Editors' Suggestion

Generalized Schur partition functions and RG flows

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(Received 14 August 2025; accepted 5 January 2026; published 11 February 2026)

We revisit a double-scaled limit of the superconformal index of $\mathcal{N}=2$ superconformal field theories (SCFTs) which generalizes the Schur index. The resulting partition function, $\hat{\mathcal{Z}}(q,\alpha)$, has a standard q-expansion with coefficients depending on a continuous parameter α. The Schur index is a special case with $\alpha=1$. Through explicit computations we argue that this partition function is an invariant of certain mass deformations and vacuum expectation value (vev) deformations of the SCFT. In particular, two SCFTs residing in different corners of the same Coulomb branch, satisfying certain restrictive conditions, have the same partition function with a nontrivial map of the parameters, $\hat{\mathcal{Z}}_1(q,\alpha_1)=\hat{\mathcal{Z}}_2(q,\alpha_2(\alpha_1))$. For example, we show that the Schur index of *all* the SCFTs in the Deligne-Cvitanović series is given by special values of α of the partition function of the $\mathrm{SU}(2) N_f=4$ $\mathcal{N}=2$ SQCD.

DOI: 10.1103/ldxd-2jm5

I. INTRODUCTION

The dynamics of quantum field theories with eight supercharges is a focal point of numerous research directions in quantum field theory and mathematical physics. An example of such a convergence of naively disparate ideas is given by the *Schur index* [1,2]. On the one hand, the Schur index is a special case of the supersymmetric index [3] of a general $\mathcal{N}=2$ superconformal field theories (SCFT) counting $\frac{1}{4}$ Bogomol'nyi-Prasad-Sommerfeld (BPS) operators. However, over the recent years numerous connections between this index and other, sometimes rather different ideas, have been observed/derived. Most notably, the Schur index is the vacuum character of a vertex operator algebra (VOA) that one can associate to an $\mathcal{N}=2$ SCFT [4]; it can be thought of as a correlation function in 2D q-deformed Yang-Mills (YM) [1]; it is directly related to Kontsevich-Soibelman wall crossing invariants of the Coulomb branch [5]; and more recently connections between Schur index and the double scaled SYK models have been observed [6]. See also [7,8].

The Schur index, naively, is an object which is naturally associated with the Higgs branch of the theory, as we will review soon. However, in a rather surprising manner it also captures some of the properties of the Coulomb branch [5,9,10]. See also [11–15] for other attempts to find connections between the Higgs and Coulomb branches. Here we will observe yet another relation between the Schur index and the Coulomb branch. In particular, we will revisit and define a generalized Schur limit of the full supersymmetric index which will depend on two parameters, $\mathcal{Z}(q,\alpha)$, such that $\alpha=1$ coincides with the Schur index. We will observe that tuning α in certain cases this partition function will also encode Schur indices of theories which can be obtained from the given theory by tuning to singular loci of its Coulomb branch.

II. BACKGROUND

Let us start by recalling the definition of the full index (a supersymmetric $\mathbb{S}^3\times\mathbb{S}^1$ partition function) of an $\mathcal{N}=2$ SCFT [2,3,16,17],

$$\mathcal{I}=\mathrm{Tr}_{\mathbb{S}^3}(-1)^F\left(\frac{qp}{t}\right)^{-r}p^{j_2-j_1}q^{j_1+j_2}t^R\prod_{i=1}^{\mathrm{rank}G_F}u_i^{F_i}. \quad (1)$$

Here r is the $\mathrm{U}(1)_r$ R-charge, R is the Cartan generator of the $\mathrm{SU}(2)_R$ R-symmetry, and j_i are the Cartan generators of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ isometry of $\mathbb{S}^3$. The charges F_i correspond to the Cartan of the flavor symmetry of the theory. Although our discussion is general, it is useful to understand various statements for a theory defined with a Lagrangian. For that, we remind the reader that the scalars in the chiral field of the vector multiplet, Φ, contribute with weight $\frac{qp}{t}$ to the index; while the scalars of the half hypermultiplets, $Q/\tilde{Q}$, have weight $t^{\frac{1}{2}}$.

