Meson octet in a uniform magnetic field

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Chiral perturbation theory is utilized to construct the renormalized magnetic masses and decay constants of the meson octet at next-to-leading order. While the neutral pion mass decreases identically to that in two-flavor chiral perturbation theory, the neutral kaon mass remains unaltered by the magnetic field. The renormalized magnetic masses for the charged mesons change identically. Meson decay constants in the axial and vector channels are constructed. Each decay constant increases monotonically in the magnetic background. Low-energy theorems—Gell-Mann-Oakes-Renner relations for the neutral mesons and their generalization for the charged mesons through the pseudoscalar coupling—are constructed and provide nontrivial cross-checks.

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I. INTRODUCTION

Magnetic fields are phenomenologically important for strongly interacting systems, particularly in astrophysical systems. Magnetars have significant surface magnetic fields, $10^{10}T$, with larger magnetic fields anticipated in the interior [1], which results in the softening of the equation of state, significant enhancement of the proton fraction, and an increase in the maximum neutron star mass [2–4]. Heavy-ion collisions at the LHC and the RHIC produce the largest known magnetic fields of size $10^{16}T$ [5–9]. Additionally, large magnetic fields may be important for big bang cosmology, in particular the electroweak transition to a confined phase of hadrons [10–14].

In addition to QCD phenomenology, the impact of strong magnetic fields on the QCD vacuum, phase diagram, and hadron properties is of significant theoretical interest [15–25]. For review articles, see Refs. [26–30]. Magnetic fields enhance the quark condensate (and the topological susceptibility) as exhibited through model-independent analyses in low-energy QCD [31–38]. Furthermore, they impact the confinement-deconfinement transition understood to be a crossover transition for weak magnetic fields. For asymptotically large magnetic fields, quarks become heavy and can be integrated out—the resulting pure glue theory is anisotropic and possesses a first-order deconfinement transition [39]. Magnetic fields also significantly alter the phase diagram of finite density QCD. For finite isospin and baryon chemical potentials, topological phases are observed including magnetic vortices and chiral soliton lattices [40–46].

Chiral perturbation theory [47,48] allows for the study of the impact of magnetic fields on mesonic properties. Since it is an effective field theory of QCD, the results are model-independent for $eB \ll \Lambda^2_{\mathrm{Hadron}} \sim 1\ \mathrm{GeV}^2$. Previously, chiral perturbation theory has been used to investigate the alteration of masses and decay constants of both the neutral and the charged pions by a background magnetic field [49] and thermal bath [50,51]. Furthermore, properties of the octet baryons including magnetic moments and polarizabilities have been previously characterized [52]. Since chiral perturbation theory encodes the long-range behavior of QCD, it aids in the investigation of finite volume corrections relevant for lattice studies of thermodynamic observables and hadronic properties in a magnetic background [53]. For magnetic fields of the size of the pion mass, $eB \sim m_\pi^2$, finite volume corrections are significant, particularly for vacuum magnetization. Furthermore, spatially inhomogeneous (periodic) structure, for the chiral condensate, is also predicted, in addition to neutral pion and neutron polarizabilities [54,55].

Magnetic masses and decay constants are inputs for the characterization of the decay width of charged and neutral pions [56]. The leptonic decay width for the charged pion is enhanced due to the increased magnetic mass and the alteration of the phase space associated with the charged decay products. On the other hand, the anomalous decay width for the neutral pion, $\pi^0 \to \gamma\gamma$, is suppressed due to the decrease in the neutral pion mass induced by the magnetic field. In this context, a novel pion decay was proposed in a pioneering lattice study [57] with a model-independent

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parametrization of the decay constants provided in a model-dependent construction of both charged and neutral pion decay widths [58–60]. A modest tension between lattice QCD and two-flavor chiral perturbation theory was noted [56]. In particular, the $(2+1)$-flavor QCD determination, using staggered fermions and at the physical point, of the pion decay constant in the vector channel is, even after accounting for error bars, not consistent with the predictions of the chiral anomaly. Additionally, the axial pion decay constant shows a qualitatively different behavior for weak fields. In two-flavor chiral perturbation theory, the decay constant increases monotonically while lattice results exhibit a modest decrease, near $B = 0$, followed by a monotonic increase. Given the tension, it is reasonable to consider the effect of finite strange quark masses on the decay constants, which is undertaken using three-flavor chiral perturbation theory. A further motivation is provided by a recent $(2+1)$-lattice QCD study [61], using a highly improved staggered quark action. It constructs the zero temperature properties of the pseudoscalar masses and decay constants at the physical point. Since the study utilizes four lattice spacings to access the continuum limit, the results are highly reliable. Furthermore, the study also determines the impact of the magnetic background on the Gell-Mann-Oakes-Renner relations for the neutral pion and neutral kaons. A lattice study of hadronic properties in two-flavor QCD is conducted in Ref. [62] while Nambu–Jona-Lasinio model studies are found in Refs. [63–68].

Three-flavor chiral perturbation theory provides a natural setting for an analogous study of magnetic masses, decay constants, and low-energy theorems, but with analytically accessible results. It has previously been utilized to explore properties of the QCD vacuum including quark condensates and vacuum magnetization [69]. However, an analysis of magnetic meson masses and decay constants, relevant for mesonic weak decay, within three-flavor chiral perturbation theory, is absent. Additionally, we consider their impact on the Gell-Mann-Oakes-Renner relations for the meson octet and construct their generalization for the charged mesons. Furthermore, we characterize the pseudoscalar coupling to the meson octet—referred to in Ref. [47] as "the coupling constant of the pseudoscalar density"—as a function of the magnetic background. Low-energy theorems involving the coupling and meson masses provide nontrivial cross-checks for our calculations. From a field theory perspective, the analysis is, in principle, straightforward, though in practice, significantly more involved than the corresponding work in two-flavor chiral perturbation theory.

The paper is organized as follows: in Sec. II, we present the chiral Lagrangian at next-to-leading order necessary for the construction of meson masses, decay constants and the pseudoscalar coupling. In Secs. III and IV, we describe the position-space construction of the renormalized masses and decay constants. In Sec. V, low-energy theorems for the octet mesons are discussed—while the zero-field Gell-Mann-Oakes-Renner relations continue to hold for the neutral mesons, their structure is modified for the charged mesons. In Sec. VI, concluding remarks are presented.

II. CHIRAL PERTURBATION THEORY INGREDIENTS

Construction of the meson octet magnetic mass and decay constants in three-flavor chiral perturbation theory [48] requires accounting for loop effects. We work up to $\mathcal{O}(p^4)$, which includes the anomalous Wess-Zumino-Witten (WZW) contribution to the chiral Lagrangian [70,71]. A scalar-pseudoscalar source is required to incorporate quark masses and construct the pseudoscalar coupling. A vector source is required to account for the magnetic field while a left source is required to construct the left current. The WZW Lagrangian encodes a vector current that couples the meson to the vacuum when a background magnetic field is introduced.

