

Yang-Mills topology on four-dimensional triangulations

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We consider 4D $SU(N)$ gauge theories coupled to gravity in the causal dynamical triangulation (CDT) approach, focusing on the topological classification of the gauge path integral over fixed triangulations. We discretize the topological charge and, after checking the emergence of topology and the continuum scaling on flat triangulations, we show that topology emerges on thermalized triangulations only in the so-called de Sitter phase of CDT, thus enforcing the link between such phase and semiclassical spacetime. We also provide a tool to visualize the topological structures.

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I. INTRODUCTION

Triangulations are useful lattice discretizations of curved spacetime [1], represented as the gluing of elementary flat regions known as simplices, like triangles in 2D, tetrahedra in 3D, and so on. They form the basis for the approach to quantum gravity (QG), known as dynamical triangulations (DT), consisting in the representation of the regularized Euclidean QG path integral in terms of such configurations, then looking for critical points within the space of bare parameters as a possible location for renormalized theory.

The version in which causality is enforced by requiring a well-defined time foliation (causal dynamical triangulation, or CDT [2,3]) is revealing particularly promising in 4D, because of a nontrivial phase diagram with critical surfaces at the boundary of regions where semiclassical features emerge. Within this context, the study of quantum field theories (QFT) coupled to triangulations is of utmost importance. First, it provides an effective tool for studying QFT on curved spacetime, which is interesting by itself. Second, the path toward renormalized QG should go, sooner or later, through a full formulation of the theory

coupled to other standard model fields. In this respect, gauge field theories are a primary goal.

While the program of coupling gauge fields to CDT has been fully accomplished in 2D [4–6], its extension to 4D is not easy, at least when building a Markov chain to explore the configuration space of the full theory [7]. In this context, the present study is an intermediate step, yet addressing an important aspect.

We investigate $SU(N)$ gauge theories discretized on quenched triangulations, meaning that the spacetime geometry stays fixed while gauge fields evolve, although we consider different triangulations, going from quasiflat¹ to those sampled from the CDT path integral. Among the possible gauge invariant observables, we focus on gauge topology.

There are at least two good reasons for that. First, the classification of the path integral into topological sectors originates many nonperturbative features of 4D $SU(N)$ gauge theories, like the nontrivial dependence on the CP -violating parameter θ . Second, such classification is intimately connected to spacetime structure, since it measures the winding over the gauge group of gauge fields on the 4D spacetime boundary. For instance, it seems hard to find nontrivial topology on triangulations with effective dimension less than four, similarly to what happens for gauge theories at asymptotically high temperatures, which are effectively three-dimensional and for which indeed topological fluctuations are strongly suppressed [8].

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¹Unlike the 2D case, it is not possible to generate a flat triangulation of a manifold via equilateral simplices, so we employ *quasiflat* triangulations, with minimal deviations from being flat.

Therefore, the study of gauge topology could better characterize the properties of spacetime geometries sampled in CDT simulations, in a novel way not considered before. From a merely algorithmic point of view, the very definition of topology will require to fix a global orientation for the triangulation, a problem not afforded in previous studies. Our program goes through three main steps: (i) implementation of gauge fields on spacetime triangulations, (ii) sampling of gauge fields to reproduce the discretized Yang-Mills path integral on some selected triangulations, which are either built *ad hoc* or sampled after CDT thermalization through the Einstein-Hilbert action, i.e., we still neglect the backreaction of gauge fields on triangulation sampling, and (iii) definition and investigation of observables capturing topological properties of gauge fields.

II. IMPLEMENTATION AND SAMPLING

In dimension $d > 2$, the only way to correctly count gauge degrees of freedom is to assign a different local gauge to each simplex [4]. Then, the elementary gauge variables are the $SU(N)$ parallel transporters connecting adjacent simplices, i.e., living on dual graph links [6,7].

The discretization of gauge invariant observables is not unique, however, it relies in general on the definition of traces of products of gauge variables around closed loops. The *plaquette* Π_b is the most elementary possibility, defined as a product of dual-link variables around a bone b of the triangulation [a $(d-2)$ -simplex, i.e., a triangle in $d=4$]. In the naive continuum limit, similarly to what happens on standard hypercubic lattices, it has the following correspondence with the field strength tensor $F_{\mu\nu}$, where $\mu\nu$ are directions orthogonal to the bone b :

$$\Pi_b \simeq \exp(ign_b \mathcal{A} F_{\mu\nu}), \quad (1)$$

where the elementary plaquette area contribution $\mathcal{A}n_b$ is given as the product of the elementary area $\mathcal{A}$ and the number of simplices around the bone n_b , while g denotes the bare gauge coupling constant.