Let us consider turning on a mass term $mQ\tilde{Q}$, or a vev $\langle \mathrm{Tr}\Phi^k \rangle \neq 0$. Generically, for the index this will imply that $t^{\frac{1}{2}} = \sqrt{qp}$ in the former case, or $(\frac{qp}{t})^k = 1$ in the latter: modulo phases that will not be important for us, these are equivalent conditions, $qp = t$. Alternatively, setting $qp = t$ in the index is equivalent to giving all the hypermultiplets a mass and all the Coulomb branch operators a vacuum expectation value (vev). It is easy to see [18] that in this case the index will have a pole singularity of order of the dimension of the Coulomb branch $\mathfrak{r}$, which will multiply $((q;q)(p;p))^{\mathfrak{r}}$. Here we define $(z;q) = \prod_{i=0}^{\infty}(1 - zq^i)$.[1]

Next, let us remind the definition of the Schur index. If we set $q = t$ the index will depend only on q (and the flavor fugacities),

$$\mathcal{I}_S = \mathrm{Tr}(-1)^F q^{R+j_1+j_2} \prod_{i=1}^{\mathrm{rank}G_F} u_i^{F_i}. \tag{2}$$

The dependence on p drops out as the charge coupled to it equals $\{\mathcal{Q}, \mathcal{Q}^\dagger\}$, with $\mathcal{Q}$ commuting with $R + j_1 + j_2$. As Coulomb branch operators of dimension n contribute to the full index as $(\frac{qp}{t})^n$, in the Schur index this contributions becomes p^n. Since the Schur limit does not depend on p this contribution has to cancel against another, fermionic, operator. For example, for field Φ this will be the gaugino λ. This is a sense in which Schur index is naturally a Higgs branch object.[2]

Setting $q = t$, the Schur index is consistent with choosing any value for p, and in particular setting $p = 1$. Although, $q = t$ and $p = 1$ satisfies $qp = t$, the index does not trivialize as above. We thus observe that we have an *order of limits issue*: first setting $qp = t$ and then $t = q$ is not the same as first setting $q = t$ and then $p = 1$.

III. GENERALIZED SCHUR PARTITION FUNCTIONS

This order of limits issue allows us to define an interesting generalized limit of the index which will interpolate between the two extreme cases discussed above.[3] We parametrize,

$$p = 1 - \epsilon, \qquad t = (qp)^{1+\frac{\alpha}{\log q}\epsilon}, \tag{4}$$

and take the limit of $\epsilon \to 0$. The case of $\alpha = 0$ corresponds to first turning on masses and then taking the Schur limit. The case of $\alpha = 1$ corresponds to first taking the Schur limit and then taking $p = 1$. This can be clearly seen by computing the (plethystic log of the) contribution of a free hypermultiplet,

$$\frac{t^{\frac{1}{2}} - \frac{qp}{t^{\frac{1}{2}}}}{(1-q)(1-p)}(z + z^{-1}) \to \alpha \frac{q^{\frac{1}{2}}}{1-q}(z + z^{-1}). \tag{5}$$

Here z denotes the $U(1)$ flavor fugacity for a full hypermultiplet. The (plethystic log of the) vector field becomes in this limit,

$$\left(-\frac{p}{1-p} - \frac{q}{1-q} + \frac{\frac{pq}{t} - t}{(1-p)(1-q)}\right)\chi_{\mathrm{adj}}(\mathbf{z}) \to \left(1 - \alpha - \frac{2q}{1-q}\alpha\right)\chi_{\mathrm{adj}}(\mathbf{z}). \tag{6}$$

