

Relativistic and recoil corrections to light-fermion vacuum polarization for bound systems of spin-0, spin-1/2, and spin-1 particles

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In bound systems whose constituent particles are heavier than the electron, the dominant radiative correction to energy levels is given by light-fermion (electronic) vacuum polarization. In consequence, relativistic and recoil corrections to the one-loop vacuum-polarization correction are phenomenologically relevant. Here, we generalize the treatment, previously accomplished for systems with orbiting muons, to bound systems of constituents with more general spins: spin-0, spin-1/2, and spin-1. We discuss the application of our more general expressions to various systems of interest, including spinless systems (pionium), muonic hydrogen and deuterium, and devote special attention to the excited non- S states of deuteronium, the bound system of a deuteron and its antiparticle. The obtained energy corrections are of order $\alpha^5 m_r$, where α is the fine-structure constant and m_r is the reduced mass.

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I. INTRODUCTION

In bound systems whose constituent particles are heavier than the electron, the dominant radiative correction to the energy of bound states is given by electronic vacuum polarization (eVP). Examples include muonic bound systems, including muonic hydrogen and deuterium [1–3], pionium (the bound $\pi^+\pi^-$ system, see Refs. [4–13]), pionic hydrogen (the bound π^-p system, see Refs. [14,15]), pionic deuterium [16,17], kaonic hydrogen [18], kaonic deuterium [19,20], antiprotonic deuterium (the bound $\bar{p}d$ system, see Refs. [21–26]), and, of course, protonium, the bound system of a proton and its antiparticle [27–30]. Recently, the bound system of a deuteron and its antiparticle has been studied in Ref. [31]. In view of a number of recent investigations that could hint at a “dark-photon” coupling to neutrons [32–37], deuteronium could present an ideal candidate for a sensitive test of the low-energy sector of the Standard Model. Yet another example consists of the bound system of a muon and its antiparticle, known as *dimuonium* (Refs. [38–40]) or *true muonium* (Refs. [41,42]).

Let us briefly review the hierarchy and scaling of the leading terms that contribute to the spectrum of the heavy bound systems, before discussing the relativistic-recoil

correction to eVP, which is the effect of interest for the current study. In all the mentioned bound systems, the leading term is the Schrödinger-Coulomb energy of order $(Z\alpha)^2 m_r$, where α is the fine-structure constant, m_r is the reduced mass, and Z is the nuclear charge number. In the heavy systems, the eVP correction to the bound state enters at the order $\alpha(Z\alpha)^2 m_r$, (i.e., at relative order α). While the dominant eVP corrections depend on the orbital angular-momentum quantum number, they are spin independent in regard to the constituent particles of the bound system.

In the heavy systems, while the leading one-loop eVP enters at order $\alpha^3 m_r$, two-loop vacuum-polarization effects (both irreducible as well as reducible diagrams) give corrections of order $\alpha^4 m_r$ [31,43–48]. Three-loop vacuum-polarization terms are of order $\alpha^5 m_r$ (see Refs. [49–57]). All of the above mentioned corrections are still spin independent in regard to the constituent particles of the bound system.

The spin dependence of the bound-state spectrum is first encountered when one considers relativistic (Breit) corrections to the bound spectrum, which enter at order $(Z\alpha)^4 m_r$ (see Chap. 11 of Ref. [58]). For the kinetic relativistic corrections, the scaling of the corrections is seen easily, in view of the fact that the atomic momentum is of order $p \sim Z\alpha m_r$. Hence, the relativistic kinetic-energy correction, given by $-\vec{p}^4/(8m^3)$, is suppressed relative to the leading nonrelativistic term [which reads as $\vec{p}^2/(2m)$], by two orders of $Z\alpha$. Spin-orbit and spin-spin interactions also lead to corrections of order $(Z\alpha)^4 m_r$. The relativistic Breit corrections are mediated through the exchange of virtual

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photons, which constitute tree-level Feynman diagrams (see Chap. 10 of Ref. [58]).

In the current investigation, we are concerned with a conceptually particularly interesting effect that combines relativistic and recoil effects and loop diagrams, i.e., the analysis of relativistic and spin-dependent corrections to the leading eVP effect in heavy systems [59–67]. One easily realizes that the relativistic and recoil corrections to the one-loop eVP effect enter at order $\alpha(Z\alpha)^4 m_r$, i.e., two orders of α higher than the leading vacuum-polarization effect. The evaluation of the latter terms, for muonic hydrogen, has a somewhat interesting history, as evident from Refs. [59–67]. For spin-1/2 bound systems, gauge invariance questions are nontrivial and have been analyzed recently [65,67]. In fact, in order to treat the problem in the computationally most effective way, one needs to use a specific gauge, which is neither the Feynman nor the Coulomb gauge, but an optimized gauge in which the time-time component of the photon propagator is free from any frequency dependence and remains static, despite the fact that a massive photon (corresponding to the integrand of the Uehling potential, see Ref. [68]) is being exchanged. This gauge has been originally introduced in Ref. [59] and recently generalized to higher orders in Ref. [67].

Our aim here is to derive general expressions for the relativistic recoil correction to eVP, valid for any bound systems with constituent spin-0, spin-1/2, and spin-1 particles. We shall devote special attention to ponium, muonic hydrogen and muonic deuterium, for both fine-structure and hyperfine-structure effects (see Sec. V), and we present a numerical evaluation for the bound system of a deuteron and its antiparticle, which we refer to as deuteronium [31] (see Sec. VI). The detailed investigation on deuteronium, which is a bound system of spin-1 particles, constitutes the most complicated spin structure among the bound systems covered in the current investigation, and is more complex than encountered for bound systems involving electrons [58]. Because the bound deuteron-antideuteron system is of particular interest for the possible detection of New Physics [31,69,70] and for measuring the tensor polarizability of the deuteron [69], it is indicated to generalize the treatment outlined in Refs. [59–67] for bound systems of spin-1/2 particles, to the spin-1 case.

We take the following steps. In Sec. II, we investigate the nonrelativistic quantum electrodynamic (NRQED) Lagrangian for particles with unit spin, and derive the corresponding Feynman rules. In Sec. III, using the interaction kernels derived from the Lagrangian, we calculate the interaction Hamiltonian for particles of spin-1 and compare our result to previous investigations recorded in the literature [71,72]. In order to calculate the relativistic and recoil corrections to the vacuum-polarization corrected photon propagator for spin-1 particles, we use the vacuum-polarization corrected photon propagator in the optimized

gauge [67] (see Sec. IV). In Sec. V we show that our expression for the vacuum-polarization corrected relativistic and recoil contributions can be used to describe the corrections for bound states of particles with any combination of spins 0, 1/2, and 1. Numerical results for the bound system of a deuteron and its antiparticle (deuteronium) are presented in Sec. VI. Conclusions are reserved for Sec. VII.

Natural units with $\hbar = c = \epsilon_0 = 1$ are used throughout this investigation. Also, we use the usual conventions $\mu, \nu = 0, 1, 2, 3$ for space-time indices, $i, j = 1, 2, 3$ for spatial indices, and West-Coast conventions for the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Values for the fundamental constants were taken from CODATA 2022 [73].

II. NRQED LAGRANGIAN AND FEYNMAN RULES

In order to find the eVP correction to the Breit Hamiltonian we shall start from a one-particle Hamiltonian (for spin-1 particles), and then derive the Breit interaction Hamiltonian from the interaction kernels describing the exchange of a massive photon, the photon mass being integrated as a spectral parameter of vacuum polarization. For the discussion of the relativistic and recoil corrections to eVP, we use the photon propagator in an optimized Coulomb gauge [67]. Formally, the photon propagator in the optimized Coulomb gauge describes the exchange of a photon of mass

$$\lambda = \frac{2m_e}{\sqrt{1-v^2}}, \quad (1)$$

where we use $v \in (0, 1)$ as the spectral parameter of vacuum polarization [67]. The first-order correction to the photon propagator in the optimized Coulomb (OC) gauge [67] reads as follows,

$$D_{\mu\nu}^{\text{OC};1}(k) = \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \begin{pmatrix} \frac{1}{\vec{k}^2 + \lambda^2} & 0 \\ 0 & \frac{\delta^{ij} - \frac{k^i k^j}{k^2 + \lambda^2}}{k^2 - \lambda^2} \end{pmatrix}. \quad (2)$$

For our purposes, it is sufficient to use the instantaneous approximation $k^2 - \lambda^2 \rightarrow -(\vec{k}^2 + \lambda^2)$ for the spatial part. The one-loop spectral function is

$$f_1(v) = \frac{v^2(1 - \frac{v^2}{3})}{1 - v^2}. \quad (3)$$

For the derivation of the Breit Hamiltonian for spin-1 particle, we use the plain Coulomb-gauge photon propagator, which we recall as follows (see Chap. 9 of Ref. [58]),

$$D_{\mu\nu}^C(k) = \begin{pmatrix} \frac{1}{k^2} & 0 \\ 0 & \frac{1}{k^2} \left(\delta^{ij} - \frac{\hat{k}^i \hat{k}^j}{k^2} \right) \end{pmatrix}. \quad (4)$$

We employ the instantaneous approximation $k^2 \rightarrow -\vec{k}^2$.

We also need the low-order Feynman rules for the spin-1 version of NRQED. We abstract these rules from the spin-1 Hamiltonian given by Zatorski and Pachucki [71] in their Eq. (1). We convert their expression for the Hamiltonian of a spin-1 particle interacting with the electromagnetic field to a Lagrangian density $\mathcal{L}$ for a particle of mass m and charge q ,

$$\begin{aligned} \mathcal{L} = & \psi^\dagger \left(i\partial_t - qA^0 - \frac{\vec{\pi}^2}{2m} + \frac{\vec{\pi}^4}{8m^3} + \frac{q}{6} r_E^2 \vec{\nabla} \cdot \vec{E} \right. \\ & + \frac{q(\tilde{g}-1)}{4m^2} \vec{S} \cdot (\vec{E} \times \vec{\pi} - \vec{\pi} \times \vec{E}) \\ & \left. + \frac{q}{2} Q_E (S^i S^j)^{(2)} \partial_j E^i + \frac{q\tilde{g}}{2m} \vec{S} \cdot \vec{B} \right) \psi, \end{aligned} \quad (5)$$

where r_E is the rms electric charge radius, Q_E is the electric quadrupole moment, and $\vec{S}$ is the spin operator. Furthermore, $\tilde{g}$ is the magnetic g factor, which parametrizes the magnetic moment $\vec{\mu}$ as $\vec{\mu} = \tilde{g}\mu_m \vec{S}$, where μ_m is the mass-scaled magneton

$$\mu_m = \frac{e\hbar}{2m} = \frac{m_p}{m} \mu_N, \quad (6)$$

where m_p is the proton mass, and μ_N is the nuclear magneton. [The relation between the usual g factor defined by $\vec{\mu} = g\mu_N \vec{S}$ and $\tilde{g}$ is $\tilde{g} = (m/m_p)g$.] The quadrupole part of the product of two vectors (which do not necessarily commute) is

$$(u^i v^j)^{(2)} \equiv \frac{u^i v^j + u^j v^i}{2} - \frac{\delta^{ij}}{3} \vec{u} \cdot \vec{v}. \quad (7)$$

We also use A^0 and $\vec{A}$ for the scalar and vector potentials and the associated electric and magnetic fields $\vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The kinetic momentum operator is

$$\vec{\pi} = \vec{p} - q\vec{A} = -i\vec{\nabla} - q\vec{A}, \quad (8)$$

where q is the particle's charge. The interaction Lagrangian (5) includes all terms involving inverse masses up to $1/m^2$. The rms electric charge radius r_E has dimension of length and is measured in units of fermi, while the quadrupole moment Q_E is canonically measured in units of fm² (the elementary charge is divided out right from the start, by convention).

TABLE I. We extract the Feynman rules for spin-1 NRQED from the Lagrangian given in Eq. (5). In the calculation of Feynman amplitudes, an unpaired spatial index is contracted with a spatial index of the photon propagator.