The second-order expansion of $\text{Tr}(\Pi_b)$ gives, apart from constant additive factors, $-Nn_b^2 g^2 \mathcal{A}^2 \text{Tr}(F_{\mu\nu}^2)/2$, so one can define

$$S_{\text{YM}} \equiv -\beta \sum_{b \in \mathcal{T}^{(d-2)}} \frac{\tilde{\Pi}_b}{n_b}, \quad \tilde{\Pi}_b \equiv \left[\frac{1}{N} \text{ReTr} \Pi_b - 1 \right]. \quad (2)$$

The factor $n_b \mathcal{A}^2$ counts the spacetime volume pertaining to each b , apart from a missing factor $\sqrt{5}$, so the whole sum returns, in the naive continuum limit, the volume integral of $\text{Tr}(F_{\mu\nu}^2)$, averaged over all possible directions, times a factor $g^2/(2N\sqrt{5})$. The usual continuum expression of the Yang-Mills action coincides with this form, apart from a

factor 6, coming from the sum over all orthogonal direction pairs. That finally fixes the relation between g and the inverse gauge coupling β ,

$$\beta = \frac{12\sqrt{5}N}{g^2}. \quad (3)$$

The sampling of gauge configurations according to $\exp(-S_{\text{YM}})$ can go through usual pure gauge algorithms: we adopted a standard heat-bath algorithm [9,10]. For quasiflat triangulations, the different spacetime regularization keeps the asymptotic behavior of the lattice spacing $a(\beta)$ unchanged up to two loops,

$$\bar{a} \equiv \Lambda_L a \simeq (\beta_0 g^2)^{\beta_1/(2\beta_0^2)} e^{-1/(2\beta_0 g^2)}, \quad (4)$$

where $3(4\pi)^2\beta_0 = 11N$ and $3(4\pi)^4\beta_1 = 34N^2$ for pure Yang-Mills. Of course, both Λ_L and the regime where such asymptotic behavior sets in are *a priori* unknown. The issue of continuum limit on thermalized triangulations is far less trivial, because of the intrinsic length scales characterizing each triangulation, and could be properly afforded only in a fully dynamical context.

III. THE OBSERVABLE

As for the definition of topological charge, its continuum expression in 4D is the following:

$$Q = \frac{g^2}{32\pi^2} \int dx \varepsilon_{\mu\nu\alpha\beta} \text{Tr}[F^{\mu\nu}(x) F^{\alpha\beta}(x)]. \quad (5)$$

We can expand the expression for the field strength tensor around the center of each simplex (i.e., a node of the dual graph) in terms of discretized variables in a local chart. Let us consider a specific simplex σ and an overcomplete set of unit vectors $\{\vec{e}_{A,\sigma}\}_{A=1,\dots,5}$, directed from the center of σ to each of its five neighbors—these vectors are related to a flat Cartesian basis at σ via a change of basis, represented by the matrix $e_{A,\sigma}^\mu$. We can then write

$$F_\sigma^{\mu\nu} = e_{A,\sigma}^\mu e_{B,\sigma}^\nu \bar{F}_\sigma^{AB}, \quad (6)$$

where $\bar{F}_\sigma^{AB}$ can now be related to the plaquette built by looping around the bone $b_{(AB\sigma)}$ identified by the neighbors A and B ,

$$\Pi_{b_{(AB\sigma)}} \simeq e^{ig\mathcal{A}n_{b_{(AB\sigma)}} \bar{F}_\sigma^{AB}} \simeq 1 + ig\mathcal{A}n_{b_{(AB\sigma)}} \bar{F}_\sigma^{AB}. \quad (7)$$

From Eq. (7) we can read off, at the leading order in a ,

$$\bar{F}_\sigma^{AB} \simeq \frac{1}{g\mathcal{A}n_{b_{(AB\sigma)}}} \Im[\Pi_{b_{(AB\sigma)}}]; \quad \Im[\Pi] \equiv \frac{\Pi - \Pi^\dagger}{2i}. \quad (8)$$

Substituting into the expression for the topological charge density [Eq. (5) before integration], we obtain

$$\rho_\sigma \simeq \rho_\sigma^{ABMN} \bar{\epsilon}_{ABMN,\sigma}, \quad (9)$$

where we defined the charge four-form density as

$$\rho_\sigma^{ABMN} \equiv \frac{1}{32\pi^2 \mathcal{A}^2} \text{Tr} \left\{ \Im \left[\frac{\Pi_{b_{(AB\sigma)}}}{n_{b_{(AB\sigma)}} \right] \Im \left[\frac{\Pi_{b_{(MN\sigma)}}}{n_{b_{(MN\sigma)}} \right] \right\}, \quad (10)$$

and the Levi-Civita symbol as

$$\bar{\epsilon}_{ABMN,\sigma} \equiv \epsilon_{\mu\nu\alpha\beta} e_{A,\sigma}^\mu e_{B,\sigma}^\nu e_{M,\sigma}^\alpha e_{N,\sigma}^\beta, \quad (11)$$

expressed in terms of a complete set of basis vectors. Finally, from Eq. (9) one can get the topological charge Q_L by integrating ρ_σ^{ABMN} over spacetime consisting of N_4 number of simplices with individual volume $V_\sigma = \sqrt{5}/96$, resulting in $V = N_4 V_\sigma a^4$ 4D volume.