Here $\chi_{\mathrm{adj}}(\mathbf{z})$ is the character of the adjoint representation of the gauge group. The first term above contributes a pole singularity of order of the rank of the gauge group. After combining the above described limits of the single particle indices and stripping off the pole singularity, the limit for a Lagrangian theory takes the form,

$$\mathcal{Z}(q,\alpha) = (q;q)^{2\mathfrak{r}} \oint d\boldsymbol{\zeta}_G \left(\frac{\prod_\beta (1 - e^{\beta(\zeta)})(qe^{\beta(\zeta)};q)^2}{\prod_\rho (q^{\frac{1}{2}} e^{\rho(\zeta)};q)}\right)^\alpha. \tag{7}$$

Here β are the nonzero roots of the gauge groups and ρ are the weights of the matter representation. For $\alpha = 1$ this is the Schur index. For $\alpha = 0$ this gives $(q;q)^{2\mathfrak{r}}$, the Schur index of the free U(1) vectors on a generic locus of the Coulomb branch. For general $\alpha \in \mathbb{R}^+$ we have an interpolating limit that is well defined and that we will study next.

Let us stress the following point. The limit we consider can be just thought of as a specialization of the full index and therefore makes sense only for conformal theories

[1] In particular if $\mathfrak{r} = 0$, the index will be just 1.

[2] Another, more invariant, way to see this is to note that the Schur index counts operators satisfying the following shortening conditions,

$$E - (j_1 + j_2) - 2R = 0, \qquad r + j_1 - j_2 = 0, \tag{3}$$

which contain as a subset, the Higgs branch operators $E = 2R, r = 0$. More fundamentally, from the point of view of the Higgs branch conjecture [19] of the VOA/SCFT correspondence [4], the Schur index is more naturally a Higgs branch quantity.

[3] Let us stress that the limit we are discussing here was originally defined and studied in [20] with the motivation to relate different *integer* values of α to traces of powers of monodromy matrix generalizing the observations of [5,9,21]. This line of thought led to various interesting observations, see, e.g., [22]. However, in our work we allow for any non-negative real value of α and do not try to connect with the monodromy matrix.

TABLE I. Values of α for the Deligne-Cvitanović rank-one series.

Theory	$h^\vee_{\mathfrak{g}}$	$12c$	$24a$	Δ	α
$\mathfrak{e}_8$	30	62	95	6	5
$\mathfrak{e}_{7\frac{1}{2}}*$	24	50	77	5	4
$\mathfrak{e}_7$	18	38	59	4	3
$\mathfrak{e}_6$	12	26	41	3	2
$\mathfrak{f}_4*$	9	20	32	$5/2$	$3/2$
$\mathfrak{d}_4$	6	14	23	2	1
$\mathfrak{g}_2*$	4	10	17	$5/3$	$2/3$
$\mathfrak{a}_2$	3	8	14	$3/2$	$1/2$
$\mathfrak{a}_1$	2	6	11	$4/3$	$1/3$
$\mathfrak{a}_{\frac{1}{2}}*$	$3/2$	5	$19/2$	$5/4$	$1/4$
$\mathfrak{a}_0$	$6/5$	$22/5$	$43/5$	$6/5$	$1/5$

(as we need both $\mathrm{SU}(2)_R$ and $\mathrm{U}(1)_r$ to define it). In particular, the index of theories related by S-dualities should be the same [23]. Moreover, the index should be expandable in terms of sets of wave functions which will in general depend on q and α and will generalize the Schur polynomials [2,18]. We comment on this more in the Appendix.[4]

IV. CASE STUDY OF RANK 1

Let us discuss the generalized Schur partition function for arguably the simplest interacting conformal $\mathcal{N}=2$ SCFT with a manifestly supersymmetric Lagrangian description, the $N_f=4$ SU(2) SQCD,

$$\hat{\mathcal{Z}}_{\mathfrak{d}_4}(q,\alpha)=\frac{(q;q)^2}{\mathtt{N}(\alpha)}\oint\frac{dz}{4\pi iz}\left(\frac{\Delta(z)(qz^{\pm 2};q)^2}{(q^{\frac{1}{2}}z^{\pm 1};q)^8}\right)^{\alpha}. \quad (8)$$