A. Chiral Lagrangian at $\mathcal{O}(p^2)$

The leading $\mathcal{O}(p^2)$ chiral Lagrangian is a function of the $SU(3)$ meson octet field, Σ,

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger + \chi \Sigma^\dagger + \Sigma \chi^\dagger \rangle. \tag{1}$$

The angled brackets represent a flavor trace. The scalar-pseudoscalar source, $\chi = 2B_0(s + ip)$, is required for the construction of quark condensates and the pseudoscalar coupling. The tree-level octet mass follows from the utilization of the coset representation, $\Sigma = \exp(\frac{i\phi_a\lambda_a}{F})$, in terms of Gell-Mann matrices, λ_a, and the quark mass matrix, $s = \mathrm{diag}(m_u, m_d, m_s)$, as the scalar source. The meson octet fields with their corresponding tree-level masses are

$$\pi^\pm = \frac{1}{\sqrt{2}}(\phi_1 \mp \phi_2), \qquad \mathring{m}^2_{\pi^\pm} = B_0(m_u + m_d), \tag{2}$$

$$\pi^0 = \phi_3, \qquad \mathring{m}^2_{\pi^0} = B_0(m_u + m_d), \tag{3}$$

$$K^\pm = \frac{1}{\sqrt{2}}(\phi_4 \mp \phi_5), \qquad \mathring{m}^2_{K^\pm} = B_0(m_u + m_s), \tag{4}$$

$$K^0, \bar{K}^0 = \frac{1}{\sqrt{2}}(\phi_6 \mp \phi_7), \qquad \mathring{m}^2_{K^0} = B_0(m_d + m_s), \tag{5}$$

$$\eta = \phi_8, \qquad \mathring{m}^2_\eta = \frac{1}{3}B_0(m_u + m_d + 4m_s). \tag{6}$$

We work in the isospin limit. Nevertheless, quark masses, m_{q_f}, are labeled uniquely, $q_f = u, d, s$. At tree level, the free energy is $\mathcal{F}_0 = -B_0F^2(m_u + m_d + m_s)$ with the

degenerate quark condensates, $\langle \bar{q}_f q_f \rangle_0 = -B_0 F^2$. The tree-level Gell-Mann-Oakes-Renner relation [72] for the pions is

$$\mathring{m}_{\pi^0}^2 F^2 = -\frac{1}{2}(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle_0. \tag{7}$$

Analogous Gell-Mann-Oakes-Renner relations hold for the kaons

$$\mathring{m}_{K^\pm}^2 F^2 = -\frac{1}{2}(m_u + m_s)\langle \bar{u}u + \bar{s}s \rangle_0, \tag{8}$$

$$\mathring{m}_{K^0}^2 F^2 = -\frac{1}{2}(m_d + m_s)\langle \bar{d}d + \bar{s}s \rangle_0, \tag{9}$$

where $\langle \bar{u}u \rangle_0$, $\langle \bar{d}d \rangle_0$, and $\langle \bar{s}s \rangle_0$ are the up, down, and strange quark condensates. As we exhibit through explicit calculation of masses and decay constants, in the following sections, only the neutral meson Gell-Mann-Oakes-Renner relations hold in a magnetic background [73]. The Gell-Mann-Oakes-Renner relation for the charged meson admits a form involving nonmagnetic masses at next-to-leading order, as we discuss in Sec. V.

The covariant derivative in the chiral Lagrangian, required for the inclusion of vector and axial sources, is

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - iR_\mu \Sigma + i\Sigma L_\mu. \tag{10}$$

The magnetic field enters through $R_\mu = L_\mu = -eA_\mu \mathcal{Q}$, with $e > 0$, where $\mathcal{Q}$ is the quark charge matrix

$$\mathcal{Q} = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8. \tag{11}$$

The quadratic Lagrangian and left-currents associated with charged mesons then depend on the covariant derivative

$$D_\mu \varphi^Q = (\partial_\mu + ieQA_\mu)\varphi^Q, \tag{12}$$

with $\varphi = \pi, K$, or η. $Q = \pm 1$ and $Q = 0$ for the charged and neutral mesons, respectively. The superscript for the octet η meson field is omitted, as is conventional. The fully antisymmetric gauge for a uniform magnetic field $\vec{B} = B\hat{z}$,

$$A_\mu = (0, -By, 0, 0), \tag{13}$$

with $B > 0$, will be utilized in the construction of the charged Green's function in Sec. III A. For this gauge, the good quantum numbers are represented by the Ritus momentum [74]. In covariant notation, it is $\tilde{P} = (P^0, P^1, 0, P^3)$, with P^0 being a function of the Landau quantum number and the z-component of momentum. The Ritus momentum is conserved in scattering and decay processes.

B. Chiral Lagrangian at $\mathcal{O}(p^4)$

The next-to-leading order Lagrangian required to renormalize the masses, decay constants, and pseudoscalar densities is

$$\begin{aligned}\mathcal{L}_4 \supset\ & L_4\langle \nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger \rangle \langle \chi \Sigma^\dagger + \chi^\dagger \Sigma \rangle \\ & + L_5 \langle \nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger (\chi \Sigma^\dagger + \chi^\dagger \Sigma) \rangle \\ & + L_6 \langle \chi \Sigma^\dagger + \chi^\dagger \Sigma \rangle^2 + L_7 \langle \chi \Sigma^\dagger - \chi^\dagger \Sigma \rangle^2 \\ & + L_8 \langle \Sigma \chi^\dagger \Sigma \chi^\dagger + \chi \Sigma^\dagger \chi \Sigma^\dagger \rangle \\ & - iL_9 \langle R_{\mu\nu} \nabla^\mu \Sigma (\nabla^\nu \Sigma)^\dagger + L_{\mu\nu} (\nabla^\mu \Sigma)^\dagger \nabla^\nu \Sigma \rangle \\ & + L_{10} \langle \Sigma L_{\mu\nu} \Sigma^\dagger R^{\mu\nu} \rangle. \end{aligned} \tag{14}$$

L_6 and L_7 terms are purely zero-field contributions while L_4 and L_5 terms are required for wave-function renormalization—the magnetic background introduces no further divergences. The L_9 and L_{10} terms give rise to purely finite field contributions to the self-energy of charged mesons. $L_{\mu\nu}$ and $R_{\mu\nu}$ are field strength tensors associated with left and right sources

$$L_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu - i[L_\mu, L_\nu], \tag{15}$$

$$R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu - i[R_\mu, R_\nu]. \tag{16}$$

In a magnetic background, $L_{\mu\nu} = R_{\mu\nu} = -e\mathcal{Q}F_{\mu\nu}$. For renormalization [48], we utilize dimensional regularization with $d = 4 - 2\epsilon$. In terms of the $\overline{\text{MS}}$ scale, Λ, the divergence in the low-energy constants, L_i, is

$$L_i = L_i^r + \Gamma_i \lambda, \qquad \lambda = -\frac{\Lambda^{-2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon} + 1\right). \tag{17}$$

Utilizing the scale independence of the bare low-energy constants, the running of the renormalized couplings, L_i^r, is encoded in the relation

$$\Lambda \frac{dL_i^r}{d\Lambda} = -\frac{\Gamma_i}{(4\pi)^2}. \tag{18}$$

It is conventional for the values of the renormalized low-energy constants, L_i^r, to be stated at the scale of the ρ meson mass, m_ρ. The constants Γ_i are

$$\Gamma_4 = \frac{1}{8}, \qquad \Gamma_5 = \frac{3}{8}, \qquad \Gamma_6 = \frac{11}{144}, \qquad \Gamma_7 = 0, \tag{19}$$

$$\Gamma_8 = \frac{5}{48}, \qquad \Gamma_9 = \frac{1}{4}, \qquad \Gamma_{10} = -\frac{1}{4}. \tag{20}$$