An additional factor $1/5$ stems from the five possible choices of the four independent directions used to build the components of the field strength tensor. Notice that this construction is very similar to the standard symmetrized clover discretization adopted on flat hypercubic lattices, where the number of links coming out from each site is 8, instead of 5, as in our case. Explicitly, Q_L reads

$$Q_L = \frac{1}{5} \sum_{\sigma \in T} \rho_\sigma V_\sigma. \quad (12)$$

Finally, a consistent ordering of the vectors $\vec{e}_{A,\sigma}$ over each simplex must be implemented, according to the global orientation of $\mathcal{T}$, in order to sum the different contributions coherently. See Appendix A for more details.

Topology is well defined only in the continuum limit. A standard procedure, adopted on regular lattices to recover an approximate topological classification, is to iteratively smooth gauge configurations so as to remove ultraviolet noise, typically by minimizing the gauge action and exploiting the fact that topological sectors are characterized by local minima, corresponding to (anti-)instantons in the continuum. We will follow this approach, exploiting in particular the so-called cooling algorithm [11–18]. We stress that, as for the discretization of other gauge invariant observables, the definition of the cooling procedure is not unique.

IV. NUMERICAL RESULTS

We start from quasiflat triangulations of a toroidal hypercube [19,20], since this important benchmark should approximate results obtained on standard hypercubic lattices. We will mainly illustrate results for $SU(3)$, with similar findings obtained for $SU(2)$.

Figure 1 shows the evolution of Q_L and S_{YM} during the action minimization for two sampled configurations: similarly to standard lattice studies, we observe intervals where both quantities experience plateaus, hinting at the presence

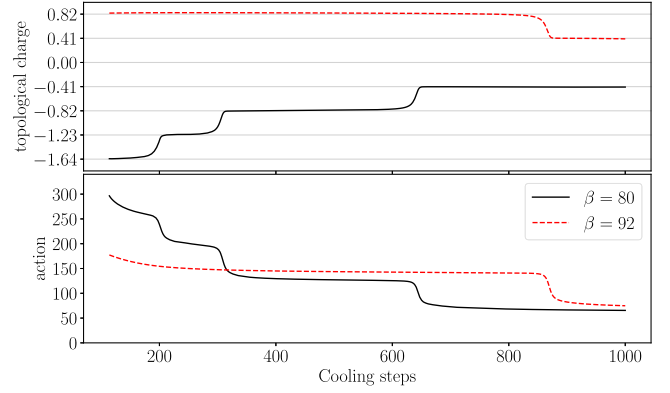


FIG. 1. Q_L and S_{YM} evolution during cooling for two $SU(3)$ configurations sampled at different β values on quasiflat triangulations with $T = 40$.

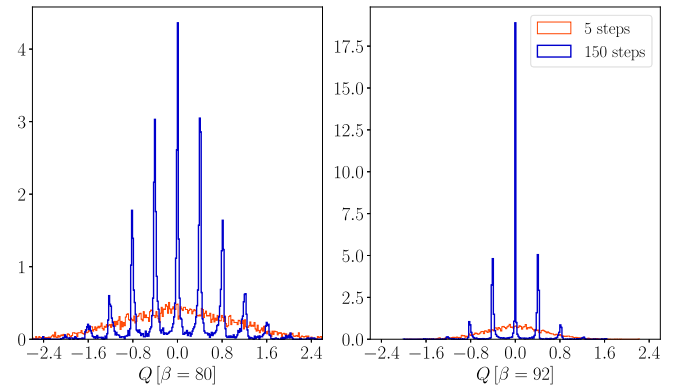


FIG. 2. Histograms of Q_L for $SU(3)$ on quasiflat $T = 40$ triangulations after 5 (red) and 150 (blue) cooling steps.

of metastable states, approximately dividing the path integral into homotopy classes characterized by different values of Q_L . The histograms of Q_L taken after a sufficient number of cooling steps (Fig. 2) reveal that the topological charge distribution peaks around integer multiples of an elementary unit Q_0 , with $Q_0 \simeq 0.41$ independently of β .

In principle, one would expect $Q_0 \simeq 1$, however, also on standard regular lattices discretization effects make $Q_0 < 1$: here, such effects could be larger, because of the approximate flatness of the triangulation. However, a well-defined topological sector Q can be assigned as follows [21,22]:

$$Q = \text{round}\{Q_L/Q_0\}, \quad (13)$$

where “round” denotes the approximation to the closest integer and Q_0 is fixed by minimizing

$$\langle (Q_L/Q_0 - \text{round}\{Q_L/Q_0\})^2 \rangle. \quad (14)$$

The distribution of Q is wider at lower β , which is expected by the running of the lattice spacing with β . The variance of Q is proportional to the topological