Here we have defined,

$$\Delta(z)=(1-z^2)(1-z^{-2}), \qquad \mathtt{N}(\alpha)=\oint\frac{dz}{4\pi iz}\Delta(z)^{\alpha}. \quad (9)$$

We also have defined $\hat{\mathcal{Z}}_{\mathfrak{d}_4}(q,\alpha)=\mathcal{Z}_{\mathfrak{d}_4}(q,\alpha)/\mathtt{N}(\alpha)$.[5] In this section, the subscript of $\hat{\mathcal{Z}}$ indicates the flavor symmetry of the theory. $N_f=4$ SQCD has a $\mathfrak{d}_4$ flavor symmetry.

One can compute explicitly this normalized partition function in series expansion in q. The coefficients will depend on α and will be in general not integer. However, we make the following "experimental" observation. Tuning α to certain rational values the expansion coefficients of this normalized partition function are integer, and moreover the partition function can be identified as the Schur index for a different SCFT, properties of which depend on α. For example,[6]

$$\begin{aligned}
\hat{\mathcal{Z}}_{\mathfrak{d}_4}\left(q,\frac{1}{5}\right)&=1+0q+q^2+q^3+\cdots=\mathcal{Z}_{\mathfrak{a}_0}(q,1),\\
\hat{\mathcal{Z}}_{\mathfrak{d}_4}\left(q,\frac{1}{3}\right)&=1+3q+9q^2+19q^3+\cdots=\mathcal{Z}_{\mathfrak{a}_1}(q,1),\\
\hat{\mathcal{Z}}_{\mathfrak{d}_4}\left(q,\frac{1}{2}\right)&=1+8q+36q^2+128q^3+\cdots=\mathcal{Z}_{\mathfrak{a}_2}(q,1),\\
\hat{\mathcal{Z}}_{\mathfrak{d}_4}(q,2)&=1+78q+2509q^2\cdots=\mathcal{Z}_{\mathfrak{e}_6}(q,1),\\
\hat{\mathcal{Z}}_{\mathfrak{d}_4}(q,3)&=1+133q+7505q^2\cdots=\mathcal{Z}_{\mathfrak{e}_7}(q,1),\\
\hat{\mathcal{Z}}_{\mathfrak{d}_4}(q,5)&=1+248q+27249q^2+\cdots=\mathcal{Z}_{\mathfrak{e}_8}(q,1). \quad (10)
\end{aligned}$$

which are precisely the Schur indices of the $\mathfrak{a}_0\simeq(A_1,A_2)$, $\mathfrak{a}_1\simeq(A_1,A_3)$, $\mathfrak{a}_2\simeq(A_1,D_4)$ Argyes-Douglas theories [26,27] and the $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ Minahan-Nemeschansky theories [28,29], respectively. More generally we find that,

$$\hat{\mathcal{Z}}_{\mathfrak{g}(h^\vee_{\mathfrak{g}})}(q,\alpha)=\hat{\mathcal{Z}}_{\mathfrak{d}_4}\left(q,\frac{h^\vee_{\mathfrak{g}}}{6}\alpha\right), \quad (11)$$

where $h^\vee_{\mathfrak{g}}$ is the dual-Coxeter number for Lie algebra $\mathfrak{g}$. The above statement is also summarized in Table I. Thus the Schur indices of the complete Deligne series of $\mathcal{N}=2$ SCFTs are encoded in the generalized Schur partition function of the SQCD.[7] The theories in the Deligne-Cvitanović series are related to each other by Coulomb branch RG flows. Let us comment that for $\alpha>1$ we obtain theories which can be deformed on their Coulomb branch to flow to the SQCD, while for $\alpha<1$ we obtain theories

[4]The limit we consider is equivalent to taking $p, q/t\to 1$ while keeping q generic and keeping track of $\log q/\log t$. If we would also take $q\to 1$, we would obtain the $\mathbb{S}^3$ partition function of the dimensionally reduced theory. Thus, in certain sense, our limit is a q-deformation of the $\mathbb{S}^3$ partition function. From this point of view α is proportional to the real mass parameter for $R-r$. See, e.g., [24,25] for some discussions of similar $3d$ limits.

