

Temperature derivative divergence of the electric conductivity and thermal photon emission rate at the critical end point from holography

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(Received 8 October 2025; accepted 25 February 2026; published 25 March 2026)

The thermal photon emission rate $\frac{d\Gamma}{dk}$ and dc electric conductivity σ_Q of the strongly coupled quark-gluon plasma are investigated around the critical end point in a $N_f = 2 + 1$ holographic QCD model with parameters obtained from machine learning. It is found that both thermal photon emission rate and electric conductivity grow most obviously around T_c , which agrees with the previous studies, and the result of electric conductivity at zero chemical potential resembles the lattice results. Moreover, it is found that both the temperature derivative of the electric conductivity and thermal photon emission rate diverge at the critical end point.

DOI: [10.1103/PhysRevD.113.056024](https://doi.org/10.1103/PhysRevD.113.056024)

I. INTRODUCTION

The existence and location of the critical end point (CEP) in the QCD phase diagram has been widely investigated theoretically [1–8]. The search for CEP serves as a prominent goal of heavy ion collision (HIC) experiments in facilities like the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL), Facility for Antiproton and Ion Research at GSI Helmholtz Centre for Heavy Ion Research, Nuclotron-based Ion Collider Facility at Dubna, and High Intensity heavy-ion Accelerator Facility in Huizhou. The nonmonotonic behavior of baryon number fluctuations has been considered as a signature of the CEP [9–12]. The latest STAR results [13,14] do not show clear evidence of the existence of CEP for collision energy $\sqrt{s} > 7.7$ GeV. From the analysis in Ref. [15], the baryon number fluctuation signature might be distorted by the hadronization process. Therefore, a different signature of the CEP which is insensitive to the hadronization would be needed.

For the quark-gluon plasma (QGP) produced in the HICs, one important class of observables is the electromagnetic probes, which mainly include photons and

dileptons. The thermal photon and dilepton produced in heavy-ion collisions are important observables in studying the properties of QGP [16,17]. Because of their colorless nature and relatively long mean free path, once generated, they pass through the fireball with barely further interaction. Thus, the information about the early stage of the collision and its subsequent evolution is well preserved by those electromagnetic probes. Photons produced in HIC can be divided into different types, including prompt (produced in early hard parton scattering), thermal (produced in QGP and latter hadron phases), and decay (from the decay of final-state hadrons) photons. The prompt and thermal photons are of particular interest experimentally and are referred to as “direct” photons. However, photons produced in various stages are detected collectively. Therefore, it is necessary to differentiate photon production from different sources. Thermal photons mainly contribute to the total production in the low transverse momentum region. The observed excess of the direct photon in Au + Au (PHENIX) [18] and Pb + Pb (ALICE) [19] is attributed to thermal photons. Much theoretical work has been devoted to calculating the production of direct photons, including perturbative QCD (pQCD) calculations for leading order [20] and next-to-leading order (NLO) [21], lattice simulations [22,23], and results from effective models such as holographic models [24–26] and linear sigma model [27]. However, a clear understanding of the possible signature of the CEP from photon emissions has not yet been concluded.

Apart from photon emission, the transport properties are necessary for both the understanding of QGP and the behavior near critical end point. Many effective models are constructed to study the real time evolution of the

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plasma (e.g., Boltzmann transport, hydrodynamics, and hadronic cascade [28–31]); the transport coefficients serve as parameters in those models. For example, it has been shown that strong electromagnetic fields are generated in the HIC process, and the electric conductivity, the transport coefficient which describes how the medium reacts to an external electric field, is crucial for the simulations. Except for phenomenological applications, the transport coefficients themselves characterize the nonequilibrium dynamics of strongly coupled plasmas and the behaviors of which can also indicate phase transition. It has been shown that an enhancement of bulk viscosity occurs around T_c [32–34], and the claim that shear viscosity can indicate phase transition has been made as well [35]. Similarly, the electric conductivity of QGP which is directly related to the photon and soft dilepton production also has the possibility to serve as a signal of phase transition. In fact, it has already been shown by both lattice results [36–38] (for a review, see Ref. [39]) and various phenomenological studies [25,40–42] that a “rapid growth” or “jump” occurs at the transition temperature T_c . For the crossover region, it is pseudotransition temperatures, but for simplicity we will still call them T_c here and below. Moreover, a series of recent work based on the Nambu–Jona-Lasinio model [43,44] predicts that the electric conductivity will diverge at the CEP. Besides, in spite of the abundant studies, there is no consensus on the magnitude of the electric conductivity yet [45]. This suggests that the research on the electric conductivity is far from settled.

Although much work has been done to investigate the photon production through pQCD and lattice calculations, as for the search for CEP, these results are very limited in the region of interest. Specifically speaking, QGP around CEP is strongly coupled, and thus the perturbative methods are not applicable, while lattice calculation is limited within zero or small baryonic chemical potential regions due to the “fermion sign” problem. On the other hand, the gauge/gravity duality [46–48] has emerged as a powerful tool to investigate the strongly coupled QGP. The duality claims that a strongly coupled gauge theory in four-dimensional spacetime is dynamically equivalent to a gravitational theory in 5 dimensional Anti-de Sitter (AdS₅) spacetime. Inspired by this, many holographic models have been proposed based on either string theory constructions (top down) [49,50] or properties of QCD (bottom up) [51–58]. The calculation of the photon emission rate using gauge/gravity duality was first investigated in Ref. [24], where the photon emission rates of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) plasma under both weakly and strongly coupled circumstances were given. The calculations for the non-conformal plasma soon followed [25,26,41]. The strong electromagnetic field can be created in the HICs; consequently, extensions of photon production in anisotropic plasmas [42] have also been explored, in which an external magnetic field induced the anisotropy. Despite all these efforts, the photon emission near the CEP remains largely

unexplored in the context of holography. Among all those bottom-up models, the Einstein-Maxwell-dilaton (EMD) model [53,59] was constructed to mimic the equations of state that capture the crossover transition. Recently, the machine-learning method [60] has been employed to determine the key parameters of the EMD model. Using this machine-learning model, we will calculate the thermal photon emission rate and electric conductivity of a strongly coupled QGP in thermal equilibrium. Actually, these two quantities can be calculated using the same retarded correlator [24]. To achieve that, we need to add a Maxwell action into the original model in the same spirit of linear-response theory, and the backreaction of the newly added action to the background should be neglected.

In this work, we use the notation $d\Gamma/dk$ as photon emission rate, where Γ denotes the number of photons emitted per unit time per unit volume and $k = \omega$ is the momentum of the photon. This follows the convention from previous holographic studies [24,26,41]. There is also a different definition $d\Gamma/d^3k$ of photon emission rate [22,23,61]. The rest of this paper is organized as follows. In Sec. II, we give a brief review of the EMD model and introduce the machine-learning model in Ref. [60]. In Sec. III A, we first present both the deduction of dc electric conductivity and the photon emission rate. We then show how the coupling function in probe action is determined in Sec. III B, where the electric conductivities computed at both zero and finite chemical potential are presented. Our numerical result of photon emission rate is given in Sec. III C. Finally, we give conclusion and discussion in Sec. IV.

