

Comment on “QCD factorization with multihadron fragmentation functions”

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We make several comments on the recent work in Rogers *et al.* [Phys. Rev. D **111**, 056001 (2025)] while also reaffirming and adding to the work in Pitonyak *et al.* [Phys. Rev. Lett. **132**, 011902 (2024)]. We show that the factorization formula for $e^+e^- \rightarrow (h_1 \cdots h_n)X$ in Rogers *et al.* is equivalent to a version one can derive using the definition of a n -hadron fragmentation function (FF) introduced in Pitonyak *et al.*. In addition, we scrutinize how to generalize the number density definition of a single-hadron FF to a n -hadron FF, arguing that the definition given in Pitonyak *et al.* should be considered the standard one while the definition in Rogers *et al.* has no clear interpretation. We also emphasize that the evolution equations for dihadron FFs (DiFFs) in Pitonyak *et al.* have the same splitting functions as those for single-hadron FFs. Therefore, the DiFF (and n -hadron FF) definitions in Pitonyak *et al.* have a natural number density interpretation and, contrary to what is stated in Rogers *et al.*, are consistent with collinear factorization using the standard hard factors and evolution kernels.

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I. INTRODUCTION

In Ref. [1], a factorization formula is put forth in e^+e^- annihilation for the production of a small-mass cluster of n hadrons, $e^+e^- \rightarrow (h_1 \cdots h_n)X$, in terms of a n -hadron fragmentation function (FF) $d(\xi, -\xi\vec{k}_T, \{P_h\})$ whose operator definition differs from the n -hadron FF $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ defined in Ref. [2].¹ The derivation in Ref. [1] is a leading-order (parton model) calculation that is promoted to a general factorization formula. No higher-order calculation was performed. Consequently, when we use the word “factorization” in this Comment, we are doing so to align with the language of Ref. [1], not as any statement or confirmation of all-orders factorization for $e^+e^- \rightarrow (h_1 \cdots h_n)X$. With regard to the fundamental definition of

a n -hadron FF as a number density, we emphasize that factorization cannot give a different definition than one based on the parton model calculations and sum rule derivations that were performed in Ref. [2]. In the following, we make several comments on Ref. [1] while reaffirming the work of Ref. [2] and adding some new calculations that further support its results. After setting up some notation and definitions in Sec. II, we show in Sec. III that the factorization formula in Ref. [1] involving $d(\xi, -\xi\vec{k}_T, \{P_h\})$ is equivalent to one involving $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ with the same standard hard factor, and we show explicitly how to connect the two. In Sec. IV, we examine the number density interpretations of $d(\xi, -\xi\vec{k}_T, \{P_h\})$ and $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$, arguing that for the latter it is unambiguous and a natural starting point from which one can define n -hadron FFs as number densities in other variables. In Sec. V, we briefly discuss the evolution equations for dihadron FFs (DiFFs) derived in Ref. [2], emphasizing that they have the same standard splitting functions as single-hadron FFs. We summarize in Sec. VI.

II. NOTATION AND FRAGMENTATION FUNCTION DEFINITIONS

We refer the reader to Refs. [1,2] for a detailed discussion of different reference frames and kinematic

¹For $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ from Ref. [2], we are using the shorthand notation $\{h_i\}_n \equiv h_1 \cdots h_n$, $\{\xi_i\}_n \equiv \xi_1, \dots, \xi_n$, $\{\vec{P}_{i\perp}\}_n \equiv \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp}$ to indicate the n -hadron FF is a number density in $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$. The $\{\vec{P}_{i\perp}\}_n$ notation is not to be confused with $\{P_h\}$ in $d(\xi, -\xi\vec{k}_T, \{P_h\})$ from Ref. [1].