[5]Note that $\mathtt{N}(1)=1$.

[6]We have computed these expressions expanding the integrand in q and performing the integrals analytically for integer α and numerically for fractional α. In the former case it is easy to compute to high orders in q, while in the latter due to numerics we could go to low orders only.

[7]Note that entries in Table I marked by $*$ do not correspond to known $4d$ SCFTs and do not show up in the rank-one classification of [30–33]. The limit of the index described does not imply existence of such SCFTs, rather the claim is that for the corresponding values of α we recover the vacuum characters of the VOA in the series. For $\alpha=4$, we find the VOA $(\mathfrak{e}_{7\frac{1}{2}})_{-5}$ [34,35] and $\alpha=\frac{1}{4}$ corresponds to $(\mathfrak{a}_{\frac{1}{2}})_{-\frac{5}{4}}$ [36] (see also [37,38]). Moreover, [38] describe an $\mathcal{N}=1$ Lagrangian which flows in the IR to an $\mathcal{N}=2$ theory, with central charges, Coulomb branch dimension, and the Schur index matching the $(\mathfrak{a}_{\frac{1}{2}})_{-\frac{5}{4}}$. The VOAs with $\mathfrak{f}_4$ and $\mathfrak{g}_2$ do not correspond to vanilla SCFTs but can be related to such by studying dual descriptions of Higgs branches of SCFTs [39].

TABLE II. Series obtained from $\mathrm{SU}(N)$ with $2N$ fundamental flavors.

Theory	$12c$	$24a$	Δ_i	α
$R_{2,2N-1}$	$2(4N^2-N-1)$	$14N^2-5N-5$	$2i+1$	2
$\mathrm{SU}(N)+2N\mathrm{F}$	$4N^2-2$	$7N^2-5$	$i+1$	1
(A_1,D_{2N})	$6N-4$	$2(6N-5)$	$\frac{i}{N}+1$	$\frac{1}{N}$
(A_1,A_{2N-1})	$\frac{6N^2-2N-2}{N+1}$	$\frac{12N^2-5N-5}{N+1}$	$\frac{i}{N+1}+1$	$\frac{1}{N+1}$

TABLE III. Series obtained from $\mathrm{USp}(2N)$ with $2N+2$ fundamental flavors.

Theory	$12c$	$24a$	Δ_i	α
$\mathrm{USp}(2N)+(2N+2)\mathrm{F}$	$2N(4N+3)$	$N(14N+9)$	$(2i-1)+1$	1
$D_2(\mathrm{SU}(2N+1))$	$4N(N+1)$	$7N(N+1)$	$\frac{(2i-1)}{2}+1$	$\frac{1}{2}$
(A_1,D_{2N+1})	$6N$	$\frac{3N(8N+3)}{2N+1}$	$\frac{(2i-1)}{2N+1}+1$	$\frac{1}{2N+1}$
(A_1,A_{2N})	$\frac{2N(6N+5)}{2N+3}$	$\frac{n(24N+19)}{2N+3}$	$\frac{(2i-1)}{2N+3}+1$	$\frac{1}{2N+3}$

into which SQCD can flow once its Coulomb branch is explored.[8]

It is interesting to note that taking the Schur limit of the integrand for the $\mathcal{N}=1$ matrix integral expression of the superconformal index [40,41] for $\mathfrak{a}_0$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ theories leads to the same expression as equation (7) for corresponding values of α.