Construction of the vector pion decay constant requires accounting for the chiral anomaly through the WZW Lagrangian,

$$\mathcal{L}_{\rm WZW} = \frac{i}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\langle \mathcal{Z}_{\mu\nu\rho\sigma}\rangle,$$
$$\begin{aligned}\mathcal{Z}_{\mu\nu\rho\sigma} \supset\ & i\partial_\mu L_\nu (L_\rho \Sigma^\dagger R_\sigma \Sigma - \Sigma^\dagger R_\rho \Sigma L_\sigma) \\ & + i\partial_\mu R_\nu (\Sigma L_\rho \Sigma^\dagger R_\sigma - R_\rho \Sigma L_\sigma \Sigma^\dagger) \\ & + (\Sigma^\dagger \partial_\mu \Sigma)(L_\nu \partial_\rho L_\sigma + \partial_\nu L_\rho L_\sigma) \\ & + \frac{1}{2}(\Sigma^\dagger \partial_\mu \Sigma)(\Sigma^\dagger \partial_\nu R_\rho \Sigma L_\sigma + \Sigma^\dagger R_\nu \Sigma \partial_\rho L_\sigma) \\ & - \frac{1}{2}(\Sigma \partial_\mu \Sigma^\dagger)(\Sigma \partial_\nu L_\rho \Sigma^\dagger R_\sigma + \Sigma L_\nu \Sigma^\dagger \partial_\rho R_\sigma). \end{aligned} \tag{21}$$

Only contributions to $\mathcal{Z}_{\mu\nu\rho\sigma}$ required for the construction of the left current are retained. The convention, $\varepsilon^{0123} = 1$, is adopted for the epsilon tensor.

III. MESON OCTET MAGNETIC MASS

A. Green's function, $G_Q(x', x)$

The construction of the meson octet magnetic mass requires the *Euclidean* Green's function, $G_Q(x', x)$. For neutral mesons ($Q = 0$), it is translationally invariant. However, the charged Green's function ($Q = \pm$), as is evident from the defining relation

$$(-D'_\mu D'_\mu + \mathring{m}^2_{\varphi^Q}) G_Q(x', x) = \delta^{(4)}(x' - x) \tag{22}$$

is not. Since the δ-function and the neutral Green's operator are translationally invariant, the neutral Green's function must be translationally invariant, $G_0(x', x) = G_0(x' - x)$. However, the covariant derivative due to the gauge potential, Eq. (13), is not translationally invariant. Therefore, the charged Green's function, $G_\pm(x', x)$, must also be translationally variant to ensure translational invariance of the product. Nevertheless, the charged Green's function is separable into translationally invariant and variant contributions

$$\begin{aligned} G_Q(x', x) = e^{ieB\Delta x_1 \bar{x}_2} \int_0^\infty \frac{ds}{(4\pi)^2 s^2} \frac{eBs}{\sinh eBs} e^{-\mathring{m}^2_{\varphi^Q} s} \\ \times \exp\left[-\frac{eB(\Delta x_1^2 + \Delta x_2^2)}{4 \tanh eBs} - \frac{\Delta x_0^2 + \Delta x_3^2}{4s} \right], \end{aligned} \tag{23}$$

with $\Delta x_\mu = x'_\mu - x_\mu$ and $\bar{x}_\mu = \frac{1}{2}(x'_\mu + x_\mu)$. The s in the expression is the Schwinger proper time and should not be conflated with the (previously utilized) strange quark field—the proper time does not appear elsewhere in the paper. The Schwinger integral is notably translationally invariant with all translational variance encoded in the Schwinger phase.

Tadpole contributions to the renormalized mass require the coincident limit of Green's function,

$$G_Q(x, x) \equiv \mathcal{D}_Q(\mathring{m}^2_{\varphi^Q}) = \mathcal{D}_0(\mathring{m}^2_{\varphi^Q}) + \mathcal{D}_B(\mathring{m}^2_{\varphi^Q}). \tag{24}$$

Despite the absence of translational invariance, the charged Green's function is coordinate independent in the coincident limit. The ultraviolet divergence (in the coincident Green's function) is handled using dimensional regularization in $d = 4 - 2\varepsilon$ dimensions. Using the $\overline{\rm MS}$ scheme with scale Λ, the B-independent contribution to Green's function is

$$\mathcal{D}_0(m^2) = -\frac{m^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 1 + \log\frac{\Lambda^2}{m^2}\right]. \tag{25}$$

The magnetic field dependent contribution

$$\mathcal{D}_B(m^2) = \frac{eB}{(4\pi)^2}\mathcal{I}\left(\frac{m^2}{eB}\right) \tag{26}$$

is a function of the dimensionless Schwinger integral $\mathcal{I}(z)$. It is finite and admits the closed form expression

$$\mathcal{I}(z) = 2\log\Gamma\left(\frac{1+z}{2}\right) + z\left(1 - \log\frac{z}{2}\right) - \log(2\pi), \tag{27}$$

in terms of the $\log\Gamma$ function. In the zero-field limit, the function vanishes, as required.

B. Two-point function and meson mass

The meson octet magnetic mass is extracted from the renormalized two-point function, $\mathcal{G}_Q(x', x)$, which, in Schwinger operator language, is defined as

$$\mathcal{G}_Q(x', x) = \langle x'|\mathcal{G}_Q|x\rangle. \tag{28}$$

In terms of the bare Green's function, $G_Q(x', x)$, the renormalized Green's function is

$$\langle x'|\mathcal{G}_Q|x\rangle = \langle x'|G_Q|x\rangle + \int_y \langle x'|G_Q|y\rangle \mathcal{O}(y) \langle y|G_Q|x\rangle, \tag{29}$$

where $\mathcal{O}(y)|y\rangle\langle y|$ represents the perturbative correction due to tadpole diagrams. $\int_y$ is compact notation for a four-dimensional Euclidean spacetime integral. In operator representation, $\mathcal{G}_Q = G_Q + G_Q \mathcal{O} G_Q$. However, because $\mathcal{O}$ can be treated perturbatively, a useful identity for the inverse renormalized Green's function follows:

$$\mathcal{G}_Q^{-1} = G_Q^{-1} - \mathcal{O}. \tag{30}$$

To extract the wave-function renormalization factor and the renormalized meson masses in position space, we separate the renormalized Green's functions into terms that contribute to wave-function renormalization and terms that do not:

$$\mathcal{G}_Q(x',x) = G_Q(x',x) + \int_y G_Q(x,y)\mathcal{O}_2(y)G_Q(y,x') + \int_y D_\mu G_Q(x,y)\mathcal{O}_1(y)D_\mu G_Q(y,x'). \tag{31}$$

The covariant derivatives act at spacetime location y and reduce to regular derivatives for $Q = 0$. $\mathcal{O}_1$ and $\mathcal{O}_2$ are constructed through Wick contractions involving the interaction Lagrangian—because the contributions arise through tadpole diagrams (coincident Green's functions), they are spacetime independent. Symmetrization of the first integral via a symmetric integration by parts is necessary to proceed

$$\mathcal{G}_Q(x',x) = G_Q(x',x) + \int_y G_Q(x,y)\mathcal{O}_2 G_Q(y,x') - \frac{1}{2}\int_y D_\mu D_\mu G_Q(x,y)\mathcal{O}_1 G_Q(y,x') - \frac{1}{2}\int_y G_Q(x,y)\mathcal{O}_1(y)D_\mu D_\mu G_Q(y,x'). \tag{32}$$