II. EINSTEIN-MAXWELL-DILATON FRAMEWORK

A. EMD model revisit

First, we give a brief introduction to the EMD framework [42,53,58,59,62–73], which has proven to be quite successful in capturing key features of hot/dense QCD matter. Following previous studies, in the Einstein frame, the general form of the EMD action, which mimics the QCD equation of state, takes the form

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - \frac{f(\phi)}{4} F^2 - \frac{1}{2} \partial_M \phi \partial^M \phi - V(\phi) \right]. \quad (1)$$

Here, G_5 denotes the five-dimensional (5D) Newton constant, and R is the Ricci scalar. The action includes a $U(1)$ Maxwell field A_M with F_{MN} as its field strength tensor, a dilaton field ϕ , and its potential $V(\phi)$. The $U(1)$ field A_M is introduced as the dual part of the baryonic current, and its temporal component A_0 on the boundary gives the baryonic chemical potential. The five-dimensional metric g_{MN} is the ($M, N = 0, 1, \dots, 5$) and is taken as

$$ds^2 = \frac{L^2 e^{A(z)}}{z^2} \left[-g(z) dt^2 + \frac{dz^2}{g(z)} + d\vec{x}^2 \right]. \quad (2)$$

Here, z represents the fifth-dimensional holographic coordinate, and L is the AdS radius. With this metric ansatz, the EMD system provides a charged black brane solution, the corresponding blackening factor is given by $g(z)$ in (2), and we use z_h to denote the horizon radius below [$g(z_h) = 0$]. The dilaton field ϕ is introduced to break the conformal invariance.

As suggested in Refs. [60,64], the deformed metric $A(z)$ in Eq. (2) takes the ansatz

$$A(z) = d \ln(az^2 + 1) + d \ln(bz^4 + 1), \quad (3)$$

and the parameters are fixed by employing machine-learning techniques to fit lattice QCD equation-of-state data at zero baryon chemical potential ($\mu_B = 0$).

The gauge kinetic function $f(\phi)$ couples to the gauge field A_μ describing the chemical potential dependent property, and following Ref. [60], it can be taken as

$$f(\phi(z)) \equiv f(z) = e^{cz^2 - A(z) + k}, \quad (4)$$

where c and k are two model parameters.

The dilaton potential $V(\phi)$ can be systematically solved from the equations of motion

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R - T_{MN} &= 0, \\ \nabla_M [f(\phi) F^{MN}] &= 0, \\ \partial_M [\sqrt{-g} \partial^M \phi] - \sqrt{-g} \left(\frac{\partial V}{\partial \phi} + \frac{F^2}{4} \frac{\partial f}{\partial \phi} \right) &= 0, \end{aligned} \quad (5)$$

with

$$T_{MN} = \frac{1}{2} \left(\partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} (\partial \phi)^2 - g_{MN} V(\phi) \right) + \frac{f(\phi)}{2} \left(F_{MP} F_N^P - \frac{1}{4} g_{MN} F^2 \right). \quad (6)$$

In the following calculation, we set the AdS radius $L = 1$. In the machine-learning EMD model [60], the specific expressions of the functions mentioned above are given by

$$g(z) = 1 - \frac{1}{\int_0^{z_h} dx x^3 e^{-3A(x)}} \left[\int_0^z dx x^3 e^{-3A(x)} + \frac{2c\mu^2 e^k}{(1 - e^{-cz_h^2})^2} \det \mathcal{G} \right], \quad (7)$$

$$\det \mathcal{G} = \begin{vmatrix} \int_0^{z_h} dy y^3 e^{-3A(y)} & \int_0^{z_h} dy y^3 e^{-3A(y) - cy^2} \\ \int_{z_h}^z dy y^3 e^{-3A(y)} & \int_{z_h}^z dy y^3 e^{-3A(y) - cy^2} \end{vmatrix}, \quad (8)$$

$$\phi'(z) = \sqrt{6(A'^2 - A'' - 2A'/z)}, \quad (9)$$

$$A_t(z) = \mu \frac{e^{-cz^2} - e^{-cz_h^2}}{1 - e^{-cz_h^2}}, \quad (10)$$

$$V(z) = -3z^2 g e^{-2A} \left[A'' + A' \left(3A' - \frac{6}{z} + \frac{3g'}{2g} \right) - \frac{1}{z} \left(-\frac{4}{z} + \frac{3g'}{2g} \right) + \frac{g''}{6g} \right], \quad (11)$$

where μ is the baryon chemical potential. The Hawking temperature is given by

$$T = \frac{z_h^3 e^{-3A(z_h)}}{4\pi \int_0^{z_h} dy y^3 e^{-3A(y)}} \left[1 + \frac{2c\mu^2 e^k (e^{-cz_h^2} \int_0^{z_h} dy y^3 e^{-3A(y)} - \int_0^{z_h} dy y^3 e^{-3A(y) - cy^2})}{(1 - e^{-cz_h^2})^2} \right], \quad (12)$$

and the baryon chemical potential $\mu = 3\mu_q$ corresponds to the UV boundary ($z = 0$) condition

$$A(0) = -\sqrt{\frac{1}{6}} \phi(0) = 0, \quad g(0) = 1, \quad (13)$$

$$A_t(0) = \mu + \rho' z^2 + \dots \quad (14)$$

Here, ρ' corresponds to a quantity proportional to the baryon number density. Following Refs. [69,74], the baryon number density and baryon chemical potential can be derived as

TABLE I. Parameters for $N_f = 2 + 1$ system from machine learning [60].

N_f	a	b	c	d	k	G_5
2 + 1	0.204	0.013	-0.264	-0.173	-0.824	0.400

$$\rho = \left| \lim_{z \rightarrow 0} \frac{\partial \mathcal{L}}{\partial (\partial_z A_t)} \right| = -\frac{1}{16\pi G_5} \lim_{z \rightarrow 0} \left[\frac{e^{A(z)}}{z} f(\phi) \frac{d}{dz} A_t(z) \right], \quad (15)$$

$$\mu = -\sqrt{\frac{-1}{\int_0^{z_h} y^3 e^{-3A} dy \int_{y_g}^y \frac{x}{e^A f} dx}} \int_0^{z_h} \frac{y}{e^A f} dy. \quad (16)$$

All parameters have already been determined by a machine-learning algorithm [60] and are listed in Table I for the $N_f = 2 + 1$ case. With these parameters, we will investigate the electromagnetic properties of the holographic QGP phase.

B. Probe action

As can be seen, the EMD action given by Eq. (1) does not contain any term involving the electromagnetic current. However, such a term is necessary to study the electromagnetic properties of the dual medium. Based on the previous calculations of thermal photon emission rate and transportation coefficients [24,25,40], we take the same method to add a probe action to (1). In the same spirit of linear response theory, the newly added action only serves as an external probe, and its backreaction to the background metric is neglected. This in fact corresponds to the case where the electric charge chemical potential vanishes, i.e., $\mu_Q = 0$. Building on previous studies, the probe action can be simply taken as

$$S_{\text{EM}} = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \frac{f_Q(\phi)}{4} F_Q^2. \quad (17)$$

Here, the additional $U(1)$ gauge field A_Q , whose field strength is F_Q , is holographically dual to the electromagnetic current, and the additional coupling function $f_Q(\phi)$ characterizes the coupling of this additional gauge field to the dilaton. We will compare the following two choices for $f_Q(\phi)$ in later calculations:

$$(1): f_Q(\phi) = \alpha_1 = \text{constant}, \quad (18)$$

$$(2): f_Q(\phi) = \alpha_1 + \alpha_2 \text{sech}[\beta\phi(z)], \quad (19)$$

with parameters $\alpha_1, \alpha_2, \beta$. The constant choice can be treated as a special case of (19), where $\alpha_2 = 0$, $f_Q(\phi)$ is fully parametrized by α_1 .

