', g' | L, G \rangle Y_{L,M}^G \langle L, M | l, m; l', m' \rangle, \quad (\text{D12a})$$

$$\begin{aligned} \mathcal{Y}_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = & -\frac{1}{2\sqrt{4\pi}} \theta C \theta \cdot \sum_{L,G,M} \left(\sqrt{(l+g)(l'-g')} \left\langle l - \frac{1}{2}, g - \frac{1}{2}; l' - \frac{1}{2}, g' + \frac{1}{2} \middle| L, G \right\rangle \right. \\ & \left. - \sqrt{(l-g)(l'+g')} \left\langle l - \frac{1}{2}, g + \frac{1}{2}; l' - \frac{1}{2}, g' - \frac{1}{2} \middle| L, G \right\rangle \right) \cdot Y_{L,M}^G \left\langle L, M \middle| l - \frac{1}{2}, m; l' - \frac{1}{2}, m' \right\rangle, \end{aligned} \quad (\text{D12b})$$

$$\begin{aligned} Y_{l,m}^g \mathcal{Y}_{l',m'}^{g'} = & \mathcal{Y}_{l',m'}^{g'} Y_{l,m}^g = \frac{1}{\sqrt{4\pi}} \sum_{L,G,M} \left(\sqrt{l'+g'} \left\langle l, g; l' - \frac{1}{2}, g' - \frac{1}{2} \middle| L, G \right\rangle \eta \right. \\ & \left. - \sqrt{l'-g'} \left\langle l, g; l' - \frac{1}{2}, g' + \frac{1}{2} \middle| L, G \right\rangle \eta^* \right) Y_{L,M}^G \left\langle L, M \middle| l, m; l' - \frac{1}{2}, m' \right\rangle. \end{aligned} \quad (\text{D12c})$$

Here, we used (4.15), (4.20) and (D6). The triple integrals on $S^{2|2}$ are derived as

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{L,M}^G \cdot Y_{l,m}^g \cdot Y_{l',m'}^{g'} = \frac{1}{\sqrt{4\pi}} \frac{L + l + l' + 1}{2L + 1} \langle l, g; l', g' | L, G \rangle \langle L, M | l, m; l', m' \rangle, \quad (\text{D13a})$$

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{L,M}^G \cdot \mathcal{Y}_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = \frac{1}{\sqrt{4\pi}} \frac{1}{2L + 1} \quad (\text{D13b})$$

$$\begin{aligned} & \cdot \left(\sqrt{(l+g)(l'-g')} \left\langle l - \frac{1}{2}, g - \frac{1}{2}; l' - \frac{1}{2}, g' + \frac{1}{2} \middle| L, G \right\rangle - \sqrt{(l-g)(l'+g')} \left\langle l - \frac{1}{2}, g + \frac{1}{2}; l' - \frac{1}{2}, g' - \frac{1}{2} \middle| L, G \right\rangle \right) \\ & \cdot \left\langle L, M \middle| l - \frac{1}{2}, m; l' - \frac{1}{2}, m' \right\rangle, \end{aligned} \quad (\text{D13c})$$

$$\begin{aligned} \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = & \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot \mathcal{Y}_{l',m'}^{g'} \cdot Y_{l,m}^g = \frac{1}{\sqrt{4\pi}} \frac{1}{2L} \\ & \cdot \left(\sqrt{(L+G)(l'+g')} \left\langle l, g; l' - \frac{1}{2}, g' - \frac{1}{2} \middle| L - \frac{1}{2}, G - \frac{1}{2} \right\rangle + \sqrt{(L-G)(l'-g')} \left\langle l, g; l' - \frac{1}{2}, g' + \frac{1}{2} \middle| L - \frac{1}{2}, G + \frac{1}{2} \right\rangle \right) \\ & \cdot \left\langle L - \frac{1}{2}, M \middle| l, m; l' - \frac{1}{2}, m' \right\rangle, \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot Y_{l',m'}^{g'} = \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot \mathcal{Y}_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = 0. \end{aligned} \quad (\text{D13d})$$

Other triple integrals are

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{L,M}^G \cdot Y_{l,m}^g \cdot Y_{l',m'}^{g'*} = \int_{S^{2|2}} d\Omega_{2|2} Y_{L,M}^G \cdot Y_{l,m}^g \cdot Y_{l',m'}^{g'}, \quad (\text{D14a})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'}, \quad (\text{D14b})$$

the $3j$ -symbols, the triple integrals are represented as

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'} = - \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{L,M}^G \cdot Y_{l,m}^g \cdot \mathcal{Y}_{l',m'}^{g'}, \quad (\text{D14c})$$