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variables.² We mention here that the hadrons carry individual momentum fractions $\xi_i = P_i^-/k^-$ and have a total momentum fraction $\xi \equiv \sum_{i=1}^n \xi_i$, where k is the momentum of the fragmenting parton, P_i is the momentum of hadron h_i , and $P_h \equiv \sum_{i=1}^n P_i$ with $P_h^2 = M_h^2$. For the case of a

dihadron, we also have $\zeta \equiv (P_1^- - P_2^-)/P_h^- = (\xi_1 - \xi_2)/\xi$ and $\vec{R}_T \equiv (\vec{P}_{1T} - \vec{P}_{2T})/2$.³ We start with the following n -hadron fragmentation correlator (which extends the dihadron correlator from Ref. [3] to include n hadrons in the final state),

$$\Delta^{\{h_i\}_n} \equiv \text{Tr} \sum_X \int \frac{dx^+ d^2 \vec{x}_\perp}{(2\pi)^3} e^{ik \cdot x} \langle 0 | \gamma^- \psi(x) | P_1, \dots, P_n; X \rangle \langle P_1, \dots, P_n; X | \bar{\psi}(0) | 0 \rangle |_{x^- = 0}. \quad (1)$$

This correlator is used in both Refs. [1] and [2] as the foundation through which n -hadron FFs are defined. In Eq. (10) of Ref. [2], a fundamental n -hadron FF $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ is introduced that is a number density in the $3n$ variables $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$,

$$D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n) = \frac{1}{4(16\pi^3)^{n-1} \xi_1 \dots \xi_n} \Delta^{\{h_i\}_n}. \quad (2)$$

A number density for any set of variables is then defined by using Eq. (13) of Ref. [2] to account for the Jacobian of the variable transformation.

In Eq. (79) of Ref. [1], a different operator definition of a n -hadron FF, denoted $d(\xi, -\xi \vec{k}_T, \{P_h\})$, was put forth,

$$d(\xi, -\xi \vec{k}_T, \{P_h\}) = \frac{1}{4\xi} \Delta^{\{h_i\}_n}, \quad (3)$$

which differs from Eq. (2) above in the prefactor and arises from the reasoning that the factorization steps for $e^+e^- \rightarrow (h_1 \dots h_n)X$ do not change from the single-hadron case [1].

In the following sections, we elaborate on the differences between $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ from Ref. [2] and $d(\xi, -\xi \vec{k}_T, \{P_h\})$ from Ref. [1]. We point out that the argument $\{P_h\}$ in $d(\xi, -\xi \vec{k}_T, \{P_h\})$ is meant to “symbolize dependence on all of the observed n hadron momenta” [1]. This is ambiguous and *a priori* prevents one from a meaningful discussion about whether $d(\xi, -\xi \vec{k}_T, \{P_h\})$ represents a number density in a specific set of $3n$ variables. The only definitive statement Ref. [1] makes in this regard is that, just like the single-hadron FF case that also has the same prefactor as Eq. (3), $d(\xi, -\xi \vec{k}_T, \{P_h\})$ is a number density in $(\xi, \xi \vec{k}_T)$ after integrating out all the relative

internal momenta of the n hadrons.⁴ However, it is important to know in the first place in which specific variables involving those internal momenta a n -hadron FF is a number density. A clear number density operator definition at the fully unintegrated level allows the most information to be gained phenomenologically from the rich structure of multihadron dynamics that exist when moving beyond single-hadron fragmentation.

For the $n = 2$ (dihadron) case, we explicitly write here for ease of reference the operator definitions for the DiFFs $D_1^{h_1 h_2}(\xi, \zeta, \vec{R}_T)$, $D_1^{h_1 h_2}(\xi_1, \xi_2, \vec{R}_T)$, and $D_1^{h_1 h_2}(\xi, M_h)$ from Ref. [2],

$$D_1^{h_1 h_2}(\xi, \zeta, \vec{R}_T) = \frac{\xi}{32\pi^3(1 - \zeta^2)} \int d^2 \vec{k}_T \Delta^{h_1 h_2}, \quad (4a)$$

$$D_1^{h_1 h_2}(\xi_1, \xi_2, \vec{R}_T) = \frac{\xi^2}{64\pi^3 \xi_1 \xi_2} \int d^2 \vec{k}_T \Delta^{h_1 h_2}, \quad (4b)$$

$$D_1^{h_1 h_2}(\xi, M_h) = \frac{\xi M_h}{64\pi^2} \int d\zeta \int d^2 \vec{k}_T \Delta^{h_1 h_2}. \quad (4c)$$

These DiFFs are number densities in $(\xi, \zeta, \vec{R}_T)$, $(\xi_1, \xi_2, \vec{R}_T)$, and (ξ, M_h) , respectively. We remark that using Eqs. (108), (105), and (81) from Ref. [1] one readily finds that $d_{\text{red},1}(\xi, M_h)$ defined there exactly matches Eq. (4c). In Ref. [1], it was implied that this definition appeared previously in the literature, e.g., in Ref. [4]. In the context of $D_1^{h_1 h_2}(\xi, M_h)$, Ref. [4] cites Ref. [5]. By using Eqs. (32), (29), (16), and (7) of Ref. [5], we attempted to write down the operator definition of $D_1^{h_1 h_2}(\xi, M_h)$ in terms of $\Delta^{h_1 h_2}$ in Eq. (1) and found

³As in Ref. [2], we use $\perp(T)$ to denote transverse components in the parton (n -hadron) frame.