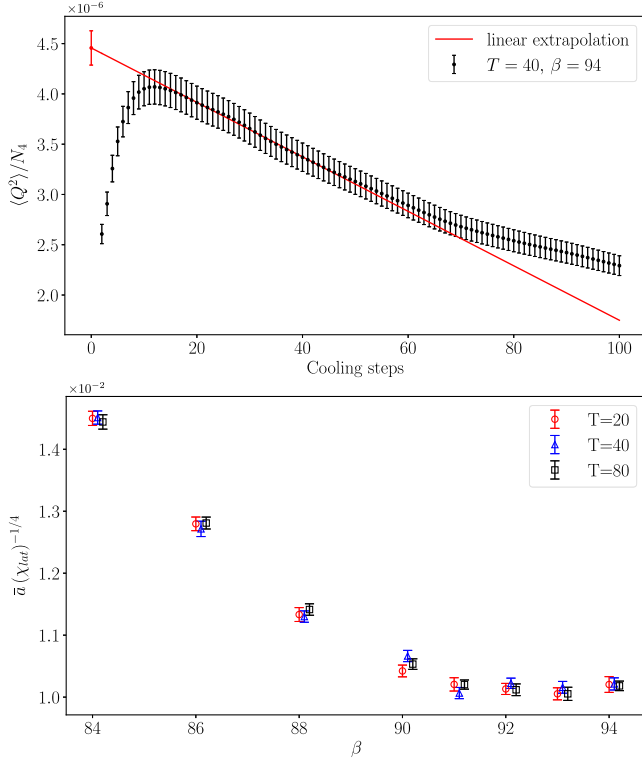


FIG. 3. Topological susceptibility for $SU(3)$ gauge fields in a quasiflat toroidal triangulation.

susceptibility $\chi \equiv \langle Q^2 \rangle / V$, therefore, with χ being a physical quantity, $\langle Q^2 \rangle$ is expected to run as a^4 at fixed triangulations. That gives us the possibility to check if the asymptotic continuum scaling of a , Eq. (4), is reached within the explored range of β .

To this end, we compute the dimensionless susceptibility $\chi_{lat} \equiv \langle Q^2 \rangle / N_4$ for each dataset. Since topological configurations are only metastable, the smoothing procedure, apart from cleaning the topological signal, leads also to a partial loss of it. For this reason, it is usual to perform an extrapolation to zero smoothing, choosing a suitable range of cooling steps for which unphysical fluctuations have already been killed, while the unwanted loss of signal proceeds linearly with the smoothing. An example is shown on the upper side of Fig. 3.

The so-extrapolated quantity is shown in the lower part of the same figure, where we report in particular $\bar{a}\chi_{lat}^{-1/4}$ as a function of β for different values of the temporal extension T , where $\bar{a}$ is the 2-loop β function in Eq. (4). This combination should approach a constant value in the small coupling regime where Eq. (4) becomes a good approximation. Our results show that, luckily enough, we have at least a small window where this is true. Moreover, the absence of T dependence, within errors, shows that also finite size effects should be reasonably under control.

Assuming the standard pure gauge value from the Witten-Veneziano mechanism [23,24], $\chi^{1/4} \sim 180$ MeV,

the measured value of χ_{lat} gives us an estimate of $a \simeq (\chi_{lat}/(\chi V_\sigma))^{1/4}$. We get $a \sim 0.2$ fm, or in terms of the dual lattice spacing $a' = a/\sqrt{10} \sim 0.06$ fm, at the border of the explored region, $\beta = 94$. It is interesting to notice that this is in the same ballpark of lattice spacings where scaling sets in for standard lattices. Moreover, that tells us *a posteriori* that the spacetime extensions are at least $O(1$ fm) for all explored values of β , which is consistent, at least for pure gauge theories, with the absence of significant finite size effects.

Another interesting question is how the topological charge density is distributed over the triangulated spacetime. Standard lattice simulations have shown that the original distribution, i.e., before prolonged smoothing is performed, differs substantially from a simple instanton picture, with a single bump for each winding unit, corresponding instead to a nontrivial delocalized structure, characterized by many local clusters (see, e.g., Refs. [25–27]). In order to investigate this aspect, we have developed a visualization tool, which is based on the definition of pseudo-Cartesian coordinates on the triangulation [28–30]. Details are reported in Appendix B and fully confirm the standard lattice picture described above.

Let us now discuss the interesting case of thermalized configurations. The observability of gauge topology is tied to the universality class and effective dimension of the underlying geometry. The phase diagram of CDT [31,32] is characterized by two bare parameters k_0 and Δ , which are related to the Newton and cosmological constants, as well as to a spacetime anisotropy parameter for the lattice spacings in the causal lattice. The four geometric phases found include a branched-polymer phase with the universality class of continuum random trees, two other phases with infinite Hausdorff dimension, and a phase representing semiclassical spacetimes [33] with emergent effective action called de Sitter phase [34].

Remarkably, we observed nontrivial topological sectors only in the de Sitter phase and, even in this case, only when the overall numerical triangulation topology was that of a torus T^4 . In Fig. 4 we show some examples of the Q_L distribution for this case: after enough smoothing, well-defined peaks appear, equally spaced by a spacing $Q_0 \sim 0.3$, which differs from the quasiflat case, probably because of the different underlying geometries.

