

Parton distribution functions in the Schwinger model from tensor network states

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Parton distribution functions (PDFs) describe the inner, nonperturbative structure of hadrons. Their computation involves matrix elements with a Wilson line along a direction on the light cone, posing significant challenges in Euclidean lattice calculations, where the time direction is not directly accessible. We propose implementing the light-front Wilson line within the Hamiltonian formalism using tensor network techniques. The approach is demonstrated in the massive Schwinger model (quantum electrodynamics in $1+1$ dimensions), a toy model that shares key features with quantum chromodynamics. We present accurate continuum results for the fermion PDF of the vector meson at varying fermion masses, obtained from first-principle calculations directly in Minkowski space. Our strategy also provides a useful path for quantum simulations and quantum computing.

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Introduction. Investigating the internal structure of protons and neutrons (nucleons), the nuclei's building blocks, is of paramount importance for modern physics. Significant progress is expected in the next years, with increased experimental efforts, such as the Electron-Ion Collider under construction at the Brookhaven National Laboratory [1]. However, a full understanding of experimental developments requires the support of a robust theoretical description, embodied in different kinds of partonic functions. These functions describe the momentum- and coordinate-space distributions of partons, i.e., quarks and gluons, inside the nucleon. The simplest, yet essential, partonic functions are parton distribution functions (PDFs). Even though, in principle, they can be obtained directly from QCD, theoretical access to them is limited by their nonperturbative nature.

In the parton picture, the PDF describes the probability of finding a constituent in a hadron with a fraction ξ of the momentum. Computing the PDF on the lattice involves the calculation of matrix elements with operators separated along a light-front direction. However, for

practical reasons, numerical calculations of lattice QCD have been predominantly performed using the path integral formalism, which requires the formulation of the theory in Euclidean spacetime. Thus, it gives no *direct* access to light-front dynamics. In the past two decades, novel approaches have been proposed [2–11] which calculate intermediate Euclidean objects that can be matched to their Minkowski counterparts. Although tremendous theoretical and numerical progress has been made, much work is still needed to achieve an accurate determination of PDFs from Euclidean lattice QCD [12–18]. Therefore, it remains desirable to have first-principle theoretical determinations of PDFs with computations performed directly in Minkowski space. Such calculations can, in principle, be achieved in the Hamiltonian formalism. However, previous works that extracted PDFs in the Hamiltonian approach [19–21] were based on few-fermion approximations or small momentum lattices and relied on exact diagonalization. These techniques are not scalable and do not allow one to control the systematic errors reliably.

Tensor network (TN) methods [22–29] provide an efficient entanglement-based ansatz for quantum many-body problems that could potentially overcome these limitations. They have been very successful in describing equilibrium states of low-dimensional lattice gauge theories (LGTs) [30–39] and even achieved the most precise continuum extrapolations for some cases. Tensor network algorithms in the Hamiltonian formalism can simulate real-time

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evolution of states with moderate entanglement [22,23,40]. Nevertheless, studies of LGTs in this regard have been less common (see references in Refs. [30,31]) and have not yet accomplished a systematic extrapolation to the continuum limit.

Here, we close this gap and demonstrate that TN methods are also suitable to determine real-time properties of gauge theories in the continuum. Our study shows that PDFs can be calculated with controlled systematic errors by a time evolution of tensor network states, in contrast to previous approaches in the Hamiltonian framework [19–21], in which uncontrolled systematic effects are present. In particular, the entanglement entropy grows during the time evolution of a state. While this can, in principle, challenge the applicability of tensor network methods, we observe that it is not a practical limitation for the calculation of PDFs in the Schwinger model.

Complementary to TN methods, quantum computation and quantum simulation are being explored in recent years as technologies that could potentially overcome the exponential complexity barrier of quantum many-body problems. Their applicability to LGTs has motivated multiple studies [31,41–45], including the possibility of extracting PDFs [43,46–55]. Our approach in the Hamiltonian formalism could also be implemented on quantum hardware.

In this work, we propose a strategy for accessing light-cone dynamics directly in Minkowski space [56]. Its feasibility is demonstrated by calculating the PDF defined in Eq. (3) of the vector meson in the massive Schwinger model for different values of the fermion mass with TN methods. We show that errors can be systematically controlled, allowing for a precise extrapolation to the continuum limit.

