

Systematics of doubly heavy strange and nonstrange tetraquarks

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Masses, magnetic moments, and color-spin structures of nonstrange and strange tetraquarks with two heavy quarks are systematically studied in a QCD string picture with a chromomagnetic interaction. Our mass computations, combined with weak and radiative decays, indicate that there are two doubly heavy tetraquarks, the bottom-charmed tetraquark $T_{cb}(7173, 01^+)^0$ and the doubly bottom tetraquark $T_{bb}(10406, 01^+)^-$, that are stable against strong decay, having lifetimes of 0.069 and 326 fs, respectively. Magnetic moments are also computed and are found to be sensitive to chromomagnetic mixing of the color-spin configurations of these tetraquarks, making them a valuable probe to experimentally detect the nontrivial color configuration $6_c \otimes \bar{6}_c$.

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I. INTRODUCTION

Soon after the birth of the quark model [1,2], exotic hadron states like tetraquarks (with quark configuration $q^2\bar{q}^2$) and pentaquarks ($q^4\bar{q}$), which are also allowed in principle by the later established quantum chromodynamics (QCD), were suggested. Yet, such an exotic state was not observed until 2003, when the Belle Collaboration reported the discovery of the first exotic hadron $X(3872)$ [3]. Since then, many exotic candidates, including $Z_c(3900)$ [4,5], $Z_c(4020)$ [6], and $X(4050)$ charmoniumlike states [7] and pentaquark states [8–11], have been observed. The first doubly charmed tetraquark $T_{cc}(3875)^+ (IJ^P = 01^+)$ was discovered in 2021 by the LHCb Collaboration [12,13], triggering considerable effort involving experimental searches and theoretical studies of multiquark states and other exotic hadrons [14–16].

Multiquarks were explored theoretically in the 1970s based on the MIT bag model [17,18]. To date, multiquark states, mainly heavy tetraquarks, have been discussed extensively based on pictures of hadronic molecular [19–28], compact tetraquark [29–34], and other explanations. To the static properties of exotic hadrons, much effort has been devoted using various approaches, including the

constituent quark model [35–38], the bag model [31,39–42], the QCD sum rule [43–47], and the relativistic three-quark model [48], and so on. See Refs. [15,49] for reviews.

To explore static properties of doubly heavy (DH) tetraquarks with and without strange quarks in a QCD string picture with a chromomagnetic interaction, we extract the model parameters from the observed mass data of the light and heavy mesons and utilize a simple mass relation with interquark binding B_{ij} and flux-tube energy attached to each quark, which we proposed previously in Ref. [50], to compute the ground-state masses of the DH tetraquarks and the chromomagnetic mixing of their color-spin structures. Further computation is given to magnetic moments of these DH tetraquarks related to their color-spin structures and the relative decay widths of the strongly stable tetraquarks.

After the Introduction, in Sec. II, a chromomagnetic interaction model in the framework of the QCD string picture is utilized to give the formulism of mass and magnetic moments of a DH tetraquark, and its parameters are determined from the established ordinary hadrons. In Sec. III, the masses and magnetic moments of nonstrange and strange DH tetraquarks T_{cc} , T_{bb} , and T_{cb} are computed, and their different decay behaviors are discussed. We end with conclusions and remarks in Sec. IV.

II. MASSES, MAGNETIC MOMENTS, AND PARAMETERS

A. Chromomagnetic interactions

For the ground state of the tetraquarks, the chromomagnetic interaction (CMI) model is given by the

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Sakharov-Zeldovich formula [50,51],

$$M = M_{\text{CE}} + \langle H_{\text{CMI}} \rangle, \quad M_{\text{CE}} = \sum_i (m_i + E_i) + B, \quad (1)$$

where M_{CE} denotes the chromoelectric energy, H_{CMI} is the CMI energy, m_i is the mass of the i th quark or antiquark, and E_i is the effective energy of the color flux attached to the i th (anti)quark. Following Refs. [50,52], we introduce an enhanced binding energy, $B = \sum_{i<j} B_{Q_i Q_j}(\bar{Q}_j)$, which arises from short-distance chromoelectric interactions between heavy quarks and heavy (anti)quarks or between heavy quarks and strange (anti)quarks [52–54].

As an effective description of QCD utilized in this work, the QCD string picture views the strong interaction among quarks and gluons as that of flux tubes of color field lines, which confines these colored sources (quarks and gluons) by exerting a linearly rising potential between them. The picture mainly describes the nonperturbative interaction of QCD and applies well in the long-distance (nonperturbative) limit, as supported by lattice QCD [55,56]. The QCD string approach becomes especially useful when one takes it as the chromoelectric flux tube tied to quarks in hadrons [56]. As we address the ground states of the DH hadrons in this work, where only the energy of the QCD string is needed, we parametrize it simply as a model parameter.

For the quark pairs cc , bb , and cb in the color antitriplet $\bar{3}_c$, we shall apply a simple relation from Ref. [50] to determine the corresponding binding energies B_{cc} , B_{bb} , and B_{cb} in the DH tetraquarks in the color representation (rep.) of $3_c \otimes \bar{3}_c$. For the quark pairs cc , bb , and cb in the color antisextet $\bar{6}_c$, we employ the scaling of color factors of rep. 6_c relative to rep. $\bar{3}_c$ to evaluate the corresponding binding energies B_{cc} , B_{bb} , and B_{cb} in DH tetraquarks in the color rep. of $6_c \otimes \bar{6}_c$. Another interquark term in Eq. (1) is that of the hyperfine chromomagnetic interaction, which is spin and color dependent [57],

$$H_{\text{CMI}} = - \sum_{i<j} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) (\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j) \frac{A_{ij}}{m_i m_j}, \quad (2)$$

where $\boldsymbol{\sigma}_i$ are the Pauli matrices for quark spin and $\boldsymbol{\lambda}_i$ are the Gell-Mann matrices for quark colors. The effective couplings A_{ij} denote the strength of the CMI between the i th and j th (anti)quarks. The color and spin factors for the quark pair ij in a DH tetraquark are defined as the respective matrix elements of $\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$ and $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ and can be evaluated via the following formulas [31,58]:

$$\begin{aligned} \langle \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \rangle_{mn} &= \sum_{a=1}^8 \text{Tr}(c_{im}^\dagger \lambda^a c_{in}) \text{Tr}(c_{jm}^\dagger \lambda^a c_{jn}), \\ \langle \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \rangle_{xy} &= \sum_{a=1}^3 \text{Tr}(\chi_{ix}^\dagger \sigma^a \chi_{iy}) \text{Tr}(\chi_{jx}^\dagger \sigma^a \chi_{jy}), \end{aligned} \quad (3)$$

where n and m denote the quark color and x and y indicate the quark spin.

For strong decay of an exotic DH tetraquark, which was discussed in Refs. [42,59,60], among other papers, we shall apply the S-wave two-body decay formula

$$\Gamma_i = \gamma_i \alpha k \cdot |c_i|^2, \quad (4)$$

where γ_i depends on the spatial wave function of final-state products, α describes the coupling strength at decay vertex, and k indicates the momentum of the final state. The coefficient c_i is the probability amplitude for the tetraquark transitioning to the i th colorless subsystem and is quantum mechanically given by the coefficients c_1 and c_8 when its color-spin wave function is expanded in terms of the color singlet 1_c and color octet 8_c components,

$$|\psi\rangle = c_1 |Q_1 \bar{Q}_2\rangle^{1_c} |Q_3 \bar{Q}_4\rangle^{1_c} + c_8 |Q_1 \bar{Q}_2\rangle^{8_c} |Q_3 \bar{Q}_4\rangle^{8_c}. \quad (5)$$

In the heavy-quark limit, the values of γ_i for the Σ_c and Σ_c^* states remain approximately the same [61]. For the same reason, one can assume the following approximate relations for the values of γ_i connecting the pseudoscalar and vector mesons [62,63]:

$$\begin{aligned} \gamma_{D_s D^*} &= \gamma_{D_s^* D}, & \gamma_{\bar{B} D^*} &= \gamma_{\bar{B}^* D}, \\ \gamma_{\bar{B}_s D^*} &= \gamma_{\bar{B}_s^* D}, & \gamma_{\bar{B} D_s^*} &= \gamma_{\bar{B}^* D_s}. \end{aligned} \quad (6)$$

B. Magnetic moments of DH tetraquarks

The magnetic moments of DH tetraquarks are one of the important electromagnetic properties of hadrons and offer, in general, a useful insight into their internal structure. In this subsection, we will explore the magnetic moment of a DH tetraquark by computing all the contributions from the magnetic moments of their constituents. At the quark level, the total magnetic moment $\hat{\mu}_{\text{TOT}}$ of a tetraquark is composed of that of total spin $\hat{\mu}_S$ and that of total orbital $\hat{\mu}_L$ of the constituent quarks in hadron. One then has, explicitly,

$$\begin{aligned} \hat{\mu}_{\text{TOT}} &= \hat{\mu}_S + \hat{\mu}_L, \\ \hat{\mu}_S &= \sum_i g_i \mu_i \hat{S}_{i,z}, \\ \hat{\mu}_L &= \sum_{i<j} \left[\frac{m'_i}{m'_i + m'_j} \mu_j + \frac{m'_j}{m'_i + m'_j} \mu_i \right] \hat{L}_{ij,z}, \end{aligned} \quad (7)$$

where the factor $g_i = 2$, $\hat{S}_{i,z}$ is the z component of the spin of the i th quark and $\hat{L}_{ij,z}$ is the z component of the relative orbital angular momentum $\hat{L}_{ij}$ between the i th and j th quarks. Here, $m'_i \equiv m_i + E_i$ stands for the effective quark mass.

Ignoring the abnormal magnetic moments of quarks, the quark with flavor f will have a magnetic moment of

$\mu_f = Q_f/2m'_f$, with Q_f the charge of its constituent quark. The magnetic moment of a DH tetraquark state is then

$$\mu = \langle \psi | \hat{\mu}_{\text{TOT}} | \psi \rangle, \quad (8)$$

where $|\psi\rangle$ stands for the hadron wave function consisting of four components,

$$\psi = \phi_{\text{color}} \otimes \chi_{\text{spin}} \otimes \eta_{\text{flavor}} \otimes R_{\text{radial}}. \quad (9)$$

Note here that we are considering the ground-state (S-wave) DH tetraquarks, for which $\hat{L}_{ij}$ is vanishing. The total magnetic moment in Eq. (8) equals the total magnetic moment $\hat{\mu}_S$ of the hadron spin, which consists of the spin magnetic moment of the quarks in the hadrons. Thus, the radial part of the wave function in Eq. (9) will be normalized to unity and the total magnetic moment in Eq. (8) will depend merely on the color-spin wave functions of the hadrons, given that its flavor content is specified.

In contrast with the conventional mesons or baryons, DH tetraquarks may have either of two independent color structures, $6_c \otimes \bar{6}_c$ and $\bar{3}_c \otimes 3_c$. Each of them is possible, except when it is dynamically prohibited. For completeness, one can write the color part of the wave function $|\psi\rangle$ of a tetraquark as

$$\phi_1 = |(Q_1 Q_2)^6 (\bar{q}_3 \bar{q}_4)^{\bar{6}}\rangle, \quad \phi_2 = |(Q_1 Q_2)^{\bar{3}} (\bar{q}_3 \bar{q}_4)^3\rangle, \quad (10)$$

and the spin part of $|\psi\rangle$ can be written as one of the following six states,

$$\begin{aligned} \chi_1^{2,2} &= |(Q_1 Q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_2, & \chi_2^{1,1} &= |(Q_1 Q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_1, \\ \chi_3^{0,0} &= |(Q_1 Q_2)_1 (\bar{q}_3 \bar{q}_4)_1\rangle_0, & \chi_4^{1,1} &= |(Q_1 Q_2)_1 (\bar{q}_3 \bar{q}_4)_0\rangle_1, \\ \chi_5^{1,1} &= |(Q_1 Q_2)_0 (\bar{q}_3 \bar{q}_4)_1\rangle_1, & \chi_6^{0,0} &= |(Q_1 Q_2)_0 (\bar{q}_3 \bar{q}_4)_0\rangle_0, \end{aligned} \quad (11)$$

or some combination, as we detail in the Appendix.