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l_1,m_1}^{g_1} \cdot Y_{l_2,m_2}^{g_2} \cdot Y_{l_3,m_3}^{g_3} = \frac{1}{\sqrt{4\pi}} (l_1 + l_2 + l_3 + 1) \begin{pmatrix} l_1 & l_2 & l_3 \\ g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad (\text{D15a})$$

where the right-hand sides are given by (D13). In terms of

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l_1,m_1}^{g_1} \cdot \mathcal{Y}_{l_2,m_2}^{g_2} \cdot \mathcal{Y}_{l_3,m_3}^{g_3} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_2 - \frac{1}{2} & l_3 - \frac{1}{2} \\ m_1 & m_2 & m_3 \end{pmatrix} \cdot \left(\sqrt{(l_2 + g_2)(l_3 - g_3)} \begin{pmatrix} l_1 & l_2 - \frac{1}{2} & l_3 - \frac{1}{2} \\ g_1 & g_2 - \frac{1}{2} & g_3 + \frac{1}{2} \end{pmatrix} - \sqrt{(l_2 - g_2)(l_3 + g_3)} \begin{pmatrix} l_1 & l_2 - \frac{1}{2} & l_3 - \frac{1}{2} \\ g_1 & g_2 + \frac{1}{2} & g_3 - \frac{1}{2} \end{pmatrix} \right), \quad (\text{D15b})$$

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l_1,m_1}^{g_1} \cdot Y_{l_2,m_2}^{g_2} \cdot \mathcal{Y}_{l_3,m_3}^{g_3} = \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l_1,m_1}^{g_1} \cdot \mathcal{Y}_{l_2,m_2}^{g_2} \cdot \mathcal{Y}_{l_3,m_3}^{g_3} = 0. \quad (\text{D15c})$$

For other supermonopole harmonics, we have

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l_1,m_1}^{g_1} \cdot \mathcal{Z}_{l_2,m_2}^{g_2} \cdot \mathcal{Z}_{l_3,m_3}^{g_3} = -\frac{1}{\sqrt{4\pi}} \left(l_1 + l_2 + l_3 + \frac{1}{2} \right) \begin{pmatrix} l_1 - \frac{1}{2} & l_2 - \frac{1}{2} & l_3 - \frac{1}{2} \\ g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} l_1 - \frac{1}{2} & l_2 - \frac{1}{2} & l_3 - \frac{1}{2} \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad (\text{D16a})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l_1,m_1}^{g_1} \cdot \mathcal{Z}_{l_2,m_2}^{g_2} \cdot \mathcal{Z}_{l_3,m_3}^{g_3} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} l_1 - \frac{1}{2} & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \cdot \left(\sqrt{\left(l_2 + g_2 + \frac{1}{2} \right) \left(l_3 - g_3 + \frac{1}{2} \right)} \begin{pmatrix} l_1 - \frac{1}{2} & l_2 & l_3 \\ g_1 & g_2 + \frac{1}{2} & g_3 - \frac{1}{2} \end{pmatrix} - \sqrt{\left(l_2 - g_2 + \frac{1}{2} \right) \left(l_3 + g_3 + \frac{1}{2} \right)} \begin{pmatrix} l_1 - \frac{1}{2} & l_2 & l_3 \\ g_1 & g_2 - \frac{1}{2} & g_3 + \frac{1}{2} \end{pmatrix} \right), \quad (\text{D16b})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l_1,m_1}^{g_1} \cdot \mathcal{Z}_{l_2,m_2}^{g_2} \cdot \mathcal{Z}_{l_3,m_3}^{g_3} = \int_{S^{2|2}} d\Omega_{2|2} \mathcal{Z}_{l_1,m_1}^{g_1} \cdot \mathcal{Z}_{l_2,m_2}^{g_2} \cdot \mathcal{Z}_{l_3,m_3}^{g_3} = 0. \quad (\text{D16c})$$

b. Low-dimensional examples

Equations (D13) yield

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l,m'}^g \cdot Y_{1,\pm 1}^0 \cdot Y_{l,m}^g = \mp \frac{1}{\sqrt{2\pi} l(2l+1)} \sqrt{l(l+1)} \mp m(m \pm 1) \delta_{m',m \pm 1}, \quad (\text{D17a})$$

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l,m'}^g \cdot Y_{1,0}^0 \cdot Y_{l,m}^g = \frac{1}{\sqrt{\pi} l(2l+1)} m \delta_{m',m}, \quad (\text{D17b})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l,m'}^g \cdot Y_{1,\pm 1}^0 \cdot \mathcal{Y}_{l,m}^g = \mp \frac{1}{\sqrt{2\pi} l(2l+1)} \sqrt{\left(l - \frac{1}{2} \right) \left(l + \frac{1}{2} \right)} \mp m(m \pm 1) \delta_{m',m \pm 1}, \quad (\text{D17c})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l,m'}^g \cdot Y_{1,0}^0 \cdot \mathcal{Y}_{l,m}^g = \frac{1}{\sqrt{\pi} l(2l+1)} m \delta_{m',m}, \quad (\text{D17d})$$