⁴Note that factors of $1/(16\pi^3 E'_i)$, where E'_i are energy variables in what Ref. [1] denotes as dY' , must also be included when integrating out those relative internal momenta—see Eqs. (74), (82), and (83) of Ref. [1].

²In Ref. [1], the fragmenting parton has a large plus-light-cone momentum k^+ , whereas here we follow Ref. [2] and take it to have a large minus-light-cone momentum k^- .

$$D_1^{h_1 h_2}(\xi, M_h)|_{\text{Ref. [5]}} = \frac{\pi \xi}{16} \int d\zeta \int d^2 \vec{k}_T \Delta^{h_1 h_2}, \quad (5)$$

which does not match Eq. (4c). Similarly, we could not find any paper from Refs. [6–9] that, after attempting to write the version of $D_1^{h_1 h_2}(\xi, M_h)$ there in terms of the correlator $\Delta^{h_1 h_2}$, gave us Eq. (4c).

III. FACTORIZATION FOR $e^+ e^- \rightarrow (h_1 \cdots h_n)X$

The key argument in Ref. [1] for the claim that $d(\xi, -\xi \vec{k}_T, \{P_h\})$ should be considered the fundamental definition of a n -hadron FF is that it arises in a derivation of factorization for $e^+ e^- \rightarrow (h_1 \cdots h_n)X$. We show here that $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ also enters a factorization formula for that process. Some relevant external kinematic variables are $z_i = 2P_i \cdot q/Q^2$, the momentum of the virtual photon q , and $Q^2 = q^2$, as well as $z \equiv \sum_{i=1}^n z_i$. We readily find Eq. (3) of Ref. [1] can equivalently be written as the following factorization formula:

$$\begin{aligned} & \left(\prod_{i=1}^n \frac{2E_i(2\pi)^3}{d^3 \vec{P}_i} \right) d\sigma \\ &= \frac{(16\pi^3)^{n-1} z_1 \cdots z_n}{z^2} \int_z^1 \frac{d\hat{z}}{\hat{z}^{n-1}} \left(\frac{2E_{\hat{k}}(2\pi)^3 d\hat{\sigma}}{d^3 \vec{k}} \right) \\ & \times \left(\frac{\xi^2}{4(16\pi^3)^{n-1} \xi_1 \cdots \xi_n} \int d^2 \vec{k}_T \Delta^{\{h_i\}_n} \right) + \text{p.s.}, \quad (6) \end{aligned}$$

where we used the fact that

$$\hat{z} = \frac{z}{\xi} + \text{p.s.}, \quad \frac{z_i}{\hat{z}} = \xi_i + \text{p.s.}, \quad (7)$$

with p.s. indicating “power-suppressed” terms $\sim \Lambda^2/Q^2$, where Λ denotes a typical hadronic scale.⁵ The factor $2E_{\hat{k}}(2\pi)^3 d\hat{\sigma}/d^3 \vec{k}$ is the usual hard factor for the inclusive production of an on-shell massless parton with 3-momentum $\vec{k}$. The last factor in parentheses on the rhs of Eq. (6) above can be written in terms of the function $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ we defined in Eq. (2), which then brings Eq. (6) into its final form,

$$\begin{aligned} & \left(\prod_{i=1}^n \frac{2E_i(2\pi)^3}{d^3 \vec{P}_i} \right) d\sigma \\ &= \frac{(16\pi^3)^{n-1} z_1 \cdots z_n}{z^2} \int_z^1 \frac{d\hat{z}}{\hat{z}^{n-1}} \left(\frac{2E_{\hat{k}}(2\pi)^3 d\hat{\sigma}}{d^3 \vec{k}} \right) \\ & \times \left(\xi^2 \int d^2 \vec{k}_T D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n) \right) + \text{p.s.} \quad (8) \end{aligned}$$

We therefore have a factorization formula involving the n -hadron FF from Ref. [2] with the usual hard factor,

contrary to what is stated in Ref. [1]. We highlight that the measure of the convolution integral in Eq. (8), which differs from Eq. (3) of Ref. [1] [see also Eq. (10) below], is consistent with the evolution equations derived in Ref. [2] for the DiFFs—see Sec. V below. The expression (8) appears in its general form here for the first time, although it follows from the same type of parton model calculations performed in Ref. [2] [as does Eq. (3) of Ref. [1]].