These two forms of f_Q are motivated by the following considerations. In Refs. [25,40], $f_Q(\phi)$ is taken to be either a single $\text{sech}(\phi)$ or a linear combination of two such terms. Here, the constant α_1 in (19) is introduced so that the dc conductivity saturates quickly enough at high temperature. Besides, one might wonder why we use a constant $f_Q(\phi)$ as in (18). Actually, using gauge/gravity duality, when one of the earliest calculations of thermal photon emission rate was carried out [24], the coupling of the probe action is a constant. Also, in several later studies, the $f_Q(\phi)$ s used are all monotonically decreasing in $z \in [0, z_h]$ from small positive values to zero with a slow speed [25,26,40]. Specifically, the plots of these choices are relatively flat and look like the ones of a constant; thus, we regard the (18) choice as not far from a reasonable one. And actually, it will be shown that the constant choice (18) does give a result that resembles the $\phi(z)$ -dependent choice (19).

As for the model parameters, we expect our thermal photon emission rate to reach the result of $\mathcal{N} = 4$ SYM plasma [24] at high temperature (above $3T_c$), and since the overall amplitude of thermal photon emission rate is proportional to the electric conductivity, our choice of $f_Q(\phi)$ should consequently coincide with the electric conductivity of $\mathcal{N} = 4$ SYM plasma as well at high T . Therefore, we shall adjust the parameters in our coupling based on the requirement that our electric conductivity matches the SYM result and lattice calculations.

Finally, we emphasize that, since the baryonic number has already been included in the EMD action (1), it is straightforward to consider the electromagnetic properties of the dual medium at finite densities. This is particularly relevant at the CEP, which has not yet been carefully analyzed using holographic models.

III. THERMAL PHOTON EMISSION RATE AND ELECTRIC CONDUCTIVITY CALCULATION

A. Derivation of thermal photon emission rate and electric conductivity

In this section, we describe the methods for calculating the thermal photon emission rate and the dc electric conductivity using holography. To study thermal photon emission, we adopt the static background metric described in Sec. II, assuming that the plasma is in local thermal equilibrium in this stationary spacetime. In this case, an early work [24] gave the thermal photon emission rate and several transport coefficients of $\mathcal{N} = 4$ super-Yang-Mills plasma based on gauge/gravity duality, and here we take similar procedures. The number of photons emitted per unit time per unit volume is denoted by Γ , and it takes the form

$$d\Gamma = \frac{d^3k}{(2\pi)^3} \frac{e^2}{2|\mathbf{k}|} \eta^{\mu\nu} C_{\mu\nu}^<(K)|_{k^0=\omega=|\mathbf{k}|}, \quad (20)$$

$$C_{\mu\nu}^<(K) = n_b(\omega)\chi_{\mu\nu}(K) = \frac{1}{e^{\omega\beta} - 1}\chi_{\mu\nu}(K), \quad (21)$$

where $C_{\mu\nu}^<(K)$ is the Wightman function of the electromagnetic currents

$$C_{\mu\nu}^<(K) = \int d^4X e^{-K \cdot X} \langle J_\mu^{\text{EM}}(0) J_\nu^{\text{EM}}(X) \rangle \quad (22)$$

and $n_b(\omega)$ denotes the Bose-Einstein distribution function and $\beta = 1/T$, while $\chi_{\mu\nu}(K)$ is the spectral density and (21) holds in the case of thermal equilibrium. Here, $X = (t, \mathbf{x})$ and $K = (\omega, \mathbf{k})$ denote spacetime coordinates and momentum, the capital letters represent 4-vectors, and the bold font like in $\mathbf{k}$ and the ones with latin index i like k^i denote 3-vectors.

For $\chi_{\mu\nu}(K)$, it is proportional to the imaginary part of the retarded current-current correlator $C_{\mu\nu}^{\text{ret}}(K)$:

$$\chi_{\mu\nu}(K) = -2 \text{Im} C_{\mu\nu}^{\text{ret}}(K). \quad (23)$$

When $T \neq 0$, we can separate $C_{\mu\nu}^{\text{ret}}(K)$ into its transverse (Π^T) and longitudinal (Π^L) parts,

$$C_{\mu\nu}^{\text{ret}}(K) = P_{\mu\nu}^T(K) \Pi^T(\omega, \mathbf{k}) + P_{\mu\nu}^L(K) \Pi^L(\omega, \mathbf{k}), \quad (24)$$

where $P_{\mu\nu}^T(K)$ and $P_{\mu\nu}^L(K)$ are transverse and longitudinal projectors:

$$P_{00}^T(K) = 0, \quad P_{0i}^T(K) = 0, \quad (25)$$

$$P_{ij}^T(K) = \delta_{ij} - k_i k_j / \mathbf{k}^2, \quad (26)$$

$$P_{\mu\nu}^L(K) \equiv P_{\mu\nu}(K) - P_{\mu\nu}^T(K), \quad (27)$$

$$P_{\mu\nu}(K) = \eta_{\mu\nu} - K_\mu K_\nu / K^2. \quad (28)$$

Here, $\eta_{\mu\nu}$ denotes the Minkowski metric $(-, +, +, +)$. Contracting the indices of $\chi_{\mu\nu}$, one can get

$$\chi^\mu{}_\mu(K) = -4 \text{Im} \Pi^T - 2 \text{Im} \Pi^L. \quad (29)$$

For photons, the longitudinal part $\text{Im} \Pi^L$ vanishes. Therefore, we only need the transverse part for the following calculations.

Now, with some simplification to (20) (see Appendix A for more details), we can finally express the photon emission rate $d\Gamma/dk$ in terms of $\text{Im} \Pi^T$,

$$\frac{d\Gamma}{dk} = -4 \frac{\alpha_{\text{EM}}}{\pi} k \frac{1}{e^{k\beta} - 1} \text{Im} \Pi^T, \quad (30)$$

where the fine structure constant $\alpha_{\text{EM}} \equiv \frac{e^2}{4\pi}$ whose value is taken as $\frac{1}{137}$ in the following calculations. Now, it becomes clear that to get the emission rate $\frac{d\Gamma}{dk}$ one only needs to calculate the correlation function $C_{\mu\nu}^{\text{ret}}(K)$ (or its transverse part Π^T) and substitute it back into (30).

TABLE II. Three different sets of parameters for $f_Q(\phi(z))$. Set 1 and set 2 are chosen to make electric conductivity reach SYM limit at high temperature above $3T_c$, while set 3 provides a better match to the lattice data within the error bars.