$\frac{i\vec{p}^4}{8m^3}$	Kinetic energy correction (K_4)
$-iq$	Coulomb interaction (C)
$\frac{iq}{6} r_E^2 \vec{k}^2$	Darwin/finite-size interaction (D)
$\frac{q(\tilde{g}-1)}{2m^2} \vec{S} \cdot (\vec{k} \times \vec{p})$	Spin-orbit interaction (SO)
$\frac{iqQ_E}{2} (S^i S^j)^{(2)} k^i k^j$	Quadrupole interaction (Q)
$\frac{iq}{2m} (p' + p)^i$	Convection interaction (conv)
$-\frac{q\tilde{g}}{2m} (\vec{S} \times \vec{k})^i$	Fermi spin interaction (FS)
$-\frac{iq^2}{m} \delta^{ij}$	Seagull interaction (SG)

We abstract Feynman rules from the Lagrangian (5) in the usual way. For derivatives, we replace ∂_i acting on an incoming spin-1 particle by ip^i , and ∂_i acting on an outgoing spin-1 particle becomes $-ip^i$. The derivative ∂_μ acting on a photon carrying momentum k into the vertex becomes $ik^\mu = i(p'^\mu - p^\mu)$.

For the construction of the Feynman rules of NRQED involving spin-1 particles, we now take the terms in $i\mathcal{L}$ from Eq. (5) one by one and determine the corresponding rules. We find the Feynman rules listed in Table I. The kinetic energy interaction acts on a single spin-1 particle line. The Coulomb, Darwin, spin-orbit, and quadrupole interactions are for a spin-1 line interacting with a Coulomb photon. The convection and Fermi spin interactions are for a spin-1 line interacting with a transverse photon. Finally, the seagull interaction is for a spin-1 particle line interacting with two transverse photons at a point.

III. SPIN-1 BREIT HAMILTONIAN

We will use the Feynman rules shown in Table I to construct the spin-1 Breit Hamiltonian. We perform this exercise to explicate the procedure and to confirm the result. In order to find energy corrections, one uses the Feynman rules to construct a two-particle to two-particle interaction kernel δK , multiplies by the imaginary unit, and finds the expectation value in momentum space. The interaction Hamiltonian is the coordinate-space version of $i\delta K$. As a first example, one considers the simple graph given in Fig. 1(a) consisting of a Coulomb interaction for each particle connected by a Coulomb photon. The energy shift is

$$\delta E = i \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \psi^\dagger(\vec{p}') (-iq_1) \frac{i}{k^2} (-iq_2) \psi(\vec{p}). \quad (9)$$

Now, we use Fourier transforms to express the wave functions in coordinate space,

$$\begin{aligned}
\delta E &= (q_1 q_2) \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \int d^3 x' \int d^3 x \\
&\quad \times \psi^\dagger(\vec{r}') e^{i\vec{p}' \cdot \vec{x}'} \frac{1}{\vec{k}^2} e^{-i\vec{p} \cdot \vec{x}} \psi(\vec{x}) \\
&= (-4\pi Z\alpha) \int d^3 x' \int d^3 x \delta^{(3)}(\vec{x}' - \vec{x}) \\
&\quad \times \psi^\dagger(\vec{x}') \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{x}}}{\vec{k}^2} \psi(\vec{x}). \quad (10)
\end{aligned}$$

It is advantageous to use the identity $\exp(i\vec{p}' \cdot \vec{x}' - i\vec{p} \cdot \vec{x}) = \exp(i\vec{k} \cdot \vec{x}' + i\vec{p} \cdot (\vec{x}' - \vec{x}))$ and to transform the integration measure $\int d^3 p' \int d^3 p \rightarrow \int d^3 k \int d^3 p$. We assume two spin-1 particles interacting, one, with a charge $q_1 = \pm|e|$, and another, with a charge $q_2 = \mp Z|e|$. Here, e is the electron charge and $|e|$ is its modulus. The charge product is $q_1 q_2 = -4\pi Z\alpha$. One finds, after a Fourier transformation,

$$\delta E = \int d^3 x \psi^\dagger(\vec{x}) \left(-\frac{Z\alpha}{r} \right) \psi(\vec{x}) \quad (11)$$

(where $r \equiv |\vec{x}|$), so we can identify the corresponding contribution to the Hamiltonian

$$\delta H_C = V_C(r) = -\frac{Z\alpha}{r}, \quad (12)$$

which is the Coulomb potential. For a kernel contribution with a factor of $\vec{p}$, say $\delta K = F^i(\vec{k})p^i$, we can replace the p^i by $i\partial_i$ acting on $e^{-i\vec{p} \cdot \vec{x}}$. Then, by partial integration, the derivative can be moved to act on the wave function $\psi(\vec{x})$ as, now, $p^i = -i\partial_i$. Similarly, for a kernel contribution

$\delta K = p'^i F^i(\vec{k})$, the p'^i can be replaced by $-i\partial'_i$ acting on $e^{i\vec{p}' \cdot \vec{x}'}$, which can be replaced by $i\partial'_i$ acting to the left on the wave function. After use of $\delta^{(3)}(\vec{x}' - \vec{x})$, another integration by parts allows the derivative to be replaced by $p^i = -i\partial_i$ acting to the right. Finally, for a kernel $\delta K(\vec{p}', \vec{p}, \vec{k})$, with $\vec{k} = \vec{p}' - \vec{p}$, all factors of $\vec{p}$ can be replaced by operators $\vec{p}$ acting on the right-hand wave function, while all factors of $\vec{p}'$ can be replaced by the operators $\vec{p}$ located just to the right of the left-hand wave function and acting to the right. In the center is the Fourier transform of the $\vec{k}$ dependence of $\delta K(\vec{p}', \vec{p}, \vec{k})$.

The Breit Hamiltonian is obtained by transforming kernels having order $1/m^2$ [which is tantamount to relative order $(Z\alpha)^2$]. These can come from $1/m^2$ interactions times a Coulomb interaction with a Coulomb propagator, or from products of the $1/m$ interactions multiplied by a transverse propagator. These graphs are shown in Figs. 1(b)–1(f). We calculate the contributions of the graphs serially. For the Darwin contribution given in Fig. 1(b), one has

$$K_{D,1}^{(4)} = \left(\frac{iq_1}{6} r_{E1}^2 \vec{k}^2 \right) \left(\frac{i}{\vec{k}^2} \right) (-iq_2) = -i \frac{2\pi Z\alpha}{3} r_{E1}^2. \quad (13)$$

Its contribution to the Breit Hamiltonian,

$$H_{D,1}^{(4)} = \mathcal{F}[i\delta K_D] = \frac{2\pi Z\alpha}{3} r_{E1}^2 \mathcal{F}[1] = \frac{2\pi Z\alpha}{3} r_{E1}^2 \delta^{(3)}(\vec{x}), \quad (14)$$

is obtained by Fourier transform $\mathcal{F}$, where

$$\mathcal{F}[f] = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} f(\vec{k}). \quad (15)$$

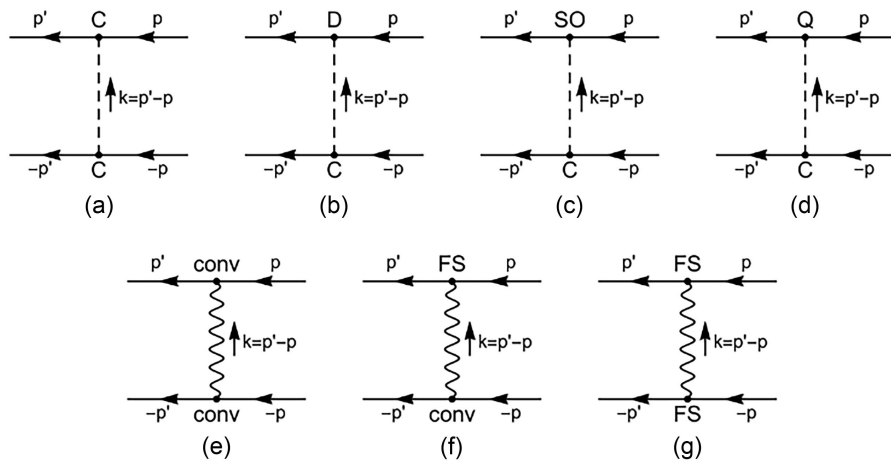


FIG. 1. Graphs contributing to the spin-1 Breit Hamiltonian. Graph (a) represents the basic Coulomb interaction. Graphs (b)–(d) also involve the Coulomb photon, and represent the Darwin, spin-orbit, and quadrupole contributions. Graphs (e)–(g) contain transverse photons, and represent the magnetic contribution (e), which involves convection currents (conv), the Fermi spin-orbit contribution (f), and the Fermi spin-spin contribution (g). Additional contributions with the vertices on the two spin-1 particles interchanged must also be included for graphs (b), (c), (d) and (f). For the notation, we refer to Table I.

A table of useful Fourier transforms is given in Appendix. The full Darwin contribution includes also the graph with particles 1 and 2 switched,

$$H_D^{(4)} = H_{D,1}^{(4)} + H_{D,2}^{(4)} = \frac{2\pi Z\alpha}{3} (r_{E1}^2 + r_{E2}^2) \delta^{(3)}(\vec{x}). \quad (16)$$

For the spin-orbit contribution of Fig. 1(c), which involves a Coulomb photon, we must remember that any factors of $\vec{p}'$ in $\delta K(\vec{p}', \vec{p}, \vec{k})$ can be pulled out of the Fourier transform and placed to its left, while any factors of $\vec{p}$ can be pulled out and placed to its right. The Coulomb spin-orbit kernel coming from Fig. 1(c) reads as follows,

$$K_{\text{CSO},1}^{(4)} = \left(\frac{q_1(\tilde{g}_1 - 1)}{2m_1^2} \vec{S}_1 \cdot (\vec{k} \times \vec{p}) \right) \left(\frac{i}{\vec{k}^2} \right) (-iq_2). \quad (17)$$

Its Breit contribution is

$$H_{\text{CSO},1}^{(4)} = i \frac{(-4\pi Z\alpha)(\tilde{g}_1 - 1)}{2m_1^2} \vec{S}_1 \cdot \left(\mathcal{F} \left[\frac{\vec{k}}{\vec{k}^2} \right] \times \vec{p} \right), \quad (18)$$

and after the Fourier transformation, one has

$$\begin{aligned} H_{\text{CSO},1}^{(4)} &= i \frac{(-4\pi Z\alpha)(\tilde{g}_1 - 1)}{2m_1^2} \vec{S}_1 \cdot \left(\frac{i\vec{r}}{4\pi r^3} \times \vec{p} \right) \\ &= \frac{Z\alpha(\tilde{g}_1 - 1)}{2m_1^2} \frac{\vec{S}_1 \cdot \vec{L}}{r^3}. \end{aligned} \quad (19)$$

Again, we must add the contribution with the identities of particles 1 and 2 interchanged,

$$H_{\text{CSO}}^{(4)} = H_{\text{CSO},1}^{(4)} + H_{\text{CSO},2}^{(4)} \quad (20)$$

$$= \frac{Z\alpha(\tilde{g}_1 - 1)}{2m_1^2} \frac{\vec{S}_1 \cdot \vec{L}}{r^3} + \frac{Z\alpha(\tilde{g}_2 - 1)}{2m_2^2} \frac{\vec{S}_2 \cdot \vec{L}}{r^3}. \quad (21)$$

The quadrupole kernel from Fig. 1(d) is

$$K_{Q,1}^{(4)} = \left(i \frac{q_1 Q_{E1}}{2} (S_1^i S_1^j)^{(2)} k^i k^j \right) \left(\frac{i}{\vec{k}^2} \right) (-iq_2). \quad (22)$$