First, we consider the magnetic moment of the isoscalar T_{cc}^+ state with $J^P = 1^+$ as a example, in which the color-spin-flavor wave function takes the form of

$$\begin{aligned} T_{cc}(01^+) &= \left[R_{24} \left| \frac{1}{\sqrt{2}} (\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow) \right\rangle |\phi_2\rangle \right. \\ &\quad \left. + R_{15} \left| \frac{1}{\sqrt{2}} (\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow) \right\rangle |\phi_1\rangle \right] |cc[\bar{u}\bar{d}] \rangle, \end{aligned} \quad (12)$$

with R_{24} the mixing coefficient for the color component $\bar{3}_c \otimes 3_c$ and R_{15} the mixing coefficient for the component $6_c \otimes \bar{6}_c$. Here, in the mixing coefficient R_{ab} (say, R_{24} and R_{15}), the subscript a stands for the index of the color basis vector and subscript b for the index of the spin basis vector. In our framework, these coefficients are determined by the color-spin wave function which diagonalize the mass matrix [Eq. (1)] and thereby diagonalize the chromomagnetic interaction matrix [Eq. (2)]. Mathematically, two mixing coefficients $R_{24,15}$ forms an eigenvector $(R_{24}, R_{15})^T$ which diagonalizes the CMI matrix [Eq. (2)].

Substitution of the wave function [Eq. (9)] solved as stated above into Eq. (8) gives rise to an expression for hadron magnetic moment in terms of the spin magnetic moment of the constituent quarks in hadrons. In the case of the T_{cc}^+ with $IJ^P = 01^+$, putting Eq. (12) into Eq. (8) leads to

$$\begin{aligned} \mu[T_{cc}(01^+)] &= \langle T_{cc}(01^+) | \hat{\mu}_S | T_{cc}(01^+) \rangle \\ &= \frac{R_{24}^2}{2} \langle cc[\bar{u}\bar{d}] (\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow) | \hat{\mu}_S | cc[\bar{u}\bar{d}] (\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow) \rangle \\ &\quad + \frac{R_{15}^2}{2} \langle cc[\bar{u}\bar{d}] (\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow) | \hat{\mu}_S | cc[\bar{u}\bar{d}] (\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow) \rangle \\ &= R_{24}^2 (\mu_c + \mu_c) + R_{15}^2 (\mu_{\bar{u}} + \mu_{\bar{d}}). \end{aligned} \quad (13)$$

One clearly sees that the magnetic moment of T_{cc} depends not only on the spin-flavor content but also on its color structures via $R_{24,15}$, which depends on internal chromodynamics of T_{cc} .

Likewise, one can apply a similar procedure to the other members of the DH tetraquark T_{QQ} with various color-spin configurations to derive a set of expressions (sum rule) for their magnetic moments, which are the linear combinations of spin magnetic moments of the individual quarks in T_{QQ} . The results obtained for T_{QQ} are

$$\begin{aligned}
\mu[\psi_1^{1,1}] &= \frac{\mu_{1234}}{2}, \\
\mu[\psi_2^{1,1}] &= R_{24}^2 \mu_{12^+} + R_{15}^2 \mu_{34^+}, \\
\mu[\psi_3^{1,1}] &= \frac{R_{22}^2}{2} \mu_{1234} + R_{24}^2 \mu_{12^+} + R_{15}^2 \mu_{34^+} + \sqrt{2} R_{22} R_{24} \mu_{34^-}, \\
\mu[\psi_4^{1,1}] &= \frac{R_{22}^2}{2} \mu_{1234} + R_{25}^2 \mu_{34^+} + R_{14}^2 \mu_{12^+} - \sqrt{2} R_{22} R_{25} \mu_{12^-}, \\
\mu[\psi_5^{1,1}] &= R_{24}^2 \mu_{12^+} + \frac{R_{12}^2}{2} \mu_{1234} + R_{15}^2 \mu_{34^+} - \sqrt{2} R_{12} R_{15} \mu_{12^-}, \\
\mu[\psi_6^{1,1}] &= \frac{R_{22}^2 + R_{12}^2}{2} \mu_{1234} + (R_{24}^2 + R_{14}^2) \mu_{12^+} + (R_{25}^2 + R_{15}^2) \mu_{34^+} \\
&\quad - \sqrt{2} R_{22} (R_{25} \mu_{12^-} - R_{24} \mu_{34^-}) - \sqrt{2} R_{12} (R_{15} \mu_{12^-} - R_{14} \mu_{34^-}), \\
\mu[\psi_1^{2,2}] &= \mu[\psi_2^{2,2}] = \mu[\psi_3^{2,2}] = \mu_{1234},
\end{aligned} \tag{14}$$

where the abbreviations $\mu_{12^\pm} = \mu_{Q_1} \pm \mu_{Q_2}$, $\mu_{34^\pm} = \mu_{\bar{q}_3} \pm \mu_{\bar{q}_4}$, and $\mu_{1234} = \mu_{12^+} + \mu_{34^+}$ are used.

Note that Eqs. (13) and (14) provide a set of sum rules for the magnetic moments of all DH tetraquark systems, including the nonstrange systems of T_{cc} , T_{bb} , and T_{cb} . As usual, we present the values of the magnetic moments in units of the nuclear magneton $\mu_N = e/2M_N$, with $M_N = 938.272$ MeV the mass of the nucleon [64].

C. Parameters

In this subsection, we determine the model parameters in Eq. (1). For this purpose, we maintain the setup of the quark mass $m_n = 230$ MeV, $m_s = 328$ MeV, $m_c = 1440$ MeV, and $m_b = 4480$ MeV, which are extracted from the Regge-like spectrum of the established heavy mesons and baryons [65]. For the chromomagnetic couplings A_{ij} , we shall resort to hyperfine mass splitting of the conventional mesons or baryons. For the CMI coupling $A_{i\bar{j}}$ of the quark-antiquark pair $i\bar{j}$, we apply mass formula (1) to the vector ($\mathbb{V}$) and pseudoscalar ($\mathbb{P}$) mesons to obtain

$$\begin{aligned}
M(\mathbb{V}) &= m'_i + m'_j + B_{i\bar{j}} + \frac{16}{3} \frac{A_{i\bar{j}}}{m_i m_j}, \\
M(\mathbb{P}) &= m'_i + m'_j + B_{i\bar{j}} - 16 \frac{A_{i\bar{j}}}{m_i m_j},
\end{aligned} \tag{15}$$

where $m'_i = m_i + E_i$ as before. From the two equations in Eq. (15), one obtains the mass splitting between vector and pseudoscalar mesons

$$M(\mathbb{V}) - M(\mathbb{P}) = \frac{64}{3} \frac{A_{i\bar{j}}}{m_i m_j}, \tag{16}$$

from which the CMI couplings $A_{i\bar{j}}$ are extractable. For the light pseudoscalar mesons, for instance, Eq. (16) becomes

$$M(\rho) - M(\pi) = \frac{64}{3} \frac{A_{n\bar{n}}}{(230 \text{ MeV})^2}, \tag{17}$$

where the lhs is 640.3 MeV and the up/down quark mass m_n is 230 MeV. This yields $A_{n\bar{n}} = 0.00159 \text{ GeV}^3$ for the up and down quarks. Likewise, one can apply Eq. (16) to other mesons, shown in Table I, and to extract the corresponding CMI couplings with the help of the measured mass splittings in Table I. The results are listed in Table II, from which the couplings $A_{i\bar{j}}$ for the quark-antiquark pair $i\bar{j}$ can be evaluated. The obtained results for $A_{i\bar{j}}$ are listed in Table II.

As for the quark-quark CMI couplings A_{ij} , one can apply Eq. (1) to the observed baryons with spin parities $J^P = 1/2^+$ and $J^P = 3/2^+$ to obtain the mass splitting between the octets and decuplets, namely, the analogy of Eq. (16). As a result, all A_{ij} are determined. In the case of the Δ and N states, for instance, one can use the following mass splitting:

$$M(\Delta) - M(N) = 16 \frac{A_{nn}}{(230 \text{ MeV})^2}, \tag{18}$$

where the lhs is 291.7 MeV. It follows that $A_{nn} = 0.00096 \text{ GeV}^3$.

In the cases where experimental data are lacking (A_{cc} , for instance), we employ an empirical scaling relation $A_{i\bar{j}}/A_{ij} = 3/2 \pm 0.3$ from Refs. [14,71] to evaluate them directly from the known quark-antiquark couplings $A_{i\bar{j}}$. The results are obtained for all CMI couplings and are listed in Table II. By the way, we find a simple function to successfully fit the obtained CMI couplings $A_{i\bar{j}}$ which depends upon the reduced mass μ_{ij} of the quark pair $i\bar{j}$. This fit function takes the form

TABLE I. The spin-averaged masses and splittings of mesons. All values are in MeV. Here, n denotes u/d quarks.

Systems	Spin-averaged masses	Values	Mass splittings	Values
$b\bar{b}$	$\frac{1}{4}[3M(\Upsilon) + M(\eta_b)]$	9445.0	$M(\Upsilon) - M(\eta_b)$	61.7
$b\bar{c}$	$\frac{1}{4}[3M(B_c^*) + M(B_c)]$	6316.9 ^a	$M(B_c^*) - M(B_c)$	56.5 ^a
$b\bar{s}$	$\frac{1}{4}[3M(B_s^*) + M(B_s)]$	5403.3	$M(B_s^*) - M(B_s)$	48.5
$b\bar{n}$	$\frac{1}{4}[3M(B^*) + M(B)]$	5313.5	$M(B^*) - M(B)$	45.0
$c\bar{c}$	$\frac{1}{4}[3M(J/\psi) + M(\eta_c)]$	3068.7	$M(J/\psi) - M(\eta_c)$	112.8
$c\bar{s}$	$\frac{1}{4}[3M(D_s^*) + M(D_s)]$	2076.2	$M(D_s^*) - M(D_s)$	143.9
$c\bar{n}$	$\frac{1}{4}[3M(D^*) + M(D)]$	1971.4	$M(D^*) - M(D)$	142.0
$s\bar{s}$	$\frac{1}{4}[3M(\phi) + M(\eta_s)]$	936.3 ^b	$M(\phi) - M(\eta_s)$	332.5 ^b
$s\bar{n}$	$\frac{1}{4}[3M(K^*) + M(K)]$	796.1	$M(K^*) - M(K)$	397.9
$n\bar{n}$	$\frac{1}{4}[3M(\rho) + M(\pi)]$	615.2	$M(\rho) - M(\pi)$	640.3

^aA mass value $M(B_c^{*-}) = 6331(4)(6)$ MeV from the lattice [66] is used here for the B_c^{*-} meson.

^bWe have excluded the $SU(3)_f$ broken effects and introduced a pure $s\bar{s}$ state with mass $M(\eta_s) = 686.92$ MeV, determined via $M^2(\eta_s) = 2M_K^2 - M_\pi^2$ [67–71].