Its inversion leads to, in operator form,

$$\mathcal{G}_Q^{-1} = G_Q^{-1} + \frac{1}{2}\left[\overleftarrow{D^2}\mathcal{O}_1 + \mathcal{O}_1\overrightarrow{D^2}\right] - \mathcal{O}_2 \tag{33}$$

$$= \left(1 - \frac{1}{2}\mathcal{O}_1\right)G_Q^{-1}\left(1 - \frac{1}{2}\mathcal{O}_1\right) - \mathring{m}^2_{\varphi^Q}\mathcal{O}_1 - \mathcal{O}_2, \tag{34}$$

where $D^2 \equiv D_\mu D_\mu$ acts directionally, as indicated. Upon performing a wave-function renormalization of the meson field,

$$\sqrt{Z_{\varphi^Q}} = 1 + \frac{1}{2}\mathcal{O}_1, \tag{35}$$

the renormalized Green's function and the corresponding renormalized mass are

$$\mathcal{G}_Q^{-1} = G_Q^{-1} + \Delta m^2_{\varphi^Q}, \tag{36}$$

$$m^2_{\varphi^Q} = \mathring{m}^2_{\varphi^Q} + \Delta m^2_{\varphi^Q} = \mathring{m}^2_{\varphi^Q}(1 - \mathcal{O}_1) - \mathcal{O}_2. \tag{37}$$

With the formal structures established, construction of $\mathcal{O}_1$ and $\mathcal{O}_2$ requires the $\mathcal{O}(p^2)$ chiral Lagrangian expanded up to quartic order in the meson fields. The quadratic Lagrangian is a function of the masses and covariant derivatives presented in Eqs. (2)–(6) and (12), respectively. These, in addition to the quartic contribution of the leading order Lagrangian, $\mathcal{L}_{2,4}$, were previously presented in Ref. [69].[1] For renormalization, the next-to-leading order contribution of $\mathcal{L}_4$ containing two meson fields is also required; see $\mathcal{L}_{4,2}$ of Eq. (B.3) in Ref. [69].

[1]There is a typo in Eq. (B.1) of Ref. [69]. While immaterial for the two-loop free energy, it impacts the calculation of renormalized masses. The coefficient in the fourth-to-last line is missing a factor of 2.

For charged mesons, contributions from charged tadpoles cancel after accounting for wave-function renormalization. Therefore, all magnetic field dependence arises through the L_9 and L_{10} terms. The L_9 term in $\mathcal{L}_{4,2}$ simplifies through integration by parts. Under the assumption of a uniform external field, the expression (in Minkowski space) is

$$-\frac{2L_9}{F^2}(eF_{\mu\nu})(eF^{\mu\nu})(\pi^+\pi^- + K^+K^-). \tag{38}$$

The identity $[D^\mu, D^\nu] = ieF^{\mu\nu}$ has been utilized in its construction. The quadratic contributions from the L_9 and L_{10} terms in the chiral Lagrangian are identical. Since $\Gamma_9 = -\Gamma_{10}$, the combination $L_9 + L_{10}$ is finite and scale-independent. The magnetic masses of the charged mesons are renormalized additively and have an identical structure[2]

$$m^2_{\varphi^\pm}(B) = m^2_{\varphi^\pm} + \frac{(eB)^2}{(4\pi F)^2}\bar{L}. \tag{41}$$

$\bar{L}$ has been introduced for compactness and is an $O(1)$ combination of renormalized low-energy constants

$$\bar{L} = 4(4\pi)^2(L_9^r + L_{10}^r). \tag{42}$$

Additionally, the expression, for the masses, is consistent with charged mesons not being pseudo-Goldstone bosons—in the chiral limit and a finite magnetic background, only the first contribution, $m^2_{\varphi^\pm}$, vanishes.[3] The renormalized masses for the charged mesons, however, do not account for Landau levels, labeled by the quantum number N in the magnetic dispersion relation below,

[2]The most precise values of the low-energy constants, L_9^r and L_{10}^r, and their corresponding uncertainties are [75]

$$L_9^r = 6.9(7)\times 10^{-3}, \qquad L_{10}^r = -5.22(6)\times 10^{-3}. \tag{39}$$

Their values are determined through the axial form factor in the decay channel $\pi^+ \to e^+\nu_e\gamma$ and the pion electromagnetic radius. The two-flavor counterpart of $\overline{L}$ is $\overline{\ell}$,

$$\overline{\ell} = \frac{1}{3}(\overline{l_6} - \overline{l_5}) = 1.0(1), \tag{40}$$

which is known at the 10% level. This is significantly smaller than the uncertainty in $\bar{L}$. Combining the uncertainties associated with L_9^r and L_{10}^r in quadrature leads to a fractional uncertainty of approximately 40% in $\bar{L}$, more specifically $\bar{L} = 1.06(0.44)$.

[3]In the presence of a magnetic field, the symmetry of the QCD Lagrangian is $U(2)_L \times U(2)_R$, which is broken down to $U(2)_V$ by the ground state. The axial fluctuations around the vacuum are generated by λ_3, $\frac{1}{\sqrt{2}}(\lambda_6 \mp \lambda_7)$, and λ_8. The $SU(2)$ subgroup is generated by $\frac{1}{\sqrt{2}}(\lambda_6 \mp \lambda_7)$ and $\frac{1}{2}(\lambda_3 - \sqrt{3}\lambda_8)$ while the $U(1)$ subgroup is generated by $\frac{\sqrt{3}}{2}(\sqrt{3}\lambda_3 - \lambda_8)$.

$$E_{N,P_z} = \sqrt{P_z^2 + m^2_{\varphi^\pm}(B) + eB(2N+1)}. \quad (43)$$

The spectrum must be interpreted as that of renormalized charged mesons propagating in a magnetic background with longitudinal momentum P_z [56]. For the lowest Landau level, $N = 0$, there is a further additive contribution, eB, to the mass squared. While both e and B are renormalized, eB is renormalization invariant [76].

For the neutral pion, the charged kaon tadpole diagrams cancel exactly. Consequently, only the charged pion tadpole contributes to the renormalized mass

$$m^2_{\pi^0}(B) = m^2_{\pi^0}\left[1 + \frac{eB}{(4\pi F)^2}\mathcal{I}\left(\frac{\mathring{m}_\pi^2}{eB}\right)\right], \quad (44)$$

with the result identical to that in two-flavor chiral perturbation theory. For the neutral kaon, on the other hand, all charged tadpole diagrams cancel, leading to no magnetic modification at next-to-leading order[4]

$$m^2_{K^0}(B) = m^2_{K^0}. \quad (45)$$

The next-to-next-to-leading order corrections to the mass are of relative $\mathcal{O}(\frac{(eB)^2}{(4\pi F)^4})$.[5] The octet eta, however, picks up corrections from both the charged pion and the charged kaon tadpoles

$$m^2_\eta(B) = m^2_\eta - \frac{\mathring{m}_\pi^2}{3}\frac{eB}{(4\pi F)^2}\mathcal{I}\left(\frac{\mathring{m}_\pi^2}{eB}\right) + \frac{2\mathring{m}_K^2}{3}\frac{eB}{(4\pi F)^2}\mathcal{I}\left(\frac{\mathring{m}_K^2}{eB}\right). \quad (47)$$

There are competing contributions: charged pion tadpoles increase the magnetic mass while charged kaon tadpoles reduce the mass.