The corresponding Hamiltonian contribution is

$$\begin{aligned} H_{Q,1}^{(4)} &= \frac{(4\pi Z\alpha) Q_{E1}}{2} (S_1^i S_1^j)^{(2)} \mathcal{F} \left[\frac{k^i k^j}{\vec{k}^2} \right] \\ &= -\frac{3Z\alpha Q_{E1}}{2r^3} (S_1^i S_1^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \end{aligned} \quad (23)$$

We have used the identity

$$\begin{aligned} \mathcal{F} \left[\frac{k^i k^j}{\vec{k}^2} \right] &= \frac{1}{3} \delta^{ij} \delta^{(3)}(\vec{x}) + \frac{1}{4\pi r^3} (\delta^{ij} - 3\hat{x}^i \hat{x}^j) \\ &= \frac{1}{3} \delta^{ij} \delta^{(3)}(\vec{x}) - \frac{3}{4\pi r^3} (\hat{x}^i \hat{x}^j)^{(2)}, \end{aligned} \quad (24)$$

with an obvious definition for the rank-2 component $(\hat{x}^i \hat{x}^j)^{(2)}$. Here, δ^{ij} times $(S_1^i S_1^j)^{(2)}$ vanishes because the quadrupole term $(S_1^i S_1^j)^{(2)}$ is traceless and we used the definition from Eq. (7). After adding the contribution from the intrinsic quadrupole moment of the second particle, one obtains

$$\begin{aligned} H_Q^{(4)} &= H_{Q,1}^{(4)} + H_{Q,2}^{(4)} \\ &= -\frac{3Z\alpha Q_{E1}}{2r^3} (S_1^i S_1^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)} \\ &\quad - \frac{3Z\alpha Q_{E2}}{2r^3} (S_2^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \end{aligned} \quad (25)$$

The remaining contributions involve a transverse photon instead of a Coulomb photon. The first is the magnetic interaction of Fig. 1(e) with two convection vertices,

$$\begin{aligned} K_M^{(4)} &= \left(\frac{iq_1}{2m_1} (p' + p)^i \right) \frac{i}{-\vec{k}^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) \\ &\quad \times \left(\frac{iq_2}{2m_2} (-p' - p)^j \right) \\ &= \frac{-i(-4\pi Z\alpha)}{4m_1 m_2} (2p')^i \frac{1}{\vec{k}^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) (2p)^j \\ &= i \frac{4\pi Z\alpha}{m_1 m_2} \frac{p'^i (\vec{k}^2 \delta^{ij} - k^i k^j) p^j}{\vec{k}^4}, \end{aligned} \quad (26)$$

where we applied the instantaneous approximation to replace k^2 by $-\vec{k}^2$ in the denominator and used the transversality of the transverse propagator to make the replacements $(p' + p)^i = (2p' - k)^i \rightarrow (2p')^i$ and $(p' + p)^j = (k + 2p)^j \rightarrow (2p)^j$. The magnetic Hamiltonian is thus

$$\begin{aligned} H_M^{(4)} &= -\frac{4\pi Z\alpha}{m_1 m_2} p^i \mathcal{F} \left[\frac{\delta^{ij}}{\vec{k}^2} - \frac{k^i k^j}{\vec{k}^4} \right] p^j \\ &= -\frac{4\pi Z\alpha}{m_1 m_2} p^i \left\{ \frac{\delta^{ij}}{4\pi r} - \frac{1}{8\pi r} (\delta^{ij} - \hat{x}^i \hat{x}^j) \right\} p^j \\ &= -\frac{Z\alpha}{2m_1 m_2} p^i \left(\frac{\delta^{ij} + \hat{x}^i \hat{x}^j}{r} \right) p^j. \end{aligned} \quad (27)$$

The Fermi spin-orbit contribution of Fig. 1(f) gives a contribution

$$\begin{aligned}
K_{\text{FSO},1}^{(4)} &= \left(-\frac{q_1 \tilde{g}_1}{2m_1} (\vec{S}_1 \times \vec{k})^i \right) \left[-\frac{i}{\vec{k}^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) \right] \\
&\times \left(\frac{i q_2}{2m_2} (-p' - p)^j \right) \\
&= \frac{(-4\pi Z\alpha) \tilde{g}_1}{4m_1 m_2} (\vec{S}_1 \times \vec{k})^i \frac{1}{\vec{k}^2} (k + 2p)^i \\
&= -\frac{2\pi Z\alpha \tilde{g}_1}{m_1 m_2} \vec{S}_1 \cdot \left(\frac{\vec{k}}{\vec{k}^2} \times \vec{p} \right). \tag{28}
\end{aligned}$$

The corresponding contribution to the Hamiltonian is

$$\begin{aligned}
H_{\text{FSO},1}^{(4)} &= -i \frac{2\pi Z\alpha \tilde{g}_1}{m_1 m_2} \vec{S}_1 \cdot \left(\mathcal{F} \left[\frac{\vec{k}}{\vec{k}^2} \right] \times \vec{p} \right) \\
&= -i \frac{2\pi Z\alpha \tilde{g}_1}{m_1 m_2} \vec{S}_1 \cdot \left(\frac{i\vec{x}}{4\pi r^3} \times \vec{p} \right) \\
&= \frac{Z\alpha \tilde{g}_1}{2m_1 m_2} \frac{\vec{S}_1 \cdot \vec{L}}{r^3}. \tag{29}
\end{aligned}$$

With particles 1 and 2 interchanged, one obtains

$$\begin{aligned}
H_{\text{FSO}}^{(4)} &= H_{\text{FSO},1}^{(4)} + H_{\text{FSO},2}^{(4)} \\
&= \frac{Z\alpha \tilde{g}_1}{2m_1 m_2} \frac{\vec{S}_1 \cdot \vec{L}}{r^3} + \frac{Z\alpha \tilde{g}_2}{2m_1 m_2} \frac{\vec{S}_2 \cdot \vec{L}}{r^3}. \tag{30}
\end{aligned}$$

Finally, the Fermi spin-spin kernel of Fig. 1(g) is

$$\begin{aligned}
K_{\text{FSS}}^{(4)} &= \left(-\frac{q_1 \tilde{g}_1}{2m_1} (\vec{S}_1 \times \vec{k})^i \right) \frac{i}{-\vec{k}^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) \\
&\times \left(-\frac{q_2 \tilde{g}_2}{2m_2} (\vec{S}_2 \times (-\vec{k}))^j \right) \\
&= i \frac{(-4\pi Z\alpha) \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{(\vec{S}_1 \times \vec{k}) \cdot (\vec{S}_2 \times \vec{k})}{\vec{k}^2} \\
&= i \frac{(-4\pi Z\alpha) \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{\vec{S}_1 \cdot \vec{S}_2 \vec{k}^2 - \vec{S}_1 \cdot \vec{k} \vec{k} \cdot \vec{S}_2}{\vec{k}^2}. \tag{31}
\end{aligned}$$

The corresponding Hamiltonian contribution is

$$\begin{aligned}
H_{\text{FSS}}^{(4)} &= \frac{\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{m_1 m_2} S_1^i \left\{ \mathcal{F} \left[\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right] \right\} S_2^j \\
&= \frac{2\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{3m_1 m_2} \vec{S}_1 \cdot \vec{S}_2 \delta^{(3)}(\vec{x}) \\
&+ \frac{3Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{S_1^i S_2^j (\hat{x}^i \hat{x}^j)^{(2)}}{r^3}. \tag{32}
\end{aligned}$$

The complete Breit Hamiltonian, which we denote by $H^{(4)}$ because the terms are of order $(Z\alpha)^4$, including all interactions on both particles, is

$$H^{(4)} = H_K^{(4)} + H_M^{(4)} + H_{\text{SO}}^{(4)} + H_{\text{FSS}}^{(4)} + H_Q^{(4)} + H_D^{(4)}, \tag{33}$$

where

$$H_{\text{SO}}^{(4)} = H_{\text{CSO}}^{(4)} + H_{\text{FSO}}^{(4)}. \tag{34}$$

In writing Eq. (33), we have adopted the following sequence: kinetic, magnetic, spin-dependent, quadrupole, and Darwin (finite size). For convenience, we list the individual terms,

$$H_K^{(4)} = -\frac{p^4}{8m_1^3} - \frac{p^4}{8m_2^3}, \tag{35a}$$

$$H_M^{(4)} = -\frac{Z\alpha}{2m_1 m_2} p^i \left(\frac{\delta^{ij} + \hat{x}^i \hat{x}^j}{r} \right) p^j, \tag{35b}$$

$$\begin{aligned}
H_{\text{SO}}^{(4)} &= \frac{Z\alpha}{2} \left(\frac{\tilde{g}_1 - 1}{m_1^2} + \frac{\tilde{g}_1}{m_1 m_2} \right) \frac{\vec{S}_1 \cdot \vec{L}}{r^3} \\
&+ \frac{Z\alpha}{2} \left(\frac{\tilde{g}_2 - 1}{m_2^2} + \frac{\tilde{g}_2}{m_1 m_2} \right) \frac{\vec{S}_2 \cdot \vec{L}}{r^3}, \tag{35c}
\end{aligned}$$

$$\begin{aligned}
H_{\text{FSS}}^{(4)} &= \frac{2\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{3m_1 m_2} \vec{S}_1 \cdot \vec{S}_2 \delta^3(\vec{x}) \\
&+ \frac{3Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{S_1^i S_2^j (\hat{x}^i \hat{x}^j)^{(2)}}{r^3}, \tag{35d}
\end{aligned}$$

$$\begin{aligned}
H_Q^{(4)} &= -\frac{3Z\alpha}{2r^3} \{ Q_{E1} (S_1^i S_1^j)^{(2)} \\
&+ Q_{E2} (S_2^i S_2^j)^{(2)} \} (\hat{x}^i \hat{x}^j)^{(2)}, \tag{35e}
\end{aligned}$$

$$H_D^{(4)} = \frac{2\pi}{3} Z\alpha (r_{E1}^2 + r_{E2}^2) \delta^3(\vec{x}). \tag{35f}$$

When applied to deuterium, our result here agrees with the Breit Hamiltonian used in Refs. [31,69]. The two radius terms r_{E1}^2 and r_{E2}^2 are in need of an interpretation. Namely, as it has been explained in Refs. [74,75], for spin-1/2 particles, the Darwin-Foldy term is inherently included in the definitions of the radii r_{E1}^2 and r_{E2}^2 of the two constituent particles. So, if r_{C1}^2 and r_{C2}^2 are the charge radii of the two particles, then

$$r_{E1}^2 \rightarrow \frac{3\delta_I}{4m_1^2} + r_{C1}^2, \tag{36a}$$

$$r_{E2}^2 \rightarrow \frac{3\delta_I}{4m_2^2} + r_{C2}^2, \tag{36b}$$

where δ_I is equal to one for particles with half-integer spin, and equal to zero for particles with integer spin.

IV. RELATIVISTIC AND RECOIL CORRECTIONS TO EVP

The relativistic and recoil corrections to the one-loop eVP (Uehling) potential are obtained from the one-loop VP corrected OC-gauge photon propagator of Eq. (2) in the same way as the Breit Hamiltonian was obtained from the uncorrected Coulomb gauge propagator. In the instantaneous limit needed here, this one-loop propagator becomes, according to Eq. (2),

$$D_{ij}^{\text{OC};1}(k) \rightarrow \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \begin{pmatrix} \frac{1}{\vec{k}^2 + \lambda^2} & 0 \\ 0 & -\frac{\delta^{ij} - \frac{k^i k^j}{\vec{k}^2 + \lambda^2}}{\vec{k}^2 + \lambda^2} \end{pmatrix}, \quad (37)$$

where f_1 has been defined in Eq. (3).