On the contrary, for triangulations thermalized in the de Sitter phase with overall $S^1 \times S^3$ topology, no signs of metastable structures were observed at all. This is best visible in Fig. 5, where we compare the gauge action descent during smoothing for a few configurations in the two cases. The completely different behavior can be hardly interpreted just in terms of the different spacetime topology, which could influence the overall stability, but not the appearance of metastable gauge configurations. We interpret this as the property of triangulations being locally mappable to 4D spacetime. Therefore, the different

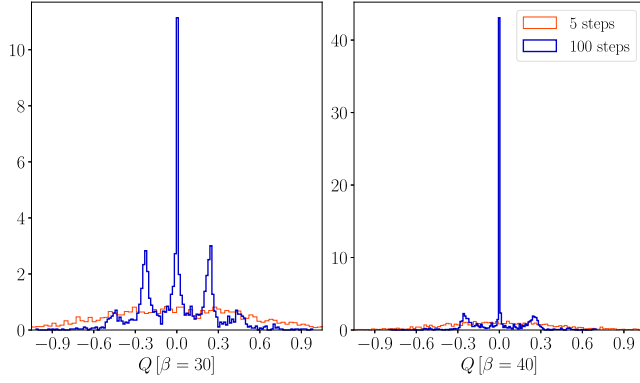


FIG. 4. Histograms of Q_L for $SU(3)$ on triangulations thermalized in the de Sitter phase, with $(S^1)^4$ overall topology, and bare parameters $\kappa_0 = 4.0$, $\Delta = 0$, $T = 20$, and $N_4 = 120k$.

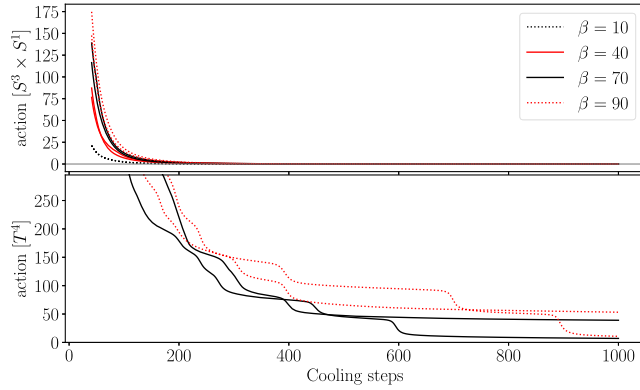


FIG. 5. Action descent during smoothing for a few configurations on triangulations thermalized in the de Sitter phase, but with different overall topologies.

behavior observed for the $S^1 \times S^3$ de Sitter phase could be due to triangulations in that phase having a local effective dimension different from four: this is a suggestion coming from our new method to look at these triangulations through the properties of gauge fields, which should be further explored in the future. We remark that, even if not shown, the action descents observed in all other CDT phases are similar to the upper side of Fig. 5.

Restricting from now on to the de Sitter phase on a torus, we remark that, along the metastable plateaus, also S_{YM} is approximately a multiple of a well-defined elementary unit S_1 , which can be interpreted as the action contribution from a single (anti-)instanton. In the continuum, the flat case one has [35] $S_1 = 8\pi^2/g^2$, which, given our definition of β in Eq. (3), means $NS_1/\beta = 2\pi^2/(3\sqrt{5}) \simeq 2.94$. In Table I we show the estimates of NS_1/β obtained for $N = 2, 3$ and for both quasiflat and thermalized triangulations in the de Sitter phase. This quantity appears to be quite stable for different values of N and β , but it significantly varies depending on the ensemble of base manifolds. In particular, the quasiflat estimate, $NS_1/\beta \simeq 2.3(3)$, is not far from the flat

TABLE I. Observed values of the action gap between consecutive topological sectors. The value of NS_1/β expected in the continuum flat case, considering Eq. (3), is about 2.94.

N	β	NS_1/β	
		Quasiflat	de Sitter phase
2	20	2.30(30)	1.10(20)
	24	2.28(25)	1.12(17)
	28	2.27(21)	1.09(18)
3	80	2.28(30)	1.07(23)
	86	2.28(28)	1.06(21)
	92	2.28(26)	1.08(20)

semiclassical expectation in the continuum 2.94, while a more significant renormalization appears for thermalized triangulations, where $NS_1/\beta \simeq 1.1(2)$.

In this case, considerations similar to those for the charge quantization Q_0 apply: it is well known that, also on standard regular lattices, the “instanton” action is typically smaller than the continuum one, because of discretization effects (see, e.g., Ref. [15]). Moreover, such effects could be enhanced in the case of thermalized configurations, because of the different underlying spacetime geometry. Overall, the most important result is the appearance of metastable gauge configurations, with topological charge and action multiple of well-defined quantities.

V. CONCLUSIONS

We successfully implemented $SU(N)$ gauge fields on fixed triangulated manifolds, going from their numerical simulations to the definition and measurement of observables related to gauge topology. First, we verified the appearance of a topological classification of gauge configurations on quasiflat triangulations, checking also the asymptotic scaling to the continuum limit of physical observables, such as the topological susceptibility, and providing a framework to visualize the nontrivial topological structures.