Model. The Schwinger model, or QED in $1 + 1$ dimensions [59,60], shares key features with QCD, including dynamical mass generation, confinement, and asymptotic freedom. Consequently, it has been widely used to study QCD-like phenomena and to test new methods in high-energy physics. The QED Lagrange density is given by [61]

$$\mathcal{L} = \bar{\Psi}(i\partial - g\mathcal{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (1)$$

Here, A_μ are $U(1)$ gauge fields with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, m is the fermion mass, and g is the coupling strength. The Hamiltonian formulation follows from a Legendre transformation. We discretize space on a finite lattice of N sites and employ staggered fermions [62] in the temporal gauge $A^0 = 0$. Using a Jordan-Wigner transformation, which maps the fermionic degrees of freedom to spins, and integrating out the gauge variables with the help of Gauss's law, the following Hamiltonian is derived (see Refs. [63–66] and Appendix A [67]):

$$H = x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] + \sum_{n=0}^{N-2} \left[\sum_{k=0}^n \left(\frac{1}{2} ((-1)^k + \sigma_k^z) + q_k \right) \right]^2. \quad (2)$$

It is equivalent to the Schwinger model in the thermodynamic and continuum limit, with the latter approached as the lattice spacing a goes to zero or, equivalently, when $x \equiv \frac{1}{a^2 g^2} \rightarrow \infty$. The dimensionless fermion mass $\mu = \frac{2m}{ag^2} - \frac{1}{4}$ includes an additive renormalization to enhance convergence to the continuum limit [71–73]. Each constant q_k represents the presence of a static charge at site k .

Implementation. For a hadron with momentum P , the unpolarized PDF $f_\Psi(\xi)$ for the fermionic parton described by the spinor field Ψ is defined as [74]

$$f_\Psi(\xi) = \int \frac{dz^-}{4\pi} e^{-i\xi P^+ z^-} \langle P | \bar{\Psi}(z^-) W_{(z^-, 0)} \gamma^+ \Psi(0) | P \rangle. \quad (3)$$

We denote the light-front components of a Lorentz vector $v = (v^0, v^1, \dots, v^d)$ in $d + 1$ dimensions as $v^\pm = (v^0 \pm v^d)/\sqrt{2}$. Here, ξ is the Bjorken scaling variable, which, in the parton model, is equivalent to the fraction of the hadron momentum carried by the fermion, and $W_{(z_2, z_1)}$ is a Wilson line connecting spacetime points z_1 and z_2 along the light-front direction.

We use the matrix product state (MPS) TN ansatz [22,23] to approximate the meson eigenstate [63], as well as the time-dependent vector resulting from the time evolution needed for the implementation of the Wilson line, in the physical subspace in which the Hamiltonian of Eq. (2) is expressed.

The matrix element in Eq. (3) involves a Wilson line along a direction on the light cone, connecting two fermionic operators. The latter, in the spin formulation discussed above, correspond to $\sigma^\pm$ operators preceded by a Jordan-Wigner string of σ^z . On the lattice, we approximate the light front by a stepwise evolution, as illustrated in Fig. 1(a). Each evolution by a timestep Δt is followed by a spatial evolution of the Wilson line by two lattice sites. Setting $\Delta t = \frac{1}{x}$ corresponds to a speed of light $c = 1$, ensuring that the light cone is approached in the continuum limit [75]. The light-cone structure that emerges is illustrated in Fig. 1(b). In the temporal gauge $A^0 = 0$, the Wilson line in time direction becomes the identity operator. Consequently, the vertical lines in Fig. 1(a) correspond solely to the application of a time evolution operator $e^{-iH\Delta t}$.

The steps along the space direction, instead, act non-trivially on the gauge (link) variables, with the effect of shifting the electric flux term of the Hamiltonian by one unit on one link at a time [65,76]. This is exactly like having a pair of charges around that link. Thus, the string