The modification required for the diagrams of Fig. 1 can be represented as

$$H_{\text{eVP}}^{(5)} = \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \mathcal{H}^{(4)}, \quad (38)$$

where $\mathcal{H}^{(4)}$ is obtained via the same matching procedure as $H^{(4)}$, but with the Coulomb-gauge photon propagator replaced with the optimized-gauge propagator from Eq. (2),

$$\mathcal{H}^{(4)} = H^{(4)}|_{D^C \rightarrow D^{\text{OC};1}}. \quad (39)$$

For terms involving the Coulomb photon, the substitution is easy,

$$\frac{1}{\vec{k}^2} \rightarrow \frac{1}{\vec{k}^2 + \lambda^2}. \quad (40)$$

For terms involving the transverse photon, a little more work is required as the numerator algebra changes, since the substitution is

$$\frac{1}{\vec{k}^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) \rightarrow \frac{1}{\vec{k}^2 + \lambda^2} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2 + \lambda^2} \right). \quad (41)$$

For example, the Coulomb potential $H_C = -\frac{Z\alpha}{r}$ becomes

$$\mathcal{H}_U = \mathcal{F} \left[\frac{-4\pi Z\alpha}{\vec{k}^2 + \lambda^2} \right] = (-4\pi Z\alpha) \frac{e^{-\lambda r}}{4\pi r} = \left(\frac{-Z\alpha}{r} \right) e^{-\lambda r},$$

so that we can easily verify the modification of the Coulomb potential to be equal to

$$\begin{aligned} \mathcal{V}_U &= \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \mathcal{H}_U \\ &= \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \left(\frac{-Z\alpha}{r} \right) e^{-\lambda r}, \end{aligned} \quad (42)$$

which is the Uehling potential.

The Coulomb-photon terms are the Darwin (finite-size) contribution [see Fig. 1(b) and Eq. (16)],

$$\begin{aligned} \mathcal{H}_{D,1}^{(4)} &= \frac{2\pi Z\alpha}{3} r_{E1}^2 \mathcal{F} \left[\frac{\vec{k}^2}{\vec{k}^2 + \lambda^2} \right] \\ &= \frac{2\pi Z\alpha}{3} r_{E1}^2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \end{aligned} \quad (43)$$

(along with the corresponding contribution with the particles interchanged), the Coulomb spin-orbit term of Fig. 1(c) and Eq. (18),

$$\begin{aligned} \mathcal{H}_{\text{CSO},1}^{(4)} &= i \frac{(-4\pi Z\alpha)(\tilde{g}_1 - 1)}{2m_1^2} \vec{S}_1 \cdot \left(\mathcal{F} \left[\frac{\vec{k}}{\vec{k}^2 + \lambda^2} \right] \times \vec{p} \right) \\ &= i \frac{(-4\pi Z\alpha)(\tilde{g}_1 - 1)}{2m_1^2} \vec{S}_1 \cdot \left(\frac{e^{-\lambda r}}{4\pi r^3} (1 + \lambda r) i\vec{r} \times \vec{p} \right) \\ &= \frac{Z\alpha(\tilde{g}_1 - 1)}{2m_1^2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_1 \cdot \vec{L}, \end{aligned} \quad (44)$$

with an obvious addition of $\mathcal{H}_{\text{CSO},2}^{(4)}$ where the particles are interchanged,

$$\begin{aligned} \mathcal{H}_{\text{CSO}}^{(4)} &= \frac{Z\alpha(\tilde{g}_1 - 1)}{2m_1^2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_1 \cdot \vec{L} \\ &\quad + \frac{Z\alpha(\tilde{g}_2 - 1)}{2m_2^2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_2 \cdot \vec{L}, \end{aligned} \quad (45)$$

and the quadrupole term of Fig. 1(d) and Eq. (25),

$$\begin{aligned} \mathcal{H}_{Q,1}^{(4)} &= \frac{(4\pi Z\alpha)Q_{E1}}{2} (S_1^i S_1^j)^{(2)} \mathcal{F} \left[\frac{k^i k^j}{\vec{k}^2 + \lambda^2} \right] \\ &= -\frac{3}{2} Z\alpha Q_{E1} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) (S_1^i S_1^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \end{aligned} \quad (46)$$

After adding the contribution from the quadrupole moment of the second particle, one obtains

$$\begin{aligned} \mathcal{H}_Q^{(4)} &= \mathcal{H}_{Q,1}^{(4)} + \mathcal{H}_{Q,2}^{(4)} \\ &= -\frac{3Z\alpha}{2} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \\ &\quad \times \{ Q_{E1} (S_1^i S_1^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)} + Q_{E2} (S_2^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)} \}. \end{aligned} \quad (47)$$

The transverse photon terms require a bit more work. The magnetic interaction kernel of Fig. 1(e) and Eq. (26),

$$\begin{aligned}
\mathcal{K}_M^{(4)} &= \left(\frac{i q_1}{2m_1} (p' + p)^i \right) \frac{i}{-(\vec{k}^2 + \lambda^2)} \\
&\times \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2 + \lambda^2} \right) \left(\frac{i q_2}{2m_2} (-p' - p)^j \right) \\
&= i \frac{4\pi Z\alpha}{4m_1 m_2} (2p' - k)^i \frac{(\vec{k}^2 + \lambda^2) \delta^{ij} - k^i k^j}{(\vec{k}^2 + \lambda^2)^2} (k + 2p)^j,
\end{aligned} \tag{48}$$

gives rise to a Hamiltonian

$$\begin{aligned}
\mathcal{H}_M^{(4)} &= -\frac{4\pi Z\alpha}{4m_1 m_2} \mathcal{F} \left[4p^i \frac{(\vec{k}^2 + \lambda^2) \delta^{ij} - k^i k^j}{(\vec{k}^2 + \lambda^2)^2} p^j \right. \\
&\quad \left. + \frac{(2p^i)^2 \lambda^2 k^i}{(\vec{k}^2 + \lambda^2)^2} - \frac{\lambda^2 k^j (2p^j)}{(\vec{k}^2 + \lambda^2)^2} - \frac{\lambda^2 \vec{k}^2}{(\vec{k}^2 + \lambda^2)^2} \right] \\
&= -\frac{Z\alpha}{2m_1 m_2} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j \\
&\quad - \frac{Z\alpha}{4m_1 m_2} \frac{\lambda^2 e^{-\lambda r}}{r} \left(1 - \frac{\lambda r}{2} \right)
\end{aligned} \tag{49}$$

after use of the identity

$$\begin{aligned}
i p^i (e^{-\lambda r} \hat{x}^i) - (e^{-\lambda r} \hat{x}^i) i p^i &= [\partial_i, e^{-\lambda r} \hat{x}^i] \\
&= \frac{2}{r} e^{-\lambda r} \left(1 - \frac{\lambda r}{2} \right).
\end{aligned} \tag{50}$$

The Fermi spin-orbit term of Fig. 1(f) and Eq. (28) is

$$\begin{aligned}
\mathcal{K}_{\text{FSO},1}^{(4)} &= \left(-\frac{q_1 \tilde{g}_1}{2m_1} (\vec{S}_1 \times \vec{k})^i \right) \frac{i}{-(\vec{k}^2 + \lambda^2)} \\
&\times \left(\delta^{ij} - \frac{k^i k^j}{(\vec{k}^2 + \lambda^2)} \right) \left(\frac{i q_2}{2m_2} (-k^j - 2p^j) \right) \\
&= -\frac{4\pi Z\alpha \tilde{g}_1}{4m_1 m_2} \frac{(\vec{S}_1 \times \vec{k}) \cdot (2\vec{p})}{(\vec{k}^2 + \lambda^2)},
\end{aligned} \tag{51}$$

so that

$$\begin{aligned}
\mathcal{H}_{\text{FSO},1}^{(4)} &= -i \frac{4\pi Z\alpha \tilde{g}_1}{4m_1 m_2} \left(\vec{S}_1 \times \mathcal{F} \left[\frac{\vec{k}}{\vec{k}^2 + \lambda^2} \right] \right) \cdot (2\vec{p}) \\
&= -i \frac{4\pi Z\alpha \tilde{g}_1}{4m_1 m_2} \left(\vec{S}_1 \times \frac{e^{-\lambda r}}{4\pi r^3} (1 + \lambda r) i \vec{x} \right) \cdot (2\vec{p}) \\
&= \frac{Z\alpha \tilde{g}_1}{2m_1 m_2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_1 \cdot \vec{L}.
\end{aligned} \tag{52}$$

There is an obvious addition for $\mathcal{H}_{\text{FSO},2}^{(4)}$, resulting in

$$\begin{aligned}
\mathcal{H}_{\text{FSO}}^{(4)} &= \frac{Z\alpha \tilde{g}_1}{2m_1 m_2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_1 \cdot \vec{L} \\
&\quad + \frac{Z\alpha \tilde{g}_2}{2m_1 m_2} \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S}_2 \cdot \vec{L}.
\end{aligned} \tag{53}$$

We summarize the sum of $\mathcal{H}_{\text{CSO}}^{(4)}$ and $\mathcal{H}_{\text{FSO}}^{(4)}$ as the total spin-orbit term

$$\mathcal{H}_{\text{SO}}^{(4)} = \mathcal{H}_{\text{CSO}}^{(4)} + \mathcal{H}_{\text{FSO}}^{(4)}. \tag{54}$$

Finally, the Fermi spin-spin contribution of Fig. 1(g) and Eq. (31) is

$$\begin{aligned}
\mathcal{K}_{\text{FSS}}^{(4)} &= \left(-\frac{q_1 \tilde{g}_1}{2m_1} (\vec{S}_1 \times \vec{k})^i \right) \frac{i}{-(\vec{k}^2 + \lambda^2)} \\
&\times \left(\delta^{ij} - \frac{k^i k^j}{(\vec{k}^2 + \lambda^2)} \right) \left(-\frac{q_2 \tilde{g}_2}{2m_2} (\vec{S}_2 \times (-\vec{k}))^j \right) \\
&= -i \frac{4\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{(\vec{S}_1 \times \vec{k}) \cdot (\vec{S}_2 \times \vec{k})}{\vec{k}^2 + \lambda^2},
\end{aligned} \tag{55}$$

so that

$$\begin{aligned}
\mathcal{H}_{\text{FSS}}^{(4)} &= \frac{4\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} S_1^i S_2^j \mathcal{F} \left[\frac{\delta^{ij} \vec{k}^2 - k^i k^j}{\vec{k}^2 + \lambda^2} \right] \\
&= \frac{4\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \left[\frac{\vec{S}_1 \cdot \vec{S}_2}{3} \delta^{ij} + (S_1^i S_2^j)^{(2)} \right] \mathcal{F} \left[\frac{\delta^{ij} \vec{k}^2 - k^i k^j}{\vec{k}^2 + \lambda^2} \right] \\
&= \frac{2\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{3m_1 m_2} \vec{S}_1 \cdot \vec{S}_2 \left\{ \delta^{(3)}(\vec{r}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \\
&\quad + \frac{3Z\alpha \tilde{g}_1 \tilde{g}_2}{4m_1 m_2} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) (S_1^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}.
\end{aligned} \tag{56}$$