Parameters	α_1	α_2	β
Set 1	1.19	0	0
Set 2	0.81	0.41	0.4
Set 3	0	1.8	0.4

On the other hand, the calculation of electric conductivity σ_Q can be expressed in terms of the same current-current correlation function [24],

$$\sigma_Q = \lim_{\omega \rightarrow 0} \frac{e^2}{4T} \eta^{\mu\nu} C_{\mu\nu}^<(K) \big|_{k^0 = \omega = |\vec{k}|}, \quad (31)$$

and it can be further simplified as

$$\sigma_Q = -\lim_{\omega \rightarrow 0} 4\pi\alpha_{\text{EM}} \text{Im} \Pi^T \frac{1}{\omega}. \quad (32)$$

Thus, once we obtain the transverse part of retarded correlator Π^T , we can get both the σ_Q and photon emission rate $d\Gamma/dk$. According to the AdS/CFT dictionary, the generating functional of the four-dimensional gauge theory can be expressed in terms of the on-shell action of the 5D gravitational theory. Following the prescription in Ref. [75], one can eventually get the transverse part of retarded correlation function (for details, please see Appendix A),

$$\Pi^T = \lim_{z \rightarrow 0} \left[-2 \cdot \frac{1}{16\pi G_5} g^{zz} \sqrt{-g} \frac{f_Q(\phi)}{4} \frac{\partial_z A_i(z, k)}{A_i(z, k)} \right], \quad (33)$$

where we choose the 4-momentum of photon to be $K = (\omega, 0, 0, k)$, and the index i should take the value of 1 or 2 (the summation over i is not implied here). The overall factor 2 here comes from the prescription. With (33), once the equation of motion is solved, one can easily obtain the electric conductivity and photon emission rate. In this paper, our choice of coupling does not provide us with an analytical solution, so we solve the equation of motion using a numerical method. Moreover, to obtain a physical solution, the incoming boundary condition should be imposed at the horizon.

B. Numerical result of electric conductivity

1. Electric conductivity at zero chemical potential

Given the metric described in Sec. II and specific values of the parameters in f_Q , it is straightforward to obtain the holographic results for the dc electric conductivity using the formula derived in the previous section. We calculate the electric conductivity at $\mu_B = 0$ using three sets of

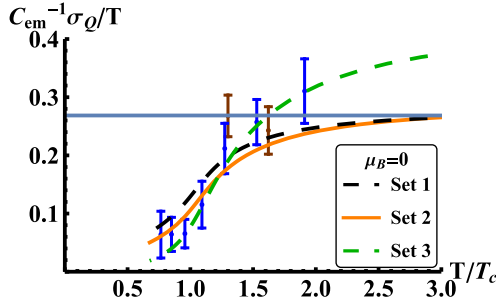


FIG. 1. Holographic results of dc electric conductivity scaled by TC_{em} as a function of T/T_c for three different coupling functions $f_Q(\phi)$, comparing with lattice results. The lattice results are marked by brown [38] and blue [36,37] with error bars, while the horizontal light-blue line corresponds to the $\mathcal{N}=4$ SYM result [24]. The other three curves represent holographic results of electric conductivity using three different sets of parameters: 1) black dashed line for $f_Q = \alpha_1 = 1.19$; 2) orange solid line for $\alpha_1 = 0.81, \alpha_2 = 0.41, \beta = 0.4$; and 3) green dashed line for $\alpha_1 = 0, \alpha_2 = 1.8, \beta = 0.4$. We also normalize the temperature using (pseudo)transition temperature T_c at zero chemical potential.

parameters listed in Table II. The numerical results from holographic model for dc electric conductivity scaled by TC_{em} are shown in Fig. 1 as a function of T/T_c , comparing with lattice results [38] and [36,37] shown in data with error bars, where the blue horizontal line is the $\mathcal{N}=4$ SYM limit (at $N_c = 3$) from Ref. [24], and three curves correspond to holographic results with three different sets of parameters, respectively. Here, $C_{em} = e^2 \sum_f q_f^2$, q_f^2 refers to the number of electric charge carried by the quark, and in the case of $N_f = 3$, one has $C_{em} = 2e^2/3$.

The lattice results [36–38] have different values of T_c at zero chemical potential; T_c is around 184 MeV in Refs. [36,37] and 153 MeV in Ref. [38]. Thus, for each type of data, we use the normalized temperature T/T_c . For the machine-learned model, $T_c = 128$ MeV when $\mu_B = 0$. In Fig. 1, the lattice data exhibits an increase in electric conductivity with temperature, and it matches the $\mathcal{N}=4$

SYM result at high temperature. It is observed that for set 1 and set 2, the dc electric conductivity scaled by TC_{em} grows fast in the region of $1 \sim 1.5T_c$, and slower above $1.5T_c$ before reaching the SYM limit.

Behaviors of electric conductivity can be modified by different parameters. Set 1 and set 2 enable the electric conductivity to reach SYM limit at high temperature (above $3T_c$). If we relax the high-temperature constraint, a better fit to the lattice data can be achieved using set 3. The result is shown by the green dashed curve in Fig. 1. Using parameters in set 1 and set 2, the resulting electric conductivity reproduces the key qualitative trends seen in lattice QCD and yields a semiquantitative agreement with existing data. As discussed before, the constant coupling (black dashed line) does give a result close to the lattice simulation. But it still deviates from the lattice error bars slightly at low T ; also, it lacks flexibility for further adjustment. Meanwhile, although set 2 (orange solid line) is very close to the constant one, it is more adjustable.

However, due to the broad error bars of the lattice result, it is always possible to find many couplings by either changing parameters or using other forms such that whose outcomes appear in “good” agreement with the lattice data. Rather than fine tuning the fit within the large lattice error bars, we focus on the generic behavior of the photon emission rate and the electric conductivity across the transition temperature. We therefore choose set 2 in the following calculations, whose photon emission rate also approaches the $\mathcal{N}=4$ SYM value at high temperatures and remains saturated thereafter. An analogous prescription was adopted in Ref. [26], and both this choice and SYM calculations were found to be close to experimental data [41]. We thus consider it a reasonable choice.

2. Electric conductivities at different chemical potentials

Since the model parameters are already fixed, finite-density effects follow directly from the holographic setup without further tuning. We take $\mu_B = 0, 0.5, 0.74$ GeV as examples; the electric conductivities (left) and their first order derivatives (right) are shown in Fig. 2. The red,

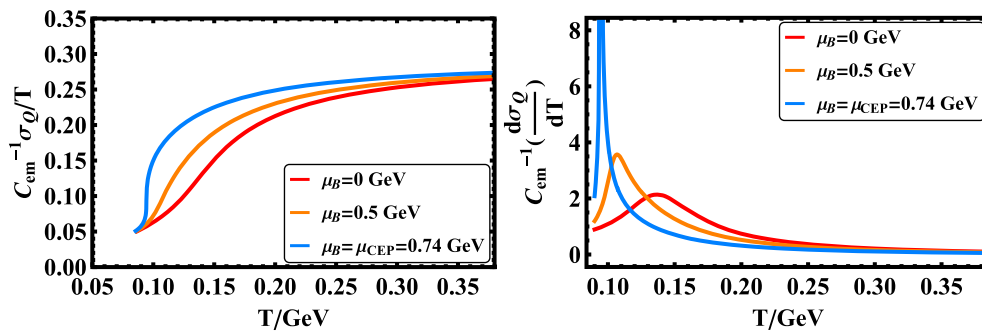


FIG. 2. Left: electric conductivities calculated using parameters in set 2 as a function of the temperature for $\mu_B = 0$ (red), 0.5 (orange) and 0.74 (blue) GeV. Right: first-order derivatives of electric conductivity at $\mu_B = 0$ (red), $\mu_B = 0.5$ GeV (orange) and $\mu_B = \mu_{CEP} = 0.74$ GeV. Each curve exhibits a peak which corresponds to the inflection point in the left panel.

