

Two faces of $\mathcal{N} = 7, 8$ superconformal mechanics

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Two variants of $\mathcal{N} = 7$ superconformal mechanics with manifest $\mathfrak{so}(7)$ and $\mathfrak{su}(2) \times \mathfrak{su}(2)$ R -symmetries possessing exceptional $G(3)$ dynamical symmetry are presented. To construct these $G(3)$ theories, we developed two new versions of covariant embedding of the exceptional $\mathfrak{g}_2$ algebra into the $\mathfrak{so}(7)$ algebra and then the embedding of $\mathfrak{so}(7)$ into $\mathfrak{so}(8)$. As a result, two $\mathcal{N} = 8$ superconformal mechanics with the $OSp(8|2)$ and $\mathfrak{F}(4)$ superconformal algebras and $\mathcal{N} = 7, G(3)$ superconformal mechanics were constructed in a uniform way. The constants of the octonion multiplication play a key role in the construction of superconformal mechanics with manifest $\mathfrak{so}(7)$ symmetry.

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I. INTRODUCTION

Superconformal mechanics with an $\mathfrak{so}(1, 2)$ conformal symmetry are potentially of great interest due to their possible relation to AdS_2 holography. In virtue of the AdS/CFT correspondence, there is by now a large amount of works on anti-de Sitter solutions of string/M-theory in diverse dimensions. The high dimensionality of the internal space of AdS_2 solutions of string/M-theory is related via duality with R -symmetry of the corresponding superconformal mechanics. However, all variants of $\mathcal{N} = 4$ superconformal mechanics possess the exceptional superalgebra $D(1, 2; \alpha)$ as dynamical symmetry. Therefore, the maximal R -symmetry for $\mathcal{N} = 4$ cases is $\mathfrak{so}(1, 2) \times \mathfrak{so}(4)$.

Passing to $\mathcal{N} = 8$ superconformal mechanics makes the situation more interesting because in contrast with the $\mathcal{N} = 4$ case in the $\mathcal{N} = 8$ supersymmetric case we have four different superconformal algebras $OSp(8|2)$, $\mathfrak{F}(4)$, $OSp(4^*|4)$, and $SU(1, 1|4)$ (see, e.g., [1]). Thus, the R -symmetries are extended to $\mathfrak{so}(8)$, $\mathfrak{so}(7)$, $\mathfrak{so}(5) \times \mathfrak{su}(2)$, and $\mathfrak{su}(4) \times \mathfrak{u}(1)$, respectively. Many variants of $\mathcal{N} = 8$ superconformal mechanics were constructed, mainly in the superfield Lagrangian formalism [2–9] and within the Hamiltonian approach as well [10–16]. In the case of

$\mathcal{N} = 8$ superconformal models, the Hamiltonian approach has several preferences, such as

- (i) In all variants of superconformal mechanics with a different number of supersymmetries the superconformal symmetry is spontaneously broken, as the result of which the dilaton u plays a master role in the form of supercharges and Hamiltonian. Roughly speaking, the supercharges have the form (cf., [11, 15–18])

$$Q \sim p_r \psi + \frac{1}{r} [R - \text{symmetry generators}] \psi. \quad (1.1)$$

Here $r = e^{u/2}$ and p_r is the corresponding momentum.

- (ii) The explanation of this form (1.1) is quite simple: due to the presence of the dilaton fields, one can always realize the dilaton $\mathcal{D}$, conformal boost $\mathcal{K}$, and the conformal supersymmetry $\mathcal{S}$ generators as

$$\mathcal{D} = \frac{1}{2} r p_r, \quad \mathcal{K} = \frac{1}{2} r^2, \quad \mathcal{S} = r \psi. \quad (1.2)$$

- (iii) Thus, the form of the supercharges (1.1) is unique to produce the main relations of any superconformal algebra

$$\{Q, \mathcal{S}\} \sim \mathcal{D} + [R - \text{symmetry generators}]. \quad (1.3)$$

As a result, all peculiarities of different variants of superconformal mechanics are hidden in the realization of R -symmetry generators, mainly in the bosonic but partially

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in the fermionic realizations too. Specializing on the first two superconformal mechanics with $OSp(8|2)$ and $\mathfrak{F}(4)$ superalgebras, we have to deal with (at least) two realizations of $\mathfrak{so}(8)$ and $\mathfrak{so}(7)$ algebras with manifest $\mathfrak{so}(7)$ and $\mathfrak{su}(2)^4$ and $\mathfrak{su}(2)^3$ symmetries. Moreover, this line can be continued as follows:

$$OSp(8|2) \rightarrow \mathfrak{F}(4) \rightarrow G(3) \leftrightarrow \mathfrak{so}(8) \rightarrow \mathfrak{so}(7) \rightarrow \mathfrak{g}_2.$$

The last case in this series is the $\mathcal{N} = 7$ superconformal mechanics which is based, as it was stated in [13], on $(1, 7, 7, 1)$ supermultiplet.¹ In the present paper we provide a detailed uniform description of these superconformal mechanics using the properly defined line of embeddings $\mathfrak{so}(8) \rightarrow \mathfrak{so}(7) \rightarrow \mathfrak{g}_2$. The manifest $\mathfrak{so}(7)$ covariant description developed in this paper reveals the tight relations of the $\mathcal{N} = 8$ superconformal algebras with the algebra of octonions. Indeed, the supercharges we constructed contain the constants of the octonion multiplication c_{ijk} . Of course, the relation of the exceptional algebras with the octonions has been known for a long time² However, explicit appearance of constants of the octonion multiplication c_{ijk} and their dual cousins f_{ijkl} (2.7) in the supercharges and Hamiltonian has not been noted yet, to the

best of our knowledge, as well as explicit expressions of embedding of $\mathfrak{g}_2$ into $\mathfrak{so}(7)$ (2.8).

The plan of our paper is as follows. In Sec. II we describe two realizations of the algebra $\mathfrak{so}(8)$ and its $\mathfrak{so}(7)$ and $\mathfrak{g}_2$ subalgebras paying much attention to the explicit embeddings $\mathfrak{g}_2 \rightarrow \mathfrak{so}(7) \rightarrow \mathfrak{so}(8)$. In Sec. III we explicitly construct supercharges and Hamiltonians for $OSp(8|2)$, $\mathfrak{F}(4)$, $G(3)$ superconformal mechanics within the $\mathfrak{so}(7)$ manifestly covariant formalism. In Sec. IV we repeat the construction of superconformal mechanics within the $\mathfrak{su}(2)$ invariant description. In the Conclusion we review the obtained results and mention possible future developments. In the Appendix, we present the explicit variant of changing variables from the $\mathfrak{su}(2)$ invariant formalism to the commonly adopted one.

II. TWO FACES OF $\mathfrak{so}(8)$ AND ITS SUBALGEBRAS

A. Standard realization

The standard realization of the algebra $\mathfrak{so}(8)$ means its realization in the sixteen-dimensional bosonic phase spaces, and eight-dimensions fermionic one. Thus, let us introduce the following set of bosonic and fermionic fields

- (i) sixteen bosonic fields: y_0, p_0 and $y_i, p_i, i, j = 1, \dots, 7$
- (ii) eight fermionic fields: ψ_0, ψ_i

which obey the following nonzero Poisson brackets:

$$\{p_0, y_0\} = 1, \quad \{p_i, y_j\} = \delta_{ij}, \quad \{\psi_0, \psi_0\} = i, \quad \{\psi_i, \psi_j\} = i\delta_{ij}. \quad (2.1)$$

1. $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$ algebras

From our fields one can construct currents spanning two different $\mathfrak{so}(7)$ algebras: bosonic and fermionic

$$\begin{aligned} W_{ij} &= p_i y_j - p_j y_i \rightarrow \{W_{ij}, W_{kl}\} = \delta_{ik} W_{jl} - \delta_{jk} W_{il} - \delta_{il} W_{jk} + \delta_{jl} W_{ik}, \\ \hat{W}_{ij} &= i\psi_i \psi_j \rightarrow \{\hat{W}_{ij}, \hat{W}_{kl}\} = \delta_{ik} \hat{W}_{jl} - \delta_{jk} \hat{W}_{il} - \delta_{il} \hat{W}_{jk} + \delta_{jl} \hat{W}_{ik}. \end{aligned} \quad (2.2)$$

To construct the generators of $\mathfrak{so}(8)$ algebras one has to add to the generators W_{ij} and $\hat{W}_{ij}$ seven bosonic V_i and seven fermionic generators $\hat{V}_i$, respectively, transforming under 7-dimensional representations of $\mathfrak{so}(7)$

$$\begin{aligned} V_i &= p_0 y_i - p_i y_0 \rightarrow \{W_{ij}, V_k\} = \delta_{ik} V_j - \delta_{jk} V_i, \quad \{V_i, V_j\} = W_{ij}, \\ \hat{V}_i &= i\psi_0 \psi_i \rightarrow \{\hat{W}_{ij}, \hat{V}_k\} = \delta_{ik} \hat{V}_j - \delta_{jk} \hat{V}_i, \quad \{\hat{V}_i, \hat{V}_j\} = \hat{W}_{ij}. \end{aligned} \quad (2.3)$$

2. $\mathfrak{g}_2$ algebra

The embedding of $\mathfrak{g}_2$ algebra into $\mathfrak{so}(7)$ can be achieved by the rank 2 totally antisymmetric tensor $\mathcal{W}_{ij}$ constructed from the $\mathfrak{so}(7)$ generators (2.2) and obeying the constraints [13]

¹In [13], the quantum version of $G(3)$ superconformal mechanics was considered.