In summary, the Hamiltonian for relativistic-recoil corrections to electronic VP is given by

$$H_{\text{eVP}}^{(5)} = H_M^{(5)} + H_{\text{SO}}^{(5)} + H_{\text{FSS}}^{(5)} + H_Q^{(5)} + H_D^{(5)}, \tag{57}$$

where the $H_X^{(5)}$ with $X \in \{M, \text{SO}, \text{FSS}, Q, D\}$ are obtained from the corresponding $\mathcal{H}_X^{(4)}$ by the integration over the spectral function of vacuum polarization, which is given in Eq. (38). We list the terms as follows,

$$\begin{aligned}
\mathcal{H}_M^{(4)} &= -\frac{Z\alpha}{2m_1 m_2} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j \\
&\quad - \frac{Z\alpha \lambda^2}{4m_1 m_2} \frac{e^{-\lambda r}}{r} \left(1 - \frac{\lambda r}{2} \right),
\end{aligned} \tag{58a}$$

$$\mathcal{H}_{\text{SO}}^{(4)} = \frac{Z\alpha e^{-\lambda r}}{2r^3} (1 + \lambda r) \left\{ \left(\frac{\tilde{g}_1 - 1}{m_1^2} + \frac{\tilde{g}_1}{m_1 m_2} \right) \vec{S}_1 \cdot \vec{L} + \left(\frac{\tilde{g}_2 - 1}{m_2^2} + \frac{\tilde{g}_2}{m_1 m_2} \right) \vec{S}_2 \cdot \vec{L} \right\}, \quad (58b)$$

$$\mathcal{H}_{\text{FSS}}^{(4)} = \frac{2\pi Z\alpha \tilde{g}_1 \tilde{g}_2}{3m_1 m_2} \vec{S}_1 \cdot \vec{S}_2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} + \frac{3Z\alpha \tilde{g}_1 \tilde{g}_2 e^{-\lambda r}}{4m_1 m_2 r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \times (S_1^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}, \quad (58c)$$

$$\mathcal{H}_{\text{Q}}^{(4)} = -\frac{3Z\alpha e^{-\lambda r}}{2r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \times \{ Q_{E1} (S_1^i S_1^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)} + Q_{E2} (S_2^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)} \}, \quad (58d)$$

$$\mathcal{H}_{\text{D}}^{(4)} = \frac{2\pi Z\alpha}{3} r_{E1}^2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} + \frac{2\pi Z\alpha}{3} r_{E2}^2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\}. \quad (58e)$$

These terms, in the limit $\lambda \rightarrow 0$, combine to give the two-particle interaction Hamiltonian of Eq. (33). The spin-orbit and magnetic terms in the limit $\tilde{g}_i \rightarrow 2$ and with λ nonzero agree with the results given in Eq. (63) of [67], provided we reinterpret the spin operators appropriately, in a form applicable for spin-1/2 particles. Namely, for spin-1/2 point particles, one needs to replace $r_{Ei}^2 \rightarrow \frac{3}{4m_i^2}$, as suggested in Eq. (6) of [72]. In the general case, one uses the spin-dependent relations given in Eq. (36) in order to relate the radius terms r_{E1}^2 and r_{E2}^2 with the mean-square charge radii of the two particles.

V. APPLICATIONS

A. Spinless bound systems: Pionium

One central point of our investigations is to derive general formulas for the relativistic-recoil correction to eVP, applicable to bound Coulomb systems with constituent spin-0, spin-1/2 and spin-1 particles. One might think that this task has already been accomplished: Simply, one replaces, in Eq. (58), the mass terms m_1 and m_2 by the masses of the constituent particles, and the spin operators accordingly, and obtains the corresponding Hamiltonian.

However, there are a few subtleties involved in this process [in part, connected with Eq. (36)], and in the current section, we shall discuss these by way of example, considering particular bound systems of special interest. First, we investigate pionium [4–13], as an example of a bound system consisting of two spinless particles ($\vec{S}_1 = \vec{S}_2 = \vec{0}$). For pionium, the two masses are equal

($m_1 = m_2 = m_\pi$), and there is no fine-structure, and no hyperfine structure (bound states are characterized by the principal quantum number n and the orbital angular momentum L). Furthermore, because of the spinless character of the pions, we can replace $\mathcal{H}_{\text{SO}}^{(4)} \rightarrow 0$, $\mathcal{H}_{\text{FSS}}^{(4)} \rightarrow 0$. Furthermore, pions do not have a intrinsic quadrupole moment, hence $Q_{E1} = Q_{E2} = 0$ and $\mathcal{H}_{\text{Q}}^{(4)} \rightarrow 0$. The magnetic interaction, in pionium, becomes

$$\mathcal{H}_{\text{M}}^{(4)} \rightarrow -\frac{Z\alpha}{2m_\pi^2} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j - \frac{Z\alpha \lambda^2 e^{-\lambda r}}{4m_\pi^2 r} \left(1 - \frac{\lambda r}{2} \right). \quad (59)$$

In view of the absence of the Darwin-Foldy term for spinless particles, the Darwin term $\mathcal{H}_{\text{D}}^{(4)} \rightarrow \mathcal{H}_{\text{FR}}^{(4)}$ is reduced to the vacuum-polarization correction to the finite-radius (FR) effect,

$$\mathcal{H}_{\text{FR}}^{(4)} \rightarrow \frac{4\pi Z\alpha}{3} r_\pi^2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\}. \quad (60)$$

In the above formulas, we leave the nuclear charge number Z in the formulas, in order to facilitate the differentiation of the Coulomb-binding parameter $Z\alpha$ from the radiative QED parameter α/π , even if, for pionium, we have $Z = 1$ (of course).

The term in Eq. (60) is the analog of the term studied in Eq. (65) of Ref. [59], in the context of muonic hydrogen (there is an additional prefactor 2 in pionium because both constituent particles have a finite radius). Lattice calculations [76] yield a value $r_\pi = 0.648(15)$ fm and we mention a recent experimental result of $r_\pi = 0.640(7)$ fm from Ref. [77]. According to the latest data compilation [78] of the Particle Data Group, the best current estimates for the mass and the charge radius of the pion are

$$m_\pi = 139.57039(18) \text{ MeV}, \quad r_\pi = 0.659(4) \text{ fm}. \quad (61)$$

We shall use these values for the numerical results given in Table II. As with all corrections studied here, we recall that it is necessary to consider a first-order perturbation-theory effect, given by the expectation value of the operators in Eq. (35), and a second-order perturbation-theory effect, given by a vacuum-polarization correction to the wave function, described by the Schrödinger-Coulomb-Green function and the corresponding Breit operator from Eq. (33).

The first-order contributions to the recoil correction to eVP are

$$E_X^{(1)}(nL) = \langle nL | H_X^{(5)} | nL \rangle, \quad (62a)$$

$$E^{(1)}(nL) = \langle nL | H_{\text{eVP}}^{(5)} | nL \rangle = E_{\text{M}}^{(1)} + E_{\text{FR}}^{(1)}, \quad (62b)$$

TABLE II. The first- and second-order recoil corrections to the eVP energy shift in pionium are given for 1S and 2S states. The total recoil correction is obtained as the sum of the first-order and second-order terms: $E_{\text{eVP}}^{(\text{RR})} = E^{(1)} + E^{(2)}$. All entries are given in units of meV.

	1S	2S
$E_{\text{M}}^{(1)}$	-0.04815468(11)	-0.006678205(16)
$E_{\text{FR}}^{(1)}$	0.02427(30)	0.003123(38)
$E^{(1)}$	-0.02389(30)	-0.003555(38)
$E_{\text{K+M}}^{(2)}$	-0.25407049(49)	-0.030929877(74)
$E_{\text{FR}}^{(2)}$	0.04495(55)	0.004720(57)
$E^{(2)}$	-0.20912(55)	-0.026209(57)
$E_{\text{eVP}}^{(\text{RR})}$	-0.23301(62)	-0.029764(69)

where $X \in \{\text{M, FR}\}$, and the Hamiltonians are given in Eqs. (38), (57), (59), and (60).

The nonretarded Breit Hamiltonian $H^{(4)}$, for pionium, is the sum of a kinetic + magnetic term $H_{\text{K+M}}^{(4)}$, and a finite-radius term $H_{\text{FR}}^{(4)}$,

$$H^{(4)} \rightarrow H_{\text{K+M}}^{(4)} + H_{\text{FR}}^{(4)}, \quad (63a)$$

$$H_{\text{K+M}}^{(4)} = -\frac{p^4}{4m_\pi^3} - \frac{Z\alpha}{2m_\pi^2} p^i \left(\frac{\delta^{ij} + \hat{x}^i \hat{x}^j}{r} \right) p^j, \quad (63b)$$

$$H_{\text{FR}}^{(4)} = \frac{4\pi Z\alpha}{3} r_\pi^2 \delta^{(3)}(\vec{x}). \quad (63c)$$

The spinless, nonretarded Breit Hamiltonian can be obtained from Eqs. (33)–(35) by setting to zero all spin operators, and setting $m_1 = m_2 = m_\pi$. Its explicit form has been given in Eq. (15) of Ref. [6]. Regarding the matrix elements of the spinless, nonretarded Breit Hamiltonian, we refer to Eq. (38) of Ref. [31], Eq. (16) of Ref. [6], Eq. (5.13) of Ref. [8], and Eqs. (5.30) and (8.2) of Ref. [10].

With the definition of the one-loop eVP potential (Uehling potential) as

$$V_{\text{eVP}}^{(1)}(r) = \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \left(-\frac{\alpha}{r} e^{-\lambda r} \right), \quad (64)$$

where $\lambda \equiv \lambda(v) = 2m_e/\sqrt{1-v^2}$ as defined in Eq. (1), one obtains a second-order effect,

$$E_X^{(2)}(nL) = 2\langle nL | H_X^{(4)} \left(\frac{1}{E_S - H_S} \right)' V_{\text{eVP}}^{(1)} | nL \rangle, \quad (65a)$$

$$\begin{aligned} E^{(2)}(nL) &= 2\langle nL | H^{(4)} \left(\frac{1}{E_S - H_S} \right)' V_{\text{eVP}}^{(1)} | nL \rangle \\ &= E_{\text{K+M}}^{(2)} + E_{\text{FR}}^{(2)}, \end{aligned} \quad (65b)$$

where $X \in \{\text{K+M, FR}\}$. The unperturbed Schrödinger energy is

$$E_S = -\frac{\alpha^2 m_\pi}{4n^2}, \quad (66)$$

and the Schrödinger-Coulomb Hamiltonian is

$$H_S = \frac{\vec{p}^2}{m_\pi} - \frac{\alpha}{r}. \quad (67)$$

Results for the first-order and second-order contributions are given in Table II, where the theoretical uncertainties correspond to the numerical values of the charged pion's mass and charge radius from Eq. (61).

B. Fine structure of muonic hydrogen

For muonic hydrogen, one encounters the masses

$$m_1 = m_\mu, \quad m_2 = m_N = m_p. \quad (68)$$

Here, m_μ is the muon mass, and $m_N = m_p$ is the nuclear mass. We attempt to reconcile the radiatively corrected Breit Hamiltonian, given in Eq. (58), with the Hamiltonian used in Refs. [59,60,63,64,67], for the evaluation of relativistic and recoil corrections to vacuum polarization in muonic hydrogenlike ions. For the discussion of the fine structure, we can exclude hyperfine effects (due to the nuclear spin).

There is slight subtlety: even for a pointlike proton, the Darwin term in Eq. (58e) gives a nonvanishing contribution originating from the proton, because of the definition of the charge radius on which Eqs. (35f) and (58e) are based. Namely, for a pointlike nucleus, one has $r_{E2}^2 \rightarrow \frac{3\delta_L}{4m_N^2}$, according to Eq. (36). The following replacements are thus relevant for the muonic systems,

$$r_{E1}^2 \rightarrow \frac{3}{4m_\mu^2}, \quad r_{E2}^2 \rightarrow \frac{3}{4m_p^2} + r_p^2, \quad \tilde{g}_1 \rightarrow 2, \quad (69)$$

where r_p^2 is the mean-square proton charge radius. One then takes into account the fact that the orbiting particle is a spin-1/2 particle, ignores hyperfine effects (one replaces $\vec{S}_2 \rightarrow \vec{0}$), and observes that the intrinsic quadrupole moments of the muon and proton vanish,

$$\vec{S}_1 \rightarrow \frac{\vec{\sigma}_\mu}{2}, \quad Q_{E1} \rightarrow 0, \quad Q_{E2} \rightarrow 0. \quad (70)$$

Here $\vec{\sigma}_\mu$ is the vector of Pauli spin matrices for the muon spin. Hence, for the fine-structure of muonic hydrogen, we can replace $\mathcal{H}_Q^{(4)} \rightarrow 0$, and we can also replace the Fermi spin-spin interaction by $\mathcal{H}_{\text{FSS}}^{(4)} \rightarrow 0$.