In Sec. VIII of Ref. [1], it is stated that the cross section results from Ref. [2] could not be reproduced. The reason is Ref. [1] did not correctly identify the DiFFs as we defined them in Eqs. (4a), (4b), and (4c). For example, what Ref. [1] calls $d_{\text{mod,red,1}}(\xi, M_h)$ in Eq. (139) is not what was defined as $D_1^{h_1 h_2}(\xi, M_h)$ in Ref. [2] [Eq. (4c) above]. Indeed, we have verified the original parton model cross section calculations in Ref. [2] for $d\sigma/dz dM_h$, $d\sigma/dz_1 dz_2 d^2 \vec{R}_T$, and $d\sigma/dz d\zeta d^2 \vec{R}_T$ starting now from Eq. (6) (for $n=2$). We also noticed that the result for $d\sigma/dz_1 dz_2 d^2 \vec{R}_T$ [see Eq. (17b) below] using Eq. (6) exactly matches the structure of the “double fragmentation” term in the next-to-leading order (NLO) calculation from Ref. [10] (see also Refs. [11,12]).

We now show how Eq. (8) above and Eq. (3) of Ref. [1] are connected. As argued in Ref. [2] (see also Sec. IV below), $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ is a number density in $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$. In order to define a n -hadron FF that is a number density in a different set of variables denoted by $(y_1, \dots, y_n, \vec{Y}_1, \dots, \vec{Y}_n)$, we must follow the simple transformation rule in Eq. (13) of Ref. [2] via the Jacobian,

$$\begin{aligned} & D_1^{\{h_i\}_n}(\{y_i\}_n, \{\vec{Y}_i\}_n) \\ &= D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n) \left| \frac{\partial(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})}{\partial(y_1, \dots, y_n, \vec{Y}_1, \dots, \vec{Y}_n)} \right|. \quad (9) \end{aligned}$$

Recall now Eq. (3) of Ref. [1] (using $d\xi/\xi^2 = -d\hat{z}/z + \text{p.s.}$),

$$\begin{aligned} & \left(\prod_{i=1}^n \frac{2E_i(2\pi)^3}{d^3 \vec{P}_i} \right) d\sigma \\ &= \frac{1}{z} \int_z^1 d\hat{z} \left(\frac{2E_{\hat{k}}(2\pi)^3 d\hat{\sigma}}{d^3 \vec{k}} \right) \left(\xi^2 \int d^2 \vec{k}_T d(\xi, -\xi \vec{k}_T, \{P_h\}) \right) \\ & + \text{p.s.} \quad (10) \end{aligned}$$

If we compare Eqs. (10) and (8), using also Eqs. (9) and (7), we conclude the factorization formula in Ref. [1] is just Eq. (8) written using a n -hadron FF that is a number density in a set of variables $(y_1, \dots, y_n, \vec{Y}_1, \dots, \vec{Y}_n)$ whose Jacobian when transforming from $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$ is

$$\left| \frac{\partial(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})}{\partial(y_1, \dots, y_n, \vec{Y}_1, \dots, \vec{Y}_n)} \right| = (16\pi^3)^{n-1} \frac{\xi_1 \cdots \xi_n}{\xi}. \quad (11)$$

For general n , due to the ambiguous nature of $d(\xi, -\xi \vec{k}_T, \{P_h\})$ from Ref. [1], where it is not specified in