TABLE III. The position of inflection points and transition temperatures are given. In the crossover region, the T_c s are very close to the inflection points, and at the CEP, the inflection point coincides with the transition temperature.

μ_B (GeV)	0	0.5	0.74
Transition temperature (GeV)	0.128	0.103	0.094
Inflection points (GeV)	0.136	0.107	0.094

orange, and blue curves correspond to $\mu_B = 0, 0.5, 0.74$ GeV cases, respectively. The machine-learning model predicts the CEP to be at $T_{\text{CEP}} = 0.094$ GeV, $\mu_{\text{CEP}} = 0.74$ GeV, so the blue line is expected to capture the CEP-induced critical behavior. In Fig. 2 (left), all curves monotonically increase with respect to T and μ_B ; they are convex below T_c and become concave above $1.5T_c$, indicating the presence of an inflection point around T_c . This can be seen more clearly from the right panel, where first-order derivatives are non-negative and approach zero at large T . The exact positions of inflection points are determined by the locations of the peaks in the derivative plot. The locations of the inflection points are given in Table III.

It is well known that the critical exponents are very important to characterize physical quantities changing around the CEP. Depending on the quark masses and flavor number in QCD theory, the QCD critical point may belong to either the Z(2) or O(4) universality class [76,77]. Although highly precise fittings to the equation of state have been achieved using these EMD models, the resulting critical exponents are still at the vicinity of mean field theory results [53,78–80]. For the model we used here, the critical exponents were obtained in Ref. [81], which are found to be very close to the mean field results.

The most rapid growth of electric conductivity occurs at the inflection points thus a correspondingly steep rise in the photon emission rate is expected, as both quantities are governed by the same current-current correlator. When increasing chemical potential, the conductivity curve becomes steeper, and eventually at CEP, the conductivity curve develops into a singular slope. From the perspective of the first-order derivative, the width of the peak shrinks, and its height rises with increasing chemical potential, while the singular slope at the CEP corresponds to a divergence of the first-order derivative. At high temperature, the increase of all three curves slows down and stops at certain values. Specifically, at $T = 0.35$ GeV, the normalized conductivity $C_{\text{EM}}^{-1}\sigma_Q/T$ of $\mu_B = 0.74$ GeV reaches 0.272 ($\mu_B = 0.74$), while for $\mu_B = 0$ and $\mu_B = 0.5$ GeV case, it is 0.261 and 0.266 respectively.

For $\mu_B = 0, 0.5$ GeV, our results for the inflection points are very close to the T_c given by the machine-learning EMD model (difference within 10 MeV). As mentioned before, in the crossover region ($\mu_B < \mu_{\text{CEP}}$), the T_c is in fact a pseudocritical temperature since the transition there is a

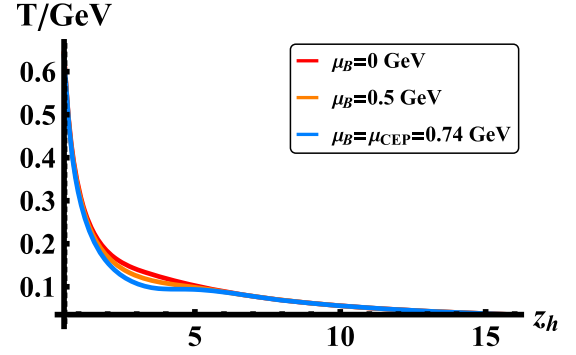


FIG. 3. Temperature dependence on z_h at $\mu_B = 0$ (red), 0.5 GeV (orange), and 0.74 GeV (blue). At both high and low temperature, all three curves converge into a single one. We only consider the region of medium temperature where the temperature changes more significantly with μ_B .

continuous type. In lattice calculations, different observables often lead to different values of T_c , and since the machine-learning EMD model we used is based on lattice result, we do not expect the inflection points of our electric conductivity to match the T_c perfectly. However, such a small difference still indicates the relatively rapid rise of electric conductivity occurs when the phase transition happens. On the other hand, the coincidence of the inflection point with transition temperature is exact at the CEP, which means that the most significant growth of dc electric conductivity will occur at the CEP.

Apart from these features, one might have noticed that the conductivity curves seem to coincide at low temperatures, and this has to do with the metric at different chemical potentials. As shown from the deduction of σ_Q in Sec. IV, the key step to obtain dc conductivity is to substitute the solution of equation of motion into the action. In this step, the only difference between all three cases lies in the metric. According to (2), (3), and (7), only the blackening factor (7) in the metric depends on μ_B and z_h . The influence of μ_B on metric can thus be investigated through the temperature-horizon plot.

In Fig. 3, we plot the temperature against the position of horizon z_h for different values of the chemical potential μ_B . The red, blue, and orange curves represent the temperature dependence on z_h at $\mu_B = 0, \mu_B = 0.5$ GeV, and $\mu_B = \mu_{\text{CEP}} = 0.74$ GeV respectively. For both very small and large z_h , corresponding to very large and small temperatures, the three curves converge into a single one. Under this circumstance, all three curves correspond to the same value of z_h for a certain temperature. This implies that the differences in the metric are negligible in these regions. Consequently, the values of σ_Q coincide at very low and high temperatures, as shown in Fig. 2. However, we would like to point out that this coincidence at low temperature might be due to some model artifact. In fact, this model is reliable only within a certain temperature range, where T is neither too high nor too low. Finally, Fig. 3 shows that the

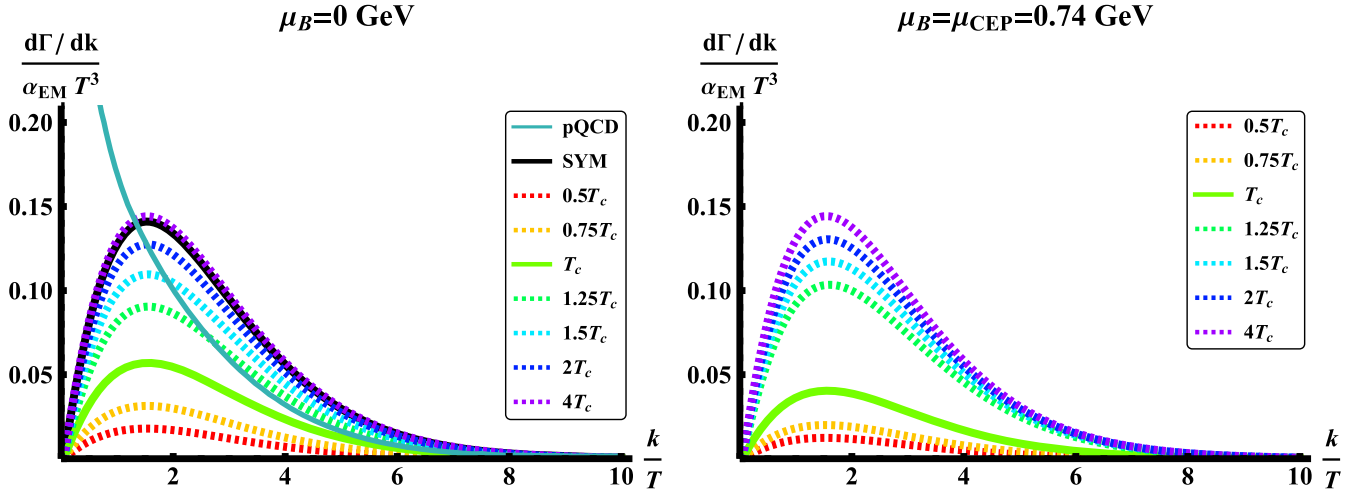


FIG. 4. Thermal photon emission rate plotted against the momentum k/T , divided by fine structure constant α_{EM} and temperature cubed. Left: $\mu_B = 0$. The dashed lines represent thermal photon emission rate calculated using parameter set 2 at different temperatures. From red to purple, the temperature increases. The SYM and pQCD (up to NLO) result are also shown for comparison, and they are plotted using black and light blue solid lines, respectively. Right: $\mu_B = \mu_{\text{CEP}} = 0.74$ GeV case. The coupling function is also parametrized by Set 2. The emission result computed at CEP is plotted by the green solid curve.