For muonic hydrogen, the magnetic term takes the form

$$\mathcal{H}_M^{(4)} \rightarrow -\frac{Z\alpha}{2m_\mu m_p} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j - \frac{Z\alpha\lambda^2}{4m_\mu m_p} \frac{e^{-\lambda r}}{r} \left(1 - \frac{\lambda r}{2}\right). \quad (71)$$

The latter form is equal to the sum $\delta\mathcal{H}_2 + \delta\mathcal{H}_3$, used in their respective forms defined in Eqs. (63d) and (63e) of Ref. [67]. The spin-orbit term takes the form

$$\mathcal{H}_{SO}^{(4)} \rightarrow Z\alpha \left(\frac{1}{4m_\mu^2} + \frac{1}{2m_\mu m_p} \right) \frac{e^{-\lambda r}}{r^3} \times (1 + \lambda r) \vec{\sigma}_\mu \cdot \vec{L} \equiv \delta\mathcal{H}_4, \quad (72)$$

where $\delta\mathcal{H}_4$ is used in the notation of Eq. (63f) of Ref. [67].

Now we investigate the Darwin term. One verifies that, for muonic hydrogenlike ions, the individual terms listed in Eq. (58e) can be expressed as a sum of two terms,

$$\mathcal{H}_D^{(4)} = \mathcal{H}_{D,1}^{(4)} + \mathcal{H}_{D,2}^{(4)}. \quad (73)$$

The first term is

$$\mathcal{H}_{D,1}^{(4)} = \frac{\pi Z\alpha}{2} \left(\frac{1}{m_\mu^2} + \frac{1}{m_p^2} \right) \left\{ \delta^{(3)}(\vec{r}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \equiv \delta\mathcal{H}_1, \quad (74)$$

where $\delta\mathcal{H}_1$ is used in the notation of Eq. (63c) of Ref. [67]. The finite-size contribution is

$$\mathcal{H}_{D,2}^{(4)} = \frac{2\pi Z\alpha}{3} r_p^2 \left\{ \delta^{(3)}(\vec{r}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \equiv \delta\mathcal{H}_{FS}, \quad (75)$$

as discussed in Eq. (65) of Ref. [59]. We have checked our results above against corresponding entries in Ref. [67], and references therein. For the comparison to the literature, it is important to realize that, e.g., the detailed breakdown of the corrections to the fine structure of muonic hydrogen in Ref. [60] excludes the finite-radius correction, which is treated, in both first- and second-order perturbation theory, in Eqs. (65) and (66) of Ref. [59].

The history of the recoil correction to eVP involves gradual progress in the calculations. For example, in Ref. [64], it was suggested that in Ref. [79], some gauge-dependent terms were present in the Hamiltonian which, along with a numerical error, prevented agreement with [60] for the $n = 2$ Lamb shift in muonic hydrogen. This fact has recently been clarified in Ref. [80], in the text preceding Eq. (21) of that reference, pointing out that a corresponding statement in [67] should properly refer only to the result of [79] and not to [63].

C. Hyperfine structure of muonic hydrogen

For the hyperfine structure of muonic hydrogen, we now investigate terms which involve the spin operator of the proton, $\vec{S}_2 = \vec{S}_p = \vec{\sigma}_p/2$. The g factor of the bound muon is $\tilde{g}_1 = \tilde{g}_\mu = 2$. Because the nuclear magneton is defined with respect to the proton mass, we have

$$\tilde{g}_2 \rightarrow \tilde{g}_p = g_p = 5.5856946893(16), \quad (76)$$

where we use the value from Ref. [73]. We recall the spin operator of the muon as $\vec{S}_\mu = \vec{\sigma}_\mu/2$. Neither the muon nor the proton have intrinsic quadrupole moments, hence, for the hyperfine structure of muonic hydrogen, we can replace $\mathcal{H}_Q^{(4)} \rightarrow 0$.

Terms that involve $\vec{\sigma}_p$ are the spin-orbit term

$$\mathcal{H}_{SO}^{(4)} \rightarrow \frac{Z\alpha e^{-\lambda r}}{4r^3} (1 + \lambda r) \left(\frac{g_p - 1}{m_p^2} + \frac{g_p}{m_\mu m_p} \right) \vec{\sigma}_p \cdot \vec{L}, \quad (77)$$

and the Fermi spin-spin term

$$\begin{aligned} \mathcal{H}_{FSS}^{(4)} \rightarrow & \frac{\pi Z\alpha g_p}{3m_\mu m_p} \vec{\sigma}_\mu \cdot \vec{\sigma}_p \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \\ & + \frac{3Z\alpha g_p}{8m_\mu m_p} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \\ & \times (\sigma_\mu^i \sigma_p^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \end{aligned} \quad (78)$$

For P states and states with higher orbital angular momenta, we recall that the fine-structure and hyperfine-structure Hamiltonians do not commute, which leads to mixing terms [see Eq. (91) of Ref. [59]]. We have checked our formulas, and corresponding results for the hyperfine structure of muonic hydrogen, for $1S$ and $2S$ states, against the literature [Eqs. (18) and (30) of Ref. [81]].

D. Fine structure of muonic deuterium

For muonic deuterium, one encounters the masses

$$m_1 = m_\mu, \quad m_2 = m_d. \quad (79)$$

Here, m_μ is the muon mass, and m_d is the deuteron mass. We first ignore the nuclear spin and concentrate on the fine structure. For muonic deuterium, with its spin-1 nucleus, we can replace [see also the discussion surrounding Eq. (69)]

$$r_{E1}^2 \rightarrow \frac{3}{4m_\mu^2}, \quad r_{E2}^2 \rightarrow r_d^2, \quad \tilde{g}_1 \rightarrow 2. \quad (80)$$

One then takes into account the fact that the orbiting particle is a spin-1/2 particle, so

$$\vec{S}_1 \rightarrow \frac{\vec{\sigma}_\mu}{2}, \quad (81)$$

ignores hyperfine effects, and observes that the intrinsic quadrupole moment of the muon vanishes while that of the deuteron is nonzero.

Since we are considering the fine structure, we can ignore terms involving the deuteron spin ($\vec{S}_2 = \vec{S}_d$). For the deuteron, we use a root-mean-square deuteron charge radius [73] of

$$r_d = 2.12778(27) \times 10^{-15} \text{ fm}. \quad (82)$$

For the deuteron's quadrupole moment (divided by the elementary charge), we use the value of [82–84])

$$Q_{Ed} = 0.285699(24) \text{ fm}^2. \quad (83)$$

For the fine structure, we can replace $\vec{S}_2 \rightarrow \vec{0}$, and we can ignore the quadrupole terms in muonic hydrogenlike ions even if the nucleus has an intrinsic quadrupole moment ($\mathcal{H}_Q^{(4)} \rightarrow 0$). The Fermi spin-spin interaction can also be ignored, $\mathcal{H}_{\text{FSS}}^{(4)} \rightarrow 0$. One verifies that, for muonic hydrogenlike ions, the individual terms listed in Eqs. (58a) and (58b) take the form

$$\begin{aligned} \mathcal{H}_M^{(4)} \rightarrow & -\frac{Z\alpha}{2m_\mu m_d} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j \\ & - \frac{Z\alpha\lambda^2}{4m_\mu m_d} \frac{e^{-\lambda r}}{r} \left(1 - \frac{\lambda r}{2}\right), \end{aligned} \quad (84)$$

$$\mathcal{H}_{\text{SO}}^{(4)} \rightarrow \frac{Z\alpha e^{-\lambda r}}{4r^3} (1 + \lambda r) \left\{ \left(\frac{1}{m_\mu^2} + \frac{2}{m_\mu m_d} \right) \vec{\sigma}_\mu \cdot \vec{L} \right\}. \quad (85)$$

The Darwin term listed in Eq. (58e) can be written as the sum of two individual contributions,

$$\mathcal{H}_D^{(4)} = \mathcal{H}_{D,1}^{(4)} + \mathcal{H}_{D,2}^{(4)}. \quad (86)$$

The first of these is a generalized form of the *zitterbewegung* term of the muon, which is obtained here as the Darwin-Foldy (DF) term contributing to the finite-radius term for spin-1/2 particles [74,75],

$$\mathcal{H}_{D,1}^{(4)} = \mathcal{H}_{\text{DF}}^{(4)} = \frac{\pi Z\alpha}{2m_\mu^2} \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\}, \quad (87)$$

while the second one is a vacuum-polarization correction to the finite-size effect, which, in the current context, we refer to as the FR effect,

$$\mathcal{H}_{D,2}^{(4)} = \mathcal{H}_{\text{FR}}^{(4)} = \frac{2\pi Z\alpha}{3} r_d^2 \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\}. \quad (88)$$

We have checked the eVP-recoil correction obtained from the above formulas (for muonic deuterium) against the corresponding entries in Table I of Ref. [67], and in Table IV of Ref. [64], for all states with principal quantum numbers $1 \leq n \leq 4$. One should remember, though, that the entries in Table I of Ref. [67], and in Table IV of Ref. [64], *exclude* the finite-radius correction due to $\mathcal{H}_{\text{FR}}^{(4)}$, and also the finite-radius term in the nonretarded Breit Hamiltonian, i.e., the term proportional to the charge radius $r_{E2}^2 = r_d^2$ in Eq. (36). We take the opportunity to point out that the entry in Eq. (43) of Ref. [85] for the second-order contribution to the $2P_{1/2}$ – $2S_{1/2}$ Lamb shift in muonic deuterium appears to be in need of a correction, replacing $0.0530 \text{ meV} \rightarrow 0.056984 \text{ meV}$. After the replacement, the sum of the results given in Eqs. (27)–(30) and (43) of Ref. [85] is in agreement with the difference of the corresponding entries in Table I of Ref. [67], $[-6.3675 \times 10^{-3} - (-2.8149 \times 10^{-2})] \text{ meV} = 0.021781 \text{ meV}$, and also with the entry in Table IV of Ref. [64].

E. Hyperfine structure of muonic deuterium

We now consider those terms in Eq. (58) which involve the spin operator of the deuteron, $\vec{S}_2 = \vec{S}_d$. The magnetic term and the Darwin term are independent of the nuclear spin, hence, for the consideration of the hyperfine structure, we can replace $\mathcal{H}_M^{(4)} \rightarrow 0$, and $\mathcal{H}_D^{(4)} \rightarrow 0$. The spin-orbit term becomes

$$\mathcal{H}_{\text{SO}}^{(4)} \rightarrow \frac{Z\alpha e^{-\lambda r}}{2r^3} (1 + \lambda r) \left\{ \left(\frac{\tilde{g}_d - 1}{m_d^2} + \frac{\tilde{g}_d}{m_\mu m_d} \right) \vec{S}_d \cdot \vec{L} \right\}, \quad (89)$$

while the Fermi spin-spin term becomes

$$\begin{aligned} \mathcal{H}_{\text{FSS}}^{(4)} \rightarrow & \frac{2\pi Z\alpha \tilde{g}_d}{3m_\mu m_d} \vec{\sigma}_\mu \cdot \vec{S}_d \left\{ \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \\ & + \frac{3Z\alpha \tilde{g}_d}{4m_\mu m_d} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \\ & \times (\sigma_\mu^i S_d^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}, \end{aligned} \quad (90)$$

and the quadrupole term becomes

$$\begin{aligned} \mathcal{H}_Q^{(4)} \rightarrow & -\frac{3Z\alpha e^{-\lambda r}}{2} \frac{r_d^2}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3} \right) \\ & \times Q_{Ed} (S_d^i S_d^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \end{aligned} \quad (91)$$

Fine-structure and hyperfine-structure effects, as well as nuclear-polarization corrections in muonic deuterium, have received considerable attention recently [85–90]. The mixing matrix of the fine-structure and hyperfine-structure components has been discussed in Eq. (45) of Ref. [91] and in Eq. (62) of Ref. [89]. We leave the nuclear charge number Z in the formulas, so that the above formulas can

easily be generalized to muonic ions with a spin-1 nucleus. We have checked our formulas above, for the hyperfine structure of muonic deuterium, against results presented in Eqs. (11), (12), (22) and (23) of Ref. [88].