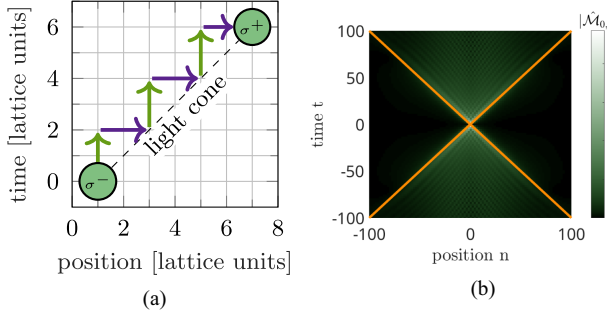


FIG. 1. Light-cone matrix elements. (a) Operators σ^- and σ^+ from Eq. (6) separated along a light front are calculated on the lattice by subsequent steps of a spatial evolution of the Wilson line (violet horizontal arrows) and time evolution steps (green vertical arrows). (b) The light-cone structure that emerges in matrix elements; shown is the modulus of an even-to-even matrix element $\hat{\mathcal{M}}_{0,0} = \langle h | e^{iH\frac{\Delta t}{2}} \prod_{k < \Delta z} (i\sigma_k^z) \sigma_{\Delta z}^+ e^{-iH_0\frac{\Delta t}{2}} \prod_{k' < 0} (-i\sigma_{k'}^z) \sigma_0^- | h \rangle$ for a light mass $\frac{m}{g} = 0.125$. $\hat{\mathcal{M}}_{0,0}$ is similar to $\mathcal{M}_{0,0}$ in Eq. (6), but with a fixed static charge to illustrate the emerging light-cone structure without enforcing a light-front direction by moving the static charge. Orange lines indicate the light-front + and - directions.

can be absorbed in the time evolution by introducing a static charge q_k that ends the Wilson line and moves along the light cone (see Appendix B [67]). Horizontal arrows in Fig. 1(a) starting at a point a with endpoint b thus correspond to decreasing the static charge q_a and increasing q_b by one unit each [77]. This formulation is most suitable for working in the physical subspace, where gauge degrees of freedom have been integrated out. Similar ideas that include static charges have been used to calculate Wilson loops in the Hamiltonian formalism [49,50].

With these techniques, the PDF in Eq. (3) can be evaluated on the lattice and becomes for a hadron of mass M at rest:

$$f_\Psi(\xi) = \frac{NM}{8\pi x} \sum_{\Delta z=0,2,4,\dots,N} e^{-i\xi\frac{M\Delta z}{2x}} \mathcal{M}(\Delta z). \quad (4)$$

If the system is translationally invariant, the matrix elements $\mathcal{M}$ will depend only on the distance between the end points of the Wilson line. Because of the staggered fermions, the Dirac structure leads, however, to four contributions to the matrix elements $\mathcal{M}(\Delta z)$, corresponding to all combinations between even (subscript f or $s = 0$) and odd (subscript $\bar{f}$ or $s = 1$) start and end sites [78]:

$$\mathcal{M}(\Delta z) = \mathcal{M}_{0,0} - \mathcal{M}_{0,1} - \mathcal{M}_{1,0} + \mathcal{M}_{1,1}, \quad (5)$$

$$\begin{aligned} \mathcal{M}_{f,s}(\Delta z) = & \langle h | e^{iH\frac{\Delta t}{2}\Delta z} \prod_{k < \Delta z+f} (i\sigma_k^z) \sigma_{\Delta z+f}^+ e^{-iH_{\Delta z-1}\Delta t} \dots \\ & \times e^{-iH_3\Delta t} e^{-iH_1\Delta t} \prod_{k' < s} (-i\sigma_{k'}^z) \sigma_s^- | h_c \rangle. \end{aligned} \quad (6)$$

The subscript c denotes the subtraction of vacuum contributions from matrix elements [57,74]: $\langle h | \hat{O} | h \rangle_c = \langle h | \hat{O} | h \rangle - \langle 0 | \hat{O} | 0 \rangle$. The ket $|h\rangle$ corresponds to the first excited state of H , a vector meson at rest, with mass $M = \langle h | H | h \rangle_c$ [63]. Hence, the first evolution operator simply introduces a phase $e^{iM\frac{\Delta t}{2}}$. Notice that for the intermediate time-evolution operators $e^{-iH_k\Delta t}$ the subindex k in the Hamiltonian indicates the inclusion of a single positive static charge at position k . For $\Delta z < 0$, the evolution in the operator is implemented in a similar way. In this case, we start with a spatial evolution in the negative direction, followed by a time evolution with a negative time step $-\Delta t$. Figure 1(b) illustrates the light cone and the spread of correlations at different times and distances, in the absence of the light-front Wilson line, and confirms that the speed of light is $c = 1$.