[4]This observation, though unpublished, was first made by B. Tiburzi. The neutral kaon electric polarizability was studied on the lattice in Ref. [77].

[5]Additionally, there are no finite volume corrections to the neutral kaon magnetic polarizability at next-to-leading order. There are two types of potential sources of finite volume corrections—the first through the finite volume contribution of each tadpole diagram that contributes in infinite volume, the second through purely finite volume diagrams. The latter arises through interactions of the form

$$\mathcal{L}_{2,4} \supset \frac{1}{3F^2}(D_\mu K_+ \partial^\mu K_0 K_- \bar{K}_0 + D_\mu K_- \partial^\mu \bar{K}_0 K_+ K_0) - \frac{1}{6F^2}(D_\mu K_+ \partial^\mu \bar{K}_0 K_- K_0 + D_\mu K_- \partial^\mu K_0 K_+ \bar{K}_0). \quad (46)$$

While they do contribute to the two-point function and the zero momentum projected two-point correlation function, upon source averaging, contributions from purely finite volume diagrams vanish [55].

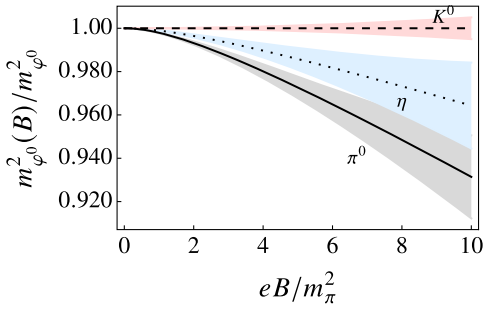


FIG. 1. The magnetic mass of neutral mesons as a function of the magnetic field. The uncertainty bands indicate next-to-next-to-leading order corrections and are generated by varying the relative mass by next-to-next-to-leading order corrections of size $\pm\frac{(eB)^2}{(4\pi F)^4}$.

In Fig. 1, the neutral meson magnetic masses (normalized by their zero-field values) are plotted as a function of the magnetic field up to $eB = 10m_\pi^2$. The neutral pion and octet eta meson magnetic masses both decrease monotonically, while the neutral kaon mass remains unmodified by the magnetic field at next-to-leading order. There is a 7% reduction in the neutral pion mass for the largest magnetic field plotted while the η mass is reduced by approximately 3.5%. Since the masses are independent of low-energy constants, error bands are generated through anticipated next-to-next-to-leading order corrections in the chiral expansion. For $eB \sim m_\pi^2$, the corrections due to higher-order effects can be estimated at approximately 0.02% while for $eB \sim m_K^2$, the estimate is 1.6%. For $eB \sim m_\eta^2$, the estimated corrections are approximately 2.4%. The size of purely magnetic corrections is larger as indicated in the plot bands for Figs. 1 and 2 with the largest magnetic fields chosen to be of size $eB \sim 10m_\pi^2$. For this value, errors are of relative order $\mathcal{O}((eB)/(4\pi F)^2)$ and estimated to be approximately 14.3%. The values of the zero-field masses and decay constants utilized in our analysis are $m_\pi = 138.57$ MeV, $m_K = 493.68$ MeV, $m_\eta = 547.86$ MeV, $\sqrt{2}F_\pi = 130.2$ MeV, and $\sqrt{2}F_K = 155.7$ MeV [78].

IV. MESON OCTET DECAY CONSTANTS

The meson decay constant for φ^Q in a zero magnetic field is defined by the meson-to-vacuum matrix element

$$\langle 0|J_{L\mu}(x)|\varphi^Q(p)\rangle|_{x=0} = iF_{\varphi^Q} p_\mu. \quad (48)$$

The left current, $J_{L\mu}(x)$, is defined as the derivative of the chiral Lagrangian,[6] $\mathcal{L}$, with respect to left-current components, $L^\mu = \sum_{a=1}^8 L^{\mu a}\lambda^a$,

[6]For a next-to-leading order analysis, contributions from $\mathcal{L}_2$, $\mathcal{L}_4$, and $\mathcal{L}_{\text{WZW}}$, presented in Eqs. (1), (14), and (21), respectively, are required.

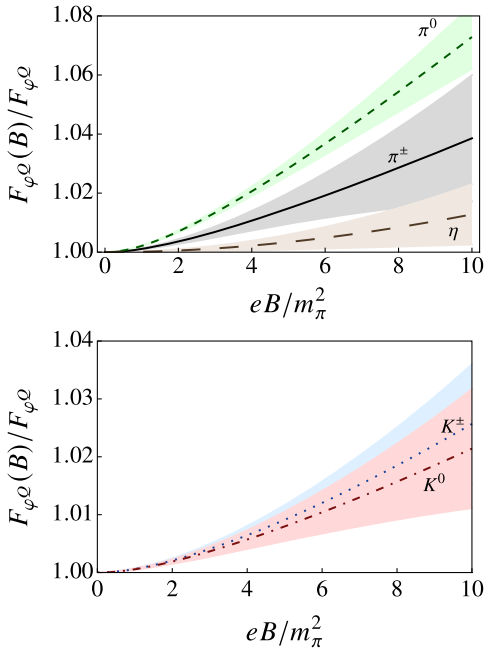


FIG. 2. The axial decay constant, $F^{(A1)}_{\varphi^Q}(B)/F_{\varphi^Q}$, for the meson octet as a function of the magnetic field. (Two plots have been utilized for clarity of presentation.) The uncertainty bands are generated by varying the relative decay constants by next-to-next-to-leading order corrections of size $\pm\frac{(eB)^2}{(4\pi F_\pi)^4}$. The $K^\pm$ decay constant uncertainties are indicated by the light blue shadow while those for K^0 are indicated by the light red shadow. While the full shadow for the charged kaon is not visible, it is identical in size to that of the neutral kaon.

$$J^a_{L\mu} = \left.\frac{\partial \mathcal{L}}{\partial L^{\mu a}}\right|_{L_\mu = R_\mu = -e\mathcal{Q}A_\mu}. \tag{49}$$

The right-hand side of the matrix element can be deduced (up to a proportionality factor) by matching the Lorentz structure. Only the four-momentum of the meson field, p_μ, provides indices with the factor, F_{φ^Q}, identified as the meson decay constant.

However, in a background magnetic field, four-momentum ceases to be a good quantum number for the charged mesons. It is, therefore, necessary to handle matrix elements in position space for the charged mesons—naturally the method is also applicable for the neutral mesons. To this end, consider the correlation function

$$\langle 0|J^{\varphi^Q}_{L\mu}(x')\varphi^Q(x)|0\rangle, \tag{50}$$

for which Lorentz structure is supplied by the covariant derivative, D_μ, Eq. (12). The magnetic field, through either the field strength tensor $F_{\mu\nu}$ or its dual field $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, provides further Lorentz indices.