VI. NUMERICAL VALUES FOR DEUTERONIUM

Let us now focus on the bound system of a deuteron and its antiparticle, which we refer to as deuteronium [31,69]. The deuteron mass is denoted as m_d , and the reduced mass of the system is $m_r = m_d/2$. Values for the deuteron radius and its quadrupole moment have been given in Eqs. (82) and (83). Appropriate quantum numbers in deuteronium are the principal quantum number n , the total orbital angular momentum L , the total spin S , and the total angular momentum quantum number J (although there is mixing of states with $L = J$, as discussed below.) We use spectroscopic notation $n^{2S+1}L_J$, where L is denoted by the letters S (for $L = 0$), P (for $L = 1$), D (for $L = 2$), and so on. As an example, for the state with quantum numbers $n = 3$, $L = 2$, $S = 1$, $J = 2$, the spectroscopic notation is 3^3D_2 .

Let us break down the individual contributions and start with the first-order perturbation theory terms, which are given by the expectation value of the radiatively corrected Breit Hamiltonian $H_{\text{evp}}^{(5)}$ given in Eqs. (38) and (57). We define the individual first-order perturbative terms [denoted with the superscript (1)] as follows,

$$E_X^{(1)}(n^{2S+1}L_J) = \frac{\alpha}{\pi} \int_0^1 dv f_1(v) \times \langle n^{2S+1}L_J | \mathcal{H}_X^{(4)} | n^{2S+1}L_J \rangle, \quad (92)$$

where we consider the terms with

$$X \in \{M1, M2, SO, FSS1, FSS2, Q, D\} \quad (93)$$

individually. For deuteronium, we have [31,69]

$$m_1 = m_2 = m_d, \quad Z = 1, \quad (94)$$

where m_d is the deuteron mass. We split the magnetic term into two contributions, $\mathcal{H}_M^{(4)} = \mathcal{H}_{M1}^{(4)} + \mathcal{H}_{M2}^{(4)}$, where

$$\mathcal{H}_{M1}^{(4)} = -\frac{\alpha}{2m_d^2} p^i \frac{e^{-\lambda r}}{r} [\delta^{ij} + (1 + \lambda r) \hat{x}^i \hat{x}^j] p^j, \quad (95a)$$

$$\mathcal{H}_{M2}^{(4)} = -\frac{\alpha \lambda^2}{4m_d^2} \frac{e^{-\lambda r}}{r} \left(1 - \frac{\lambda r}{2}\right). \quad (95b)$$

The spin-orbit contribution is

$$\mathcal{H}_{SO}^{(4)} = \frac{\alpha}{2m_d^2} (2\tilde{g}_d - 1) \frac{e^{-\lambda r}}{r^3} (1 + \lambda r) \vec{S} \cdot \vec{L}, \quad (96)$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the total spin operator. We consider the two parts of the Fermi spin-spin term separately, $\mathcal{H}_{\text{FSS}}^{(4)} = \mathcal{H}_{\text{FSS1}}^{(4)} + \mathcal{H}_{\text{FSS2}}^{(4)}$, where

$$\mathcal{H}_{\text{FSS1}}^{(4)} = \frac{2\pi\alpha\tilde{g}_d^2}{3m_d^2} \left\{ \delta^3(\vec{r}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\} \vec{S}_1 \cdot \vec{S}_2, \quad (97a)$$

$$\mathcal{H}_{\text{FSS2}}^{(4)} = \frac{3\alpha\tilde{g}_d^2}{4m_d^2} \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3}\right) (S_1^i S_2^j)^{(2)} (\hat{x}^i \hat{x}^j)^{(2)}. \quad (97b)$$

The quadrupole term is

$$\mathcal{H}_Q^{(4)} = -\frac{3\alpha}{2m_d^2} \tilde{Q}_d \frac{e^{-\lambda r}}{r^3} \left(1 + \lambda r + \frac{\lambda^2 r^2}{3}\right) \times [(S_1^i S_1^j)^{(2)} + (S_2^i S_2^j)^{(2)}] (\hat{x}^i \hat{x}^j)^{(2)}, \quad (98)$$

where the dimensionless quadrupole moment factor $\tilde{Q}_d$ is defined as

$$\tilde{Q}_d \equiv m_d^2 Q_d = \left(\frac{m_d c^2}{\hbar c}\right)^2 Q_d = 25.8120(22). \quad (99)$$

Finally, the Darwin (finite size) contribution is

$$\mathcal{H}_D^{(4)} = \frac{4\pi\alpha}{3m_d^2} \tilde{r}_d^2 \left\{ \delta^3(\vec{r}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r} \right\}, \quad (100)$$

where the dimensionless radius is

$$\tilde{r}_d \equiv m_d r_d = \frac{m_d c^2}{\hbar c} r_d = 20.2248(26). \quad (101)$$

The matrix elements are to be taken with the states $\langle \vec{r} | nLSJM_J \rangle = \psi_{nLSJM_J}(\vec{r})$, where the wave functions are

$$\psi_{nLSJM_J}(\vec{r}) = R_{nL}(r) \Xi_{JM_J}^{LS}(\theta, \varphi), \quad (102)$$

and where $R_{nL}(r)$ is the nonrelativistic Schrödinger-Coulomb wave function for the appropriate reduced mass $m_r = m_d/2$ and nuclear charge number $Z = 1$ (see Chap. 4 of Ref. [58]). Furthermore, the spin-angular function $\Xi_{JM_J}^{LS}(\theta, \varphi)$ can be expressed as

$$\Xi_{JM_J}^{LS}(\theta, \varphi) = \sum_{M_L, M_S, m_a, m_b} C_{LM_L M_S}^{JM_J} C_{1m_a 1m_b}^{SM_S} \times Y_{LM_L}(\theta, \varphi) \chi_{m_a}^{(1)} \chi_{m_b}^{(2)} \quad (103)$$

where the Clebsch-Gordan coefficients $C_{j_1 m_1 j_2 m_2}^{JM_J}$ are used in the notation of Chap. 6 of Ref. [58], and $\chi_{m_a}^{(1)}$ is the spin state for particle 1 (deuteron) with magnetic projection m_a , and $\chi_{m_b}^{(2)}$ is the spin state for particle 2 (antideuteron) with magnetic projection m_b . Expressions for the matrix

elements of the various angular operators appearing in the terms of $\mathcal{H}^{(4)}$ were given in Appendix B of [31]. We observe that the results for the first-order matrix elements, given by $E_X^{(1)}$ as outlined in Eq. (92), are independent of the projection M_J .

Deuteronium states with $S = 0, L = J$ and $S = 2, L = J$ are mixed at the level of the Breit corrections (that is, at order $m_d\alpha^4$), as discussed in [31]. The mixed states and corresponding Breit energy contributions were found by diagonalizing the mixing matrix

$$\langle nLS'JM_J | H^{(4)} | nLSJM_J \rangle = m_d\alpha^4 M_{S'S}^{nLJ}, \quad (104)$$

as given explicitly in (37) of [31]. (These matrix elements are independent of M_J .) The only off-diagonal matrix elements have $S = 0, L = J$ and $S = 2, L = J$. So, for each value of $L = J$, there are two mixed states. The state with the smaller (larger) eigenvalue is labeled n^-J_J (n^+J_J). For the order $m_d\alpha^5$ relativistic recoil corrections, we have

$$E_{\pm}^{(5)} = \xi_{\pm}^* H^{(5)} \xi_{\pm} = \sum_{\substack{S'=0,2 \\ S=0,2}} \xi_{\pm}^*(S') \langle nJS'JM_J | H^{(5)} | nJSJM_J \rangle \xi_{\pm}(S), \quad (105)$$

where the eigenvectors of the mixing matrix $\xi_{\pm}$ satisfy

$$m_d\alpha^4 \sum_{S=0,2} M_{S'S}^{nLJ} \xi_{\pm}(S) = E_{\pm}^{(4)} \xi_{\pm}(S'). \quad (106)$$

The Breit energies $E^{(4)}$ are given in Table III of [31].

The $3D$ and $4D$ states of deuteronium have been identified as particularly interesting for spectroscopic investigations on New Physics effects [69]. We here selected the states 3^3D_2 and 4^3D_2 with spin $S = 1$ and total angular momentum $J = 2$ for our illustrative calculations of the individual contributions. For the state of deuteronium with the principal quantum number $n = 3$, and the spectroscopic notation 3^3D_2 , the individual results are

$$E_{M1}^{(1)}(3^3D_2) = -0.003537 \text{ meV}, \quad (107a)$$

$$E_{M2}^{(1)}(3^3D_2) = 0.000105 \text{ meV}, \quad (107b)$$

$$E_{SO}^{(1)}(3^3D_2) = -0.002201 \text{ meV}, \quad (107c)$$

$$E_{FSS1}^{(1)}(3^3D_2) = 0.000900 \text{ meV}, \quad (107d)$$

$$E_{FSS2}^{(1)}(3^3D_2) = 0.001782 \text{ meV}, \quad (107e)$$

$$E_Q^{(1)}(3^3D_2) = 0.031309(3) \text{ meV}, \quad (107f)$$

$$E_D^{(1)}(3^3D_2) = -0.250691(64) \text{ meV}, \quad (107g)$$

$$E^{(1)}(3^3D_2) = -0.22233(7) \text{ meV}. \quad (107h)$$

For the state with $n = 4$, and $S = 1, L = J = 2$, with spectroscopic notation 4^3D_2 , the individual contributions are

$$E_{M1}^{(1)}(4^3D_2) = -0.001954 \text{ meV}, \quad (108a)$$

$$E_{M2}^{(1)}(4^3D_2) = 0.000038 \text{ meV}, \quad (108b)$$

$$E_{SO}^{(1)}(4^3D_2) = -0.001043 \text{ meV}, \quad (108c)$$

$$E_{FSS1}^{(1)}(4^3D_2) = 0.000391 \text{ meV}, \quad (108d)$$

$$E_{FSS2}^{(1)}(4^3D_2) = 0.000826 \text{ meV}, \quad (108e)$$

$$E_Q^{(1)}(4^3D_2) = 0.0145171(12) \text{ meV}, \quad (108f)$$

$$E_D^{(1)}(4^3D_2) = -0.108783(28) \text{ meV}, \quad (108g)$$

$$E^{(1)}(4^3D_2) = -0.09601(3) \text{ meV}. \quad (108h)$$

The total first-order contribution is

$$E^{(1)}(n^{2S+1}L_J) = \langle n^{2S+1}L_J | H_{\text{evp}}^{(5)} | n^{2S+1}L_J \rangle, \quad (109)$$

where $H_{\text{evp}}^{(5)}$ is given in Eq. (57). Results for the total first-order terms for $3P, 3D, 4D$ and $4F$ states of deuteronium are given in Table III.

Now, we investigate the second-order terms. We define the individual contribution as follows,

$$E_X^{(2)}(n^{2S+1}L_J) = 2 \langle n^{2S+1}L_J | H_X^{(4)} \left(\frac{1}{E_S - H_S} \right)' V_{\text{evp}}^{(1)} | n^{2S+1}L_J \rangle, \quad (110)$$

where $\lambda \equiv \lambda(v) = 2m_e/\sqrt{1-v^2}$ as defined in Eq. (1). Here

$$E_S = -\frac{\alpha^2 m_d}{4n^2} \quad (111)$$

is the unperturbed Schrödinger-Coulomb energy and the Schrödinger-Coulomb Hamiltonian is

$$H_S = \frac{\vec{p}^2}{m_d} - \frac{\alpha}{r}. \quad (112)$$

The individual terms H_X with

$$X \in \{K, M, SO, FSS, Q\}, \quad (113)$$

TABLE III. Expectation values $E^{(1)}$ of the radiatively corrected Breit Hamiltonian from Eq. (57) (in meV) for $3P$, $3D$, $4D$, and $4F$ states of deuterium. The dominant uncertainty affecting the $E^{(1)}$ energy contributions is due to the uncertainty in the radius r_d .