TABLE II. The chromomagnetic couplings A_{ij} between any two quarks. All values are in GeV^3 .

$A_{n\bar{n}}$	$A_{s\bar{n}}$	$A_{s\bar{s}}$	$A_{c\bar{n}}$	$A_{c\bar{s}}$	$A_{c\bar{c}}$	$A_{b\bar{n}}$	$A_{b\bar{s}}$	$A_{b\bar{c}}$	$A_{b\bar{b}}$
0.00159	0.00141	0.00168	0.0022	0.00318	0.01096	0.00217	0.00334	0.01709	0.05805
A_{nn}	A_{ns}	A_{ss}	A_{nc}	A_{sc}	A_{cc}	A_{nb}	A_{sb}	A_{cb}	A_{bb}
0.00096	0.00094	0.00112	0.00147	0.00212	0.00731	0.00145	0.00223	0.0114	0.0387

$$A_{ij}(\mu_{ij}) = a + b \frac{\mu_{ij}^2}{\Lambda^2} + c \ln\left(\frac{\mu_{ij}^2}{\Lambda^2}\right), \quad (19)$$

with $\Lambda = 0.2$ GeV, $a = 0.00197$, $b = 0.00041$, and $c = 0.00104$, as shown in Fig. 1.

To evaluate the interquark binding energies $B_{Q\bar{Q}'}$, which is relevant (nonvanishing) for the hadrons containing two or more heavy (strange) quarks, we use the relations proposed in Ref. [50], which are to extract the binding energies $B_{Q\bar{Q}'}$, stemming mainly from the short-distance color interaction, by concealing the strings connecting the quarks $Q\bar{Q}'$ responsible for long-distance color interaction and the effective quark masses in hadrons based on the QCD picture of the hadrons. For the quark pair $Q\bar{Q}'$ in the color singlet 1_c , these relations take the form of

$$\begin{aligned}
B_{c\bar{s}} &= M_{c\bar{s}} - M_{c\bar{n}} - M_{s\bar{n}} + M_{n\bar{n}} = -76.0 \text{ MeV}, \\
B_{b\bar{s}} &= M_{b\bar{s}} - M_{b\bar{n}} - M_{s\bar{n}} + M_{n\bar{n}} = -91.1 \text{ MeV}, \\
B_{c\bar{c}} &= M_{c\bar{c}} - 2M_{c\bar{n}} + M_{n\bar{n}} = -258.8 \text{ MeV}, \\
B_{b\bar{b}} &= M_{b\bar{b}} - 2M_{b\bar{n}} + M_{n\bar{n}} = -566.8 \text{ MeV}, \\
B_{c\bar{b}} &= M_{c\bar{b}} - M_{b\bar{n}} - M_{c\bar{n}} + M_{n\bar{n}} = -352.8 \text{ MeV}. \quad (20)
\end{aligned}$$

Here, the energy of string connecting two heavy quarks is assumed to be negligible and the energy of the string connecting the heavy quark and the light antiquarks $Q\bar{n}'$ is

half that of the string connecting the nonstrange quark and antiquarks $n\bar{n}'$ (due to the difference of the local position of the center of mass of the hadron between the two situations).

For the quark-quark pair QQ' in other color configurations, we expect $B_{QQ'}$ to be governed by the short-distance interquark force, which is Coulomb-like, according to the perturbative QCD, and thereby scales like the color factor $\langle \lambda_Q \cdot \lambda_{Q'} \rangle$ given in Eq. (3). We apply then the scaling of the color factor in Ref. [31] to estimate the binding energies for the heavy-quark pair QQ' based on the binding $B_{Q\bar{Q}'}$ for $Q\bar{Q}'$ in the color singlet in Eq. (20). For the color rep. R_c of the quark pair QQ' or $Q\bar{Q}'$, one can first compute the scaling factor [31]

$$\begin{aligned}
g(R_c) &\equiv \frac{\langle \lambda_i \cdot \lambda_k \rangle_{R_c}}{\langle \lambda_i \cdot \lambda_j \rangle_{1_c}} = \frac{(B_{ik})_{R_c}}{(B_{ij})_{1_c}}, \\
i &= Q, \quad j = \bar{Q}', \quad k = Q'(\bar{Q}'), \quad (21)
\end{aligned}$$

and then use it to scale $(B_{Q\bar{Q}'})_{1_c}$ to either $(B_{QQ'})_{R_c}$ or $(B_{Q\bar{Q}'})_{R_c}$. In the case of the heavy pair QQ' in the color antitriplet $\bar{3}_c$, for instance, the scaling factor $g(\bar{3}_c)$ is $1/2$, which leads to $B_{\bar{3}_c}/B_{1_c} = 1/2$ for a given QQ' . One can then estimate the quark-quark binding energies in $\bar{3}_c$ to be [half of the respective values in Eq. (20)]

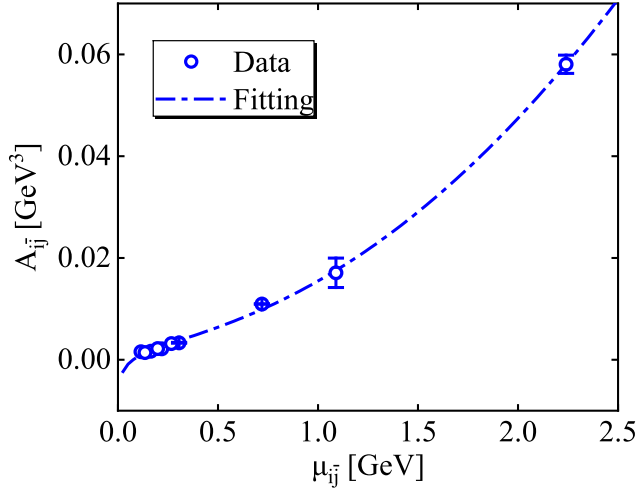


FIG. 1. Dependence of the reduced mass on the chromomagnetic coupling (blue point).

$$\begin{aligned} B_{cc} &= -129.4 \text{ MeV}, & B_{bb} &= -283.4 \text{ MeV}, \\ B_{cb} &= -176.4 \text{ MeV}. \end{aligned} \quad (22)$$

The third parameter to be determined is the string energy E_i , which stems from the long-distance interaction and should be flavor independent in principle. In hadrons with finite size, however, the string lives between the quark sources and is constrained by the hadron size due to the kinematics of the sources (with finite masses). Thus, string energy E_i displays a slight dependence on flavor of the quarks i when they are averaged in hadron. In DH tetraquark states, we assume that the string energy E_n is the same for the u and d quarks, while the string energies E_s , E_c , and E_b differ from E_n and differ from each other as the s , c , and b quarks, with one heavier than the other. In the following, we fix these parameters by applying Eq. (1) to the relevant hadrons.

- (i) E_n : Experimentally, the spin-averaged mass of the nucleons Δ and N is

$$\frac{4}{6}M(\Delta) + \frac{2}{6}M(N) = 1134.1 \text{ MeV}. \quad (23)$$

Application of Eq. (1) to the Δ and N composed of the nnn system with spins $3/2$ and $1/2$ leads to

$$\begin{aligned} M(\Delta) &= 3m'_n + 8 \frac{A_{nn}}{m_n m_n}, \\ M(N) &= 3m'_n - 8 \frac{A_{nn}}{m_n m_n}, \end{aligned} \quad (24)$$

respectively. Putting the above relations into Eq. (23) and solving for E_n , one obtains $E_n = 131.8 \text{ MeV}$.

- (ii) E_s : Averaging the masses of the strange baryons Σ^* and Σ gives, experimentally,

$$\frac{4}{6}M(\Sigma^*) + \frac{2}{6}M(\Sigma) = 1320.0 \text{ MeV}. \quad (25)$$

Application of Eq. (1) to the strange baryons Σ^* and Σ of the nns system with spins $3/2$ and $1/2$ gives rise to

$$\begin{aligned} M(\Sigma^*) &= 2m'_n + m'_s + \frac{8}{3} \left[\frac{A_{nn}}{m_n m_n} + 2 \frac{A_{ns}}{m_n m_s} \right], \\ M(\Sigma) &= 2m'_n + m'_s + \frac{8}{3} \left[\frac{A_{nn}}{m_n m_n} - 4 \frac{A_{ns}}{m_n m_s} \right], \end{aligned} \quad (26)$$

Solving Eq. (25) combined with Eq. (26) for E_s , one gets $E_s = 219.8 \text{ MeV}$.

- (iii) E_c, E_b : On average, the observed charmed baryon $\Sigma_c^{(*)} = nnc$ and the bottom baryon $\Sigma_b^{(*)} = nnb$ have the masses

$$\frac{4}{6}M(\Sigma_c^*) + \frac{2}{6}M(\Sigma_c) = 2496.9 \text{ MeV}, \quad (27)$$

$$\frac{4}{6}M(\Sigma_b^*) + \frac{2}{6}M(\Sigma_b) = 5826.1 \text{ MeV}, \quad (28)$$

respectively. A similar procedure applying Eq. (1) to them gives $E_c = 284.7 \text{ MeV}$ and $E_b = 573.8 \text{ MeV}$, respectively.

III. DOUBLY HEAVY TETRAQUARKS

Given the parameters determined, we apply our approach in Sec. II to the nonstrange and strange DH tetraquarks of the four-quark system $QQ'\bar{q}\bar{q}$, which consist of two heavy quarks ($Q, Q' = c, b$) and two light antiquarks $\bar{q}\bar{q}$, with $q = u, d, s$. For short, we use the notation $T_{QQ'}(M, IJ^P)$ to denote a DH tetraquark with mass M and the quantum number of the isospin and spin-parity IJ^P .

We solve the CMI model [Eq. (1)] first for nonstrange DH tetraquarks $QQ'\bar{n}\bar{n}$ and then for strange DH tetraquarks $QQ'\bar{s}\bar{s}$ and $QQ'\bar{s}\bar{s}$. As stated in Sec. II, for this, one has to use the proper wave function [Eq. (9)] constructed in the color-spin-flavor space, as illustrated in Eq. (12), to diagonalize the mass Hamiltonian, that is, to diagonalize $B + H_{\text{CMI}}$ with the H_{CMI} given in Eq. (2). Next, one can substitute the obtained eigenvalues of the Hamiltonian $B + H_{\text{CMI}}$ into Eq. (1) to obtain the 1S-wave masses of the relevant DH tetraquark systems with given quantum numbers. The obtained eigenvectors provide the respective mixing coefficients R_{ab} of the given configurations and can be used to compute via Eq. (14) the magnetic moments of the hadron systems.

When the DH tetraquarks are above the two-meson thresholds, one can employ the two-body formula in Eq. (4) to explore their relatively strong decay widths. In addition, when the DH tetraquarks, some of the doubly bottom and bottom-charmed tetraquarks, in our case, are below the

two-meson thresholds. They decay mainly via weak decay or radiative decay. Thus, the lifetime of these doubly bottom and bottom-charmed tetraquarks is determined by the shorter of the lifetimes of the weak decay and the radiative decay. The lifetime can stem from the radiative decay, when it is more likely to occur, to end the life of the DH tetraquarks before the weak decay.

For the weak decays, a crude estimate of lifetime is available using the procedure in Refs. [52,72–75] for the weak transition involving the most-favored Cabibbo-Kobayashi-Maskawa matrix elements (at $b \rightarrow cW^{*-}$)

$$\Gamma(T_{bb}) = \frac{18G_F^2 M_i^5}{192\pi^3} F(x) |V_{cb}|^2, \quad (29)$$

which depends on the squared mass ratio $x = [M_f/M_i]^2$ of the final state and initial state via a kinematic suppression factor [75,76]

$$F(x) = 1 - 8x + 8x^3 - x^4 + 12x^2 \ln(1/x), \quad (30)$$

with $|V_{cb}| = 0.042$ the Cabibbo-Kobayashi-Maskawa matrix element [64] and $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ [64] the Fermi constant.