²In the present context such relations were discussed in [12, 13, 17].

$$c_{ijk}\mathcal{W}_{jk} = 0. \quad (2.4)$$

Here the rank-3 totally antisymmetric tensor c_{ijk} is the structure constant of the octonionic multiplication. Namely, if the $e_i, i = 1, \dots, 7$ are seven octonions with the following multiplication:

$$e_i e_j = -\delta_{ij} + c_{ijk} e_k, \quad (2.5)$$

the tensor c_{ijk} has the following nonzero components³

$$c_{123} = c_{147} = c_{165} = c_{246} = c_{257} = c_{354} = c_{367} = 1. \quad (2.6)$$

We will also need the dual, rank-4 totally antisymmetric tensor f_{ijkl} defined as follows

$$f_{ijkl} = \frac{1}{6} \epsilon_{ijklmnp} c_{mnp}, \quad \epsilon_{1234567} = 1. \quad (2.7)$$

It is rather easy to check that the solution of Eq. (2.4) $\mathcal{W}_{ij}$ can be constructed as follows [14]:

$$\mathcal{W}_{ij} = W_{ij} + \frac{1}{4} f_{ijkl} W_{kl}. \quad (2.8)$$

The tensor $\mathcal{W}_{ij}$ obeys the following Poisson brackets:

$$\{\mathcal{W}_{ij}, \mathcal{W}_{kl}\} = -\frac{3}{2} (f_{ijkm} \mathcal{W}_{lm} - f_{ijlm} \mathcal{W}_{km}) + c_{ijm} c_{kln} \mathcal{W}_{mn}. \quad (2.9)$$

Thus, fourteen currents $\mathcal{W}_{ij}$ span $\mathfrak{g}_2$ algebra.

such that satisfy the following conjugation rule:

$$(Z^{iA}_m)^\dagger = Z_{iAm}, \quad (B^{a\alpha}_m)^\dagger = B_{aam}, \quad (\phi^{iA})^\dagger = \phi_{iA}, \quad (\chi^{a\alpha})^\dagger = \chi_{a\alpha} \quad (2.13)$$

and for which the Poisson brackets hold

$$\begin{aligned} \{Z^{iA}_m, Z^{jB}_n\} &= 2\epsilon^{ij}\epsilon^{AB}\epsilon_{mn}, & \{\phi^{iA}, \phi^{jB}\} &= 2i\epsilon^{ij}\epsilon^{AB}, \\ \{B^{a\alpha}_m, B^{b\beta}_n\} &= 2\epsilon^{ab}\epsilon^{\alpha\beta}\epsilon_{mn}, & \{\chi^{a\alpha}, \chi^{b\beta}\} &= 2i\epsilon^{ij}\epsilon^{AB}. \end{aligned} \quad (2.14)$$

Note, the indices m, n are not the indices of any $\mathfrak{su}(2)$ algebras, instead these indices just count the fields involved. However, the corresponding metric is ϵ^{nm} , i.e., $A^n B_n$ means $\epsilon^{nm} A_m B_n$.

³Here we followed the notation adopted in [13].

3. Nonstandard embedding of $\mathfrak{so}(7)$ into $\mathfrak{so}(8)$

With the help of the constants c_{ijk} and f_{ijkl} one can define two different embeddings of $\mathfrak{so}(7)$ into $\mathfrak{so}(8)$

$$\begin{aligned} \tilde{W}_{ij} &= \frac{1}{2} \left(W_{ij} + \alpha c_{ijk} V_k + \frac{1}{2} f_{ijkl} W_{kl} \right), \quad \alpha^2 = 1 \\ \{\tilde{W}_{ij}, \tilde{W}_{kl}\} &= \delta_{ik} \tilde{W}_{jl} - \delta_{jk} \tilde{W}_{il} - \delta_{il} \tilde{W}_{jk} + \delta_{jl} \tilde{W}_{ik}. \end{aligned} \quad (2.10)$$

The additional generator

$$\tilde{V}_i = \frac{1}{2} \left(V_i + \frac{\alpha}{2} c_{ijk} W_{jk} \right) \quad (2.11)$$

with the new defined generators $\tilde{W}_{ij}$ form the $\mathfrak{so}(8)$ algebra

$$\{\tilde{W}_{ij}, \tilde{V}_k\} = \delta_{ik} \tilde{V}_j - \delta_{jk} \tilde{V}_i, \quad \{\tilde{V}_i, \tilde{V}_j\} = \tilde{W}_{ij}. \quad (2.12)$$

B. Realization with manifest $\mathfrak{su}(2)$ symmetries

Such realization was used in the papers [10,15,16]. To construct the generators of the $\mathfrak{so}(8)$ algebra, we again need sixteen bosonic and eight fermionic fields but now with different symmetry properties—all fields are supposed to transform under fundamental representations of different $\mathfrak{su}(2)$ algebras. Thus, we will consider the following set of bosonic and fermionic fields:

- (i) sixteen bosonic fields: $B^{a\alpha}_m$ and Z^{iA}_m , $\{a, \alpha, i, A, m\} = 1, 2$
- (ii) eight fermionic fields $\phi^{iA}, \chi^{a\alpha}$

1. $\mathfrak{so}(8)$ algebra

From our fields one can construct bosonic and fermionic currents spanning four different $\mathfrak{su}(2)$ algebras: four bosonic

$$\begin{aligned} J^{ij} &= \frac{1}{4} Z^{iam} Z^j_{am} \rightarrow \{J^{ij}, J^{kl}\} = -\epsilon^{ik} J^{jl} - \epsilon^{jl} J^{ik}, \\ W^{AB} &= \frac{1}{4} B^{aAm} B^B_{am} \rightarrow \{W^{AB}, W^{CD}\} = -\epsilon^{AC} W^{BD} - \epsilon^{BD} W^{AC}, \\ X^{ab} &= \frac{1}{4} B^{aAm} B^b_{Am} \rightarrow \{X^{ab}, X^{cd}\} = -\epsilon^{ac} X^{bd} - \epsilon^{bd} X^{ac}, \\ Y^{\alpha\beta} &= \frac{1}{4} Z^{iam} Z^\beta_{im} \rightarrow \{Y^{\alpha\beta}, Y^{\gamma\delta}\} = -\epsilon^{\alpha\gamma} Y^{\beta\delta} - \epsilon^{\beta\delta} Y^{\alpha\gamma}, \end{aligned} \quad (2.15)$$

and four fermionic ones

$$\begin{aligned} \hat{J}^{ij} &= \frac{i}{4} \phi^{iA} \phi^j_A \Rightarrow \{\hat{J}^{ij}, \hat{J}^{kl}\} = -\epsilon^{ik} \hat{J}^{jl} - \epsilon^{jl} \hat{J}^{ik}, \\ \hat{W}^{AB} &= \frac{i}{4} \phi^{iA} \phi_i^B \Rightarrow \{\hat{W}^{AB}, \hat{W}^{CD}\} = -\epsilon^{AC} \hat{W}^{BD} - \epsilon^{BD} \hat{W}^{AC}, \\ \hat{X}^{ab} &= \frac{i}{4} \chi^{a\alpha} \chi^b_{\alpha} \Rightarrow \{\hat{X}^{ab}, \hat{X}^{cd}\} = -\epsilon^{ac} \hat{X}^{bd} - \epsilon^{bd} \hat{X}^{ac}, \\ \hat{Y}^{\alpha\beta} &= \frac{i}{4} \chi^{a\alpha} \chi_a^{\beta} \Rightarrow \{\hat{Y}^{\alpha\beta}, \hat{Y}^{\gamma\delta}\} = -\epsilon^{\alpha\gamma} \hat{Y}^{\beta\delta} - \epsilon^{\beta\delta} \hat{Y}^{\alpha\gamma}. \end{aligned} \quad (2.16)$$