⁵Note that Eq. (7) also implies $(z_1 - z_2)/z = \zeta + \text{p.s.}$

which set of $3n$ variables $d(\xi, -\xi\vec{k}_T, \{P_h\})$ is a number density, we could not readily identify exactly what set of variables $(y_1, \dots, y_n, \vec{Y}_1, \dots, \vec{Y}_n)$ gives this Jacobian. For the special case $n = 2$, we were able to use the discussion in Ref. [1] around Eqs. (104)–(108), where (for the first time) explicit information can be found about what actually is meant by $d(\xi, \{P_h\})$ for the $n = 2$ case, to recognize that

$$\left| \frac{\partial(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})}{\partial(\xi, \zeta, \xi\vec{k}_T, \vec{M}_h)} \right| = 4\pi^3 \xi (1 - \zeta^2) = 16\pi^3 \frac{\xi_1 \xi_2}{\xi}, \quad (12)$$

where $\vec{M}_h$ is a vector with magnitude $|\vec{M}_h| = M_h / \sqrt{32\pi^3}$ and the same azimuthal angle ϕ_R as $\vec{R}_T$. That is, for $n = 2$, $d(\xi, -\xi\vec{k}_T, \{P_h\})$ is a number density in $(\xi, \zeta, \xi\vec{k}_T, \vec{M}_h)$.⁶

Connected to this, Ref. [2] was able to explicitly define DiFFs that are number densities in a given set of variables *before* performing parton model calculations. Then, it was shown that one obtains the expected result for the $e^+e^- \rightarrow (h_1 h_2)X$ cross section differential in a given set of variables. On the other hand, Ref. [1] must define a “reduced” DiFF *in the middle of* a parton model calculation in such a way to ensure the expected expression for the cross section is achieved. This is a consequence of the fact that Ref. [1] is not explicit in which set of $3n$ variables $d(\xi, -\xi\vec{k}_T, \{P_h\})$ is a number density.

To summarize, Eq. (8) is a factorization formula for $e^+e^- \rightarrow (h_1 \cdots h_n)X$ involving the fundamental n -hadron FF $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ from Ref. [2], which has an unambiguous interpretation as a number density in $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$. In contrast, the factorization formula (10) above derived in Ref. [1] involves a n -hadron FF $d(\xi, -\xi\vec{k}_T, \{P_h\})$ that is a number density in some other set of $3n$ variables which, for arbitrary n , was not specified in Ref. [1], and we cannot readily identify them either. We have inferred, though, from comparing Eqs. (8) and (10), that the Jacobian of the transformation from the variables $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$ to the variables used in $d(\xi, -\xi\vec{k}_T, \{P_h\})$ is given by Eq. (11).

IV. NUMBER DENSITY INTERPRETATION

In Ref. [2], justification was given for $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ being a number density in $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$ through a number sum rule, and for $n = 2$ also a momentum sum rule and cross section calculations for $e^+e^- \rightarrow (h_1 h_2)X$ [see Supplemental Material Secs. S1 and S2 and the discussion around Eq. (20) in Ref. [2]]. In Ref. [13], there was criticism that number sum rules for FFs do not hold in QCD, and Ref. [1] argues therefore that factorization is the rigorous way in which one can develop the fundamental operator

⁶Any set of variables with unit Jacobian relative to $(\xi, \zeta, \xi\vec{k}_T, \vec{M}_h)$ could also be used.

definition of n -hadron FFs. In Appendix B of Ref. [1], an attempt is made to impose the number sum rule (B13) on a DiFF called $D_1(\xi, \zeta, \vec{k}_T, M_h^2)$, and a “paradox” is found with Eq. (B9) calculated from a cross section. Eventually Ref. [1] “resolves” this by stating Eq. (B13) is not a valid relation. However, the correct use of the number sum rule in Eq. (6) of Ref. [2] is for the integration measure in Eq. (B13) of Ref. [1] to be $d\xi d\zeta d^2(\xi\vec{k}_T) d^2\vec{M}_h$, since, as mentioned above, $D_1(\xi, \zeta, \vec{k}_T, M_h^2)$ is a number density in $(\xi, \zeta, \xi\vec{k}_T, \vec{M}_h)$. This then gives exactly the lhs of Eq. (B9). Therefore, there is no contradiction between an operator definition of a n -hadron FF deduced from factorization and one based on a “naïve” number sum rule. The work here and in Ref. [2] shows the two are fully compatible.