In addition to Lorentz index matching, parity matching aids the construction of the correlation function. Since the left-current contains both parity-odd and parity-even contributions, it is preferable to consider the parity sectors separately. For the axial current, the matrix element is

$$\langle 0|J^{\varphi^Q}_{A\mu}(x)\varphi^Q(0)|0\rangle = -\left[F^{(A1)}_{\varphi^Q}D_\mu - ieF^{(A2)}_{\varphi^Q}F_{\mu\nu}D^\nu + \cdots\right] \times \langle 0|\varphi^Q(x)\varphi^Q(0)|0\rangle. \tag{51}$$

$F^{(A1)}_{\varphi^Q}$ first appears at tree level while $F^{(A2)}_{\varphi^\pm}$ at next-to-leading order. Since the correlation function is parity-odd, the contributions on the right-hand side must be proportional to the parity-odd covariant derivative. Each term may contain one or more of the parity-even field strength tensors. More generally, the axial pion decay constant $F^{(An)}_{\varphi^Q}$ for $n \geq 1$ is associated with the term containing $n-1$ field strength tensors.

The parity-even contribution to the correlation function, Eq. (50), arises through the vector current,

$$\langle 0|J^{\varphi^Q}_{V\mu}(x)\varphi^Q(0)|0\rangle = -[eF^{(V)}_{\varphi^Q}\tilde{F}^{\mu\nu}D_\mu + \cdots]\langle 0|\varphi^Q(x)\varphi^Q(0)|0\rangle. \tag{52}$$

Since the dual field is parity-odd for a magnetic background, the right-hand side is parity-even in spite of the covariant derivative.

A. Axial meson decay constant, $F^{(A1)}_{\varphi^Q}$

The axial decay constant, $F^{(A1)}_{\varphi^Q}$, is most conveniently constructed through the renormalization of the axial current. Accounting for the wave-function renormalization factor, Z_{φ^Q}, the renormalized axial current admits the structure, $J_{A\mu} \sim [F\sqrt{Z_{\varphi^Q}} + \mathcal{T}_{\varphi^Q} + \mathcal{C}_{\varphi^Q}]D_\mu\varphi^Q$. $\mathcal{T}_{\varphi^Q}$ accounts for tadpole contributions, while $\mathcal{C}_{\varphi^Q}$ accounts for counterterm contributions proportional to L_4 and L_5. Since the tadpoles introduce no magnetic field dependent divergences, the resulting decay constants are devoid of low-energy-constant contributions. The magnetic corrections are proportional to tadpole contributions of charged mesons:

$$\frac{F^{(A1)}_{\pi^\pm}(B)}{F_\pi} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{2}\left\{\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \frac{1}{2}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right], \tag{53}$$

$$\frac{F^{(A1)}_{\pi^0}(B)}{F_\pi} = 1 - \frac{eB}{(4\pi F)^2}\left[\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \frac{1}{4}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right], \tag{54}$$

$$\frac{F^{(A1)}_{K^\pm}(B)}{F_K} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{2}\left\{\frac{1}{2}\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right], \tag{55}$$

$$\frac{F_{K^0}^{(A1)}(B)}{F_K} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{4}\left\{\mathcal{I}\left(\frac{\mathring{m}_\pi^2}{eB}\right) + \mathcal{I}\left(\frac{\mathring{m}_K^2}{eB}\right)\right\}\right], \quad (56)$$

$$\frac{F_{\eta}^{(A1)}(B)}{F_\eta} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{3}{4}\mathcal{I}\left(\frac{\mathring{m}_K^2}{eB}\right)\right]. \quad (57)$$

The magnetic decay constant, $F_{\varphi^Q}^{(A1)}(B)$, has been normalized by the zero-field counterparts at next-to-leading order. For each of the pions, the decay constants, F_π, are degenerate—similarly, the kaon decay constants, F_K. With the exception of the octet η, for which only charged kaon tadpoles contribute, each magnetic decay constant admits corrections through charged pion and kaon tadpoles.

In Fig. 2, we plot these ratios as a function of the magnetic field. Each decay constant increases monotonically with the neutral pion exhibiting the largest increase followed by the charged pions, the charged kaons, the neutral kaons, and the octet η. The increase in the decay constant for the neutral pion is approximately 7% for the largest magnetic field, $eB = 10m_\pi^2$, plotted. For the charged pions, the increase is approximately 4% while the increase for the charged kaons is 2.5%. For the neutral kaons, the increase is approximately 2.0%. The increase for the η is modest, approximately 1%, due to the absence of pion tadpoles.

B. Axial meson decay constant, $F_{\varphi^Q}^{(A2)}$

The magnetic decay constant, $F_{\varphi^Q}^{(A2)}$, is nonzero only for the charged mesons. Additionally, it is a pure finite field contribution. As is evident from the correlation function, Eq. (50), $F_{\varphi^Q}^{(A2)}$ arises through a next-to-leading order contribution with all magnetic field dependence entering the correlation function through the explicit $F_{\mu\nu}$. The contribution to the left current arises through the L_9 and L_{10} terms in the chiral Lagrangian, $\mathcal{L}_4$ [see Eq. (14)],

$$\begin{aligned} J_{L\mu}^a = &-eF_{\mu\nu}L_9\langle\lambda^a\{(\nabla^\nu\Sigma)^\dagger Q\Sigma + \Sigma^\dagger Q\nabla^\nu\Sigma \\ &- \Sigma^\dagger\nabla^\mu\Sigma Q - Q(\nabla^\mu\Sigma)^\dagger\Sigma\}\rangle \\ &- 2eF_{\mu\nu}L_{10}\langle\lambda^a(\nabla^\nu\Sigma^\dagger Q\Sigma + \Sigma^\dagger Q\nabla^\nu\Sigma)\rangle. \end{aligned} \quad (58)$$

The resulting contribution to the left current is nonzero only for the charged mesons. Additionally, their values are identical

$$J_{L\mu}^{\varphi^\pm}\Big|_{\text{NLO}} = -F_{\varphi^\pm}^{(A2)}F_{\mu\nu}D^\nu\varphi^\pm \quad \text{with} \quad F_{\varphi^\pm}^{(A2)} = \frac{\bar{L}}{16\pi^2 F}. \quad (59)$$

The combination, $\bar{L}$, defined in Eq. (42), previously appeared in the charged meson magnetic mass, Eq. (41).

C. Vector meson decay constant, $F_{\varphi^Q}^{(V)}$

The contribution to the vector meson decay constant, $F_{\varphi^Q}^{(V)}$, arises through the WZW Lagrangian, (21). For the charged mesons, the left current at next-to-leading order is gauge covariant

$$J_{L\mu}^{\varphi^\pm}\Big|_{\text{WZW}} = -eF_{\varphi^\pm}^{(V)}\tilde{F}_{\mu\nu}D^\nu\varphi^\pm \quad \text{where } F_{\varphi^\pm}^{(V)} = \frac{1}{8\pi^2 F}, \quad (60)$$

and $\tilde{F}_{\mu\nu}$ is the dual field strength tensor. For the neutral kaon, the structure is identical with the covariant derivative replaced by a regular derivative

$$J_{L\mu}^{K^0}\Big|_{WZW} = -eF_{K^0}^{(V)}\tilde{F}_{\mu\nu}\partial^\nu K^0 \quad \text{with} \quad F_{K^0}^{(V)} = \frac{1}{16\pi^2 F}. \quad (61)$$