CEP in this EMD model can be attributed to the flat shape of the $T - z_h$ curve near the CEP. Consequently, the divergence of the temperature derivative may also manifest in many other thermal quantities. Thus, we expect that this divergence is independent of the holographic settings chosen here.

C. Numerical result of thermal photon emission rate

Similar to the conductivity, the photon emission rate can also be obtained from the current-current correlator. In Fig. 4, we show the normalized photon emission rate $\frac{d\Gamma/dk}{\alpha_{\text{EM}} T^3}$ against the momentum of photon k/T (normalized by temperature). The analytic expression for the SYM and pQCD result are given in Refs. [24,21], respectively; for the SYM emission rate, we take $N_c = 3$. With our normalization notation, both the SYM and pQCD emission rate $\frac{d\Gamma/dk}{\alpha_{\text{EM}} T^3}$ are functions of k/T only and independent of temperature. In Fig. 4, the pQCD result [21] corresponds to the $g_s = 2$ and $N_f = N_c = 3$ case, whose temperature is estimated to be over 2 GeV ($\Lambda_{\text{QCD}} = 200$ MeV). At large k/T region, our result resembles the pQCD calculation, while at small k/T region, the pQCD result deviates from ours greatly and eventually diverges at $k/T = 0$.

In this figure, the left panel shows the result of zero chemical potential, while the right one represents the $\mu_B = \mu_{\text{CEP}}$ case. In both plots, each curve corresponds to a certain temperature, and they are chosen to be $T = 0.5, 0.75, 1, 1.25, 1.5, 2, 4 T_c$ (the first five temperatures increase linearly). The emission rate at CEP is marked by a solid green line. Our results share some features with those from previous holographic studies, which include the following:

- (1) For each curve, the normalized photon emission rate first increases and then falls with respect to k/T (momentum of photon normalized by temperature). The maximum value of the normalized emission rate is located around $k = 1.5T$ [41] (for our result, maximum values sit at $k = 1.55T$).
- (2) The overall amplitude of $\frac{d\Gamma/dk}{\alpha_{\text{EM}} T^3}$ monotonically increases as one increase the temperature or μ_B (as in Refs. [25,41]); at high enough T , it eventually saturates and almost stops at a certain distribution (as in Refs. [26,41]). With our choice of $f_Q(\phi)$, $\frac{d\Gamma/dk}{\alpha_{\text{EM}} T^3}$ reaches the $\mathcal{N} = 4$ SYM result [24] at high T and eventually saturates at a value slightly higher than SYM result.

Apart from these common features, in the left panel of Fig. 4, one can observe that the emission rate at $T = T_c$ is relatively far from its neighboring curves ($T = 0.75T_c, 1.25T_c$). Furthermore, the low temperature curves ($T = 0.5T_c$ and $T = 0.75T_c$) and high temperature curves ($T = 2T_c$ and $T = 4T_c$) are rather close to each other. This implies that the emission rate increases faster around T_c , while at both relatively lower ($< 0.8T_c$) or higher ($> 2T_c$) temperature, the increase of $\frac{d\Gamma/dk}{\alpha_{\text{EM}} T^3}$ is much slower. In the right panel ($\mu_B = \mu_{\text{CEP}}$), the same feature becomes more obvious.

This behavior can be checked more easily in Fig. 5, where both temperature and momentum dependence of the emission rate are shown. The same as in Fig. 4, each colored curve represents the emission rate at a specific temperature. In Fig. 5, the x and y axes denote momentum and temperature, respectively, and the z direction stands for the emission rate. The result at CEP is marked by a solid

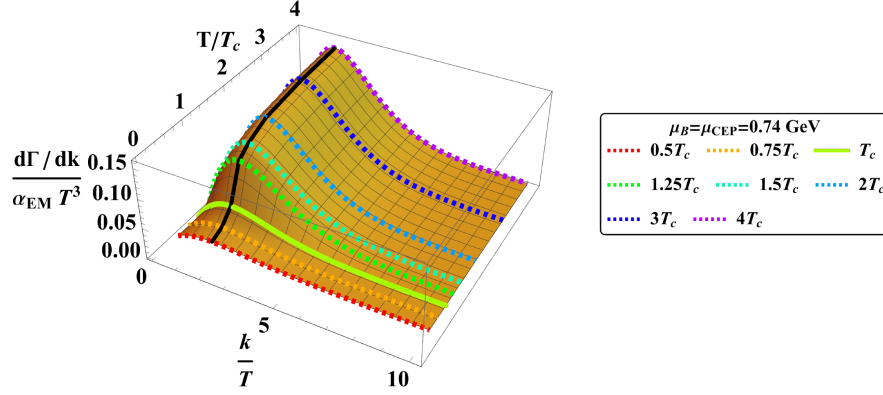


FIG. 5. Photon emission rate at $\mu_B = \mu_{\text{CEP}} = 0.74$ GeV. Each curve corresponds to one particular temperature. From red to purple, the corresponding temperatures rise from $0.5T_c$ to $4T_c$. In particular, the green solid line represents the photon emission rate at CEP. The overall amplitude of photon emission rate increases with temperature. Similar to the conductivity before, the most rapid growth occurs at the critical temperature as shown by the obvious slope in this 3D figure. To show the slope tendency more clearly, a black line connecting the peaks of all curves is depicted.

green line. The fast growth of the emission rate is characterized by the slope in Fig. 5. To see how the emission rate varies according to the temperature more clearly, a black line connecting the maximum values of the emission rate is also shown in Fig. 5. Actually, from the deduction, one can see that the photon emission rate depends on z_h continuously. Therefore, the divergence of $dz_h/dT = 1/(dT/dz_h)$ at CEP might cause the divergence of the photon emission rate, since $d/dT = d/dz_h \times dz_h/dT$. Correspondingly, we also found a divergence in the temperature derivative of the photon emission rate at the CEP, which can be considered as a generic prediction from the EMD model. To demonstrate the divergence, we calculated the derivative of photon emission rate fixing $k/T = 1.55$, which corresponds to the black curve in Fig. 5. But because the figure is too similar to the electric conductivity plot Fig. 2, we leave this figure in Appendix B. Additionally, we also found that the photon emission rate shows a jump when passing through of first order transition line, which is presented in Fig. 6 of Appendix B.