Even though Ref. [1] relies on the factorization formula in Eq. (10) above to motivate its operator definition of a n -hadron FF, its number density interpretation still must be justified. This is done by attempting to generalize the number density operator definition of a single-hadron FF discussed in Sec. 12.4 of Ref. [14]. If one looks at Eq. (12.35) of Ref. [14] [see also Eq. (4.5) of Ref. [15]], we can write⁷

$$\sum_h D_1^h(\xi, \vec{P}_\perp) \langle \vec{k}_1 | \vec{k}_2 \rangle \equiv \langle \vec{k}_1 | \frac{d\hat{N}}{d\xi d^2\vec{P}_\perp} | \vec{k}_2 \rangle, \quad (13)$$

where $|\vec{k}_{1(2)}\rangle$ are quark states and $\hat{N}$ is the total number operator,

$$\hat{N} \equiv \sum_h \int \frac{d\xi d^2\vec{P}_\perp}{2\xi(2\pi)^3} \hat{a}_h^\dagger \hat{a}_h, \quad (14)$$

with $\hat{a}_h^\dagger$ and $\hat{a}_h$ equal time creation and annihilation operators, respectively, for a hadron h with momentum $\vec{P}$. We highlight that the single-hadron FF $D_1^h(\xi, \vec{P}_\perp)$ (summed over h) is defined as a number density in $(\xi, \vec{P}_\perp)$ by writing the total number operator as differential in those variables. Starting from the operator definition of $D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$ in Eq. (2), and using expressions for the quark field operator $\psi(x)$ in terms of (quark) light cone creation and annihilation operators as well as Eq. (S30) of Ref. [2], we find

$$\begin{aligned} & \sum_{h_1} \cdots \sum_{h_n} D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n) \langle \vec{k}_1 | \vec{k}_2 \rangle \\ &= \langle \vec{k}_1 | \frac{d(\prod_{k=0}^{n-1} (\hat{N} - k))}{d\xi_1 \cdots d\xi_n d^2\vec{P}_{1\perp} \cdots d^2\vec{P}_{n\perp}} | \vec{k}_2 \rangle. \end{aligned} \quad (15)$$

We recognize on the rhs the operator for the total number of n -hadrons differential in $(\xi_1, \dots, \xi_n, \vec{P}_{1\perp}, \dots, \vec{P}_{n\perp})$, which is

⁷In Eq. (12.35) of Ref. [14], there is no $\sum_h$. We have added it here for reasons that will be apparent when we generalize to n hadrons.

exactly what one wants for the definition of a n -hadron FF that is a number density in those variables. This is an additional result to what was already derived in Ref. [2] to justify the number density interpretation of $D_1^{h_1 h_2}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$. If we follow the same steps for the operator definition of $d(\xi, -\xi\vec{k}_T, \{P_h\})$ in Eq. (3), we instead find the rhs of Eq. (15) is modified by the Jacobian factor in Eq. (11) [see also the second line of Eq. (77) in Ref. [1]], which for $n \geq 2$ does not give the standard expression we expect for a number density. We therefore disagree that Eq. (77) of Ref. [1] is how to generalize the fundamental single-hadron FF to the n -hadron case.

V. DIHADRON FRAGMENTATION EVOLUTION EQUATIONS

In the “Evolution of Extended DiFFs” section of Ref. [2] as well as Supplemental Material Sec. S4, it was explained how to derive the evolution equations for the DiFF definitions there, and, contrary to what is stated in Ref. [1], it was found that the splitting functions are the same as for the single-hadron case. The only potential change is in the measure of the convolution integral, depending on which DiFF is under consideration. To highlight this, here are the evolution equations [to $\mathcal{O}(\alpha_s)$] derived in Ref. [2] for $D_1^{h_1 h_2}(\xi, M_h)$ and $D_1^{h_1 h_2}(\xi_1, \xi_2, \vec{R}_T)$,

$$\frac{\partial D_1^{h_1 h_2/i}(\xi, M_h; \mu)}{\partial \ln \mu^2} = \sum_{i'} \int_{\xi}^1 \frac{d\hat{z}}{\hat{z}} D_1^{h_1 h_2/i'}\left(\frac{\xi}{\hat{z}}, M_h; \mu\right) P_{i \rightarrow i'}(\hat{z}), \quad (16a)$$

$$\frac{\partial D_1^{h_1 h_2/i}(\xi_1, \xi_2, \vec{R}_T; \mu)}{\partial \ln \mu^2} = \sum_{i'} \int_{\xi}^1 \frac{d\hat{z}}{\hat{z}^2} D_1^{h_1 h_2/i'}\left(\frac{\xi_1}{\hat{z}}, \frac{\xi_2}{\hat{z}}, \vec{R}_T; \mu\right) P_{i \rightarrow i'}(\hat{z}), \quad (16b)$$