Let us define sixteen bosonic currents $V^{iA\alpha\alpha}$ and sixteen fermionic currents $\hat{V}^{iA\alpha\alpha}$

$$V^{iA\alpha\alpha} = Z^{iam} B^{aA}_{am}, \quad \hat{V}^{iA\alpha\alpha} = i\phi^{iA} \chi^{\alpha\alpha}_A. \quad (2.17)$$

These fields transform under four $\mathfrak{su}(2)$ generators (2.15) as follows

$$\begin{aligned} \{J^{ij}, V^{kA\alpha\alpha}\} &= -\frac{1}{2} (\epsilon^{ik} V^{jA\alpha\alpha} + \epsilon^{jk} V^{iA\alpha\alpha}), & \{W^{AB}, V^{iC\alpha\alpha}\} &= -\frac{1}{2} (\epsilon^{AC} V^{iB\alpha\alpha} + \epsilon^{BC} V^{iA\alpha\alpha}), \\ \{X^{ab}, V^{iA\alpha\alpha}\} &= -\frac{1}{2} (\epsilon^{ac} V^{iAb\alpha\alpha} + \epsilon^{bc} V^{iAa\alpha\alpha}), & \{Y^{\alpha\beta}, V^{iA\alpha\gamma}\} &= -\frac{1}{2} (\epsilon^{\alpha\gamma} V^{iA\alpha\beta} + \epsilon^{\beta\gamma} V^{iA\alpha\alpha}). \end{aligned} \quad (2.18)$$

These sixteen generators, $V^{kA\alpha\alpha}$ form the $\mathfrak{so}(8)$ algebra together with four $\mathfrak{su}(2)$ algebras (2.15). The main Poisson brackets read

$$\{V^{iA\alpha\alpha}, V^{jB\beta\beta}\} = -4(\epsilon^{AB} \epsilon^{ab} \epsilon^{\alpha\beta} J^{ij} + \epsilon^{ij} \epsilon^{ab} \epsilon^{\alpha\beta} W^{AB} + \epsilon^{ij} \epsilon^{AB} \epsilon^{\alpha\beta} X^{ab} + \epsilon^{ij} \epsilon^{AB} \epsilon^{\alpha\beta} Y^{\alpha\beta}). \quad (2.19)$$

The hatted pure fermionic generators obey the same Poisson brackets.

2. $\mathfrak{so}(7)$ algebra

Now the situation with embedding of the algebra $\mathfrak{so}(7)$ into the algebra $\mathfrak{so}(8)$ is more subtle than it was in the previous case in Sec. II 1. To provide such embedding, one has

(i) to identify two $\mathfrak{su}(2)$ subalgebras, for example

$$W^{AB} \oplus X^{ab} \rightarrow Z^{ab}_{\mathfrak{so}(7)} = W^{ab} + X^{ab} \quad (2.20)$$

(ii) to symmetrize the generators $V^{iA\alpha\alpha}$ over the corresponding indices

$$V^{iaba}_{\mathfrak{so}(7)} = V^{iaba} + V^{ibaa}. \quad (2.21)$$

Thus, we have nine generators in three $\mathfrak{su}(2)$ subalgebras $J^{ij}, Z^{ab}_{\mathfrak{so}(7)}, Y^{\alpha\beta}$ and twelve generators $V^{iaba}_{\mathfrak{so}(7)}$ which together give a full 21-dimensional set of $\mathfrak{so}(7)$ generators.

The new Poisson brackets read

$$\begin{aligned} \{Z_{\mathfrak{so}(7)}^{ab}, V_{\mathfrak{so}(7)}^{icda}\} &= -\epsilon^{ac} V_{\mathfrak{so}(7)}^{ibda} - \epsilon^{bd} V_{\mathfrak{so}(7)}^{icda}, \\ \{V_{\mathfrak{so}(7)}^{iaba}, V_{\mathfrak{so}(7)}^{jcd\beta}\} &= -8[\epsilon^{ij}\epsilon^{\alpha\beta}(\epsilon^{ac}Z_{\mathfrak{so}(7)}^{bd} + \epsilon^{bd}Z_{\mathfrak{so}(7)}^{ac}) + (\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{bc})(\epsilon^{ij}Y^{\alpha\beta} + \epsilon^{\alpha\beta}J^{ij})]. \end{aligned} \quad (2.22)$$

The hatted pure fermionic generators have the same structure and obey the same Poisson brackets.

3. $\mathfrak{G}_2$ algebra

To extract the generators of the $\mathfrak{g}_2$ algebra from the generators of $\mathfrak{so}(7)$, one can do the following:

- (i) The rank of the $\mathfrak{g}_2$ algebra is equal to 2. Thus, one has to pick up two $\mathfrak{su}(2)$ subalgebras from the $\mathfrak{so}(7)$ algebra. So let us identify the indices a, b, α . Therefore, instead of three $\mathfrak{su}(2)$ algebras with the generators $J^{ij}, Z_{\mathfrak{so}(7)}^{ab}, Y^{\alpha\beta}$ we have two $\mathfrak{su}(2)$ algebras with the generators

$$J^{ij} \quad \text{and} \quad Z_{\mathfrak{g}_2}^{ab} = Z_{\mathfrak{so}(7)}^{ab} + Y^{ab} = X^{ab} + W^{ab} + Y^{ab} \quad (2.23)$$

- (ii) In the generators $V_{\mathfrak{so}(7)}^{iaba}$ one has to identify and then symmetrize the indices a, b, α

$$V_{\mathfrak{g}_2}^{iabc} = V_{\mathfrak{so}(7)}^{iabc} + V_{\mathfrak{so}(7)}^{ibca} + V_{\mathfrak{so}(7)}^{icab} + V_{\mathfrak{so}(7)}^{ibac} + V_{\mathfrak{so}(7)}^{iacb} + V_{\mathfrak{so}(7)}^{icba} \quad (2.24)$$

- (iii) The generators $J^{ij}, Z_{\mathfrak{g}_2}^{ab}$, and $V_{\mathfrak{g}_2}^{iabc}$ form the $\mathfrak{g}_2$ algebra.

The main Poisson brackets read

$$\begin{aligned} \{Z_{\mathfrak{g}_2}^{ab}, V_{\mathfrak{g}_2}^{icde}\} &= -\frac{1}{2}(\epsilon^{ac}V_{\mathfrak{g}_2}^{ibde} + \epsilon^{bc}V_{\mathfrak{g}_2}^{iade} + \epsilon^{ad}V_{\mathfrak{g}_2}^{icbe} + \epsilon^{bd}V_{\mathfrak{g}_2}^{icae} + \epsilon^{ae}V_{\mathfrak{g}_2}^{icdb} + \epsilon^{be}V_{\mathfrak{g}_2}^{icda}), \\ \{V_{\mathfrak{g}_2}^{iabc}, V_{\mathfrak{g}_2}^{jdef}\} &= -8\epsilon^{ij}[(\epsilon^{be}\epsilon^{cf} + \epsilon^{bf}\epsilon^{ce})Z_{\mathfrak{g}_2}^{ad} + (\epsilon^{bd}\epsilon^{cf} + \epsilon^{bf}\epsilon^{cd})Z_{\mathfrak{g}_2}^{ae} + (\epsilon^{bd}\epsilon^{ce} + \epsilon^{cd}\epsilon^{be})Z_{\mathfrak{g}_2}^{af} \\ &\quad + (\epsilon^{ae}\epsilon^{cf} + \epsilon^{af}\epsilon^{ce})Z_{\mathfrak{g}_2}^{bd} + (\epsilon^{ad}\epsilon^{cf} + \epsilon^{af}\epsilon^{cd})Z_{\mathfrak{g}_2}^{be} + (\epsilon^{ad}\epsilon^{ce} + \epsilon^{cd}\epsilon^{ae})Z_{\mathfrak{g}_2}^{bf} \\ &\quad + (\epsilon^{be}\epsilon^{af} + \epsilon^{bf}\epsilon^{ae})Z_{\mathfrak{g}_2}^{cd} + (\epsilon^{bd}\epsilon^{af} + \epsilon^{bf}\epsilon^{ad})Z_{\mathfrak{g}_2}^{ce} + (\epsilon^{bd}\epsilon^{ae} + \epsilon^{ad}\epsilon^{be})Z_{\mathfrak{g}_2}^{cf}] \\ &\quad - 8J^{ij}[(\epsilon^{be}\epsilon^{cf} + \epsilon^{bf}\epsilon^{ce})\epsilon^{ad} + (\epsilon^{bd}\epsilon^{cf} + \epsilon^{bf}\epsilon^{cd})\epsilon^{ae} + (\epsilon^{bd}\epsilon^{ce} + \epsilon^{cd}\epsilon^{be})\epsilon^{af} \\ &\quad + (\epsilon^{ae}\epsilon^{cf} + \epsilon^{af}\epsilon^{ce})\epsilon^{bd} + (\epsilon^{ad}\epsilon^{cf} + \epsilon^{af}\epsilon^{cd})\epsilon^{be} + (\epsilon^{ad}\epsilon^{ce} + \epsilon^{cd}\epsilon^{ae})\epsilon^{bf} \\ &\quad + (\epsilon^{be}\epsilon^{af} + \epsilon^{bf}\epsilon^{ae})\epsilon^{cd} + (\epsilon^{bd}\epsilon^{af} + \epsilon^{bf}\epsilon^{ad})\epsilon^{ce} + (\epsilon^{bd}\epsilon^{ae} + \epsilon^{ad}\epsilon^{be})\epsilon^{cf}]. \end{aligned} \quad (2.25)$$