$$\int_{S^{2|2}} d\Omega_{2|2} Y_{l,m'}^g \cdot \mathcal{Y}_{1,\pm \frac{1}{2}}^0 \cdot \mathcal{Y}_{l,m}^g = -\frac{1}{\sqrt{\pi} 2l(2l+1)} \sqrt{l \pm m + \frac{1}{2}} \delta_{m',m \pm \frac{1}{2}}, \quad (\text{D17e})$$

$$\int_{S^{2|2}} d\Omega_{2|2} \mathcal{Y}_{l,m'}^g \cdot \mathcal{Y}_{1,\pm\frac{1}{2}}^0 \cdot Y_{l,m}^g \\ = \mp \frac{1}{\sqrt{\pi}} \frac{g}{2l(2l+1)} \sqrt{l \mp m} \delta_{m',m\pm\frac{1}{2}}. \quad (\text{D17f})$$

These formulas are used in Sec. VII B 1.

APPENDIX E: THE LOWEST LANDAU LEVEL SUPERGEOMETRY

Here, we provide concise derivations of the fuzzy supersphere geometry for the LLL, employing the Lagrangian and Hamiltonian formalisms, respectively.

1. From Lagrange formalism

The SUSY Lagrangian for the present system may be given by

$$L = \frac{M}{2} (\dot{x}_i \dot{x}_i + C_{\alpha\beta} \dot{\theta}_\alpha \dot{\theta}_\beta) - \dot{x}_i A_i - \dot{\theta}_\alpha A_\alpha. \quad (\text{E1})$$

In the LLL, the kinetic terms are quenched, and the Lagrangian reduces to a first-order form:

$$L_{\text{LLL}} = -\dot{x}_i A_i - \dot{\theta}_\alpha A_\alpha = -2ig\psi^\dagger \dot{\psi}. \quad (\text{E2})$$

The canonical momenta for $\psi_a = (u, v, \eta)$ are derived as

$$\pi_a = \frac{\partial}{\partial \dot{\psi}_a} L_{\text{LLL}} = -2ig\psi_a^*. \quad (\text{E3})$$

Imposing the canonical supercommutation relations $[\pi_a, \psi_b] = -i\delta_{ab}$, we obtain

$$\pi_a = -i \frac{\partial}{\partial \psi_a} \quad (\text{E4})$$

or

$$\psi_a^* = \frac{1}{2g} \frac{\partial}{\partial \psi_a}. \quad (\text{E5})$$

By substituting (E5) into the graded Hopf map (2.6), we promote the coordinates of the supersphere to operators:

$$X_i = \frac{r}{g} \psi^\dagger l_i \frac{\partial}{\partial \psi}, \quad \Theta_\alpha = \frac{r}{g} \psi^\dagger l_\alpha \frac{\partial}{\partial \psi}. \quad (\text{E6})$$

They obviously satisfy the superalgebra of the fuzzy supersphere;

$$[X_i, X_j] = i \frac{r}{g} \epsilon_{ijk} X_k, \quad [X_i, \Theta_\alpha] = \frac{r}{2g} (\sigma_i)_{\beta\alpha} \Theta_\beta, \\ \{\Theta_\alpha, \Theta_\beta\} = \frac{r}{2g} (C\sigma_i)_{\alpha\beta} X_i, \quad (\text{E7})$$

and

$$X_i X_i + C_{\alpha\beta} \Theta_\alpha \Theta_\beta = \left(\frac{r}{g}\right)^2 L \left(L + \frac{1}{2}\right) = r^2 \left(1 + \frac{1}{2|g|}\right), \quad (\text{E8})$$

where L is given by (3.18) with eigenvalue $|g|$.

2. From Hamilton formalism

In the LLL, the kinetic part of the Hamiltonian is quenched, which effectively sets the covariant derivatives to zero: $D_i \rightarrow 0$ and $D_\alpha \rightarrow 0$. With this replacement, the L_A (2.43) reduce to the following forms [29]:

$$L_i \rightarrow r^2 B_i = \frac{g}{r} x_i, \quad L_\alpha \rightarrow r^2 B_\alpha = \frac{g}{r} \theta_\alpha, \quad (\text{E9})$$

implying that the superspin index l reduces to $|g|$. Then, the coordinates on the supersphere can be identified with the $UOSP(1|2)$ generators:

$$X_i = \frac{r}{g} L_i, \quad \Theta_\alpha = \frac{r}{g} L_\alpha, \quad (\text{E10})$$

which obey the supercommutation relations (E7) and satisfy the fuzzy supersphere constraint:

$$X_i X_i + C_{\alpha\beta} \Theta_\alpha \Theta_\beta = \left(\frac{r}{g}\right)^2 (L_i L_i + C_{\alpha\beta} L_\alpha L_\beta) \\ = r^2 \left(1 + \frac{1}{2|g|}\right). \quad (\text{E11})$$

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