Then, we explored triangulations sampled from the CDT path integral. Remarkably, nontrivial gauge topology appears only in the de Sitter phase realized on a spacetime torus, thus enforcing the idea that such phase correctly reproduces semiclassical spacetime, capable of sustaining nontrivial field-theoretic features, such as gauge topology. We have argued that the absence of such features in the corresponding de Sitter phase realized on $S^1 \times S^3$, as well as in other CDT phases, could reveal different properties of the corresponding triangulations at a local level, such as an effective local dimension different from four, i.e., the absence of effectively four-dimensional regions capable of sustaining gauge topology even at a metastable level. We stress that this is one of the main outcomes of our study that should be further explored in the future, i.e., providing a

new tool to explore the geometric properties of triangulations via the nontrivial properties of gauge fields living on them.

Future efforts should also be devoted to the definition of further gauge observables, as well as, from a long-term perspective, to the numerical simulation of the full gravity + gauge path integral.

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APPENDIX A: REORDERING PROCEDURE

In order to have a meaningful implementation of the topological charge, the simplices of the triangulation must be properly oriented so that the pseudoscalarity of Q holds. In the particular case of simplicial manifolds, this means that for any pair of adjacent simplices $\sigma_{1,2} \in \mathcal{T}$, the local frames $\{\vec{e}_{\sigma_{1,2},A}\}$ already mentioned before must coincide up to a rototranslation,

$$\vec{e}_{A,\sigma_2} = (R_{\sigma_2} \circ t_{\sigma_2 \leftarrow \sigma_1}) \vec{e}_{A,\sigma_1} \quad \forall A \in \{1, \dots, d\}, \quad (\text{A1})$$

with $R_{\sigma_2} \in SO(d)$ and $t_{\sigma_2 \leftarrow \sigma_1}$, respectively, a rotation defined in the center of σ_2 and a translation that moves the center of σ_1 to the center of σ_2 . In other words, Eq. (A1) just states that the two frames must have the same parity. In the following, we will show a possible strategy to fix a global orientation of $\mathcal{T}$. This method is simply based on setting the orientation of $\{\vec{e}_{A,\sigma_2}\}$ assuming that $\{\vec{e}_{A,\sigma_1}\}$ has already been fixed. To do so, we will make use of the vectors $\{\vec{v}_{A,\sigma_{1,2}}\}_{A=1,\dots,d}$, which represent the set of vectors connecting the centers of $\sigma_{1,2}$ to their vertices. Let us start by first fixing $\{\vec{v}_{A,\sigma_1}\}$. This can be done, for instance, by symmetrizing and scaling² $\{\vec{e}_{A,\sigma_1}\}$ with respect to the center of σ_1 ,

$$\vec{v}_{A,\sigma_1} = -\|\vec{v}\| \vec{e}_{A,\sigma_1} \quad \forall A \in \{1, \dots, d\}. \quad (\text{A2})$$

²We are assuming that $\{\vec{e}_{A,\sigma}\}$ are unitary vectors.

In this way, the two frames in the expression above have the same parity in even dimensions and opposite parity otherwise. At this point one just needs to fix $\{\vec{v}_{A,\sigma_2}\}$ by requiring that the latter has the same orientation of $\{\vec{v}_{A,\sigma_1}\}$ and use again (A2) to retrieve $\{\vec{e}_{A,\sigma_2}\}$, which eventually will satisfy (A1). For this purpose, let us label with $\{P_i\}_{i=1,\dots,d-1}$ the vertices of the $d-1$ -dimensional subsimplex shared by σ_1, σ_2 : if labels are chosen so that the association between such points and the first $d-1$ vectors of $\{\vec{v}_{A,\sigma_1}\}$ is

$$\{\vec{v}_{A,\sigma_1}\} = \{\vec{v}_{1,\sigma_1}(P_1), \dots, \vec{v}_{d-1,\sigma_1}(P_{d-1}), \vec{v}_{d,\sigma_1}\}, \quad (\text{A3})$$

then $\{\vec{v}_{A,\sigma_2}\}$ can be obtained by exchanging any two directions after that the common vertices have been associated with the vectors of σ_2 in the same way as for σ_1 . Explicitly, it reads

$$\{\vec{v}_{A,\sigma_2}\} = \tau_{jk} \{\vec{v}_{1,\sigma_2}(P_1), \dots, \vec{v}_{d-1,\sigma_2}(P_{d-1}), \vec{v}_{d,\sigma_2}\}, \quad (\text{A4})$$

where τ_{jk} is a transpose operator acting as

$$\begin{aligned} \tau_{jk} \{ \dots \vec{v}_{j,\sigma_2}(P_j), \dots, \vec{v}_{k,\sigma_2}(P_k), \dots \} \\ = \{ \dots \vec{v}_{j,\sigma_2}(P_k), \dots, \vec{v}_{k,\sigma_2}(P_j), \dots \}. \end{aligned} \quad (\text{A5})$$

Indeed, since the uncommon vertices (one for each simplex) belong to each of the two separated space regions defined by the shared subsimplex, $\{\vec{v}_{A,\sigma_1}\}$ and $\{\vec{v}_{A,\sigma_2}\}$ would have opposite parity if the action of a discrete operator like τ was neglected. Such a procedure, sketched in Fig. 6 for the 2D case, can be iterated for all the nearest neighbors of σ_1 and, therefore, extended to all the simplices of the triangulation. Of course, this can be done consistently, up to pure rotations, only if the given triangulation is globally orientable.