When the radiative decay dominates for the states considered, one can use the M1 radiative decay formula [36],

$$\begin{aligned} \Gamma &= \frac{\omega^3}{M_p^2} \frac{\alpha_{\text{EM}}}{2J_i + 1} \frac{|\mu_{T_i \rightarrow T_f}|^2}{\mu_N^2}, \\ J_z &= \text{Min}[J_i, J_f], \\ \mu_{T_i \rightarrow T_f} &= \langle J_f, J_z | \hat{\mu}_{\text{TOT}} | J_i, J_z \rangle, \end{aligned} \quad (31)$$

where α_{EM} is the fine-structure constant, $\mu_{T_i \rightarrow T_f}$ is the electromagnetic transition magnitude, J_i is the total angular momentum of the initial state T_i and J_f is that of the final state T_f , and $\omega = \frac{M_i^2 - M_f^2}{2M_i}$ is the momentum magnitude of the outgoing photon during the radiative decay.

A. The doubly charmed states

1. Nonstrange tetraquarks

Consider first the DH tetraquark $T_{cc} = cc\bar{n}\bar{n}$, which is composed of two charm quarks cc and two light nonstrange antiquarks $\bar{n}\bar{n}$. As stated at the beginning of Sec. III, we utilize the wave functions in Eq. (12) and others detailed in the Appendix for the DH tetraquark without strange quarks to diagonalize the Hamiltonian $B + H_{\text{CMI}}$ and put the obtained eigenvalues of $B + H_{\text{CMI}}$ into Eq. (1) to give the masses of the relevant tetraquark systems. The numerical results for the CMI matrices and the entire Hamiltonian are given in Table III, and the masses and magnetic moments obtained therein are listed in Table IV for the various states of the nonstrange T_{cc} systems, including the cases of color-spin states mixed by chromomagnetic interaction. We compute similar results for all other T_{cc} with strangeness $S = 0, -1, -2$ and list them in Table IV.

To illustrate the computation, we choose the T_{cc} state with $I(J^P) = 0(1^+)$ as an example. Using the parameters determined in Sec. II, one obtains numerically for the matrix H and H_{CMI}

TABLE III. The computed matrices of the CMI and the entire Hamiltonian of the doubly charmed tetraquarks $cc\bar{n}\bar{n}$, $cc\bar{s}\bar{n}$, and $cc\bar{s}\bar{s}$.

Systems	$I(J^P)$	CMI matrices	Hamiltonians
$cc\bar{n}\bar{n}$	$0(1^+)$	$\begin{bmatrix} -136.5 & -75.3 \\ -75.3 & -10.2 \end{bmatrix}$	$\begin{bmatrix} -265.9 & -75.3 \\ -75.3 & 54.5 \end{bmatrix}$
	$1(0^+)$	$\begin{bmatrix} -13.0 & 130.4 \\ 130.4 & 87.0 \end{bmatrix}$	$\begin{bmatrix} -142.4 & 130.4 \\ 130.4 & 151.7 \end{bmatrix}$
	$1(1^+)$	22.5	-106.9
	$1(2^+)$	93.5	-35.9
$cc\bar{s}\bar{n}$	$1/2(0^+)$	$\begin{bmatrix} -28.9 & 131.3 \\ 131.3 & 63.8 \end{bmatrix}$	$\begin{bmatrix} -234.3 & 131.3 \\ 131.3 & -61.4 \end{bmatrix}$
	$1/2(1^+)$	$\begin{bmatrix} 6.8 & 0.3 & 0.7 \\ 0.3 & -90.1 & -75.8 \\ 0.7 & -75.8 & -2.5 \end{bmatrix}$	$\begin{bmatrix} -198.6 & 0.3 & 0.7 \\ 0.3 & -295.5 & -75.8 \\ 0.7 & -75.8 & -127.7 \end{bmatrix}$
	$1/2(2^+)$	78.3	-127.1
	$0(0^+)$	$\begin{bmatrix} -34.8 & 132.1 \\ 132.1 & 55.7 \end{bmatrix}$	$\begin{bmatrix} -316.2 & 132.1 \\ 132.1 & -259.6 \end{bmatrix}$
$cc\bar{s}\bar{s}$	$0(1^+)$	1.1	-280.2
	$0(2^+)$	73.1	-208.3

TABLE IV. Masses (in MeV), mixing weights of the configuration $\bar{3}_c \otimes 3_c$, and magnetic moments [for $I = 0$, $I = 1/2(1/2, -1/2)$, or $I = 1(1, 0, -1)$ in the nuclear magneton] of the doubly charmed tetraquark T_{cc} with strangeness $S = 0, -1, -2$. The mass spacing dM is relative to the relevant lowest rearrangement decay two-meson threshold.

Systems	I	J^P	Masses	dM	Mixings	Magnetic moments
$cc\bar{n}\bar{n}$	0	1^+	3890 ± 13	$14(DD)$	0.95	0.65
			4244 ± 15	$368(DD)$	0.05	-0.79
	1	0^+	3981 ± 14	$246(DD^*)$	0.87	
			4374 ± 14	$636(DD^*)$	0.13	
		1^+	4066 ± 13	$190(DD^*)$	1	1.23, -0.07, -1.37
		2^+	4137 ± 13	$120(D^*D^*)$	1	2.45, -0.14, -2.73
$cc\bar{s}\bar{n}$	$\frac{1}{2}$	0^+	4054 ± 23	$218(DD_s)$	0.77	
			4368 ± 26	$532(DD_s)$	0.23	
		1^+	4034 ± 31	$57(D^*D_s)$	0.87	0.82, 0.50
			4160 ± 4	$183(D^*D_s)$	1	1.08, -0.24
			4260 ± 14	$283(D^*D_s)$	0.13	1.34, -0.91
			4232 ± 22	$111(DD_s^*)$	1	2.16, -0.43
$cc\bar{s}\bar{s}$	0	0^+	4122 ± 22	$185(D_sD_s)$	0.60	
			4392 ± 23	$455(D_sD_s)$	0.40	
		1^+	4265 ± 20	$184(D_sD_s^*)$	1	0.93
		2^+	4337 ± 20	$113(D_s^*D_s^*)$	1	1.87

$$H = B + H_{\text{CMI}} = \begin{bmatrix} -265.9 & -75.3 \\ -75.3 & 54.5 \end{bmatrix},$$

$$H_{\text{CMI}} = \begin{bmatrix} -136.5 & -75.3 \\ -75.3 & -10.2 \end{bmatrix}, \quad (32)$$

the former of which gets two eigenvalues $h_{1,2}$ and the respective eigenvectors $R_{1,2}$ for H ,

$$h_1 = -282.7, \quad R_1 = \begin{bmatrix} R_{24} \\ R_{15} \end{bmatrix} = \begin{bmatrix} -0.976 \\ -0.218 \end{bmatrix},$$

$$h_2 = 71.3, \quad R_2 = \begin{bmatrix} R_{24} \\ R_{15} \end{bmatrix} = \begin{bmatrix} 0.218 \\ -0.976 \end{bmatrix}. \quad (33)$$

With $2(m'_c + m'_n) = 2(1440 + 284.7) + 2(230 + 131.8) = 4173.0$ MeV, the substitution of two eigenvalues $h_{1,2}$ into Eq. (1), namely, $M = 2(m'_c + m'_n) + \text{Eig}(H)$, gives rise to two masses for two color-spin multiplets T_{cc} and T'_{cc} ,

$$M(T_{cc}, 01^+) = 4173.0 \text{ MeV} - 282.7 \text{ MeV} = 3890.3 \text{ MeV},$$

$$M(T'_{cc}, 01^+) = 4173.0 \text{ MeV} + 71.3 \text{ MeV} = 4244.3 \text{ MeV}. \quad (34)$$

The lower color-spin state $T_{cc}(01^+)$ is near the DD^* mass threshold $M(DD^*) = 3875.8$ MeV from the above with a spacing about 15 MeV. With Eq. (12), its wave function takes the form of

$$T_{cc}(01^+) = \left[-0.976 \left| \frac{1}{\sqrt{2}} (\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow) \right\rangle |\phi_2\rangle \right. \\ \left. - 0.218 \left| \frac{1}{\sqrt{2}} (\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow) \right\rangle |\phi_1\rangle \right] |cc[\bar{u}\bar{d}] \rangle, \quad (35)$$

in which $R_{24} = -0.976$ is the amplitude of the color configuration $\bar{3}_c \otimes 3_c$ and $R_{15} = -0.218$ is the amplitude for the $6_c \otimes \bar{6}_c$ mixed in $T_{cc}(01^+)$. This gives a mixing weight $(R_{24})^2 = 0.95$ for the exotic component $\bar{3}_c \otimes 3_c$ mixed in $T_{cc}(01^+)$.

Considering the uncertainty of about 13 MeV in our parameters, our mass prediction in Eq. (34) is not precise enough to exclude the possibility that it is a narrow resonance structure. This is similar to the situation of a tetraquark $T_{cc}(3875)^+$ having mass and width [12,13]

$$M = 3874.8 \pm 0.1 \text{ MeV}, \quad \Gamma = 0.4 \pm 0.2 \text{ MeV}, \quad (36)$$

as reported by LHCb. Here, the uncertainty consists of two parts: one is the intrinsic uncertainty stemming from the model approximation and the other is that from the parameter uncertainty. The former (about 11 MeV) is estimated from the mismatch in mass of Ξ_{cc} predicted by our model relative to the measured mass, while the latter is estimated by uncertainty stemming from that of the CMI couplings A_{ij} . For instance, the mass of $T_{cc}(3890)$ containing uncertainty is estimated to be $3890 \pm 11 \pm 2 \text{ MeV} = 3890 \pm 13 \text{ MeV}$.

Given $R_{24,15} = -0.976(-0.218)$, one has for the magnetic moments

TABLE V. Comparison of predictions for the magnetic moments (in μ_N) of the $T_{cc}^+(01^+)$ in the compact and molecular pictures.

References	Magnetic moments	Methods
Compact		
Present work	0.65	
[31]	0.88	MIT bag model
[32]	0.18	Color screening confinement + one-gluon exchange
[45]	$0.66^{+0.34}_{-0.23}$	QCD sum rule
[77]	0.73	Constituent quark model
[78]	0.28	Diffusion Monte Carlo
Molecular		
[32]	0.13	One-boson exchange
[38]	-0.09	Constituent quark model
[45]	$0.43^{+0.23}_{-0.22}$	QCD sum rule

$$\begin{aligned}\mu[T_{cc}^+, 01^+] &= 0.65\mu_N, \\ \mu[T_{cc}^{+'}, 01^+] &= -0.79\mu_N,\end{aligned}\quad (37)$$

which differs in chromodynamics: the former contributes 95% from two charm quarks, while the latter contributes 95% from the $\bar{u}$ and $\bar{d}$ antiquarks. Our prediction of the magnetic moments for the tetraquark T_{cc}^+ with $I(J^P) = 0(1^+)$ are compared to other computations cited in the compact and molecule pictures (see Table V).