The significant difference between our result and the ones from pQCD is not surprising, since the pQCD method is reliable only at sufficiently high temperature where the QGP is weakly coupled. On the contrary, the EMD model was constructed to capture the feature of a strongly coupled QGP and is thus limited to a lower temperature range. Therefore, one should not expect that the result obtained by this model can be continuously extended to weakly coupled regimes. Moreover, as mentioned above, our result saturates at a certain distribution at high enough temperatures. However, a true holographic dual theory to QCD should be able to reproduce the pQCD predictions. This suggests that for the emission rate given by such a complete dual theory the peak in the plot should increase in height and shift toward the low k/T region as the temperature rises. We will leave this task to the future.

IV. CONCLUSIONS

In a machine-learning holographic EMD model with a CEP at finite baryon density, we calculate the thermal photon emission rate and the dc electric conductivity in thermal equilibrium at different temperatures and baryonic chemical potentials, especially around the CEP. We choose the overall normalization factor to make our thermal photon emission rate reach the SYM limit above $3T_c$. Our result on photon emission rate captures some common features of previous holographic calculations, and our calculation shows that the thermal photon emission rate experiences the most rapid growth at the CEP with the increase of temperature. Using linear sigma model, similar result has also been obtained, indicating a jump of photon emission rate [27]. Moreover, in previous works, the electric conductivity is described as exhibiting a “jump” or a “rapid growth” around the transition temperature. In this work, we explore the behavior of electric conductivity around the CEP in detail. It is found that, within the model we used, the electric conductivity reaches its inflection point around the transition temperature in the crossover region (with a difference less than 10 MeV). It is worth mentioning that the temperature derivative of both the dc electric conductivity and photon emission rate show divergence at the CEP, which corresponds to a singular slope at $\mu_B = \mu_{\text{CEP}}$. One might ask whether the choice of $f_Q(\phi)$ affects our results. In fact, as pointed out in Sec. II, $f_Q(\phi)$ typically varies slowly in the region of interest. Therefore, our conclusion remains general, and changing $f_Q(\phi)$ should not cause the transition temperature to deviate significantly from the inflection point of the $\sigma_Q/T - T$ plot. Similarly, the critical behavior of thermal photon should not be affected either. We regard the singular slope of dc electric conductivity as a signature of the CEP.

The behaviors of photon emission rate and electric conductivity are not unanimous for different models.

Although our result matches other previous holographic and lattice result, another recent paper using the Nambu–Jona-Lasinio model [43] gave different predictions of dc electric conductivity. In their work, instead of a significant increase, a divergence in dc electric conductivity is expected to occur at the CEP. A possible mechanism for this enhancement might be due to the nonequilibrium properties near the critical point [61]. These properties cannot be captured in our near-equilibrium analysis. Therefore, although the current analysis has already provided partial enhancement of the photon emission rate near the CEP (the divergence of the growth rate), further investigations, especially including coupling with the critical long-wavelength modes, are still needed to achieve a full understanding through holography. However, as the electrical conductivity and photon emission rate are responses of the vector current—which does not directly couple to the scalar order parameter—it is reasonable that these quantities do not diverge at the CEP. Instead, the divergence of their temperature derivatives captures key features of the CEP. It deserves further studies in the future to clarify the feature of the electric conductivity and photon emission rate at CEP from effective models as well as from other holographic framework.

ACKNOWLEDGMENTS

We thank Zhibin Li, Hong-An Zeng, and Ruixiang Chen for helpful discussions. Yi-Ping Si also wants to thank Xun Chen for providing model related *Mathematica* file. This work is supported in part by the National Natural Science Foundation of China (NSFC) Grants No. 12235016, No. 12221005, and No. 12275108 and the Fundamental Research Funds for the Central Universities.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DETAILS IN THE DEDUCTION OF THE PHOTON EMISSION RATE AND ELECTRIC CONDUCTIVITY

1. Photon emission rate

One can integrate over k in (20), and since $d^3k = k^2 dk d\Omega$, we have

$$\Gamma = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{e^2}{2k} \eta^{\mu\nu} C_{\mu\nu}^< = \int \frac{e^2 dk}{4\pi^2} k \eta^{\mu\nu} C_{\mu\nu}^<.$$

Taking the derivative to k , one can get

$$\frac{d\Gamma}{dk} = \frac{e^2}{4\pi^2} k \eta^{\mu\nu} C_{\mu\nu}^< = \frac{\alpha_{\text{EM}}}{\pi} k \eta^{\mu\nu} C_{\mu\nu}^< \quad (\text{A1})$$

$$= \frac{\alpha_{\text{EM}}}{\pi} k \frac{1}{e^{k\beta} - 1} \eta^{\mu\nu} \chi_{\mu\nu} = -2 \frac{\alpha_{\text{EM}}}{\pi} k \frac{1}{e^{k\beta} - 1} \eta^{\mu\nu} \text{Im} C_{\mu\nu}^{\text{ret}}(K) \quad (\text{A2})$$

$$= -4 \frac{\alpha_{\text{EM}}}{\pi} k \frac{1}{e^{k\beta} - 1} \text{Im} \Pi^T. \quad (\text{A3})$$

2. Electric conductivity

The simplification from (31) to (32) is straightforward:

$$\sigma_Q = \lim_{\omega \rightarrow 0} \frac{e^2}{4T} n_b(k^0) \chi^\mu{}_\mu|_{k^0=\omega=|\vec{k}|} \quad (\text{A4})$$

$$= -\lim_{\omega \rightarrow 0} \frac{e^2}{T} \text{Im} \Pi^T n_b(k^0)|_{k^0=\omega=|\vec{k}|} \quad (\text{A5})$$

$$= -\lim_{\omega \rightarrow 0} \frac{e^2}{T} \text{Im} \Pi^T \frac{1}{e^{k/T} - 1} \Big|_{k^0=\omega=|\vec{k}|} \quad (\text{A6})$$

$$\Rightarrow \sigma_Q = -\lim_{\omega \rightarrow 0} 4\pi\alpha_{\text{EM}} \text{Im} \Pi^T \frac{1}{\omega}. \quad (\text{A7})$$

3. Equations of motion and on-shell action

To simplify the deduction, we rewrite the metric (2) as

$$ds^2 = \frac{1}{\zeta^2(z)} \left[-g(z) dt^2 + \frac{dz^2}{g(z)} + d\vec{x}^2 \right], \quad \frac{L^2 e^{2A(z)}}{z^2} \equiv \frac{1}{\zeta^2(z)}. \quad (\text{A8})$$

The equation of motion of the probe action (17) is almost the Maxwell equation; the only difference is the additional coupling $f_Q(\phi)$:

$$\partial_M (\sqrt{-g} f_Q(\phi) F^{MN}) = 0. \quad (\text{A9})$$

Using (A9), one can get the equations of motion of every component (with gauge condition $A_z \equiv 0$),

$$\omega(\partial_z A_0) + [g(z)q(\partial_z A_3)] = 0 \quad (\text{A10})$$

$$\partial_z \left[\frac{f_Q}{\zeta(z)} (\partial_z A_0) \right] - \left[\frac{q f_Q}{g(z)\zeta(z)} (q A_0 + \omega A_3) \right] = 0 \quad (\text{A11})$$

$$\frac{g(z)\zeta(z)}{f_Q} \partial_z \left[\frac{f_Q g(z)}{\zeta(z)} (\partial_z A_i) \right] + [\omega^2 - q^2 g(z)] A_i = 0, \quad i = 1, 2 \quad (\text{A12})$$

$$\partial_z \left[\frac{f_Q g(z)}{\xi(z)} (\partial_z A_3) \right] + \frac{\omega f_Q}{g(z) \xi(z)} [(qA_0 + \omega A_3)] = 0; \quad (\text{A13})$$

as mentioned above, only the transverse part (A12) is what we really needed (as $\text{Im} \Pi^L$ vanishes for photons).