where $P_{i \rightarrow i'}(\hat{z})$ are the standard unpolarized time-like splitting functions [16]. We note that the integration measure in (16a) is $d\hat{z}/\hat{z}$ while in (16b) it is $d\hat{z}/\hat{z}^2$. The latter matches the first line in Eq. (30) of Ref. [10] and the first and fourth lines of Eq. (23) of Ref. [11] as well as Eq. (24) of Ref. [17]. We also mention that (16b) follows from (16a) (or vice versa) using Eqs. (4b) and (4c), where the fact that ξ_1, ξ_2 in the definition of $D_1^{h_1 h_2}(\xi_1, \xi_2, \vec{R}_T)$ are fractions of the parton’s momentum carried by the hadrons is crucial in this regard. [They were instead identified as external kinematic variables in Eq. (130) of Ref. [1].] Appendix C of Ref. [1] essentially recognized that a simple change in integration measure of the evolution equations occurs. However, it was interpreted as modifying the splitting

function because an incorrect factorization formula was being used with the DiFFs from Ref. [2]. The integration measures in a NLO calculation of our DiFF correlators would be the same as our evolution equations (16a) and (16b) and are consistent with the integration measure in the factorization formula, Eq. (8), which is needed in order to be able to cancel the collinear singularities in the cross section that arise at NLO. Namely, we find

$$\frac{d\sigma}{dz dM_h} = \int_z^1 \frac{d\hat{z}}{\hat{z}} \frac{d\hat{\sigma}}{d\hat{z}} D_1^{h_1 h_2}\left(\frac{z}{\hat{z}}, M_h\right), \quad (17a)$$

$$\frac{d\sigma}{dz_1 dz_2 d^2 \vec{R}_T} = \int_z^1 \frac{d\hat{z}}{\hat{z}^2} \frac{d\hat{\sigma}}{d\hat{z}} D_1^{h_1 h_2}\left(\frac{z_1}{\hat{z}}, \frac{z_2}{\hat{z}}, \vec{R}_T\right). \quad (17b)$$

The change in integration measure compared to the single-hadron case was also discussed previously for $d\sigma/dz_1 dz_2$ in Ref. [11] [see after Eq. (23)]. Note that if one demands that the convolution integral in Eq. (17b) be $\int_z^1 \frac{d\hat{z}}{\hat{z}} \frac{d\hat{\sigma}}{d\hat{z}} \dots$, as it is for the single-hadron case, then one is forced to define a DiFF that is not a number density in $(\xi_1, \xi_2) = (\frac{z_1}{\hat{z}}, \frac{z_2}{\hat{z}})$ (or to modify the hard factor by including $1/\hat{z}$ in it). We also remark that one can derive Eq. (17a) starting from Eq. (17b) by rewriting the lhs of (17b) using the Jacobian for the variable transformation from $(z, \xi, \vec{M}_h)$ to $(z_1, z_2, \vec{R}_T)$ and then solving for $d\sigma/dz dM_h$ ($\vec{M}_h$ is a vector with magnitude M_h and azimuthal angle ϕ_{R_T}), making use of the definition of $D_1^{h_1 h_2}(\xi, M_h)$ in Eq. (4c).

VI. SUMMARY AND CONCLUSIONS

We have reconfirmed that the DiFF (and n -hadron FF) definitions in Ref. [2] have a natural number density interpretation and, contrary to what is stated in Ref. [1], are consistent with collinear factorization using the standard hard factors and evolution kernels, with Eqs. (8) and (15) here adding to and supporting the results from Ref. [2]. We have also corrected other erroneous claims in Ref. [1] about our work in Ref. [2]. We emphasize that the approach in Ref. [2] of using sum rules and parton model calculations to support the definition of DiFFs (or n -hadron FFs) as number densities remains valid. There have been no new insights offered in that regard by the analysis of (leading order) factorization in Ref. [1] [and the introduction of an ambiguous n -hadron FF $d(\xi, -\xi\vec{k}_T, \{P_h\})$]. With respect to factorization, establishing a proof for $e^+e^- \rightarrow (h_1 \cdots h_n)X$ to all orders, valid for all M_h , that includes contributions involving FFs for less than n hadrons still remains an open problem.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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