The hatted pure fermionic generators have the same structure and obey the same Poisson brackets.

III. SUPERCONFORMAL MECHANICS WITH MANIFEST $\mathfrak{so}(7)$ SYMMETRY

With our choice of the fermions ψ_0, ψ_i , the basic Poisson brackets between the generators of $\mathcal{N} = 8$ Poincaré supersymmetry read

$$\{Q_0, Q_0\} = 2iH, \quad \{Q_0, Q_i\} = 0, \quad \{Q_i, Q_i\} = 2i\delta_{ij}H, \quad (3.1)$$

To extend this algebra to the superconformal one, we need the generators of dilatation $\mathcal{D}$, conformal boost $\mathcal{K}$, and superconformal transformations S_i, S_0 defined now as follows:

$$\mathcal{D} = \frac{1}{2}rp_r, \quad \mathcal{K} = \frac{1}{2}r^2, \quad S_i = r\psi_i, \quad S_0 = r\psi_0, \quad (3.2)$$

where the new bosonic field r and its momentum p_r obey the standard brackets

$$\{p_r, r\} = 1. \quad (3.3)$$

The basic Poisson brackets have the form

$$\{S_i, S_j\} = 2i\delta_{ij}K, \quad \{S_0, S_0\} = 2iK, \quad \{S_i, S_0\} = 0. \quad (3.4)$$

Depending on the structure of superconformal symmetry, the generators of R -symmetry include the bosonic $\{V_i, W_{ij}\}$ and/or fermionic $\mathfrak{so}(8)$ generators $\{\hat{V}_i, \hat{W}_{ij}\}$ or some subset of these generators.

In full correspondence with our ansatz for the $\mathcal{N} = 8$ supercharges⁴ [15,16], the supercharges must have the form

$$Q_i = p_r \psi_i + \frac{\Theta_i}{r}, \quad Q_0 = p_r \psi_0 + \frac{\Theta_0}{r},$$

The composites Θ^{iA} and $\Upsilon^{a\alpha}$ have the structure “ $(R - \text{symmetry generators}) \times \text{fermions}$.” Thus, the terms cubic in fermions in the supercharges correspond to fermionic realizations of the R -symmetry generators while the terms linear in fermions in the supercharges correspond to the bosonic R -symmetry generators.

A. $OSp(8|2)$ superconformal mechanics

The supercharges generating the $OSp(8|2)$ superconformal mechanics read

$$\begin{aligned} Q_0 &= p_r \psi_0 - \frac{1}{4r} (c_{ijk} W_{ij} \psi_k + 2V_i \psi_i), \\ Q_i &= p_r \psi_i + \frac{1}{4r} (2V_i \psi_0 - 2W_{ij} \psi_j + 2c_{ijk} V_j \psi_k + c_{ijk} W_{jk} \psi_0 - f_{ijkl} W_{jk} \psi_l). \end{aligned} \quad (3.5)$$

The supercharges form the $\mathcal{N} = 8$ super Poincaré algebra

$$\{Q_0, Q_0\} = 2iH, \quad \{Q_0, Q_i\} = 0, \quad \{Q_i, Q_i\} = 2i\delta_{ij}H, \quad (3.6)$$

with the Hamiltonian

$$H = \frac{1}{2} p_r^2 + \frac{1}{r^2} \left[\frac{1}{8} V_i (V_i - 4\hat{V}_i) + \frac{1}{16} W_{ij} (W_{ij} - 4\hat{W}_{ij}) - \frac{1}{4} c_{ijk} (V_i \hat{W}_{jk} + \hat{V}_i W_{jk}) - \frac{1}{8} f_{ijkl} W_{ij} \hat{W}_{kl} \right] \quad (3.7)$$

The bosonic part of the Hamiltonian (3.7) reads

$$H_{\text{bos}} = \frac{1}{2} p_r^2 + \frac{1}{16r^2} \sum_{\mu, \nu=0}^7 W_{\mu\nu} W_{\mu\nu}. \quad (3.8)$$

It describes the particle on the eight-dimensional Euclidean cone, since the angular part of this Hamiltonian is the seven-dimensional sphere [16].

To understand the full dynamical symmetry of the system, one has to calculate the brackets between the Poincaré (3.5) and conformal supersymmetry generators (3.2)

$$\begin{aligned} \{Q_0, S_0\} &= 2iD, \\ \{Q_0, S_i\} &= -i \left(\hat{V}_i + \frac{1}{2} V_i + \frac{1}{4} c_{ijk} W_{jk} \right) \equiv -i\mathbf{V}_i, \quad \{Q_i, S_0\} = i\mathbf{V}_i, \\ \{Q_i, S_j\} &= 2i\delta_{ij}D - i \left(\frac{1}{2} W_{ij} + \frac{1}{2} c_{ijk} V_k + \frac{1}{4} f_{ijkl} W_{kl} + \hat{W}_{ij} \right) \equiv 2i\delta_{ij}D - i\mathbf{W}_{ij}. \end{aligned} \quad (3.9)$$

Thus we see that we have $\mathfrak{so}(8)$ R -symmetry spanned by the generators

$$\mathbf{V}_i = \hat{V}_i + \frac{1}{2} \left(V_i + \frac{1}{2} c_{ijk} W_{jk} \right), \quad \mathbf{W}_{ij} = \hat{W}_{ij} + \frac{1}{2} \left(W_{ij} + c_{ijk} V_k + \frac{1}{2} f_{ijkl} W_{kl} \right), \quad (3.10)$$

⁴Compare with those for the $\mathcal{N} = 4$ case [17,18].

$$\{\mathbf{W}_{ij}, \mathbf{W}_{kl}\} = \delta_{ik}\mathbf{W}_{jl} - \delta_{jk}\mathbf{W}_{il} - \delta_{il}\mathbf{W}_{jk} + \delta_{jl}\mathbf{W}_{ik}, \quad \{\mathbf{W}_{ij}, \mathbf{V}_k\} = \delta_{ik}\mathbf{V}_j - \delta_{jk}\mathbf{V}_i, \quad \{\mathbf{V}_i, \mathbf{V}_j\} = \mathbf{W}_{ij}. \quad (3.11)$$

The generators $\mathbf{V}_i, \mathbf{W}_{ij}$ are conserved and they have the following Poisson brackets with the Hamiltonian (3.7) and supercharges (3.5)

$$\begin{aligned} \{H, \mathbf{V}_i\} &= \{H, \mathbf{W}_{ij}\} = 0, & \{\mathbf{V}_i, Q_0\} &= Q_i, & \{\mathbf{V}_i, Q_j\} &= -\delta_{ij}Q_0, \\ \{\mathbf{W}_{ij}, Q_k\} &= \delta_{ik}Q_j - \delta_{jk}Q_i, & \{\mathbf{W}_{ij}, Q_0\} &= 0. \end{aligned} \quad (3.12)$$