Likewise, the matrix H_{CMI} and H for other higher nonstrange T_{cc} states with $I(J^P) = 1(0, 1, 2)^+$ can be computed similarly, with results for H_{CMI} and H shown in Table III. As such, the masses and mixing weights can be solved numerically; they are listed explicitly in Table IV. The magnetic moments for other higher nonstrange T_{cc} states are listed in Table IV. Clearly, the magnetic moment depends crucially on the color structure of the tetraquarks, which is helpful for revealing the internal structure of the hadrons.

2. Strange tetraquarks

For the strange tetraquark $T_{cc,s} = cc\bar{s}\bar{n}$ and $T_{cc,ss} = cc\bar{s}\bar{s}$, the masses, mixing weights, and magnetic moments are computed and listed in Table IV, based on the mass matrices in Table III. Clearly, most of the strange tetraquarks are well above the relevant two-meson thresholds. Among them, the lowest state, $T_{cc,s}(4034, (1/2)1^+)$, lies 55 ± 31 MeV above the DD_s^* mass threshold and 57 ± 31 MeV above the D^*D_s mass threshold. For its strong decay channels D_s^*D and D_sD^* , one can use Eq. (4) combined with Eq. (6) to obtain the width ratio between the two channels,

$$\frac{\Gamma(T_{cc,s}(4034, (1/2)1^+) \rightarrow D_sD^*)}{\Gamma(T_{cc,s}(4034, (1/2)1^+) \rightarrow D_s^*D)} = \frac{59.21\gamma_{D_sD^*}\alpha}{56.68\gamma_{D_s^*D}\alpha} = 1.04, \quad (38)$$

and find them to be close to each other in strong interaction.

Note that the meson D^* has a strong tendency, with largest branching fraction $\text{Br} = (67.7 \pm 0.5)\%$ [64], to decay to the final state $D\pi$. This allows the strong decay process $T_{cc,s} \rightarrow D_s(D\pi)^{J=1}$ in the P-wave. However, this channel is suppressed by the kinematic factor and thus is not the dominant decay channel for the strange $T_{cc,s}$. We suggest that the strange tetraquark $T_{cc,s}(4034, (1/2)1^+)$ is more likely to be found in the channels of $T_{cc,s} \rightarrow D_sD^*$ and D_s^*D , and this state has two magnetic moments (in μ_N), $\mu[T_{cc,s}^+, (1/2)1^+] = 0.82$ and $\mu[T_{cc,s}^+, (1/2)1^+] = 0.50$, with respect to two isospin components $I_z = 1/2, -1/2$, respectively.

B. The doubly bottom states

We consider the doubly bottom tetraquark state, denoted as $T_{bb}(M, IJ^P)$ when it has the mass M and isospin I and spin-parity J^P . In the case of the tetraquark T_{cc} , a similar procedure applies to the tetraquark $T_{bb} = bb\bar{n}\bar{n}$, which gives rise to numerical masses, mixing weights, and magnetic moments of the doubly bottom systems. The computed results for the CMI matrices and the entire Hamiltonian are shown in Table VI. The resulted masses, mixing weights, and magnetic moments of the T_{bb} tetraquarks, based on diagonalization, are listed in Table VII for the states with strangeness $S = 0, -1, -2$.

1. Nonstrange tetraquarks

In the case of nonstrange T_{bb} , the hadron mass lies in the range from 10.4 to 11.1 GeV, with a gap of about 0.65 GeV. In contrast with the doubly charmed tetraquarks, all of which are above their relevant two-meson thresholds,

TABLE VI. Computed matrices of the CMI and the entire Hamiltonian of the doubly bottom tetraquarks $bb\bar{n}\bar{n}$, $bb\bar{s}\bar{n}$, and $bb\bar{s}\bar{s}$.

Systems	$I(J^P)$	CMI matrices	Hamiltonians
$bb\bar{n}\bar{n}$	$0(1^+)$	$\begin{bmatrix} -140.7 & -23.9 \\ -23.9 & -16.6 \end{bmatrix}$	$\begin{bmatrix} -424.1 & -23.9 \\ -23.9 & 125.1 \end{bmatrix}$
	$1(0^+)$	$\begin{bmatrix} 31.2 & 41.4 \\ 41.4 & 80.6 \end{bmatrix}$	$\begin{bmatrix} -252.2 & 41.4 \\ 41.4 & 222.4 \end{bmatrix}$
	$1(1^+)$	42.5	-240.9
	$1(2^+)$	65.0	-218.4
	$1/2(0^+)$	$\begin{bmatrix} 14.9 & 42.9 \\ 42.9 & 57.5 \end{bmatrix}$	$\begin{bmatrix} -359.6 & 42.9 \\ 42.9 & -28.6 \end{bmatrix}$
$bb\bar{s}\bar{n}$	$1/2(1^+)$	$\begin{bmatrix} 26.6 & 0.6 & 1.3 \\ 0.6 & -94.3 & -24.8 \\ 1.3 & -24.8 & -8.9 \end{bmatrix}$	$\begin{bmatrix} -347.9 & 0.6 & 1.3 \\ 0.6 & -468.8 & -24.8 \\ 1.3 & -24.8 & -94.9 \end{bmatrix}$
	$1/2(2^+)$	50.0	-324.5
	$0(0^+)$	$\begin{bmatrix} 8.6 & 44.5 \\ 44.5 & 49.3 \end{bmatrix}$	$\begin{bmatrix} -457.0 & 44.5 \\ 44.5 & -264.4 \end{bmatrix}$
$bb\bar{s}\bar{s}$	$0(1^+)$	20.7	-444.8
	$0(2^+)$	45.0	-420.6

some of the doubly bottom tetraquark T_{bb} binds more strongly to lie below their relevant two-meson thresholds. The typical examples are $T_{bb}(10406, 01^+)^-$ and $T_{bb}(10613, 12^+)$, as well as the strange tetraquarks $T_{bb,s}(10547, (1/2)1^+)$, $T_{bb,s}(10693, (1/2)2^+)$, and $T_{bb,ss}(10783, 02^+)^0$, which bound deeply and are in the color rep. $\bar{3}_c \otimes 3_c$. This can be due to the enhanced deep binding B_{bb} between the bottom quarks [79] compared to the binding B_{cc} between the two charm quarks. As shown in Table VII, the binding is more deep for the isoscalar tetraquark $T_{bb}(10406, 01^+)^-$, 198 MeV below the $\bar{B}\bar{B}^*$ threshold, which makes it more stable against the strong decays. As shown in Table VIII, we have compared the predicted mass of this state to the results from other works.

To estimate the lifetime of the T_{bb} , we focus on the charged vertex $b \rightarrow cW^{*-}$, where the W^{*-} couples to the products $e\nu$, $\mu\nu$, and $\tau\nu$, or the color charges of $d\bar{u}$ and $s\bar{c}$ [52]. We consider results from the initial state to the lowest state $T_{bb}(10406, 01^+)$ so that $M_i = 10406$ MeV. The final state is $\bar{B}D$, with the mass $M_f = M(\bar{B}) + M(D) = 7147$ MeV. Applying Eq. (29) to $T_{bb}(10406, 01^+)$ decaying to $\bar{B}D$ gives the weak decay width

$$\begin{aligned} \Gamma(T_{bb}(10406, 01^+)^-) &= \frac{18G_F^2 M_i^5}{192\pi^3} F(x_{\bar{B}D}) |V_{cb}|^2 \\ &= 2.02 \times 10^{-12} \text{ GeV}, \end{aligned} \quad (39)$$

where the matrix element $|V_{cb}| = 0.042$ [64], the Fermi constant $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ [64], $x_{\bar{B}D} = (M_f/M_i)^2 = 0.472$, and $F(x_{\bar{B}D}) = 0.023$ via Eq. (30). This amounts to a lifetime of 326 fs for $T_{bb}(10406, 01^+)$, which is consistent with the predicted results from Refs. [79,94].

To find the dominant decay channels, we consider the following characteristics of the weak decays:

- (i) Weak processes involving the charged vertex $b \rightarrow cW^{*-}$. In this situation, the processes $T_{bb}(10406, 01^+)^- \rightarrow \bar{B}^0 D^0 \pi^-$ or $B^- D^+ \pi^-$ are more likely to occur, suggesting that we search for the T_{bb}^- structure in these two channels. Other relevant processes contain the decay processes $T_{bb}(10406, 01^+)^- \rightarrow J/\psi B^- \bar{K}^0$, $J/\psi \bar{B}^0 K^-$, $J/\psi J/\psi K^- \bar{K}^0$, and $D^0 D^+ \pi^- \pi^-$, and the estimates of their branching fractions will not exceed that of the aforementioned two processes.
- (ii) Alternative channel involving the scattering between the $\bar{d}$ and b quarks, which is associated with the mesons $\bar{B}/D$. In the heavy-light systems, however, the spatial wave function $\lim_{x_i \rightarrow x_j} |\psi_{ij}(x_i - x_j)|^2$ overlap of these two quarks is quite small; thus, it is difficult for the decay channels $T_{bb}(10406, 01^+)^- \rightarrow J/\psi B^-$ and $B^- D^0$ to occur and they cannot be the dominant decay channels.

2. Strange tetraquark

In the case of the strange state $T_{bb,s} = bb\bar{s}\bar{n}$, the mass range is from 10.5 to 11.0 GeV, with a mass gap of 0.45 GeV, whereas for the strange $T_{bb,ss} = bb\bar{s}\bar{s}$, the mass range is from 10.7 to 10.9 GeV, with a mass gap of 0.21 GeV. Clearly, as the number of strange quarks increases, the mass gap of the T_{bb} system becomes significantly narrower. Among these strange states, there is an analog of the state $T_{bb}(10406)^-$, denoted as $T_{bb,s}(10547, (1/2)1^+)$, which lies about 145 MeV below

TABLE VII. Masses (in MeV), mixing weights of the configuration $\bar{3}_c \otimes 3_c$, and magnetic moments [for $I = 0$, $I = 1/2(1/2, -1/2)$, or $I = 1(1, 0, -1)$ in the nuclear magneton] of the doubly bottom tetraquark T_{bb} with strangeness $S = 0, -1, -2$. The mass spacing dM is relative to the relevant lowest rearrangement decay two-meson threshold.

Systems	I	J^P	Masses	dM	Mixings	Magnetic moments
$bb\bar{n}\bar{n}$	0	1^+	10406 ± 17	$-198(\bar{B}\bar{B}^*)$	1	-0.13
			10957 ± 18	$353(\bar{B}\bar{B}^*)$		-0.86
	1	0^+	10576 ± 17	$17(\bar{B}\bar{B})$	0.99	
			11057 ± 18	$498(\bar{B}\bar{B})$	0.01	
		1^+	10590 ± 17	$-14(\bar{B}\bar{B}^*)$	1	0.80, -0.49, -1.79
		2^+	10613 ± 17	$-37(\bar{B}^*\bar{B}^*)$	1	1.61, -0.99, -3.58
$bb\bar{s}\bar{n}$	$\frac{1}{2}$	0^+	10652 ± 26	$5(\bar{B}\bar{B}_s)$	0.98	
			10994 ± 30	$347(\bar{B}\bar{B}_s)$	0.02	
		1^+	10547 ± 7	$-145(\bar{B}^*\bar{B}_s)$	1	-0.12, -0.11
			10669 ± 26	$-23(\bar{B}^*\bar{B}_s)$	1	0.66, -0.66
			10924 ± 15	$232(\bar{B}^*\bar{B}_s)$		1.43, -1.15
			10693 ± 26	$-47(\bar{B}^*\bar{B}_s^*)$	1	1.31, -1.28
$bb\bar{s}\bar{s}$	0	0^+	10736 ± 25	$2(\bar{B}_s\bar{B}_s)$	0.95	
			10949 ± 28	$215(\bar{B}_s\bar{B}_s)$	0.05	
		1^+	10758 ± 24	$-24(\bar{B}_s\bar{B}_s^*)$	1	0.51
		2^+	10783 ± 24	$-48(\bar{B}_s^*\bar{B}_s^*)$	1	1.02

TABLE VIII. Comparison of predictions for $M(T_{bb}^-)$. $M(\bar{B}\bar{B}^*) = 10604$ MeV. $M(\bar{B}\bar{B}\pi) = 10699$ MeV. NBS, Nambu-Bethe-Salpeter.