We use the variational optimization described in Ref. [63] to obtain a MPS approximation to the ground state $|0\rangle$ and the first excited state in the chargeless sector $|h\rangle$ of Eq. (2) for a finite system with open boundary conditions. For the time evolution, we use a second-order Suzuki-Trotter decomposition [79,80] and divide each time step into $N_\tau = \frac{\Delta t}{\tau}$ smaller evolution steps that can be applied efficiently to the MPS via the tMPS method [22,23,40]. We closely follow the approach in Ref. [81], where the time evolution operator for the electric component of the Hamiltonian, expressed as a matrix product operator, is truncated to allow for only a change in the electric flux by at most $L_{\text{cut}} = 10$. By varying L_{cut} , we confirm that the results do not depend on the cutoff (see Appendix D [67]).

We set the bond dimension of the initial MPS (i.e., the one used to approximate the eigenstate) and that of the MPS during the time evolution equally to a value D . For all the analyzed cases, we checked that our results show no significant dependence for $D \geq 40$, and thus we choose $D = 80$. The convergence with such moderate values of the bond dimensions indicates that the ansatz is remarkably adequate to capture the properties of this problem. We observe that a large Trotter step introduces a phase error in the matrix elements, causing a shift of the fermion PDF toward smaller ξ . With the choice $N_\tau = 100$, this effect remains insignificant compared to other errors in our simulations. We observe that the dominant error contributions in regions where the PDF is sizable [e.g., $\xi \in (0.3, 0.7)$ at $\tilde{m} = 10$] arise from the finite lattice spacing and the finite-volume effects, the latter dominating close to $\xi = 0.5$. A detailed error analysis is provided in Appendix D [67].

Results. The real part of the matrix element $\mathcal{M}(\Delta z)$ in Eq. (6) can be shown to be symmetric, the imaginary part antisymmetric with respect to $\xi = 0$; see Appendix C [67]. We observe that the real part is several orders of magnitude smaller than the imaginary part, making its contribution to the PDF negligible. As a result, $f_\Psi(\xi)$ is antisymmetric.

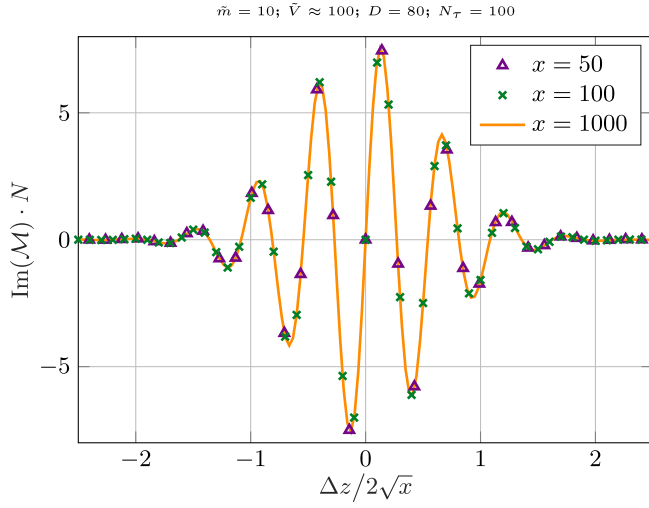


FIG. 2. Imaginary part of matrix element. The values for $x = 1000$ are discrete but shown as a line for clarity.

The imaginary part exhibits oscillatory behavior and vanishes for large $|\Delta z|$. Figure 2 shows the imaginary part of the matrix element for a fixed fermion mass $\tilde{m} \equiv \frac{m/g}{m_v/g} = \frac{5.6419}{1/\sqrt{\pi}} \approx 10$ (where $m_v/g = 1/\sqrt{\pi}$ is the vector meson mass in the $m \rightarrow 0$ limit) and volume $\tilde{V} = \tilde{m} \cdot \frac{N}{\sqrt{x}} \approx 100$, for several values of the lattice spacing $a = \frac{1}{g\sqrt{x}}$. For each value of x , the PDF is obtained according to Eq. (4) from $\mathcal{M}(\Delta z)$ by a discrete Fourier transform. The results are shown in Fig. 3.