B. $\mathfrak{F}(4)$ superconformal mechanics

The supercharges generating the $\mathfrak{F}(4)$ superconformal mechanics can be constructed with the help of $\mathfrak{so}(7)$ generators W_{ij} and $\hat{W}_{ij}$ (2.2) as follows

$$\begin{aligned} Q_0 &= p_r\psi_0 - \frac{1}{3r}c_{ijk}\left(W_{ij} - \frac{1}{6}\hat{W}_{ij}\right)\psi_k, \\ Q_i &= p_r\psi_i - \frac{1}{3r}\left(2W_{ij}\psi_j - c_{ijk}\left(W_{jk} - \frac{1}{2}\hat{W}_{jk}\right)\psi_0 + f_{ijkl}\left(W_{jk} + \frac{1}{6}\hat{W}_{jk}\right)\psi_l\right). \end{aligned} \quad (3.13)$$

The supercharges form the $\mathcal{N} = 8$ super Poincaré algebra

$$\{Q_0, Q_0\} = 2iH, \quad \{Q_0, Q_i\} = 0, \quad \{Q_i, Q_i\} = 2i\delta_{ij}H, \quad (3.14)$$

with the Hamiltonian

$$H = \frac{1}{2}p_r^2 + \frac{1}{r^2}\left[\frac{1}{9}W_{ij}(W_{ij} - 3\hat{W}_{ij}) - \frac{1}{6}f_{ijkl}\left(W_{ij} + \frac{1}{12}\hat{W}_{ij}\right)\hat{W}_{kl} - \frac{1}{3}c_{ijk}\left(W_{ij} - \frac{1}{6}\hat{W}_{ij}\right)\hat{V}_j\right]. \quad (3.15)$$

The bosonic part of the Hamiltonian (3.15) reads

$$H_{\text{bos}} = \frac{1}{2}p_r^2 + \frac{1}{9r^2}\sum_{i,j=1}^7 W_{ij}W_{ij}, \quad (3.16)$$

In contrast with (3.8), it describes the particle on the seven-dimensional Euclidean cone, since the angular part of this Hamiltonian is the six-dimensional sphere [16].

To understand the full dynamical symmetry of the system, one has to calculate the brackets between the Poincaré (4.12) and conformal supersymmetry generators (3.2)

$$\begin{aligned} \{Q_0, S_0\} &= 2iD, \\ \{Q_0, S_i\} &= -\frac{i}{3}c_{ijk}\hat{W}_{jk}, & \{Q_i, S_0\} &= \frac{i}{3}c_{ijk}\hat{W}_{jk}, \\ \{Q_i, S_j\} &= 2i\delta_{ij}D - \frac{i}{3}f_{ijkl}\hat{W}_{kl} - \frac{2i}{3}\hat{W}_{ij}, \end{aligned} \quad (3.17)$$

where

$$\hat{W}_{ij} = W_{ij} + \frac{1}{2}\left(\hat{W}_{ij} + c_{ijk}\hat{V}_k + \frac{1}{2}f_{ijkl}\hat{W}_{kl}\right). \quad (3.18)$$

Thus we see that we have $\mathfrak{so}(7)$ R -symmetry spanned by the generators $\mathbf{W}_{ij}$. These generators obey the following brackets:

$$\{H, \hat{W}_{ij}\} = 0, \quad \{\hat{W}_{ij}, Q_0\} = \frac{1}{2}c_{ijk}Q_k, \quad \{\hat{W}_{ij}, Q_k\} = \frac{1}{2}(\delta_{ik}Q_j - \delta_{jk}Q_i) - \frac{1}{2}c_{ijk}Q_0 + \frac{1}{2}f_{ijkl}Q_l. \quad (3.19)$$

C. $G(3)$ superconformal mechanics with $\mathcal{N}=7$ supersymmetry

To construct $G(3)$ superconformal mechanics, one has to explore the fermionic version of the $\mathfrak{g}_2$ generators (2.8)

$$\hat{\mathcal{W}}_{ij} = \hat{W}_{ij} + \frac{1}{4}f_{ijkl}\hat{W}_{kl}. \quad (3.20)$$

With their help the unique solution for supercharges spanning $N=7$ super-Poincaré algebra reads

$$Q_i = p_r\psi_i - \frac{1}{6r}f_{ijkl}\hat{\mathcal{W}}_{jk}\psi_l = p_r\psi_i - \frac{i}{12r}f_{ijkl}\psi_j\psi_k\psi_l. \quad (3.21)$$

The supercharges (3.21) form the $\mathcal{N}=7$ super Poincaré algebra

$$\{Q_i, Q_i\} = 2i\delta_{ij}H, \quad (3.22)$$

with the Hamiltonian

$$H = \frac{1}{2}p_r^2 - \frac{1}{18r^2}\hat{\mathcal{W}}_{ij}\hat{\mathcal{W}}_{ij} = \frac{1}{2}p^2 + \frac{1}{48x^2}f_{ijkl}\psi_i\psi_j\psi_k\psi_l. \quad (3.23)$$

Again, to clarify the full dynamical symmetry of the system, one has to calculate the brackets between the Poincaré (3.21) and conformal supersymmetry generators S_i (3.2)

$$\{Q_i, S_j\} = 2i\delta_{ij}D - i\hat{\mathcal{W}}_{ij}. \quad (3.24)$$

Thus we see that we have $\mathfrak{g}_2$ R -symmetry spanned by the generators $\hat{\mathcal{W}}_{ij}$. These generators obey the following brackets

$$\{H, \hat{\mathcal{W}}_{ij}\} = 0, \quad \{\hat{\mathcal{W}}_{ij}, Q_k\} = \delta_{ik}Q_j - \delta_{jk}Q_i + \frac{1}{2}f_{ijkl}Q_l, \quad (3.25)$$

$$\begin{aligned} \{\hat{\mathcal{W}}_{ij}, S_k\} &= \delta_{ik}S_j - \delta_{jk}S_i + \frac{1}{2}f_{ijkl}S_l, \\ \{\hat{\mathcal{W}}_{ij}, \hat{\mathcal{W}}_{kl}\} &= -\frac{3}{2}(f_{ijkm}\hat{\mathcal{W}}_{lm} - f_{ijlm}\hat{\mathcal{W}}_{km}) + c_{ijm}c_{kln}\hat{\mathcal{W}}_{mn}. \end{aligned} \quad (3.26)$$

Alas, we did not find any possibility to include in the game more bosonic fields. So, for the time being, we have $\mathcal{N}=7$ superconformal mechanics with $G(3)$ superconformal dynamical symmetry with one bosonic and seven fermionic components.

IV. SUPERCONFORMAL MECHANICS WITH MANIFEST $\mathfrak{su}(2)$ SYMMETRY

With our choice of the fermions (2.13), the basic Poisson brackets between the generators of $\mathcal{N}=8$ Poincaré supersymmetry read

$$\{Q^{iA}, Q^{jB}\} = 4\epsilon^{ij}\epsilon^{AB}\mathcal{H}, \quad \{q^{a\alpha}, q^{b\beta}\} = 4\epsilon^{ab}\epsilon^{\alpha\beta}\mathcal{H}, \quad \{Q^{iA}, q^{a\alpha}\} = 0. \quad (4.1)$$

To extend this algebra to the superconformal ones, we need the generators of dilatation $\mathcal{D}$, conformal boost $\mathcal{K}$, and superconformal transformations $S^{iA}, s^{a\alpha}$ defined as follows:

$$\mathcal{D} = \frac{1}{2}rp_r, \quad \mathcal{K} = \frac{1}{2}r^2, \quad S^{iA} = r\phi^{iA}, \quad s^{a\alpha} = r\chi^{a\alpha}. \quad (4.2)$$

The basic Poisson brackets have the form [15,16]

$$\{S^{iA}, S^{jB}\} = 4\epsilon^{ij}\epsilon^{AB}\mathcal{K}, \quad \{s^{a\alpha}, s^{b\beta}\} = 4\epsilon^{ab}\epsilon^{\alpha\beta}\mathcal{K}, \quad \{S^{iA}, s^{a\alpha}\} = 0. \quad (4.3)$$

Depending on the structure of superconformal symmetry, the generators of R -symmetry include the bosonic $\{J^{ij}, W^{AB}, X^{ab}, Y^{\alpha\beta}, V^{iA\alpha\alpha}\}$ (2.15), and/or fermionic $\mathfrak{so}(8)$ generators $\{\hat{J}^{ij}, \hat{W}^{AB}, \hat{X}^{ab}, \hat{Y}^{\alpha\beta}, \hat{V}^{iA\alpha\alpha}\}$ (2.16), (2.17) or some subset of these generators.