V. HIGHER RANK AND OTHER GENERALIZATIONS

In the previous section, we have described the generalized Schur limit in the case of the rank-one theories. This can be extended to theories of higher rank. Studying various examples, we propose the following conjectural statement.

A necessary condition for the partition function of two theories $\mathcal{T}_1$ and $\mathcal{T}_2$ to be related to each other as follows

$$\hat{\mathcal{Z}}_{\mathcal{T}_2}(q,1)=\hat{\mathcal{Z}}_{\mathcal{T}_1}(q,\alpha) \tag{12}$$

is that their central charges $c^{(1)}$, $c^{(2)}$ and the Coulomb branch scaling dimensions $\Delta^{(1)}_{i=1,2,\ldots,\mathfrak{r}}$, $\Delta^{(2)}_{i=1,2,\ldots,\mathfrak{r}}$ are related as follows

$$c^{(2)}=c^{(1)}\alpha+\frac{1}{6}(1-\alpha)\mathfrak{r},\qquad \Delta^{(2)}_i=(\Delta^{(1)}_i-1)\alpha+1, \tag{13}$$

The matching of the $a-c$ anomaly is natural from the point of view of the index. The leading behavior of a variety of partition functions in the Cardy limit is determined by $a-c$ under some assumptions [42–46]. Thus $a-c$ of the putative theory $\mathcal{T}_2$ is determined by that of $\mathcal{T}_1$ and by the value of α. In fact looking at the subleading behavior this is also true for the c central charge itself (under certain assumptions) [43].[9] Taking the integrand to the power α multiplies c by α, and further there are $2(1-\alpha)\mathfrak{r}$ q-Pochhamer symbols, $(q;q)^{2(1-\alpha)\mathfrak{r}}$, which are the index of free vector multiplets contributing $\frac{1}{6}(1-\alpha)\mathfrak{r}$ to c. The combined contribution equals the central charge of the theory obtained in this limit. The above is consistent with the Shapere-Tachikawa relation $4(2a-c)=\sum_{i=1}^{\mathfrak{r}}(2\Delta_i-1)$ [48], and then $a^{(2)}=a^{(1)}\alpha+\frac{5}{24}(1-\alpha)\mathfrak{r}$. The relation between the Coulomb branch scaling dimensions is an "experimental" observation.

We have tested the above conjecture in various computations. For example, the central charges and Coulomb branch scaling dimensions of $\mathrm{SU}(N)$ SQCD with $2N$ flavors and $\mathrm{USp}(2N)$ SQCD with $2N+2$ flavors satisfy the above relations and the values are tabulated in Tables II and III. For low ranks, the generalized Schur limit matches the Schur indices of theories in Tables II and III for the corresponding value of α. The entries $\mathfrak{a}_0\simeq(A_1,A_2)$, $\mathfrak{a}_1\simeq(A_1,A_3)\simeq(A_1,D_3)$, $\mathfrak{a}_2\simeq D_2(\mathrm{SU}(3))\simeq(A_1,D_4)$, $\mathfrak{e}_6\simeq R_{2,3}$ in the Deligne-Cvitanović series are special cases of Tables II and III.[10] For USp(4) SQCD with 6 flavor, in addition to the entries in Table III we also found that the index for $\alpha=3$ coincides with the index of $D_1^{20}(\mathrm{E}_8)$ theory.[11]

There are numerous examples of theories related by Coulomb branch deformations for which one might expect the indices to be related via this limit, but they are not. These cases do not satisfy the conditions that we have outlined. The simplest ones are the theories which are

[8]The Schur indices of the Deligne-Cvitanović rank-one SCFTs solve a second order differential equation parametrized by a single parameter $h^\vee_{\mathfrak{g}}$ [19]. Our results suggest that this can be interpreted as the generalized Schur partition function solving an MLDE with α as a parameter. We have made sporadic checks of this statement.

[9]The matching of c is also expected from the modular properties of the index [19,47].