The periodic nature of the Fourier transform imposes a bound on the range of ξ that can be studied at finite x , given

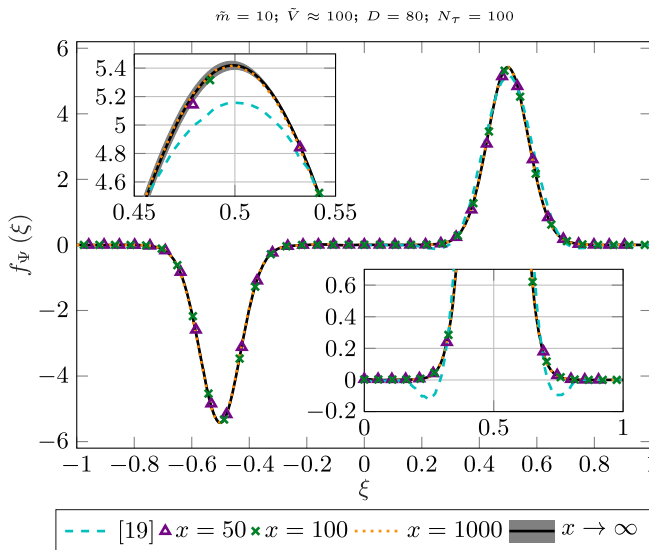


FIG. 3. PDF for different lattice spacings, obtained by a discrete Fourier transform. The values for $x = 1000$ are shown as a dotted line for clarity. Solid line with error bands: extrapolation to the continuum limit at fixed volume. Dashed line: results from Ref. [19] (normalized).

by $|\xi| \leq \xi_{\max} = \frac{\pi}{M}x$. Therefore, x should be chosen large enough such that $\xi_{\max} \geq 1$. Furthermore, the lattice spacing must be fine enough to resolve the oscillations of the matrix elements (see Fig. 2). With these conditions met, the results are close to the continuum limit, allowing us to extrapolate the PDF to this limit (solid line in Fig. 3) using simulation results at different values of x (more details in Appendix D [67]). The volume must be large enough for the matrix element to vanish at large $|\Delta z|$ within the system size. In this case, ξ in Eq. (4) can take real values $|\xi| \leq \xi_{\max}$, which results in a continuous curve for the PDF (as seen in the extrapolation in Fig. 3 and all lines in Fig. 4).

The distribution function of an antifermion $\bar{\Psi}$ [74] can be obtained from $f_{\bar{\Psi}}(\xi) = -f_{\Psi}(-\xi)$. Because of the antisymmetry of the matrix elements and thus of $f_{\Psi}(\xi)$, we numerically find $f_{\Psi}(\xi) \equiv f_{\bar{\Psi}}(\xi)$, which is expected in the continuum and thermodynamic limit due to charge conjugation symmetry (see Appendix C [67]). The average momentum fractions carried by fermions and antifermions are within $\langle \xi \rangle = \int_0^1 d\xi (\xi f_{\Psi/\bar{\Psi}}(\xi)) = 0.50 \pm 0.02$ for all cases in Fig. 4. Thus, fermions and antifermions each carry half of the hadron's momentum, and both have the same PDF. These results agree with the expectations for a meson in the Schwinger model [82].

We compare the PDF for the fermion mass $\tilde{m} = 10$ with the work of Mo and Perry [19], where a finite basis in the two- and four-fermion approximation in light-front field theory was used. We find good agreement of the overall shape. Notably, our calculated $f_{\Psi}(\xi)$, within error bands, remains real and non-negative for $0 \leq \xi \leq 1$ and vanishes

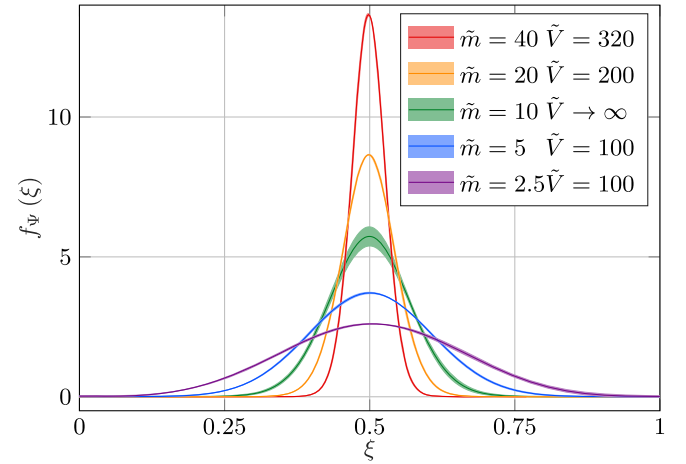


FIG. 4. PDFs for different fermion masses $\tilde{m} = \frac{m\sqrt{\pi}}{g}$. The continuum limit $x \rightarrow \infty$ is taken with the volume $\tilde{V} = \tilde{m} \cdot \frac{N}{\sqrt{x}}$ kept fixed. The error bands denote the uncertainties due to the continuum extrapolations and are smaller than the line width of the plot in most regions. For $\tilde{m} = 10$, instead, we estimate the errors due to finite D , N_τ , and x and take the infinite volume limit. The error band in this case includes all uncertainties, dominated mostly by the volume effects. See the main text and Appendix D [67] for details.

for $|\xi| > 1$, all as required for distribution functions and the probability interpretation in the parton model. In contrast, the PDF calculated by Mo and Perry exhibits regions with negative values of $f_\Psi(\xi)$ (see Ref. [19] and inset on lower right in Fig. 3).