In full correspondence with our ansatz for the $\mathcal{N} = 8$ supercharges [15,16] (see [17,18] for the $\mathcal{N} = 4$ case), the supercharges have the form

$$Q^{iA} = p_r \phi^{iA} + \frac{\Theta^{iA}}{r}, \quad q^{a\alpha} = p_r \chi^{a\alpha} + \frac{\Upsilon^{a\alpha}}{r}.$$

The composites Θ^{iA} and $\Upsilon^{a\alpha}$ have the structure “ $(R - \text{symmetry generators}) \times \text{fermions}$.”

A. $OSp(8|2)$ superconformal mechanics

This superconformal mechanics was constructed in [10] at the level of superfields and components. The corresponding supercharges and Hamiltonian for this case were constructed in [15,16]. Their have the form

$$\begin{aligned} Q^{iA} &= p_r \phi^{iA} + \frac{1}{r} \left(J^{ij} \phi_j^A + W^{AB} \phi_B^i - \frac{1}{2} V^{iA\alpha\alpha} \chi_{\alpha\alpha} \right), \\ q^{a\alpha} &= p_r \chi^{a\alpha} + \frac{1}{r} \left(X^{ab} \chi_b^{\alpha} + Y^{\alpha\beta} \chi_{\beta}^a + \frac{1}{2} V^{iA\alpha\alpha} \phi_{iA} \right). \end{aligned} \quad (4.4)$$

The corresponding Hamiltonian reads

$$H_{Osp(8|2)} = \frac{1}{2} p_r^2 + \frac{1}{r^2} \left(-\frac{1}{2} \mathcal{C}_{\mathfrak{so}(8)} + \frac{5}{8} (\mathcal{C}_{\mathfrak{so}(8)}|_{\text{fermions} \rightarrow 0}) \right). \quad (4.5)$$

The $\mathfrak{so}(8)$ Casimir operator $\mathcal{C}_{\mathfrak{so}(8)}$ is defined in a standard way as

$$\begin{aligned} \mathcal{C}_{\mathfrak{so}(8)} &\equiv (J^{ij} + \hat{J}^{ij})(J_{ij} + \hat{J}_{ij}) + (W^{AB} + \hat{W}^{AB})(W_{AB} + \hat{W}_{AB}) + (X^{ab} + \hat{X}^{ab})(X_{ab} + \hat{X}_{ab}) \\ &\quad + (Y^{\alpha\beta} + \hat{Y}^{\alpha\beta})(Y_{\alpha\beta} + \hat{Y}_{\alpha\beta}) + \frac{1}{4} (V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha})(V_{iA\alpha\alpha} + \hat{V}_{iA\alpha\alpha}). \end{aligned} \quad (4.6)$$

The generators of dynamical $\mathfrak{so}(8)$ R -symmetry of the system appear in the brackets between the Poincaré (4.4) and conformal supersymmetry generators (4.2)

$$\begin{aligned} \{Q^{iA}, S^{jB}\} &= 2i\epsilon^{ij}(W^{AB} + \hat{W}^{AB}) + 2i\epsilon^{AB}(J^{ij} + \hat{J}^{ij}) + 4i\epsilon^{ij}\epsilon^{AB}D, \\ \{Q^{iA}, s^{a\alpha}\} &= -i(V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha}), \quad \{q^{a\alpha}, S^{iA}\} = i(V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha}), \\ \{q^{a\alpha}, s^{b\beta}\} &= 2i\epsilon^{ab}(Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}) + 2i\epsilon^{\alpha\beta}(X^{ab} + \hat{X}^{ab}) + 4i\epsilon^{ij}\epsilon^{\alpha\beta}D. \end{aligned} \quad (4.7)$$

The supercharges Q^{iA} and $q^{a\alpha}$ form the fundamental representations with respect to the corresponding $\mathfrak{su}(2)$ algebras, i.e.,

$$\begin{aligned} \{J^{ij} + \hat{J}^{ij}, Q^{kA}\} &= -\frac{1}{2}(\epsilon^{ik}Q^{jA} + \epsilon^{jk}Q^{iA}), & \{W^{AB} + \hat{W}^{AB}, Q^{iC}\} &= -\frac{1}{2}(\epsilon^{AC}Q^{iB} + \epsilon^{BC}Q^{iA}), \\ \{X^{ab} + \hat{X}^{ab}, Q^{iA}\} &= \{Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, Q^{iA}\} = 0, & \{J^{ij} + \hat{J}^{ij}, q^{a\alpha}\} &= \{W^{AB} + \hat{W}^{AB}, q^{a\alpha}\} = 0, \\ \{X^{ab} + \hat{X}^{ab}, q^{c\alpha}\} &= -\frac{1}{2}(\epsilon^{ac}q^{b\alpha} + \epsilon^{bc}q^{a\alpha}), & \{Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, q^{a\gamma}\} &= -\frac{1}{2}(\epsilon^{\alpha\gamma}q^{a\beta} + \epsilon^{\beta\gamma}q^{a\alpha}). \end{aligned} \quad (4.8)$$

In addition, these currents commute with the Hamiltonian (4.5)

$$\{H_{Osp(8|2)}, J^{ij} + \hat{J}^{ij}\} = \{H_{Osp(8|2)}, W^{AB} + \hat{W}^{AB}\} = \{H_{Osp(8|2)}, X^{ab} + \hat{X}^{ab}\} = \{H_{Osp(8|2)}, Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}\} = 0. \quad (4.9)$$

Finally, we have

$$\begin{aligned} \{V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha}, Q^{jB}\} &= 2\epsilon^{ij}\epsilon^{AB}q^{a\alpha}, & \{V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha}, q^{b\beta}\} &= -2\epsilon^{ab}\epsilon^{a\beta}Q^{iA}, \\ \{H_{Osp(8|2)}, V^{iA\alpha\alpha} + \hat{V}^{iA\alpha\alpha}\} &= 0. \end{aligned} \quad (4.10)$$

The bosonic part of the Hamiltonian (4.5) then coincides with (3.8)⁵

$$H_{\text{bos}} = \frac{1}{2}p_r^2 + \frac{1}{8r^2}C_{\mathfrak{so}(8)}|_{\text{fermions} \rightarrow 0} = \frac{1}{2}p^2 + \frac{1}{16r^2} \sum_{\mu,\nu=1}^8 \ell_{\mu,\nu} \ell_{\mu,\nu}. \quad (4.11)$$

B. $\mathfrak{F}(4)$ supersymmetric mechanics

Now to construct the supercharges we have to use the generators spanning $\mathfrak{so}(7)$ algebra, i.e.,

$$J^{ij}, Y^{\alpha\beta}, Z_{\mathfrak{so}(7)}^{ab}, V_{\mathfrak{so}(7)}^{iaba}$$

with $Z_{\mathfrak{so}(7)}^{ab}$ and $V_{\mathfrak{so}(7)}^{iaba}$ defined in (2.20), (2.21). In addition, the fermions ϕ^{iA} carrying out now the indices $\{i, a\}$, i.e., $\phi^{iA} \rightarrow \phi^{ia}$.

The $\mathfrak{F}(4)$ supersymmetric mechanics is described by the following supercharges:

$$\begin{aligned} Q^{ia} &= p_r \phi^{ia} - \frac{1}{r} \left[\left(\frac{2}{3} Z_{\mathfrak{so}(7)}^{ab} + 2\hat{Z}_{\mathfrak{so}(7)}^{ab} \right) \phi^i_b + \frac{4}{9} (3J^{ij} + 5\hat{J}^{ij}) \phi_j^a + \frac{1}{3} (V_{\mathfrak{so}(7)}^{iaba} + \hat{V}_{\mathfrak{so}(7)}^{iaba}) \chi_{aa} \right], \\ q^{a\alpha} &= p_r \chi^{a\alpha} + \frac{1}{r} \left[\left(\frac{2}{3} Z_{\mathfrak{so}(7)}^{ab} + 2\hat{Z}_{\mathfrak{so}(7)}^{ab} \right) \chi_b^a + \frac{4}{9} (3Y^{\alpha\beta} + 5\hat{Y}^{\alpha\beta}) \chi_\beta^a - \frac{1}{3} (V_{\mathfrak{so}(7)}^{jaba} + \hat{V}_{\mathfrak{so}(7)}^{jaba}) \phi_{jb} \right], \end{aligned} \quad (4.12)$$

and by the Hamiltonian

$$H_{\mathfrak{F}(4)} = \frac{1}{2}p_r^2 + \frac{1}{r^2} \left[-\frac{2}{3}C_{so(7)} + \frac{8}{9}(C_{so(7)}|_{\text{fermions} \rightarrow 0}) + \frac{4}{9}(C_{so(7)}|_{\text{bosons} \rightarrow 0}) \right]. \quad (4.13)$$