[10]We note that (A_1,D_{2N}), (A_1,A_{2N-1}) theories are obtained by an $\mathcal{N}=1$ Lagrangian obtained from deforming $\mathcal{N}=2$ $\mathrm{SU}(N)$ gauge theory with $2N$ flavors [41,49]. Similarly (A_1,D_{2N+1}), (A_1,A_{2N}) are obtained starting from $\mathcal{N}=2$ $\mathrm{USp}(2N)$ $2N+2$ flavors [41,49].

[11]In fact the expression with $\alpha=3$ of the USp(4) model can be thought of as a definition of the Schur index of the $D_1^{20}(\mathrm{E}_8)$ theory. Here the independent check we performed is that of anomalies, Coulomb branch dimensions, and the dimension of the flavor symmetry group as read from order q term in the index (and in some cases we could go to higher orders in q). For the $\mathfrak{e}_8$ theory the expression using $\alpha=5$ of the SU(2) SQCD is a novel prediction for a two parameter slice of the index (beyond the Macdonald index which is explicitly known). We would also like to note that similar to the rank-one case, for higher rank cases, the order of modular linear differential equation (MLDE) solved by the Schur indices is the same for theories related by an α, in all cases we could check.

related by Coulomb branch deformations to the $\mathcal{N}=4$ SYM, see e.g. [50,51].

VI. SUMMARY AND DISCUSSION

Supersymmetric partition functions of theories related to each other by RG flows are often equal. For example if theory $\mathcal{T}_1$ when deformed by a superpotential leads to theory $\mathcal{T}_2$, the index of theory $\mathcal{T}_2$ refined with symmetries visible in the UV is the same as the index of theory $\mathcal{T}_1$ when one switches off the fugacities corresponding to symmetries broken by the superpotential [52,53]. Similarly, the index of a theory $\mathcal{T}_2$ obtained by vev deformation of theory $\mathcal{T}_1$ is given by a certain residue of the index of $\mathcal{T}_1$ [18]. Moreover, two (IR or conformally) dual theories also should have the same index once a proper map between parameters corresponding to global symmetries is found.[12]

Our main result is a version/generalization of this statement. Here we find that certain classes of theories which are related by RG flows have the same partition function albeit with a non-trivial map between the parameters *including the R-symmetries*. The relevant RG flows correspond to fine tuning various physical parameters such that without the fine tuning the theories are massive. For example, exploring the Coulomb branch of the theory the index, as we have discussed, trivializes and is just given by the U(1) vectors. However, in the examples we have considered the generalized Schur indices of two theories connected by such Coulomb branch deformation, and satisfying conditions we have outlined, are mapped to each other with different values of their α parameters.

Our observation is specific to $\mathcal{N}=2$ SCFTs, and exploits an orders of limits tension between the Schur index and the index obtained turning on Coulomb branch vevs. This tension was phrased in the past as an observation that the Schur index is consistent with mass terms in Lagrangian but nevertheless captures more than just the vacuum. See. e.g., [55].

Once we understand that starting from an $\mathcal{N}=2$ SCFT and computing its generalized Schur partition function with α we can obtain this partition function for a different theory with $\alpha'(\alpha)$, we can iterate this procedure by, e.g., adding matter and gauging the symmetry (assuming one can compute the index refined by flavor fugacities, which is true in various cases) with parameter α'' to obtain partition functions of new theories. For example, the (A_3, A_3) AD theory can be obtained by a diagonal SU(2) gauging of two (A_1, D_4) AD theories along with free hypermultiplets [49,56]. Another example, is the fact that the $(A_2, D_4) \simeq \hat{E}_6(\mathrm{SU}(2))$ can be obtained by a diagonal gauging of three copies of (A_1, A_3) theories [57]. On the other hand $(A_1, D_4)/(A_1, A_3)$ can be thought of as being obtained in the $\alpha' = \frac{\alpha}{2}/\alpha' = \frac{\alpha}{3}$ limit of the SU(2) SQCD. Further gauging (along with hypers when needed) can be thought of as a standard $\alpha = 1$ gauging. Thus one can write the partition function as a matrix integral with different fields taken into different powers.[13]

The results of this note open many venues for connecting partition functions of different SCFTs. It will be extremely interesting to find a first principle derivation of the "experimental" observations we report here.