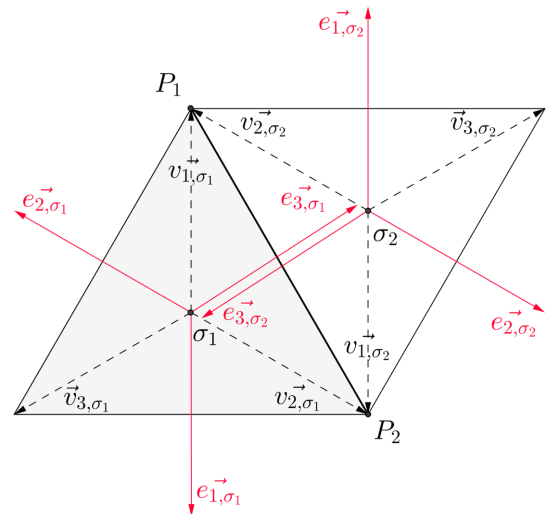


FIG. 6. Two-dimensional sketch of the ordering procedure.

APPENDIX B: VISUALIZATION OF NONTRIVIAL TOPOLOGICAL STRUCTURES

By definition, compact triangulations do not admit a natural coordinate system. In [28–30], a method was introduced to define pseudo-Cartesian coordinates in CDT. A curved Riemannian manifold M can be mapped to another manifold N with a flat metric using a mapping represented by a scalar field ϕ . Associating a field with every coordinate direction, the values of the four fields $\phi_{x,y,z,t}$ represent a new field coordinate system. By choosing a suitable boundary condition, a useful metric structure can be associated with the triangulation, which can be used to map distances and structures different from that of the dual lattice.

We used the extracted scalar field values as a map and plotted the topological charge density within each simplex, with red for positive and blue for negative charges, as shown in Fig. 7 for the snapshot of a specific configuration, which is classified in the $Q = 1$ topological sector, after 10 cooling steps. It is interesting to notice that, at this early stage of cooling, the topological charge is still substantially delocalized, i.e., many clusters appear. This is similar to what is observed in standard lattice simulations (see, e.g., Refs. [25–27]). On the other hand, the same picture after prolonged cooling (see Fig. 8) appears as a single and well-defined cluster.

Let us now illustrate some more technical details about how the pseudo-Cartesian coordinates are actually implemented. One way to add matter fields into a triangulated manifold is to treat the field as a map between the triangulated spacetime $\mathcal{M}_H(g_{\mu\nu})$ and a target space $\mathcal{N}(h_{\alpha\beta})$

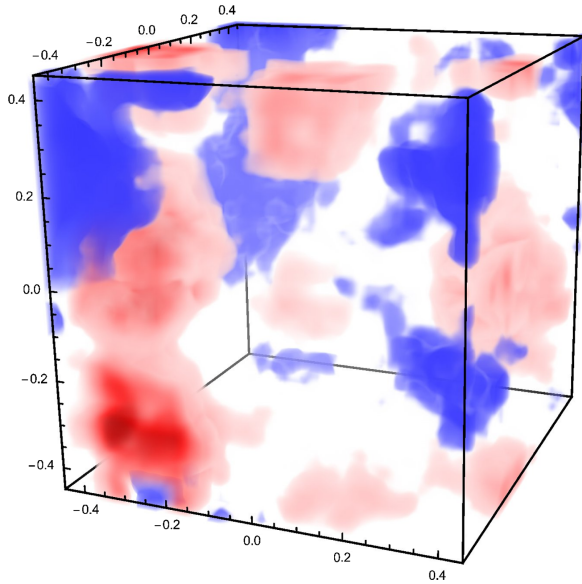


FIG. 7. A visualization of the charge density in ϕ_x, ϕ_y, ϕ_z scalar field coordinate representation after 10 cooling steps for $\beta^{SU(2)} = 30$, using the quasiflat configuration. The topological charge density has been integrated in the time direction.

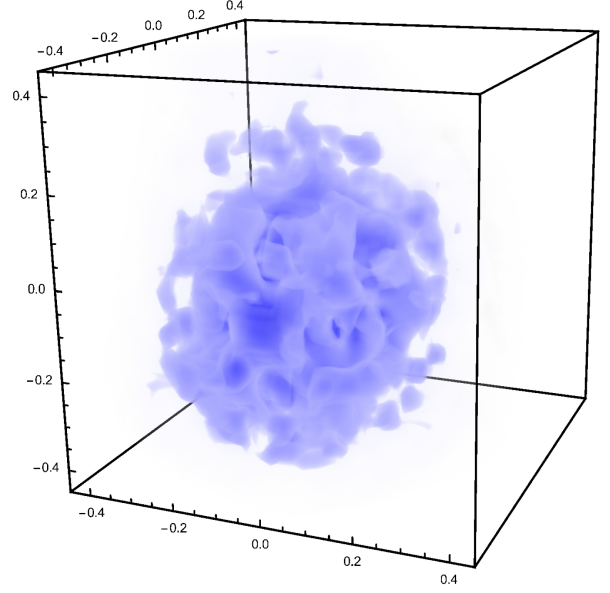


FIG. 8. Charge density visualization for the same triangulation and gauge configuration reported in Fig. 7, but after prolonged cooling (150 steps). Also in this case, the topological charge density has been integrated in the time direction.