References	Masses (MeV)	Method
Present work	10406 ± 17	
[31]	10654	MIT bag model
[50]	10298	Chromomagnetic interactions model
[58]	10686	Same as above
[79]	10389 ± 12	Quark model
[80]	10552	BO bound
[81]	10466(10448)	Hydrogen bond of QCD
[82]	10482	Heavy-quark symmetry relations
[83]	10530	Regge relation + mass scaling
[84]	10502	Relativistic quark model
[85]	10471 ± 25	Color-Coulomb potential + heavy-quark limit
[86]	10550	Relativistic quark model
[87]	10200 ± 300	QCD sum rule
[88]	10520 ± 80	Same as above
[89]	10426	Constituent quark model
[90]	10261	Chiral constituent quark model
[91]	10489(17)	Lattice QCD
[92]	$10504 \pm 10^{+36}_{-51}$	Lattice QCD
[93]	$10449 \pm 17 \pm 20$	Lattice QCD + NBS wave function

the $\bar{B}^*\bar{B}_s$ threshold and is stable against both the strong and electromagnetic decays. It can decay via the standard weak process $b \rightarrow cW^{*-}$ to the three color singlets of $\bar{c}s$ and $\bar{u}d$, as well as those of $e\nu$, $\mu\nu$, and $\tau\nu$.

Assuming the initial-state mass to be $M_i = 10547$ MeV and the final-state mass to be $M_f = M(\bar{B}_s) + M(D) = 7234$ MeV, which gives the kinematic suppression factor $x_{\bar{B}_s D} = 0.47$, one can estimate the total width to be

$$\begin{aligned}\Gamma(T_{bb,s}(10547, (1/2)1^+)) &= \frac{18G_F^2 M_l^5}{192\pi^3} F(x_{\bar{B},D}) |V_{cb}|^2 \\ &= 2.19 \times 10^{-12} \text{ GeV},\end{aligned}\quad (40)$$

with the factor of 2 accounting for the two bottom quarks. From this, we can roughly estimate its lifetime of the $T_{bb,s}(10547, (1/2)1^+)$, which is 301 fs. Here, we have taken the small Cabibbo-Kobayashi-Maskawa (CKM) matrix element $V_{ub} = 0$, which does not significantly shorten the predicted lifetime of $T_{bb,s}^-$.

C. The bottom-charmed states

Now let us consider the bottom-charmed tetraquarks, denoted as $T_{cb}(M, IJ^P)$ when it has the mass M and quantum number IJ^P . The procedure stated in Sec. II applies for exploring the bottom-charmed tetraquarks $T_{cb} = bc\bar{n}\bar{n}$. As such, having the CMI matrices and the entire Hamiltonian for them computed, as shown in Table IX, one can compute numerical masses, mixing weights, and magnetic moments of these bottom-charmed systems, with the results listed in Table X. As we see, the bottom-charmed tetraquark mass is 7.14–7.69 GeV in the nonstrange T_{cb} states region, 7.37–7.86 GeV in the strange

$T_{cb,s}$ states region, and 7.66–8.03 GeV in the doubly strange $T_{cb,ss}$ states region.

1. Nonstrange tetraquarks

As shown in Table X, most of the tetraquark T_{cb} unbind as they lie above the two-meson threshold $M(\bar{B}^*D) = 7192$ MeV. Two exceptions are the $0(0^+)$ state at 7142 MeV and the $0(1^+)$ state at 7173 MeV, which are likely to be loosely bound states similar to the T_{cc} states. The mass spacing $dM = M(T_{cb}) - M(D\bar{B}^{(*)})$ of two states relative to $D\bar{B}^{(*)}$ are

$$\begin{aligned}dM(T_{cb}, 00^+) &= -5 \pm 13 \text{ MeV}, \\ dM(T_{cb}, 01^+) &= -19 \pm 16 \text{ MeV},\end{aligned}\quad (41)$$

where the uncertainty stems largely from that due to the model parameters.

Our computations imply that the binding depths of the T_{cc} , T_{cb} , and T_{bb} obey $T_{cc} < T_{cb} < T_{bb}$ in size (see Fig. 2), which is consistent with the values of the binding energy $B_{QQ'}$ in Eq. (22). One anticipates then the state $T_{cb}(7173, 01^+)^0$ with a mass of 7173 ± 16 MeV to be a state below the $\bar{B}^*D$ mass threshold. For the below-threshold binding of the T_{cb} and the T_{bb} , one may expect that two

TABLE IX. The computed matrices of the CMI and the entire Hamiltonian of the bottom-charmed tetraquarks $bc\bar{n}\bar{n}$, $bc\bar{s}\bar{n}$, and $bc\bar{s}\bar{s}$.

Systems	$I(J^P)$	CMI matrices	Hamiltonians
$bc\bar{n}\bar{n}$	$0(0^+)$	$\begin{bmatrix} -160.0 & 85.9 \\ 85.9 & -143.6 \end{bmatrix}$	$\begin{bmatrix} -336.4 & 85.9 \\ 85.9 & -55.4 \end{bmatrix}$
	$0(1^+)$	$\begin{bmatrix} -141.2 & 36.4 & -49.6 \\ 36.4 & -85.1 & 42.9 \\ -49.6 & 42.9 & -17.2 \end{bmatrix}$	$\begin{bmatrix} -317.5 & 36.4 & -49.6 \\ 36.4 & 3.1 & 42.9 \\ -49.6 & 42.9 & 71.0 \end{bmatrix}$
	$0(2^+)$	31.8	120.0
	$1(0^+)$	$\begin{bmatrix} 6.6 & 85.9 \\ 85.9 & 80.0 \end{bmatrix}$	$\begin{bmatrix} -169.8 & 85.9 \\ 85.9 & 168.2 \end{bmatrix}$
	$1(1^+)$	$\begin{bmatrix} 30.0 & 17.1 & 36.4 \\ 17.1 & 34.5 & -49.6 \\ 36.4 & -49.6 & 70.6 \end{bmatrix}$	$\begin{bmatrix} -146.4 & 17.1 & 36.4 \\ 17.1 & -141.9 & -49.6 \\ 36.4 & -49.6 & 158.8 \end{bmatrix}$
	$1(2^+)$	76.7	-99.7
	$1/2(0^+)$	$\begin{bmatrix} -9.5 & -0.2 & 0.4 & 87.1 \\ -0.2 & -113.6 & 87.1 & 0.0 \\ 0.4 & 87.1 & -137.5 & -0.4 \\ 87.1 & 0.0 & -0.4 & 56.8 \end{bmatrix}$	$\begin{bmatrix} -269.5 & -0.2 & 16.4 & 87.1 \\ -0.2 & -373.5 & 87.1 & 16.0 \\ 16.4 & 87.1 & -258.1 & -0.4 \\ 87.1 & 16.0 & -0.4 & -63.8 \end{bmatrix}$
$bc\bar{s}\bar{n}$	$1/2(1^+)$	$\begin{bmatrix} 14.2 & 0.5 & 17.0 & 0.2 & 36.1 & 1.0 \\ 0.5 & -94.8 & 0.1 & 36.1 & 0.0 & -50.3 \\ 17.0 & 0.1 & 19.0 & 0.9 & -50.3 & 0.0 \\ 0.2 & 36.1 & 0.9 & -78.2 & 1.2 & 42.5 \\ 36.1 & 0.0 & -50.3 & 1.2 & 47.4 & 0.2 \\ 1.0 & -50.3 & 0.0 & 42.5 & 0.2 & -9.5 \end{bmatrix}$	$\begin{bmatrix} -245.8 & 0.5 & 17.0 & 16.2 & 36.1 & 1.0 \\ 0.5 & -354.7 & 0.1 & 36.1 & 16.0 & -50.3 \\ 17.0 & 0.1 & -240.9 & 0.9 & -50.3 & 16.0 \\ 16.2 & 36.1 & 0.9 & -198.9 & 1.2 & 42.5 \\ 36.1 & 16.0 & -50.3 & 1.2 & -73.3 & 0.2 \\ 1.0 & -50.3 & 16.0 & 42.5 & 0.2 & -130.2 \end{bmatrix}$
	$1/2(2^+)$	$\begin{bmatrix} 61.6 & -0.2 \\ -0.2 & 40.3 \end{bmatrix}$	$\begin{bmatrix} -198.3 & 15.8 \\ 15.8 & -80.3 \end{bmatrix}$
	$0(0^+)$	$\begin{bmatrix} -15.7 & 88.3 \\ 88.3 & 48.6 \end{bmatrix}$	$\begin{bmatrix} -359.1 & 88.3 \\ 88.3 & -280.8 \end{bmatrix}$
	$0(1^+)$	$\begin{bmatrix} 8.4 & 16.9 & 35.8 \\ 16.9 & 13.6 & -51.0 \\ 35.8 & -51.0 & 39.2 \end{bmatrix}$	$\begin{bmatrix} -335.1 & 16.9 & 35.8 \\ 16.9 & -329.9 & -51.0 \\ 35.8 & -51.0 & -290.3 \end{bmatrix}$
$bc\bar{s}\bar{s}$	$0(2^+)$	56.5	-287.0

TABLE X. Masses (in MeV), mixing weights of the configuration $\bar{3}_c \otimes 3_c$, and magnetic moments [for $I = 0$, $I = 1/2(1/2, -1/2)$, or $I = 1(1, 0, -1)$ in the nuclear magneton] of the bottom-charmed tetraquark T_{cb} with strangeness $S = 0, -1$, and -2 . The mass spacing dM is relative to the relevant lowest rearrangement decay two-meson threshold.