ACKNOWLEDGMENTS

We would like to thank Leonardo Rastelli, Matteo Sacchi, Wenbin Yan, and Gabi Zafrir for enlightening and useful discussions. S. S. R. is grateful to the SCGP for hospitality during the program "Supersymmetric Quantum Field Theories, Vertex Operator Algebras, and Geometry," where this project was initiated. The work of A. D. is supported in part by NSF Grant No. PHY-2210533 and by the Simons Foundation Grant No. 681267 (Simons Investigator Award). The research of S. S. R. is supported in part by the Planning and Budgeting committee, by the Israel Science Foundation under Grant No. 2159/22, and by BSF-NSF Grant No. 2023769.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: TQFT STRUCTURE OF $\mathcal{Z}(q, \alpha)$

Superconformal indices are invariants of S-duality. In the framework of theories of class $\mathcal{S}$ [59,60] this leads to a mathematical statement that the index should be invariant under exchange of puncture symmetries [23]. For example, the SU(2) SQCD has $\mathrm{SU}(2)^4 \subset \mathrm{SO}(8)$ global symmetry with the four SU(2) factors corresponding to four punctures. S-duality then implies that,

$$\oint \frac{dz}{4\pi i z}\left(\frac{\Delta(z)(qz^{\pm 2};q)^2}{(q^{\frac{1}{2}}z^{\pm 1}a^{\pm 1}b^{\pm 1};q)(q^{\frac{1}{2}}z^{\pm 1}c^{\pm 1}d^{\pm 1};q)}\right)^{\alpha}$$
$$= \oint \frac{dz}{4\pi i z}\left(\frac{\Delta(z)(qz^{\pm 2};q)^2}{(q^{\frac{1}{2}}z^{\pm 1}a^{\pm 1}c^{\pm 1};q)(q^{\frac{1}{2}}z^{\pm 1}b^{\pm 1}d^{\pm 1};q)}\right)^{\alpha}, \quad \text{(A1)}$$

[12] Note that to match the indices of a pair of dual theories one needs to map correctly the R-symmetry of the two. This is usually done using a-maximization arguments [54].

[13] Let us also mention here that the $\alpha = 2$ limit for (A_3, A_3) AD theory gives the Schur index for $[1] - \mathrm{SU}(2) - \mathrm{SU}(3) - [4]$ and $\alpha = 3$ limit for (A_2, D_4) AD theory gives the Schur index for $[\mathrm{SO}(8)] - \mathrm{USp}(4) - \mathrm{SO}(4)$ theory. In [58], an $\mathcal{N}=1$ deformation of the quivers flows to the respective AD theories described. These are a special case of the general AD theories described in [58] and we expect that the limit of the index described here also relates those theories.

for any value of α. This equality follows mathematically from [61,62] and can be explicitly verified in series expansion. Moreover, one should be able to expand the index in eigenfunctions [18] generalizing the Schur polynomials. For example, using the techniques of [63] one can obtain that the index of A_1 class $\mathcal{S}$ theories on genus g surface with n punctures in the large g limit and $\alpha = 2$ as,

$$\mathcal{Z} \sim C(g, n) \prod_{i=1}^{n} \psi_0(a_i), \tag{A2}$$

with,

$$\frac{\psi_0(a) - 1}{(a - a^{-1})^2} = \frac{8q}{3} + \frac{(135a^2 - 74 + 135a^{-2})q^2}{27} + \frac{8(243a^4 - 216a^2 + 254 - 216a^{-2} + 243a^{-4})q^3}{243} + \cdots. \tag{A3}$$

Here the wave function was normalized so that $\psi_0(1) = 1$.

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