Additionally, the vector decay constant is smaller by a factor of 2 compared to that of the charged mesons—the factor is explained at a technical level by the structure of the current, $J_L^{a\nu} \sim -\frac{3e}{(4\pi)^2 F}\langle\{\lambda^a, Q\}\partial^\nu\phi^b\lambda^b\rangle$. Unlike the left current for the neutral kaon, those for the neutral pion and the octet eta are proportional to the derivative of both the neutral pion and eta fields

$$J_{L\mu}^{\pi^0}\Big|_{\text{WZW}} = -eF_{\pi^0}^{(V)}\tilde{F}_{\mu\nu}(\partial^\nu\pi^0 + \sqrt{3}\partial^\nu\eta), \quad (62)$$

$$J_{L\mu}^{\eta}\Big|_{\text{WZW}} = -eF_{\eta}^{(V)}\tilde{F}_{\mu\nu}(\partial^\nu\eta - \sqrt{3}\partial^\nu\pi^0). \quad (63)$$

In the absence of isospin breaking, the neutral pion and eta do not mix, and therefore the neutral pion-to-vacuum matrix element is independent of the eta contribution to the vector current—similarly, for the eta-to-vacuum matrix element. The decay constants can then be identified with the respective neutral mesons

$$F_{\pi^0}^{(V)} = F_\eta^{(V)} = \frac{1}{8\pi^2 F}. \quad (64)$$

They are identical to those of the charged mesons.

V. LOW-ENERGY THEOREMS

Low-energy theorems provide nontrivial cross-checks for the derived renormalized magnetic masses and decay constants (in the axial channel). Explicit chiral symmetry breaking for the neutral mesons is evident from the divergence of the respective axial current components

$$\partial^\mu J_{A\mu}^{\varphi^0} = F_{\varphi^0}^{(A1)}(B)m_{\varphi^0}^2(B)\varphi^0. \quad (65)$$

The corresponding relationship in QCD for the neutral mesons is

$$\partial^\mu\left(\bar{q}\gamma_\mu\gamma_5\frac{1}{2}\lambda_a q\right) = i\bar{q}\gamma_5\left\{\frac{1}{2}\lambda_a, s\right\}q, \tag{66}$$

where s is the quark mass matrix and $q(x)$ is the quark field $q^T(x) = (u(x), d(x), s(x))$. Comparison of the correlation function in chiral perturbation theory

$$\begin{aligned}&\langle 0|\partial^\mu J^{\varphi^0}_{A\mu}(x)\varphi^0(0)|0\rangle \\ &\quad = F^{(A1)}_{\varphi^0}(B)m^2_{\varphi^0}(B)\langle 0|\varphi^0(x)\varphi^0(0)|0\rangle\end{aligned} \tag{67}$$

with the corresponding relation in QCD leads to the low-energy theorems

$$F_{\pi^0}(B)m^2_{\pi^0}(B) = \frac{1}{2}(m_u + m_d)G_{\pi^0}(B), \tag{68}$$

$$F_{K^0}(B)m^2_{K^0}(B) = \frac{1}{2}(m_d + m_s)G_{K^0}(B). \tag{69}$$

For compactness, we have adopted the notation $F^{(A1)}_{\varphi^0}(B) = F_{\varphi^0}(B)$.[7] The pseudoscalar coupling, $G_{\varphi^0}(B)$, is defined in terms of the pseudoscalar matrix element

$$\langle 0|P_{\varphi^0}(x)\varphi^0(0)|0\rangle = G_{\varphi^0}(B)\langle 0|\varphi^0(x)\varphi^0(0)|0\rangle, \tag{73}$$

with the corresponding pseudoscalar operator in QCD defined as $P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x)$. The couplings are readily constructed through the pseudoscalar current [47]

$$J_{Pa} = \left.\frac{\partial\mathcal{L}}{\partial p_a}\right|_{p=0}, \tag{74}$$

which is a partial derivative of the chiral Lagrangian, $\mathcal{L}$, with respect to the pseudoscalar source, p_a. It is proportional to the corresponding pseudoscalar field, $J_P \sim G_\varphi\varphi$. The resulting relations

$$\frac{G_{\pi^0}(B)}{G_\pi} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{4}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right], \tag{75}$$

$$\frac{G_{K^0}(B)}{G_K} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{4}\left\{\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right] \tag{76}$$

are consistent and readily verified using the renormalized meson masses and decay constants. For the neutral pion, the contribution of the charged pion tadpole cancels in the product $F_{\pi^0}(B)m^2_{\pi^0}(B)$, consistent with the expression for $G_{\pi^0}(B)$. For the neutral kaon, since the mass is not modified, the modification of the axial decay constant and the pseudoscalar coupling are identical.

Additionally, the Gell-Mann-Oakes-Renner relations must be obeyed for the neutral pion, Eq. (7), and the neutral kaon, Eq. (8). The shift in the quark condensates in the magnetic background is deduced through the magnetic contribution to the free energy, which is due to the charged mesons. The shift in the up-quark condensate arises through the charged pions and kaons; that of the down-quark condensate arises through the charged pion; and that of the strange-quark condensate arises through the charged kaons

$$\frac{\langle\bar{u}u\rangle_B}{\langle\bar{u}u\rangle_0} = -\frac{eB}{(4\pi F)^2}\left[\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right], \tag{77}$$

$$\frac{\langle\bar{d}d\rangle_B}{\langle\bar{d}d\rangle_0} = -\frac{eB}{(4\pi F)^2}\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right), \tag{78}$$

$$\frac{\langle\bar{s}s\rangle_B}{\langle\bar{s}s\rangle_0} = -\frac{eB}{(4\pi F)^2}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right). \tag{79}$$

$\langle\bar{q}_f q_f\rangle_0 = -B_0F^2$ are the tree-level quark condensates. A low-energy theorem $\langle\bar{d}d\rangle_B + \langle\bar{s}s\rangle_B = \langle\bar{u}u\rangle_B$ naturally follows [38]. It holds up to next-to-leading order and is violated by next-to-next-to-leading order corrections [69]. Utilizing the neutral pion mass, Eq. (44), and the axial decay constant, Eq. (54), we find that

[7] An analogous relation for the octet eta,

$$F^{(A1)}_\eta(B)m^2_\eta(B) = \frac{1}{6}(m_u + m_d + 4m_s)G_\eta(B), \tag{70}$$

does not appear to hold in chiral perturbation theory. Unlike the magnetic corrections to the masses of the neutral pion and the neutral kaons, the magnetic corrections to the octet eta mass (squared) are not the product of the zero-field mass (squared) and a magnetic tadpole. Amusingly, however, in the limit of equal quark masses Eq. (47) reduces to

$$m^2_\eta(B) = m^2_\eta\left[1 + \frac{eB}{(4\pi F)^2}\left\{-\frac{1}{3}\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \frac{2}{3}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right]. \tag{71}$$

The coefficients of $\mathcal{I}$ are then precisely the ones required to satisfy the low-energy theorem for the eta. The renormalized pseudoscalar coupling is found to be

$$\frac{G_\eta(B)}{G_\eta} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{3}\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \frac{1}{12}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right]. \tag{72}$$

The coefficient in the magnetic eta mass of $-\frac{1}{3}$ for the magnetic pion tadpole is consistent with the coefficient above, and the coefficient of $\frac{2}{3}$ for the magnetic kaon tadpole combines with that in the axial decay constant to give $-\frac{1}{12}$, which is precisely the factor present in the pseudoscalar coupling.

$$\frac{m^2_{\pi^0}(B)F^2_{\pi^0}(B)}{m^2_{\pi^0}F^2_{\pi^0}} = 1 + \frac{\langle \bar{u}u + \bar{d}d\rangle_B}{\langle \bar{u}u + \bar{d}d\rangle_0}. \tag{80}$$

In the limit, $B = 0$, the right-hand side reduces to one. Using the neutral kaon decay constant and the observation that the neutral kaon mass is unaltered by the magnetic field at next-to-leading order, the relations

$$\frac{m^2_{K^0}(B)F^2_{K^0}(B)}{m^2_{K^0}F^2_{K^0}} = \frac{F^2_{K^0}(B)}{F^2_{K^0}} = 1 + \frac{\langle \bar{d}d + \bar{s}s\rangle_B}{\langle \bar{d}d + \bar{s}s\rangle_0} \tag{81}$$

exhibit the Gell-Mann-Oakes-Renner relation for the neutral kaon.