Here $C_{\mathfrak{so}(7)}$ is the Casimir operator commuting with the generators $\{J^{ij} + \hat{J}^{ij}, Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}, Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, V_{\mathfrak{so}(7)}^{iaba} - \hat{V}_{\mathfrak{so}(7)}^{iaba}\}$

$$\begin{aligned} C_{so(7)} &= (J^{ij} + \hat{J}^{ij})(J_{ij} + \hat{J}_{ij}) + \frac{1}{2}(Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab})((Z_{\mathfrak{so}(7)})_{ab} + (\hat{Z}_{\mathfrak{so}(7)})_{ab}) + (Y^{\alpha\beta} + \hat{Y}^{\alpha\beta})(Y_{\alpha\beta} + \hat{Y}_{\alpha\beta}) \\ &+ \frac{1}{16}(V_{\mathfrak{so}(7)}^{iaba} - \hat{V}_{\mathfrak{so}(7)}^{iaba})((V_{\mathfrak{so}(7)})_{iaba} - (\hat{V}_{\mathfrak{so}(7)})_{iaba}). \end{aligned} \quad (4.14)$$

To understand the full dynamical symmetry of the system, one has to calculate the Poisson brackets between the Poincaré (4.12) and conformal supersymmetry generators (4.2) (with the indices A, B replaced by the indices a, b):

$$\begin{aligned} \{Q^{ia}, S^{jb}\} &= \frac{4}{3}i\epsilon^{ij}(Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}) + \frac{8}{3}i\epsilon^{ab}(J^{ij} + \hat{J}^{ij}) + 4i\epsilon^{ij}\epsilon^{ab}D, \\ \{Q^{ia}, s^{b\alpha}\} &= \frac{2}{3}i(V_{\mathfrak{so}(7)}^{iaba} - \hat{V}_{\mathfrak{so}(7)}^{iaba}), & \{q^{b\alpha}, S^{ia}\} &= -\frac{2}{3}i(V_{\mathfrak{so}(7)}^{iaba} - \hat{V}_{\mathfrak{so}(7)}^{iaba}), \\ \{q^{a\alpha}, s^{b\beta}\} &= \frac{4}{3}i\epsilon^{\alpha\beta}(Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}) + \frac{8}{3}i\epsilon^{ab}(Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}) + 4i\epsilon^{ij}\epsilon^{ab}D. \end{aligned} \quad (4.15)$$

Thus, we see that instead of $\mathfrak{so}(8)$ symmetry we indeed have its subalgebra spanned by the generators

$$\{J^{ij} + \hat{J}^{ij}, Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}, Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, V_{\mathfrak{so}(7)}^{iaba} - \hat{V}_{\mathfrak{so}(7)}^{iaba}\}. \quad (4.16)$$

These 21 generators span $\mathfrak{so}(7)$ subalgebra in $\mathfrak{so}(8)$.

⁵The $\mathfrak{so}(8)$ operators $\ell_{\mu,\nu}$ are defined in (A4). Note, the term $\sum_{\mu,\nu=1}^8 \ell_{\mu,\nu} \ell_{\mu,\nu}$ does not fix the norm $y_\mu y_\mu$, which therefore can be identified with r^2 reducing “evident” nine bosonic coordinates to eight.

Thus, we deal with $\mathfrak{F}(4)$ superconformal symmetry. Note, the pure fermionic parts of the supercharges in (4.12) coincide with those obtained in [8].

The generators (4.16) are the conserved quantities

$$\{H_{\mathfrak{F}(4)}, \{J^{ij} + \hat{J}^{ij}, Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}, Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, V_{\mathfrak{so}(7)}^{iabc} - \hat{V}_{\mathfrak{so}(7)}^{iabc}\}\} = 0 \quad (4.17)$$

and they rotate the supercharges as follows:

$$\begin{aligned} \{J^{ij} + \hat{J}^{ij}, Q^{k,a}\} &= -\frac{1}{2}(\epsilon^{ik}Q^{ja} + \epsilon^{jk}Q^{ia}), & \{J^{ij} + \hat{J}^{ij}, q^{aa}\} &= 0, \\ \{Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}, Q^{ic}\} &= -\frac{1}{2}(\epsilon^{ac}Q^{ib} + \epsilon^{bc}Q^{ia}), & \{Z_{\mathfrak{so}(7)}^{ab} + \hat{Z}_{\mathfrak{so}(7)}^{ab}, q^{c\alpha}\} &= -\frac{1}{2}(\epsilon^{ac}q^{b\alpha} + \epsilon^{bc}q^{a\alpha}), \\ \{Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, Q^{ia}\} &= 0, & \{Y^{\alpha\beta} + \hat{Y}^{\alpha\beta}, q^{a\gamma}\} &= -\frac{1}{2}(\epsilon^{\alpha\gamma}q^{a\beta} + \epsilon^{\beta\gamma}q^{a\alpha}), \\ \{V_{\mathfrak{so}(7)}^{iabc} - \hat{V}_{\mathfrak{so}(7)}^{iabc}, Q^{jc}\} &= -2\epsilon^{ij}(\epsilon^{ac}q^{ba} + \epsilon^{bc}q^{aa}), & \{V_{\mathfrak{so}(7)}^{iabc} - \hat{V}_{\mathfrak{so}(7)}^{iabc}, q^{c\beta}\} &= 2\epsilon^{\alpha\beta}(\epsilon^{ac}Q^{ib} + \epsilon^{bc}Q^{ia}). \end{aligned} \quad (4.18)$$

The bosonic part of the Hamiltonian (4.13) then coincides with (3.16)

$$H_{\text{bos}} = \frac{1}{2}p_r^2 + \frac{2}{9r^2}\mathcal{C}_{so(7)}|_{\text{fermions} \rightarrow 0} = \frac{1}{2}p_r^2 + \frac{1}{9r^2}\sum_{\mu,\nu=1}^7 \ell_{\mu,\nu}\ell_{\mu,\nu}. \quad (4.19)$$

C. $G(3)$ superconformal algebra

Now to construct the supercharges spanning the $G(3)$ superconformal algebra, we have to use the generators spanning $\mathfrak{g}_2$ algebra, i.e.,

$$J^{ij}, Z_{\mathfrak{g}_2}^{ab}, V_{\mathfrak{g}_2}^{iabc}$$

with $Z_{\mathfrak{g}_2}^{ab}$ and $V_{\mathfrak{g}_2}^{iabc}$ defined in (2.23), (2.24). In addition, the fermions ϕ^{iA} carrying now the indices $\{i, a\}$, i.e., $\phi^{iA} \rightarrow \phi^{ia}$, and the fermions χ^{ab} carrying the indices $\{a, b\}$ are symmetric over these indices, i.e.,

$$\chi^{iA} \rightarrow \chi_{\mathfrak{g}_2}^{ab} = \chi^{ab} + \chi^{ba}. \quad (4.20)$$

Correspondingly, we have the supercharges $Q^{ia}, q^{ab} = q^{ba}$ with the following Poisson brackets

$$\{Q^{ia}, Q^{jb}\} = 4i\epsilon^{ij}\epsilon^{ab}H, \quad \{q^{ab}, q^{cd}\} = 8i(\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{cb})H, \quad \{Q^{ia}, q^{bc}\} = 0. \quad (4.21)$$

The conformal supercharges (3.2) can also be redefined as

$$S^{ia} = r\phi^{ia}, \quad s^{ab} = r\chi_{\mathfrak{g}_2}^{ab}, \quad (4.22)$$

$$\{S^{ia}, S^{jb}\} = 4i\epsilon^{ij}\epsilon^{ab}K, \quad \{s^{ab}, s^{cd}\} = 8i(\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{cb})K, \quad \{S^{ia}, s^{bc}\} = 0. \quad (4.23)$$

The supercharges spanning $\mathcal{N} = 7$ super Poincaré algebra have the form

$$\begin{aligned} Q^{ia} &= p_r\phi^{ia} + \frac{1}{r}\left[\frac{1}{3}\hat{J}^{ij}\phi_j^a - \frac{1}{32}\hat{V}_{\mathfrak{g}_2}^{iabc}(\chi_{\mathfrak{g}_2})_{bc}\right], \\ q^{ab} &= p_r\chi_{\mathfrak{g}_2}^{ab} + \frac{1}{8r}\hat{V}_{\mathfrak{g}_2}^{jcab}\phi_{jc}. \end{aligned} \quad (4.24)$$

while the Hamiltonian H reads

$$H_{\mathfrak{G}(3)} = \frac{1}{2}p_r^2 - \frac{1}{r^2} \left[\frac{5}{12} \hat{J}^{ij} \hat{J}_{ij} + \frac{1}{4} \hat{Z}_{\mathfrak{g}_2}^{ab} (\hat{Z}_{\mathfrak{g}_2})_{ab} \right] = \frac{1}{2}p_r^2 + \frac{1}{192r^2} [3\phi^{ia}\phi_i^b(\chi_{\mathfrak{g}_2})_a{}^c(\chi_{\mathfrak{g}_2})_{bc} + 2\phi^{ia}\phi_i^b\phi^j{}_a\phi_{jb}] \quad (4.25)$$