$3^{2S+1}P_J$	$E^{(1)}$	$3^{2S+1}D_J$	$E^{(1)}$
3^-P_1	-0.51655(5)	3^-D_2	-0.062271(5)
3^+P_1	-2.3859(8)	3^+D_2	-0.46433(13)
3^3P_0	-2.4681(4)	3^3D_1	-0.29292(7)
3^3P_1	-1.2899(4)	3^3D_2	-0.22233(7)
3^3P_2	-1.6910(4)	3^3D_3	-0.25828(7)
3^5P_2	-2.1630(4)	3^5D_0	-0.20918(7)
3^5P_3	-1.4601(4)	3^5D_1	-0.23650(7)
		3^5D_3	-0.28877(7)
		3^5D_4	-0.22935(7)

$4^{2S+1}D_J$	$E^{(1)}$	$4^{2S+1}F_J$	$E^{(1)}$
4^-D_2	-0.028705(2)	4^-F_3	-0.0049864(4)
4^+D_2	-0.20130(6)	4^+F_3	-0.042495(11)
4^3D_1	-0.12878(3)	4^3F_2	-0.025005(6)
4^3D_2	-0.09601(3)	4^3F_3	-0.020829(6)
4^3D_3	-0.11261(3)	4^3F_4	-0.023167(6)
4^5D_0	-0.08996(3)	4^5F_1	-0.020935(6)
4^5D_1	-0.10261(3)	4^5F_2	-0.022960(6)
4^5D_3	-0.12674(3)	4^5F_4	-0.024756(6)
4^5D_4	-0.09910(3)	4^5F_5	-0.021000(6)

in the Breit Hamiltonian are given in Eq. (35). Again, we devote special attention to the 3^3D_2 and 4^3D_2 states. The individual results for 3^3D_2 are

$$E_K^{(2)}(3^3D_2) = -0.0053598 \text{ meV}, \quad (114a)$$

$$E_M^{(2)}(3^3D_2) = -0.0132351 \text{ meV}, \quad (114b)$$

$$E_{SO}^{(2)}(3^3D_2) = -0.0043041 \text{ meV}, \quad (114c)$$

$$E_{FSS}^{(2)}(3^3D_2) = 0.0026039 \text{ meV}, \quad (114d)$$

$$E_Q^{(2)}(3^3D_2) = 0.0457556(39) \text{ meV}, \quad (114e)$$

$$E^{(2)}(3^3D_2) = 0.025460(4) \text{ meV}. \quad (114f)$$

For the 4^3D_2 state, the individual contributions are

$$E_K^{(2)}(4^3D_2) = -0.0025677 \text{ meV}, \quad (115a)$$

$$E_M^{(2)}(4^3D_2) = -0.0055379 \text{ meV}, \quad (115b)$$

$$E_{SO}^{(2)}(4^3D_2) = -0.0015290 \text{ meV}, \quad (115c)$$

$$E_{FSS}^{(2)}(4^3D_2) = 0.0009250 \text{ meV}, \quad (115d)$$

TABLE IV. Expectation values $E^{(2)}$ of the second-order matrix elements [see Eq. (116)] are given in meV for the $3P$, $3D$, $4D$ and $4F$ states of deuterium. The main uncertainty affecting the $E^{(2)}$ corrections is from the uncertainty in Q_d .

$3^{2S+1}P_J$	$E^{(2)}$	$3^{2S+1}D_J$	$E^{(2)}$
3^-P_1	-0.56801(5)	3^-D_2	-0.082692(7)
3^+P_1	0.85626(10)	3^+D_2	0.014097(5)
3^3P_0	-1.12341(8)	3^3D_1	-0.079867(4)
3^3P_1	0.34352(4)	3^3D_2	0.025460(4)
3^3P_2	-0.141812(8)	3^3D_3	-0.02238038(11)
3^5P_2	-0.72461(6)	3^5D_0	0.041884(8)
3^5P_3	0.164931(15)	3^5D_1	0.003036(4)
		3^5D_3	-0.067911(5)
		3^5D_4	0.023279(2)

$4^{2S+1}D_J$	$E^{(2)}$	$4^{2S+1}F_J$	$E^{(2)}$
4^-D_2	-0.029411(3)	4^-F_3	-0.0047254(4)
4^+D_2	0.002043(2)	4^+F_3	-0.0011946(3)
4^3D_1	-0.0298719(14)	4^3F_2	-0.0047244(2)
4^3D_2	0.0075449(14)	4^3F_3	0.0005208(2)
4^3D_3	-0.0099559(4)	4^3F_4	-0.00195927(7)
4^5D_0	0.013379(3)	4^5F_1	0.0000252(4)
4^5D_1	-0.0004211(14)	4^5F_2	-0.00224033(8)
4^5D_3	-0.025625(2)	4^5F_4	-0.0038294(2)
4^5D_4	0.0067701(8)	4^5F_5	0.00107371(14)

$$E_Q^{(2)}(4^3D_2) = 0.0162544(14) \text{ meV}, \quad (115e)$$

$$E^{(2)}(4^3D_2) = 0.0075449(14) \text{ meV}. \quad (115f)$$

Results for the total second-order terms $E^{(2)}$,

$$E^{(2)} = E_K^{(2)} + E_M^{(2)} + E_{SO}^{(2)} + E_{FSS}^{(2)} + E_Q^{(2)} \quad (116)$$

for the $3P$, $3D$, $4D$ and $4F$ states of deuterium, are given in Table IV.

VII. CONCLUSIONS

The calculation of the spectrum of heavy bound systems of particles heavier than the electron constitutes an involved task because radiative and recoil corrections are more intertwined than for lighter systems (for details, see Ref. [3]). In the heavy systems, electronic vacuum polarization takes the role of the dominant radiative correction to energy levels, being larger than self-energy effects (see Refs. [59–67] as well as Refs. [31,69]).

In our investigations, we have generalized the calculation of relativistic-recoil corrections to electronic vacuum polarization to bound systems where the constituent particles are either spinless or have spin 1/2 or spin 1. To this end, we have used a version of NRQED where the mass scale separating soft from hard energies is much larger than the

electron mass [67]. This extension of NRQED has been identified in Ref. [92] and discussed in the context of muonic atoms in Ref. [67]. From the interaction kernels, we have verified the Breit (interaction) Hamiltonian for particles of spin-1, in Sec. III, and discussed the generalization to spins 0 and 1/2. The Breit Hamiltonian is based on the exchange of massless photons [for the propagator, see Eq. (4)]. The result is given in Eq. (33). For the treatment of vacuum polarization, we have generalized the derivation of the effective Hamiltonian (see Sec. IV) to the exchange of a massive photon, formulated in the one-loop optimized gauge [see Eq. (2)]. We have derived the Hamiltonian for the massive exchange in Eq. (58) and have discussed its application to systems composed of particles having spins 0, 1/2, and 1.

The application of our formalism to bound systems of interest has been discussed in Sec. V, with a special emphasis on ponium, muonic hydrogen and muonic deuterium. In Sec. VI, a bound system consisting of two spin-1 particles is investigated, namely, the bound system of a deuteron and its antiparticle, referred to as deuteronium. The results reported here remove an obstacle in the process of advancing the theory of the bound-state spectrum of deuteronium [31,69] to the $\alpha^5 m_d$ order. Furthermore, in the context of heavy bound systems, the full consideration of the $\alpha^5 m_r$ corrections, for the bound-state spectra of bound systems with spins 0, 1/2 and 1, enables sensitive tests for possible low-energy extensions of the Standard Model [32–37].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: FOURIER TRANSFORMS

Some useful three-dimensional Fourier transforms are

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{1}{\vec{k}^2 + \lambda^2} = \frac{e^{-\lambda r}}{4\pi r}, \quad (\text{A1a})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k}^2 e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + \lambda^2} = \delta^{(3)}(\vec{x}) - \lambda^2 \frac{e^{-\lambda r}}{4\pi r}, \quad (\text{A1b})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\lambda^2 e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 (\vec{k}^2 + \lambda^2)} = \frac{e^{-\lambda r}}{4\pi r} (e^{\lambda r} - 1). \quad (\text{A1c})$$

For integrals with the square of $(\vec{k}^2 + \lambda^2)$ in the denominator, we obtain

$$\int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{(\vec{k}^2 + \lambda^2)^2} = \frac{e^{-\lambda r}}{8\pi\lambda}, \quad (\text{A1d})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k}^2 e^{i\vec{k}\cdot\vec{x}}}{(\vec{k}^2 + \lambda^2)^2} = \frac{e^{-\lambda r}}{4\pi r} \left(1 - \frac{\lambda r}{2}\right), \quad (\text{A1e})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\lambda^4 e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 (\vec{k}^2 + \lambda^2)^2} = \frac{e^{-\lambda r}}{4\pi r} \left(e^{\lambda r} - 1 - \frac{\lambda r}{2}\right). \quad (\text{A1f})$$

The following vector-valued integrals are also relevant,

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i}{\vec{k}^2 + \lambda^2} = \frac{e^{-\lambda r}}{4\pi r^2} (1 + \lambda r) i\hat{x}^i, \quad (\text{A2a})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\lambda^2 k^i e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 (\vec{k}^2 + \lambda^2)} = \frac{e^{-\lambda r}}{4\pi r^2} (e^{\lambda r} - 1 - \lambda r) i\hat{x}^i, \quad (\text{A2b})$$

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i}{(\vec{k}^2 + \lambda^2)^2} = \frac{e^{-\lambda r}}{8\pi} i\hat{x}^i. \quad (\text{A2c})$$

In these calculations, the general identity for Fourier transforms $\mathcal{F}[k^i f(\vec{k})](\vec{x}) = -i\partial_i \mathcal{F}[f(\vec{k})](\vec{x})$ was used, along with partial-fraction decompositions such as

$$\frac{1}{\vec{k}^2 + \lambda^2} = \frac{1}{\vec{k}^2} - \frac{\lambda^2}{\vec{k}^2 (\vec{k}^2 + \lambda^2)}. \quad (\text{A3})$$

The relevant integrals of second-rank tensors are

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i k^j}{\vec{k}^2} = \frac{1}{3} \delta^{ij} \delta^{(3)}(\vec{x}) + \frac{1}{4\pi r^3} (\delta^{ij} - 3\hat{x}^i \hat{x}^j), \quad (\text{A4})$$

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i k^j}{\vec{k}^4} = \frac{1}{8\pi r} (\delta^{ij} - \hat{x}^i \hat{x}^j), \quad (\text{A5})$$

$$\int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i k^j}{(\vec{k}^2 + \lambda^2)^2} = \frac{e^{-\lambda r}}{8\pi r} (\delta^{ij} - (1 + \lambda r) \hat{x}^i \hat{x}^j). \quad (\text{A6})$$

The ones here without λ factors were evaluated as the $D \rightarrow 3$ limits of the general- D results [93]. Two further tensor integrals have more complicated expressions. The first of these is

$$\begin{aligned} \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^i k^j}{\vec{k}^2 + \lambda^2} \\ = \frac{1}{3} \delta^{ij} \delta^{(3)}(\vec{r}) + \frac{e^{-\lambda r}}{4\pi r^3} \{ (1 + \lambda r) \delta^{ij} - (3 + 3\lambda r + \lambda^2 r^2) \hat{x}^i \hat{x}^j \}, \end{aligned} \quad (\text{A7})$$

while the second is

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\lambda^2 k^i k^j}{\vec{k}^2 (\vec{k}^2 + \lambda^2)} = \frac{e^{-\lambda r}}{4\pi r^3} \left\{ (e^{\lambda r} - 1 - \lambda r) \delta^{ij} - 3 \left(e^{\lambda r} - 1 - \lambda r - \frac{\lambda^2 r^2}{3} \right) \hat{x}^i \hat{x}^j \right\}. \quad (\text{A8})$$

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