Systems	I	J^P	Masses	dM	Mixings	Magnetic moments
$bc\bar{n}\bar{n}$	0	0^+	7142 ± 13	$-5(\bar{B}D)$	0.93	
			7471 ± 14	$324(\bar{B}D)$	0.07	
		1^+	7173 ± 16	$-19(\bar{B}^*D)$	0.97	0.26
			7494 ± 15	$302(\bar{B}^*D)$	0.03	-0.56
			7596 ± 16	$404(\bar{B}^*D)$	0.01	-0.54
			7622 ± 14	$289(\bar{B}^*D^*)$		-0.56
	1	0^+	7312 ± 16	$165(\bar{B}D)$	0.95	
			7691 ± 17	$544(\bar{B}D)$	0.05	
		1^+	7330 ± 14	$138(\bar{B}^*D)$	0.97	1.04, -0.83, -2.69
			7375 ± 14	$183(\bar{B}^*D)$	1	1.67, -0.28, -2.23
			7672 ± 14	$480(\bar{B}^*D)$	0.03	0.33, 0.26, 0.19
			7402 ± 16	$69(\bar{B}^*D^*)$	1	2.03, -0.56, -3.16
		2^+				
$bc\bar{s}\bar{n}$	$\frac{1}{2}$	0^+	7266 ± 8	$32(\bar{B}_sD)$	0.77	
			7387 ± 24	$153(\bar{B}_sD)$	0.89	
			7477 ± 8	$243(\bar{B}_sD)$	0.22	
			7657 ± 28	$423(\bar{B}_sD)$	0.12	
		1^+	7310 ± 7	$27(\bar{B}_s^*D)$	0.88	0.34, 0.33
			7409 ± 22	$126(\bar{B}_s^*D)$	0.92	0.83, -1.17
			7457 ± 20	$174(\bar{B}_s^*D)$	0.92	1.44, -0.66
			7490 ± 11	$207(\bar{B}_s^*D)$	0.16	0.72, -0.58
			7584 ± 13	$301(\bar{B}_s^*D)$	0.03	1.53, -0.73
			7635 ± 26	$352(\bar{B}_s^*D)$	0.09	0.36, 0.24
		2^+	7488 ± 24	$64(\bar{B}_s^*D^*)$	0.98	1.74, -0.86
			7610 ± 12	$186(\bar{B}_s^*D^*)$	0.02	1.74, -0.86
$bc\bar{s}\bar{s}$	0	0^+	7457 ± 24	$122(\bar{B}_sD_s)$	0.70	
			7651 ± 26	$316(\bar{B}_sD_s)$	0.30	
		1^+	7486 ± 21	$102(\bar{B}_s^*D_s)$	0.71	0.56
			7557 ± 21	$173(\bar{B}_s^*D_s)$	0.99	1.16
			7624 ± 23	$240(\bar{B}_s^*D_s)$	0.30	0.45
		2^+	7587 ± 23	$59(\bar{B}_s^*D_s^*)$	1	1.44

heavy quarks form a core with size of about 0.3 fm or below, quite smaller than the size of hadrons and different in size depending on the flavor contents [82]. It has also been suggested [79] that the interquark potential is singular at the origin, leading to a rapid increase in the QQ' binding energy as the distance decreases.

To estimate the lifetime of the loosely bound state $T_{cb}(7173, 01^+)^0$ in Table X, one has to consider, in addition to the contributions from weak decays, the contributions from electromagnetic transition. The initial state $T_{cb}(7173, 1^+)^0$ decays into the final state $T_{cb}(0^+)^0\gamma$, assuming the T_{cb}^0 to have a zero lifetime [see Eq. (41), and the $0(0^+)$ state is more loosely bound than the $0(1^+)$ state]. For the state $T_{cb}(7173, 1^+)^0$, one can first extract the mixing coefficients directly from the solved eigenvectors with respect to the eigenvalue 7173 MeV in Table IX for the Hamiltonian of the system $bc\bar{n}\bar{n}$ with the quantum number $0(1^+)$. The numerical results for these coefficients (in

Table X, we omit them and all others for brevity) are

$$(R_{26}, R_{13}) = (-0.96, 0.27),$$

$$(R_{24}, R_{12}, R_{15}) = (0.98, -0.13, 0.14). \quad (42)$$



Distances from threshold for T_{bb}^- , T_{cb}^0 , and T_{cc}^+ [MeV]

FIG. 2. Distances relative to the two-meson thresholds with $\bar{B}\bar{B}^*$, $D\bar{B}^*$, and DD^* are shown for the T_{bb}^- , T_{cb}^0 , and T_{cc}^+ states (green lines).

As stated at the beginning of Sec. III, one can employ the wave function of the $T_{cb}(7173, 1^+)^0$ to derive the transition magnetic moment

$$\begin{aligned} \mu[T_{cb}(7173, 1^+)^0 \rightarrow T_{cb}(0^+)^0] &= \langle \psi_2^{J=0, J_z=0} | \hat{\mu}_{\text{TOT}} | \psi_5^{J=1, J_z=0} \rangle \\ &= R_{26}R_{24}(\mu_c - \mu_b) + \frac{R_{13}R_{15}}{\sqrt{3}}(\mu_b - \mu_c) \\ &\quad + \frac{R_{13}R_{12}}{\sqrt{3}}\sqrt{2}(\mu_b + \mu_c - \mu_{\bar{u}} - \mu_{\bar{d}}) \\ &= -0.443. \end{aligned} \quad (43)$$

Given the initial- and final-state masses $M_i = 7173$ MeV and $M_f = M(\bar{B}) + M(D) = 7147$ MeV, the momentum of the outgoing photon $\omega = 26$ MeV, as well as other parameters given above, one can apply Eq. (31) to get the M1 decay width

$$\Gamma(T_{cb}(7173, 1^+)^0 \rightarrow T_{cb}(0^+)^0 \gamma) = 9.58 \times 10^{-9} \text{ GeV}. \quad (44)$$

Next, we consider the weak decays associated with the charged vertex $c \rightarrow sW^{*+}$, where W^{*+} couples to $e\nu$, $\mu\nu$, and three colors of $u\bar{d}$. In this situation, the transitions $T_{cb}(7173, 01^+)^0 \rightarrow B^-\bar{K}^0\pi^+$ and $\bar{B}^0K^-\pi^+$ are more likely to occur, which suggests searching for the T_{cb}^0 in these two channels. Using the initial- and final-state masses $M_i = 7173$ MeV and $M_f = M(\bar{B}) + M(K) = 5775$ MeV, the squared ratio $x_{\bar{B}K} = 0.648$ of the two masses, and the CKM matrix element $|V_{cs}| = 0.973$ [64], one has

$$\begin{aligned} \Gamma(T_{cb}(7173, 01^+)^0) &= \frac{5G_F^2 M_i^5}{192\pi^3} F(x_{\bar{B}K}) |V_{cs}|^2 \\ &= 5.40 \times 10^{-12} \text{ GeV}. \end{aligned} \quad (45)$$

A combination of the contributions from Eqs. (44) and (45) yields the lifetime of $T_{cb}(7173, 01^+)^0$, which is about 0.069 fs. This lifetime is much shorter than that of the states $T_{bb}(10406, 01^+)^-$ and $T_{bb,s}(10547, (1/2)1^+)$. Note that for states that allow electromagnetic decays, the contribution of weak decay processes to lifetime becomes negligible (4 orders of magnitude smaller). Moreover, the decays associated with charged vertices which we have not yet considered include those associated with transitions $b \rightarrow cW^{*-}$, $c \rightarrow dW^{*+}$, and $bc \rightarrow cs$. The latter transitions correspond to channels like $D^0\bar{K}^0$, D^+K^- , and $\bar{B}^0\pi^+\pi^-$, which contribute negligibly to the lifetime of the T_{cb} state.

There are some multiplets, such as $T_{cb}(7330, 11^+)$ and $T_{cb}(7375, 11^+)$, which are near each other in weight. The ratios of the strong decay partial widths of them with respect to different channels are

$$\frac{\Gamma(T_{cb}(7330, 11^+) \rightarrow \bar{B}D^*)}{\Gamma(T_{cb}(7330, 11^+) \rightarrow \bar{B}^*D)} = \frac{0.09\gamma_{\bar{B}D^*}\alpha}{194.75\gamma_{\bar{B}^*D}\alpha} = 0.0005, \quad (46)$$

$$\frac{\Gamma(T_{cb}(7375, 11^+) \rightarrow \bar{B}D^*)}{\Gamma(T_{cb}(7375, 11^+) \rightarrow \bar{B}^*D)} = \frac{129.32\gamma_{\bar{B}D^*}\alpha}{3.33\gamma_{\bar{B}^*D}\alpha} = 38.83. \quad (47)$$

Clearly, they differ significantly, by 3 orders of magnitude, and are then easy to distinguish in experiments. We suggest searching for these two states in the decay channels of $\bar{B}D^*$ and $\bar{B}^*D$.

2. Strange tetraquarks

In the case of the strange tetraquark $T_{cb,s} = bc\bar{s}\bar{n}$ and $T_{cb,ss} = bc\bar{s}\bar{s}$, the masses and magnetic moments are similarly computable; they are numerically listed in Table X. In the case of the bottom-charmed tetraquark states with one or two strange quarks, all states are above their respective mass thresholds. There are three multiplets with masses near 7409, 7457, and 7490 MeV. Similar computations give the partial width ratios for the different channels shown below,

$$\frac{\Gamma(T_{cb,s}(7409, (1/2)1^+) \rightarrow \bar{B}D_s^*)}{\Gamma(T_{cb,s}(7409, (1/2)1^+) \rightarrow \bar{B}^*D_s)} = \frac{2.20\gamma_{\bar{B}D_s^*}\alpha}{171.20\gamma_{\bar{B}^*D_s}\alpha} = 0.013, \quad (48)$$

$$\frac{\Gamma(T_{cb,s}(7457, (1/2)1^+) \rightarrow \bar{B}D_s^*)}{\Gamma(T_{cb,s}(7457, (1/2)1^+) \rightarrow \bar{B}^*D_s)} = \frac{46.16\gamma_{\bar{B}D_s^*}\alpha}{4.50\gamma_{\bar{B}^*D_s}\alpha} = 10.26, \quad (49)$$

$$\frac{\Gamma(T_{cb,s}(7490, (1/2)1^+) \rightarrow \bar{B}D_s^*)}{\Gamma(T_{cb,s}(7490, (1/2)1^+) \rightarrow \bar{B}^*D_s)} = \frac{157.02\gamma_{\bar{B}D_s^*}\alpha}{324.48\gamma_{\bar{B}^*D_s}\alpha} = 0.484. \quad (50)$$

For the $\bar{B}_s D^*$ and $\bar{B}_s^* D$ channels, the ratios corresponding to the states $T_{cb,s}(7409, (1/2)1^+)$, $T_{cb,s}(7457, (1/2)1^+)$, and $T_{cb,s}(7490, (1/2)1^+)$, as shown in Eqs. (48)–(50), become 0.002, 1.907, and 0.472, respectively. These ratios clearly differ and are helpful for identifying or distinguishing these states in experiments.

IV. CONCLUSIONS AND REMARKS

This work is devoted to systematically study the main static properties and inner structure of nonstrange and strange tetraquarks with two heavy quarks within the picture of a QCD string with a chromomagnetic interaction. The masses, magnetic moments, and color-spin structures are computed for all DH (doubly charmed, doubly bottom, and bottom-charmed) tetraquarks, and the hadron lifetime is estimated for those tetraquarks that are doubly bottom and bottom charmed and lie below the relevant two-meson thresholds by analyzing the weak decay and the radiative decay. The two-body formula is applied to explore the relative strong decay widths of the doubly charmed and bottom-charmed tetraquarks.

Our mass analysis indicates that all nonstrange doubly charmed tetraquarks T_{cc} are above the relevant two-meson thresholds and are unstable against strong decays to two-meson dissociation. There is a relatively stable state among the doubly charmed tetraquarks with strangeness values of -1 and -2 , which is the lowest state, $T_{cc,s}(4034, (1/2)1^+)$, 57 ± 31 MeV above the D^*D_s mass threshold. We anticipate that the strange tetraquark $T_{cc,s}(4034, (1/2)1^+)$ will more likely be discovered in the channels of $T_{cc,s} \rightarrow D_s D^*$ and $D_s^* D$. In contrast, some of the doubly bottom tetraquark T_{bb} binds more strongly to lie below their relevant two-meson thresholds, which are the nonstrange state $T_{bb}(10406, 01^+)^-$ and the strange state $T_{bb,s}(10547, (1/2)1^+)$, all in the color rep. $\bar{3}_c \otimes 3_c$. We find that two doubly bottom states, $T_{bb}(10406, 01^+)^-$ and $T_{bb,s}(10547, (1/2)1^+)$, are stable against the radiative decay and strong decay, with lifetimes of 326 and 301 fs, respectively.