For charged mesons, the divergence of the axial current is defined covariantly

$$D^\mu J^{\varphi^\pm}_{A\mu} = -\left[F^{(A1)}_{\varphi^\pm} D_\mu D^\mu + \frac{1}{2}e^2 F^{(A2)}_{\varphi^\pm} F_{\mu\nu}F^{\mu\nu}\right]\varphi^\pm \tag{82}$$

with the corresponding relation from QCD arising upon the replacement of the derivative with a covariant one, $\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + ie\mathcal{Q}A_\mu$, in Eq. (66). The resulting correlation function

$$\langle 0|D^\mu J^{\varphi^\pm}_{A\mu}(x)\varphi^\pm(0)|0\rangle = F_{\varphi^\pm}(B)m^2_{\varphi^\pm}\langle 0|\varphi^\pm(x)\varphi^\pm(0)|0\rangle \tag{83}$$

follows upon utilizing Green's function relation. $F_{\varphi^\pm}(B)$ is defined through the relation

$$F_{\varphi^\pm}(B)m^2_{\varphi^\pm} = F^{(A1)}_{\varphi^\pm}(B)m^2_{\varphi^\pm}(B) - F^{(A2)}_{\varphi^\pm}(eB)^2. \tag{84}$$

The contribution proportional to $F^{(A2)}_{\varphi^\pm}$ [see Eq. (59)] cancels with that from the magnetic mass, $m^2_{\varphi^\pm}(B)$. Consequently, the low-energy theorems for charged mesons are modified

$$F_{\pi^\pm}(B)m^2_{\pi^\pm} = \frac{1}{2}(m_u + m_d)G_{\pi^\pm}(B), \tag{85}$$

$$F_{K^\pm}(B)m^2_{K^\pm} = \frac{1}{2}(m_d + m_s)G_{K^\pm}(B), \tag{86}$$

and consistent with the magnetic pseudoscalar densities

$$\frac{G_{\pi^\pm}(B)}{G_\pi} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{2}\left\{\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \frac{1}{2}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right], \tag{87}$$

$$\frac{G_{K^\pm}(B)}{G_K} = 1 - \frac{eB}{(4\pi F)^2}\left[\frac{1}{2}\left\{\frac{1}{2}\mathcal{I}\left(\frac{\mathring{m}^2_\pi}{eB}\right) + \mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right)\right\}\right]. \tag{88}$$

Moreover, the Gell-Mann-Oakes-Renner relation for the charged mesons generalizes

$$\frac{m^2_{\pi^\pm}F^2_{\pi^\pm}(B)}{m^2_\pi F^2_\pi} = \frac{G^2_{\pi^\pm}(B)}{G^2_\pi} = 1 + \frac{\langle \bar{u}u + \bar{d}d\rangle_B}{\langle \bar{u}u + \bar{d}d\rangle_0}, \tag{89}$$

$$\frac{m^2_{K^\pm}F^2_{K^\pm}(B)}{m^2_K F^2_K} = \frac{G^2_{K^\pm}(B)}{G^2_K} = 1 + \frac{\langle \bar{u}u + \bar{s}s\rangle_B}{\langle \bar{u}u + \bar{s}s\rangle_0}. \tag{90}$$

For the charged pion, the quark condensate ratio is identical to that of the neutral counterpart. However, for the charged kaon, the condensate ratio involves the up-quark condensate.

VI. CONCLUSION

In this paper, we have studied the impact of magnetic fields on the meson octet masses and decay constants using three-flavor chiral perturbation theory. There are some surprises—at next-to-leading order, the magnetic mass for the neutral pion is identical to the result in two-flavor chiral perturbation theory, and the magnetic field does not modify the mass of the neutral kaon. Additionally, no finite volume corrections are realized in the neutral kaon polarizability at this order. The magnetic mass of the octet eta decreases monotonically with the magnetic field. For the charged mesons, the corrections to the renormalized mass and pole mass are identical. The first axial decay constant, $F^{(A1)}_{\varphi^\mathcal{Q}}$, monotonically increases for each of the mesons. The neutral pion shows the most significant increase, followed by the charged pions, the charged kaons, the neutral kaons, and the octet eta. The second axial decay constant, $F^{(A2)}_{\varphi^\pm}$, is nonzero for the charged mesons and degenerate. The vector decay constant, $F^{(V)}_{\varphi^\mathcal{Q}}$, is identical for each of the mesons with the exception of the neutral kaon for which it is smaller by a factor of 2.

The renormalized magnetic masses and decay constants are inputs in the construction of magnetic decay widths, which may be explored further in future work, particularly for the kaons. For the pions, decay width modifications were constructed in two-flavor chiral perturbation theory in Ref. [56] for both the nonleptonic, purely anomalous channel $\pi^0 \to \gamma\gamma$ and the leptonic channel $\pi^+ \to \ell^+\nu_\ell$. Among the inputs required for the analysis, only $F^{(A1)}_{\pi^\pm}(B)$ is modified in three-flavor chiral perturbation theory. The ratio of the three-flavor axial decay constant to its two-flavor counterpart

$$\frac{F^{(A1)}_{\pi^\pm}(B)|_{N_f=3}}{F^{(A1)}_{\pi^\pm}(B)|_{N_f=2}} = -\frac{eB}{(4\pi F)^2}\frac{1}{4}\mathcal{I}\left(\frac{\mathring{m}^2_K}{eB}\right) \tag{91}$$

suggests a 0.3% larger value of the decay constant at $eB = 10m^2_\pi$. Consequently, the charged decay width in three-flavor chiral perturbation theory is not significantly modified by the inclusion of the strange mass.

For the neutral mesons, Gell-Mann-Oakes-Renner relations provide nontrivial tests of the validity of the magnetic masses and decay constants—their alteration must be consistent with the enhancement of the corresponding quark condensates. Furthermore, the relations generalize for the charged mesons with the replacement of the magnetic mass by the renormalized mass at zero field. There are further low-energy theorems involving the pseudoscalar coupling, neutral meson masses, and decay constants. As the calculations of the hadronic properties are nontrivial, these theorems provide handy cross-checks of the renormalized quantities. Future work could investigate finite temperature effects on the magnetic properties of the meson octet—this may be of interest in heavy-ion collisions in which hadron properties are modified not only by the large magnetic fields but also by the temperature of the resulting fireball. A heat bath, like the magnetic field, additionally alters the phase space of the decay products. Another line of investigation could include the impact on the properties of the η' meson, accessed through a large N_c analysis.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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