with fixed metric and topology. In the case where the target space is Euclidean $\mathbb{R}^d$ or S^d with a diagonal and flat metric on $\mathcal{N}$, then the continuum action

$$\begin{aligned} S_M[\phi, g] &= \frac{1}{2} \int d^4x \sqrt{g(x)} g^{\mu\nu}(x) h_{\rho\sigma}(\phi^\rho(x)) \\ &\quad \times \partial_\mu \phi^\rho(x) \partial_\nu \phi^\sigma(x) \\ &= \frac{1}{2} \sum_{\sigma=1}^d \int d^4x \sqrt{g(x)} \partial^\nu \phi^\sigma(x) \partial_\nu \phi^\sigma(x) \end{aligned} \quad (\text{B1})$$

can be separated for each direction σ . The addition of a boundary allows the field values to technically obtain a winding number, but the added boundary jump condition (with amplitude δ) renders the mapping to remain continuous by identifying

$$\phi_i^\sigma \equiv \phi_i^\sigma + n \cdot \delta, \quad n \in \mathbb{Z} \quad (\text{B2})$$

at the boundary, mapping the field values to a circle with circumference δ . The discrete form of the matter field action with this boundary term is

$$\begin{aligned} S_M^{CDT}[\phi, T] &= \frac{1}{2} \sum_{\sigma=1}^d \sum_{i \leftrightarrow j} (\phi_i^\sigma - \phi_j^\sigma - \delta \cdot \mathbf{B}_{ij}^\sigma)^2 \\ &= \sum_{\sigma=1}^d \left(\sum_{i,j} \phi_i^\sigma \mathbf{L}_{ij}(T) \phi_j^\sigma - 2\delta \sum_i \phi_i^\sigma b_i^\sigma + \delta^2 \cdot V^\sigma \right), \end{aligned} \quad (\text{B3})$$

where $\sum_{i \leftrightarrow j}$ counts all of the neighbors of the discrete building blocks and $\mathbf{L}_{ij}(T)$ is defined as the discrete Laplace-Beltrami operator for the underlying dual graph. The boundary term $\mathbf{B}_{ij}^\sigma$ gives a $\pm\delta$ contribution if the particular simplex pair $\{i, j\}$ lies on the boundary. Minimizing the action leads to

$$\mathbf{L}\phi^\sigma = \delta \cdot b^\sigma, \quad (\text{B4})$$

which is the inhomogeneous Laplace equation for the scalar field ϕ^σ with a boundary term b^σ with scale δ . Decomposing the scalar field as $\phi^\sigma = \bar{\phi}^\sigma + \xi^\sigma$ and expanding the action leads to the form

$$\begin{aligned} S_M^{CDT}[\phi, \xi, T] &= \sum_{\sigma, i, j} \xi_i^\sigma \mathbf{L}_{ij}(T) \xi_j^\sigma + \sum_{\sigma=1}^d \left(\sum_{i, j} \bar{\phi}_i^\sigma \mathbf{L}_{ij}(T) \bar{\phi}_j^\sigma - 2\delta \sum_i \bar{\phi}_i^\sigma b_i^\sigma + \delta^2 \cdot V^\sigma \right) \\ &= \sum_{\sigma, i, j} \xi_i^\sigma \mathbf{L}_{ij}(T) \xi_j^\sigma + S_M^{CDT}[\bar{\phi}, T]. \end{aligned} \quad (\text{B5})$$

The classical solution $\bar{\phi}^\sigma$ of the scalar fields obtains a winding number 1 and the quantum fields ξ^σ take real values with winding number 0.

We took thermalized pure gravity geometries with defined boundaries as a technical tool. For fixed geometries (in the quenched approximation), setting the value of $\delta > 0$ and solving the inhomogeneous Laplace equation, Eq. (B4), by cooling the dynamical scalar fields, we effectively forced the quantum fields ξ^σ to align with the classical solution $\bar{\phi}^\sigma$. Then, extracting the scalar field coordinates for each simplex

in directions (ϕ_x, ϕ_y, ϕ_z) gave us a set of spatial coordinates, and using the foliation t as a fourth coordinate and rescaling all coordinate values to 1 we obtained a $[0, 1]^4$ box with a density inside, corresponding to the map from the underlying geometry to the flat box. Associating to each simplex the topological charge density, we can visualize it in this representation as the function of the scalar field coordinates, which is analogous to a Laplacian embedding of a graph. It was shown that this map exhibits homogeneous large-scale properties only in the de Sitter phase.

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