To understand the full dynamical symmetry of the system, one has to calculate the Poisson brackets between the Poincaré (4.24) and conformal supersymmetry generators (4.22)

$$\begin{aligned} \{Q^{ia}, S^{jb}\} &= -i\epsilon^{ij}\hat{Z}_{\mathfrak{g}_2}^{ab} - 3i\epsilon^{ab}\hat{J}^{ij} + 4i\epsilon^{ij}\epsilon^{ab}D, \\ \{q^{ab}, S^{ic}\} &= \frac{1}{2}i\hat{V}_{\mathfrak{g}_2}^{iabc}, \quad \{Q^{ia}, S^{bc}\} = -\frac{1}{2}i\hat{V}_{\mathfrak{g}_2}^{iabc}, \\ \{q^{ab}, S^{cd}\} &= 4i(\epsilon^{ac}\hat{Z}_{\mathfrak{g}_2}^{bd} + \epsilon^{bd}\hat{Z}_{\mathfrak{g}_2}^{ac}) + 8i(\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{bc})D. \end{aligned} \quad (4.26)$$

Thus we see that we have R -symmetry spanned by $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebras with the generators $\hat{Z}_{\mathfrak{g}_2}^{ab} = \hat{X}^{ab} + \hat{W}^{ab} + \hat{Y}^{ab}$ and $\hat{J}^{ij}$ and the generators $\hat{V}_{\mathfrak{g}_2}^{iabc}$ from the coset $\mathfrak{g}_2/\mathfrak{su}(2) \times \mathfrak{su}(2)$. Therefore, we are dealing with $G(3)$ superconformal symmetry.

The generators $\hat{J}^{ij}, \hat{Z}_{\mathfrak{g}_2}^{ab}$ are the conserved quantities, i.e., they commute with the Hamiltonian (4.25)

$$\{H_{\mathfrak{G}(3)}, \hat{J}^{ij}\} = \{H_{\mathfrak{G}(3)}, \hat{Z}_{\mathfrak{g}_2}^{ab}\} = 0. \quad (4.27)$$

The Poisson brackets between the generators $\hat{J}^{ij}, \hat{Z}_{\mathfrak{g}_2}^{ab}$ and the supercharges (4.24) read

$$\begin{aligned} \{\hat{J}^{ij}, Q^{ka}\} &= -\frac{1}{2}(\epsilon^{ik}Q^{ja} + \epsilon^{jk}Q^{ia}), \quad \{\hat{J}^{ij}, q^{ab}\} = 0, \\ \{\hat{Z}_{\mathfrak{g}_2}^{ab}, Q^{ic}\} &= -\frac{1}{2}(\epsilon^{ac}Q^{ib} + \epsilon^{bc}Q^{ia}), \quad \{\hat{Z}_{\mathfrak{g}_2}^{ab}, q^{cd}\} = -\frac{1}{2}(\epsilon^{ac}q^{bd} + \epsilon^{ad}q^{bc} + \epsilon^{bc}q^{ad} + \epsilon^{bd}q^{ac}), \\ \{\hat{V}_{\mathfrak{g}_2}^{iabc}, Q^{jd}\} &= 2\epsilon^{ij}(\epsilon^{ad}q^{bc} + \epsilon^{bd}q^{ac} + \epsilon^{cd}q^{ab}), \\ \{\hat{V}_{\mathfrak{g}_2}^{iabc}, q^{de}\} &= -4[\epsilon^{ad}(\epsilon^{be}Q^{ic} + \epsilon^{ce}Q^{ib}) + \epsilon^{bd}(\epsilon^{ae}Q^{ic} + \epsilon^{ce}Q^{ia}) + \epsilon^{cd}(\epsilon^{ae}Q^{ib} + \epsilon^{be}Q^{ia})]. \end{aligned} \quad (4.28)$$

V. CONCLUSION

In the present paper we have constructed the supercharges and Hamiltonians for two variants of superconformal mechanics associated with the superalgebras $OSP(8|2)$, $\mathfrak{F}(4)$ and $G(3)$. The fermionic and bosonic fields involved were arranged into the generators spanning $\mathfrak{so}(8)$, $\mathfrak{so}(7)$ and $\mathfrak{g}_2$ R -symmetries of the corresponding superconformal algebras. The bosonic and fermionic parts of these R -symmetry generators define the constants of motion and form the same algebras. We have also presented the explicit embedding of $\mathfrak{g}_2$ into $\mathfrak{so}(7)$ for both realizations, with manifest $\mathfrak{so}(7)$ and $\mathfrak{su}(2)$ symmetries. These embeddings have never been presented in such form to the best of our knowledge.

Similarly to the results of our previous papers [15–17], the angular parts of supercharges defining the system have the structure “(R – symmetry generators) $\times$ fermions,” while the angular part of the Hamiltonian is just a sum of full Casimir operators and its purely bosonic and/or fermionic parts. We explicitly demonstrated that the constructed system describes the specific $\mathcal{N} = 7, 8$ supersymmetric extension of free particles on some cone embedded in (pseudo-)Euclidean space with the fermionic part that can be

interpreted as the spin-orbit coupling terms. The $\mathcal{N} = 7$ superconformal mechanics with dynamical $G(3)$ symmetry describe the first example of the classical system constructed on the nonminimal supermultiplet (1, 7, 7, 1). It is very interesting to construct a superspace description of this system. All our mechanics admit the reduction: *bosonic R symmetry generators* $\rightarrow 0$. Probably, these cases are the simplest ones for the search of superfield description.

Finally, we would like to stress that the explicit form of the supercharges and Hamiltonians contains the generators of the bosonic parts of R -symmetry. The structure of these bosonic generators is more or less arbitrary. Thus, similarly to the $\mathcal{N} = 4$ supersymmetric case [11] we have the possibility to describe a different bosonic system simply by choosing different realizations of these symmetries. Clearly, our supercharges in this case will provide the $\mathcal{N} = 8$ superextension for these systems. We are planning to consider this situation in more detail in forthcoming publications.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

To treat the bosonic variables $\{B^{aA}_m, Z^{i\alpha}_m\}$ as additional coordinates and momenta in the first order Lagrangians, one has to introduce these new coordinates and momenta in the supersymmetric mechanics we are dealt with. The simplest way is to associate the fields $\{B^{aA}_2, Z^{i\alpha}_2\}$ with lower index $m = 2$ with the momenta and the fields with lower index $m = 1$, i.e., $\{B^{aA}_1, Z^{i\alpha}_1\}$, with the coordinates. One of the possible associations read

$$\begin{aligned} B^{11}_1 &= y^1 + iy^2, & B^{12}_1 &= y^3 + iy^4, & B^{21}_1 &= -y^3 + iy^4, & B^{22}_1 &= y^1 - iy^2, \\ B^{11}_2 &= -p_1 - ip_2, & B^{12}_2 &= -p_3 - ip_4, & B^{21}_2 &= p_3 - ip_4, & B^{22}_2 &= -p_1 + ip_2, \end{aligned} \quad (A1)$$

$$\begin{aligned} Z^{11}_1 &= y^5 + iy^6, & Z^{12}_1 &= y^8 + iy^7, & Z^{21}_1 &= -y^8 + iy^7, & Z^{22}_1 &= y^5 - iy^6, \\ Z^{11}_2 &= -p_5 - ip_6, & Z^{12}_2 &= -p_8 - ip_7, & Z^{21}_2 &= p_8 - ip_7, & Z^{22}_2 &= -p_5 + ip_6. \end{aligned} \quad (A2)$$

One can check that with the standard brackets

$$\{p_\mu, y_\nu\} = \delta_{\mu\nu}, \quad \mu, \nu = 1, \dots, 8 \quad (A3)$$

the brackets (2.14) are correctly reproduced.

In terms of the coordinates y^μ and momenta p_μ , the $\mathfrak{so}(8)$ generators have the standard form

$$\ell_{\mu,\nu} = p_\mu y_\nu - p_\nu y_\mu. \quad (A4)$$

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