In the bottom-charmed sector of the DH tetraquarks T_{cb} , there is only one tetraquark, a nonstrange state $T_{cb}(7173)^0$ found to lie below the $\bar{B}^*D$ threshold, that is therefore stable against strong interactions of two-meson dissociation. We analyze the decay behaviors of this tetraquarks via weak and electromagnetic interactions and find that the nonstrange state $T_{cb}(7173)^0$ has a lifetime of 0.069 fs.

Two color-spin multiplets of the doubly charmed tetraquark T_{cc} are predicted, namely, the lower state $T_{cc}(3890.3, 01^+)$ at 3890.3 MeV, with mixing weight $(R_{24})^2 = 0.95$ for the exotic component $\bar{3}_c \otimes 3_c$, and the higher state $T'_{cc}(4244.3, 01^+)$ at 4244.3 MeV. The lower one is consistent with the LHCb-reported tetraquark $T_{cc}(3875)^+$ with 01^+ , which is close to the DD^* mass threshold $M(DD^*) = 3875.8$ MeV. Furthermore, two heavy-flavor counterparts of $T_{cc}(3875)^+$, namely, $T_{bb}^-(01^+)$ and $T_{cb}^0(01^+)$, are found to lie below the two-meson thresholds. We suggest experimental searches for these two states in the decay channels $T_{bb}^- \rightarrow \bar{B}^0 D^0 \pi^-$ or $B^- D^+ \pi^-$ and $T_{cb}^0 \rightarrow B^- D^+ \gamma$ or $\bar{B}^0 D^0 \gamma$, respectively.

The calculated chromomagnetic mixing of the color-spin states of the DH tetraquarks implies that the mixing is deep for the doubly charmed and bottom-charmed tetraquarks, especially for the strange case of these systems. We find that the magnetic moments are sensitive to chromomagnetic mixing of the color-spin states of these tetraquarks, suggesting a valuable probe to experimentally detect the nontrivial color configuration $6_c \otimes \bar{6}_c$.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: WAVE FUNCTIONS OF TETRAQUARKS

For numerical calculations of masses, mixing weights, and magnetic moments, it is essential to employ color-spin wave functions to describe the chromomagnetic structure of a given tetraquark state. In the case of a ground-state tetraquark, its total angular momentum should be determined entirely by the intrinsic angular momentum of its constituent quarks. For the spin configuration of the DH tetraquark $QQ\bar{q}\bar{q}$, the possible wave functions are

$$\begin{aligned}
 \chi_1^{2,2} &= \uparrow\uparrow\uparrow\uparrow, \\
 \chi_2^{1,1} &= \frac{1}{2}(\uparrow\uparrow\uparrow\downarrow + \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow), \\
 \chi_3^{0,0} &= \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow\downarrow + \downarrow\downarrow\uparrow\uparrow) \\
 &\quad - \frac{1}{2\sqrt{3}}(\uparrow\downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow), \\
 \chi_4^{1,1} &= \frac{1}{\sqrt{2}}(\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow), \\
 \chi_5^{1,1} &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow), \\
 \chi_6^{0,0} &= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow), \tag{A1}
 \end{aligned}$$

where the up and down arrows denote the z components of the quark spins.

Based on the color $SU(3)_c$ symmetry, one can obtain two combinations of color singlets $6_c \otimes \bar{6}_c$ and $\bar{3}_c \otimes 3_c$ for DH tetraquarks, that is,

$$\begin{aligned}
 \phi_1 &= \frac{1}{\sqrt{6}}(rr\bar{r}\bar{r} + gg\bar{g}\bar{g} + bb\bar{b}\bar{b}) \\
 &\quad + \frac{1}{2\sqrt{6}}(rb\bar{b}\bar{r} + br\bar{b}\bar{r} + gr\bar{g}\bar{r} + rg\bar{g}\bar{r} + gb\bar{b}\bar{g} + bg\bar{b}\bar{g} + gr\bar{r}\bar{g} + rg\bar{r}\bar{g} + gb\bar{g}\bar{b} + bg\bar{g}\bar{b} + rb\bar{r}\bar{b} + br\bar{r}\bar{b}), \\
 \phi_2 &= \frac{1}{2\sqrt{3}}(rb\bar{b}\bar{r} - br\bar{b}\bar{r} - gr\bar{g}\bar{r} + rg\bar{g}\bar{r} + gb\bar{b}\bar{g} - bg\bar{b}\bar{g} + gr\bar{r}\bar{g} - rg\bar{r}\bar{g} - gb\bar{g}\bar{b} + bg\bar{g}\bar{b} - rb\bar{r}\bar{b} + br\bar{r}\bar{b}). \tag{A2}
 \end{aligned}$$

We can combine the flavor, color, and spin wave functions provided that the constraints from the Pauli principle are applied. In the diquark-antidiquark picture, there are 12 possible bases for the wave function,

$$\begin{aligned}
\phi_1 \chi_1^{2,2} &= |[Q_1 Q_2]_1^{\bar{6}_c} [\bar{q}_3 \bar{q}_4]_1^{\bar{6}_c} \rangle_2, & \phi_2 \chi_1^{2,2} &= |\{Q_1 Q_2\}_1^{\bar{3}_c} \{\bar{q}_3 \bar{q}_4\}_1^{\bar{3}_c} \rangle_2, \\
\phi_1 \chi_2^{1,1} &= |[Q_1 Q_2]_1^{\bar{6}_c} [\bar{q}_3 \bar{q}_4]_1^{\bar{6}_c} \rangle_1, & \phi_2 \chi_2^{1,1} &= |\{Q_1 Q_2\}_1^{\bar{3}_c} \{\bar{q}_3 \bar{q}_4\}_1^{\bar{3}_c} \rangle_1, \\
\phi_1 \chi_3^{0,0} &= |[Q_1 Q_2]_1^{\bar{6}_c} [\bar{q}_3 \bar{q}_4]_1^{\bar{6}_c} \rangle_0, & \phi_2 \chi_3^{0,0} &= |\{Q_1 Q_2\}_1^{\bar{3}_c} \{\bar{q}_3 \bar{q}_4\}_1^{\bar{3}_c} \rangle_0, \\
\phi_1 \chi_4^{1,1} &= |[Q_1 Q_2]_1^{\bar{6}_c} [\bar{q}_3 \bar{q}_4]_0^{\bar{6}_c} \rangle_1, & \phi_2 \chi_4^{1,1} &= |\{Q_1 Q_2\}_1^{\bar{3}_c} [\bar{q}_3 \bar{q}_4]_0^{\bar{3}_c} \rangle_1, \\
\phi_1 \chi_5^{1,1} &= |\{Q_1 Q_2\}_0^{\bar{6}_c} [\bar{q}_3 \bar{q}_4]_1^{\bar{6}_c} \rangle_1, & \phi_2 \chi_5^{1,1} &= |[Q_1 Q_2]_0^{\bar{3}_c} \{\bar{q}_3 \bar{q}_4\}_1^{\bar{3}_c} \rangle_1, \\
\phi_1 \chi_6^{0,0} &= |\{Q_1 Q_2\}_0^{\bar{6}_c} \{\bar{q}_3 \bar{q}_4\}_0^{\bar{6}_c} \rangle_0, & \phi_2 \chi_6^{0,0} &= |[Q_1 Q_2]_0^{\bar{3}_c} [\bar{q}_3 \bar{q}_4]_0^{\bar{3}_c} \rangle_0,
\end{aligned} \tag{A3}$$

where the symbol $|(\cdots)_{\text{spin}}^{\text{color}} (\cdots)_{\text{spin}}^{\text{color}} \rangle_{\text{spin}}$ represents the wave function with specified color and spin configurations and the brackets $\{\cdots\}$ and $[\cdots]$ denote the symmetric and antisymmetric diquarks, respectively. One can combine these basis vectors to construct the wave functions of the DH tetraquarks for all possible quantum numbers (as listed in Table XI). In addition, when discussing the decay behavior of these states, the following two sets of wave functions are useful (in this work). They consist of the color wave functions $8_c \otimes 8_c$ and $1_c \otimes 1_c$ given by

$$\begin{aligned}
\phi'_1 &= \frac{1}{3\sqrt{2}}(r\bar{r}r\bar{r} + b\bar{b}b\bar{b} + g\bar{g}g\bar{g}) + \frac{1}{2\sqrt{2}}(b\bar{r}r\bar{b} + r\bar{b}b\bar{r} + g\bar{r}r\bar{g} + r\bar{g}g\bar{r} + g\bar{b}b\bar{g} + b\bar{g}g\bar{b}) \\
&\quad - \frac{1}{6\sqrt{2}}(r\bar{r}g\bar{g} + g\bar{g}r\bar{r} + b\bar{b}g\bar{g} + g\bar{g}b\bar{b} + b\bar{b}r\bar{r} + r\bar{r}b\bar{b}), \\
\phi'_2 &= \frac{1}{3}(r\bar{r}r\bar{r} + b\bar{b}b\bar{b} + g\bar{g}g\bar{g} + r\bar{r}b\bar{b} + r\bar{r}g\bar{g} + b\bar{b}r\bar{r} + b\bar{b}g\bar{g} + g\bar{g}r\bar{r} + g\bar{g}b\bar{b}),
\end{aligned} \tag{A4}$$

as well as the three spin wave functions

$$\begin{aligned}
\chi_2^{1,0} &= \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\uparrow), \\
\chi_4^{1,0} &= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow), \\
\chi_5^{1,0} &= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow).
\end{aligned} \tag{A5}$$

TABLE XI. The wave function that we constructed for the DH tetraquarks with given flavor, color, and spin configurations. The mixing coefficient R_{ab} is consistent with that in Sec. II B.

0^+	$\psi_1^{0,0} = R_{23}\phi_2\chi_3^{0,0} + R_{16}\phi_1\chi_6^{0,0}$ $\psi_2^{0,0} = R_{26}\phi_2\chi_6^{0,0} + R_{13}\phi_1\chi_3^{0,0}$ $\psi_3^{0,0} = R_{23}\phi_2\chi_3^{0,0} + R_{26}\phi_2\chi_6^{0,0} + R_{13}\phi_1\chi_3^{0,0} + R_{16}\phi_1\chi_6^{0,0}$
1^+	$\psi_1^{1,1} = R_{22}\phi_2\chi_2^{1,1}$ $\psi_2^{1,1} = R_{24}\phi_2\chi_4^{1,1} + R_{15}\phi_1\chi_5^{1,1}$ $\psi_3^{1,1} = R_{22}\phi_2\chi_2^{1,1} + R_{24}\phi_2\chi_4^{1,1} + R_{15}\phi_1\chi_5^{1,1}$ $\psi_4^{1,1} = R_{22}\phi_2\chi_2^{1,1} + R_{25}\phi_2\chi_5^{1,1} + R_{14}\phi_1\chi_4^{1,1}$ $\psi_5^{1,1} = R_{24}\phi_2\chi_4^{1,1} + R_{12}\phi_1\chi_2^{1,1} + R_{15}\phi_1\chi_5^{1,1}$ $\psi_6^{1,1} = R_{22}\phi_2\chi_2^{1,1} + R_{24}\phi_2\chi_4^{1,1} + R_{25}\phi_2\chi_5^{1,1} + R_{12}\phi_1\chi_2^{1,1} + R_{14}\phi_1\chi_4^{1,1} + R_{15}\phi_1\chi_5^{1,1}$
2^+	$\psi_1^{2,2} = R_{21}\phi_2\chi_1^{2,2}$ $\psi_2^{2,2} = R_{11}\phi_1\chi_1^{2,2}$ $\psi_3^{2,2} = R_{21}\phi_2\chi_1^{2,2} + R_{11}\phi_1\chi_1^{2,2}$

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