Repeating the procedure for different values of the volume, we extrapolate the result for $\tilde{m} = 10$ to the thermodynamic limit (green curve in Fig. 4). We carry out a systematic error analysis (more details in Appendix D [67]). The error band for the case $\tilde{m} = 10$ in the figure accounts for uncertainties due to D , N_τ , x , and $\tilde{V}$, with the latter two contributions dominating. We obtain a 6% uncertainty at the peak $\xi = 0.5$.

Additionally, we compute the PDF for different fermion masses at fixed but large enough $\tilde{V}$. The results, already extrapolated to the continuum limit, are shown in Fig. 4. We observe that larger volumes $\tilde{V}$ are necessary as $\tilde{m}$ increases in order to resolve more oscillations of the matrix elements that arise due to the steeper features of $f_\Psi(\xi)$. The previous findings remain qualitatively unchanged across all masses, but the peak around $\xi = 0.5$ broadens for smaller $\tilde{m}$ and sharpens for larger $\tilde{m}$. This is in good agreement with previous results [19–21], and with the limits of a step function $f_\Psi(\xi) \equiv 1$ for $0 \leq \xi \leq 1$ at $m = 0$ [82,83] and a delta function at $\xi = 0.5$ when the fermion mass is infinite.

The PDFs for these additional masses are shown at finite volume, but extrapolated to the continuum limit, with error bands (mostly smaller than the line width) indicating the effect of uncertainties due to the extrapolation in x . Even though these PDFs are computed at fixed finite volume, since the value of $\tilde{V}$ is kept at least as large as in the simulations for the $\tilde{m} = 10$ case, we expect that finite-volume effects are similar. This can be tested with further data available in the future.

Conclusion. We have demonstrated how tensor networks can be used to directly compute PDFs from light-cone matrix elements in Minkowski space. In our approach, the light-front Wilson line becomes a stepwise evolution in space and time on the lattice that can be applied on the Hamiltonian eigenstate representing a hadron at rest. Tensor networks provide an efficient representation of eigenstates and enable simulating the necessary time evolution. This framework allows us to compute the PDFs of the fermion and the antifermion in the vector meson of the Schwinger model from first principles, yielding accurate results that fulfill the physical constraints. We present the PDFs for several fermion masses in the continuum limit. In previous TN calculations, the continuum limit of lattice gauge theories was only taken for time-independent properties. Our work extends this to a simulation including a time evolution for the first time, further establishing TNs for the direct calculation of dynamical quantities in lattice gauge theories.

Tensor network states can be applied for higher-dimensional lattice gauge theories. Even though the numerical costs are higher, several works have successfully demonstrated the applicability of projected entangled pair states, multiscale entanglement renormalization ansatz, and tree tensor networks for lattice gauge theories; see Refs. [30,31] for reviews. While the Schwinger model is a $1+1$ -dimensional model, our results demonstrate that light-cone physics can be directly addressed with TNs in a well-controlled way and motivate further investigations in higher dimensions and with non-Abelian gauge theories toward QCD. Our study demonstrates that the relevant light-cone observables can be computed with moderate bond dimensions, which is a promising foundation for higher-dimensional systems. The stepwise evolution of the Wilson line can be adapted there as well, and static charges can be utilized to mimic the Wilson line if the Hilbert space is restricted to the physical space obeying the constraints of Gauss's law. Although further theoretical investigations are needed, notably to address renormalization effects [84], this would open the path on spin- and transverse-momentum-dependent partonic functions, as well as gluon PDFs.

As a related and complementary approach, tensor network formulations like the one presented in this work can also be realized on quantum simulation platforms [31]. The hadronic state that was represented by a tensor network in our work can be transformed into a quantum circuit which prepares the initial state on a quantum computer; see, for example, Ref. [85] and references therein. The stepwise evolution of the Wilson line through a time evolution with a Hamiltonian containing static charges can be readily implemented with established time-evolution techniques on quantum devices [86–88].

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Data availability. The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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