In the probe action, the only term that is related to our correlator is

$$\int dz d^4x \frac{1}{16\pi G_5} g^{zz} \sqrt{-g} \frac{f_Q(\phi)}{4} \partial_z A_i(z, x) \partial_z A_i(z, x), \quad i = 1, 2, \quad (\text{A14})$$

and after Fourier transformation and integration by parts, we have

$$\int d^4k \frac{1}{16\pi G_5} g^{zz} \sqrt{-g} \frac{f_Q(\phi)}{4} A_i(z, -k) \partial_z A_i(z, k) \Big|_{z=0}^{z=z_h}, \quad (\text{A15})$$

where

$$\begin{aligned} A_i(z, x) &= \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} A_i(z, k) \\ &\equiv \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} A_i(0, k) f_k(z) \end{aligned} \quad (\text{A16})$$

and $f_k(0) = 1$.

According to (2), $g^{zz} \propto g(z)$, on the other hand, $g(z_h) = 0$ at $z = z_h$; thus, Eq. (A15) becomes

$$\int d^4k \frac{-1}{16\pi G_5} \frac{f_Q(\phi)}{4} g^{zz} \sqrt{-g} A_i(z, -k) \partial_z A_i(z, k) \Big|_{z=0}. \quad (\text{A17})$$

Using this and the prescription in Ref. [75], one can eventually get

$$\Pi^T = \lim_{z \rightarrow 0} \left[-2 \cdot \frac{1}{16\pi G_5} g^{zz} \sqrt{-g} \frac{f_Q(\phi)}{4} \frac{\partial_z A_i(z, k)}{A_i(z, k)} \right], \quad (\text{A18})$$

where $f_{-k}(z)$ is implicit due to $f_{-k}(0) = 1$ and the overall factor 2 here comes from the prescription.

APPENDIX B: SUPPLEMENT: CRITICAL BEHAVIOR OF PHOTON EMISSION RATE

Figure 6 shows the photon emission rate fixing $k/T = 1.55$. Corresponding to the black curve in Fig. 5, the blue curve here gives the photon emission rate at the CEP, while the red and orange lines stand for the $\mu_B = 0, 0.5$ GeV case separately. One can clearly see from this figure that photon emission rate increases most significantly at the critical temperature and exhibits a singular slope at CEP, which corresponds to a divergence of first-order derivative of emission rate to temperature, as mentioned in Sec. III C. Additionally, the emission rate crossing the first-order transition line is plotted in purple, whose chemical potential $\mu_B = 0.9$ GeV. This result is obtained after removing the unstable and metastable states across the phase transition. The overall increase of the emission rate with respect to temperature and chemical potential still exists. More importantly, the jump occurring at the CEP remains evident along the first-order transition line, which is qualitatively consistent with Ref. [27].

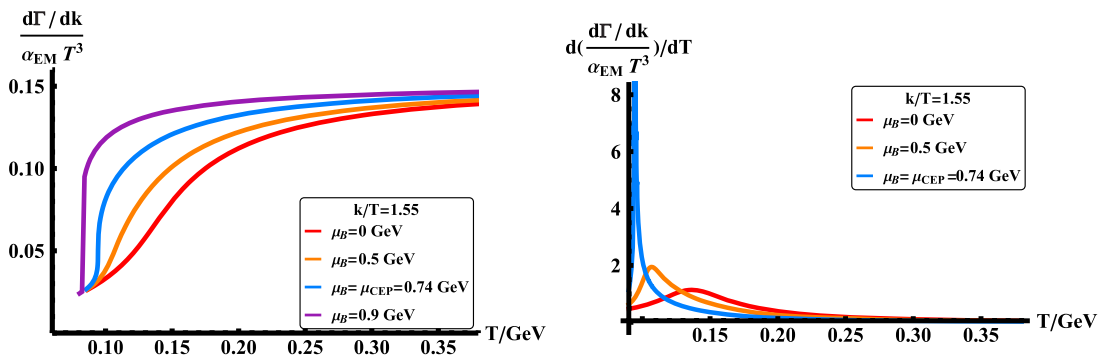


FIG. 6. Left: photon emission rate at $\mu_B = 0, 0.5, 0.74$ GeV, while the momentum over temperature is fixed as $k/T = 1.55$, which makes the blue curve here correspond to the black line in Fig. 5. The red and orange curves correspond to the $\mu_B = 0$ and $\mu_B = 0.5$ GeV circumstances, respectively. The purple curves correspond to the $\mu_B = 0.9$ GeV case, whose transition temperature is located on the first order transition line. Right: the temperature derivative of photon emission rate at $k/T = 1.55$, using same notation as the left panel.

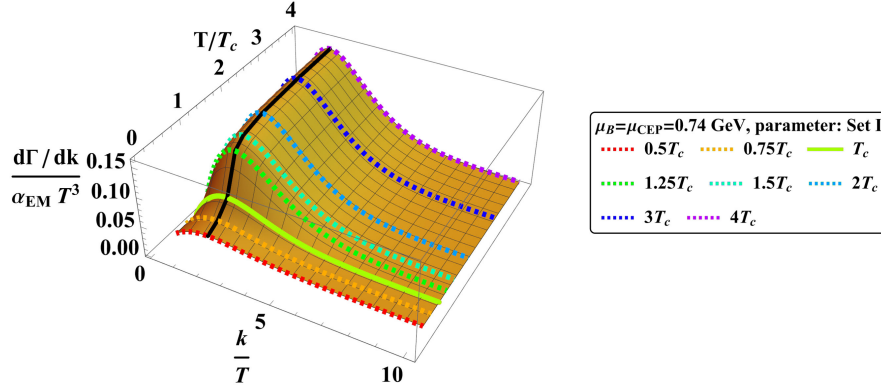


FIG. 7. Photon emission rate at $\mu_B = \mu_{\text{CEP}} = 0.74$ GeV, obtained using parameter set 1, different from the one we used in Fig. 5. The photon emission rate at CEP is still marked by solid green curve. The 2D surface is plotted by connecting all the emission rate curves.

APPENDIX C: PHOTON EMISSION RATE USING ANOTHER SET OF PARAMETERS

As argued above, even though the parameters choice for $f_Q(\phi)$ in (19) can influence emission rate, the existence of a pronounced rise across the critical temperature is unaffected. We present a concrete example in Fig. 7 below.

In comparison with Fig. 5, we plotted Fig. 7 using parameter set 1 in Table II. The most prominent growth is still located at T_c as marked by the solid green curve, and the overall dependence of emission rate on temperature is the same as Fig. 5. This verifies our argument that the qualitative behavior of photon emission rate across the transition temperature is unaffected by the parameters in $f_Q(\phi)$.

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