

Higher dimensional quantum Hall effect and the analog of W_∞ -algebra

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(Received 17 July 2026; accepted 3 September 2026; published 23 September 2026)

We show that Abelian and non-Abelian gauge transformations are the analog of W_∞ transformations for higher dimensional quantum Hall effect. The commutator anomaly and the extended algebra of such transformations on the edge modes of a droplet are obtained by purely topological arguments, basically utilizing the two-cocycle in the descent procedure for anomalies and using the fact that there is anomaly cancellation between the bulk and boundary actions. The method relies on the fact that bulk actions are easily constructed in general using the Dolbeault index theorem. The resulting algebras are shown to agree with explicit edge mode calculations for cases where they are available. We also comment on the similarities and differences in the nature of these transformations between two and higher dimensions.

DOI: [10.1103/l8nj-kpgg](https://doi.org/10.1103/l8nj-kpgg)

I. INTRODUCTION

The quantum Hall effect has been the subject of abiding research interest for a few decades, both experimentally and theoretically [1]. Although much of the research has been on the two-dimensional case, many features of this effect and the mathematical structures involved admit interesting extensions to higher dimensions [2–6]. Interestingly, QHE in higher dimensions is also experimentally realizable using the idea of synthetic dimensions [7]. Edge excitations of a droplet of fermions is one of the many interesting features of QHE. It is perhaps easiest to characterize these for quantum Hall states with filling fraction equal to one. Since inter-Landau level transitions are energetically disfavored for high values of the magnetic field, one may consider the low energy dynamics as confined to a particular Landau level, say, the lowest one. In two dimensions, the degeneracy of the level is given by the area of the sample in units of an elemental area defined by the magnetic length, with each elemental area corresponding to a single quantum state. Since the exclusion principle forbids multiple occupancy, each fermion takes up an elemental unit of area. They also cluster towards the minimum of any additional confining potential, thus forming a droplet of area proportional to the number of fermions. This droplet behaves as a fluid in the limit of large number of fermions, it is

also incompressible by virtue of the exclusion principle. Since the occupied area has to be preserved, the low energy excitations of this droplet will correspond to area-preserving diffeomorphisms of the droplet, manifesting as chiral waves on the edge of the droplet. Quantum theoretically, each fermion occupies a state in the single-particle Hilbert space $\mathcal{H}$. Excitations correspond to unitary transformations on $\mathcal{H}$, modulo the invariances of the density matrix of the droplet.

The canonical or symplectic structure for the single-particle dynamics is proportional to the geometric area, by virtue of projection to a single Landau level. Therefore area-preserving diffeomorphisms correspond to canonical transformations; in particular they can be viewed as $U(1)$ -valued gauge transformations of the symplectic potential or one-form. In the quantum theory, they are therefore naturally elevated to unitary transformations of the single-particle Hilbert space.

The edge excitations can be described by a chiral boson field, in terms of which one can define a current operator which is part of the generator of the $U(1)$ transformations mentioned above. The key defining feature for the edge modes is that there is a central extension for the algebra of currents given by a two-cocycle which is related to the anomaly structure of the theory. The purpose of this paper is to show how this arises from the bulk action via the descent method for anomalies, a method which can then be extended to the higher dimensional cases.

It is useful to recall some of the key features familiar from two dimensions before proceeding to the case of arbitrary even dimensions. The transformation of interest is the $U(1)$ gauge transformation of the electromagnetic vector potential, which is also accompanied by the transformation $\psi \rightarrow e^{i\theta}\psi$, $\psi^\dagger \rightarrow e^{-i\theta}\psi^\dagger$ for the fermion field

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operators. The canonical generator of this transformation on ψ , $\psi^\dagger$ is the density operator $\rho(\theta) = \int \theta \psi^\dagger \psi$, with the commutation rule $[\rho(\theta), \rho(\theta')] = 0$. The fermion field operator restricted to the lowest Landau level (LLL) has the expansion

$$\psi = \sum_k c_k \Psi_k(x), \quad \psi^\dagger = \sum_k c_k^\dagger \Psi_k^*(x), \quad (1)$$

where $\{\Psi_k\}$ are a basis of single-particle LLL wave functions and $c_k, c_k^\dagger$ are the annihilation and creation operators. With this restriction ρ does not commute with itself for different test functions, but obeys the rule

$$[\rho(\theta), \rho(\theta')] = \rho(\{\{\theta, \theta'\}\}), \quad (2)$$

where $\{\{\theta, \theta'\}\}$ denotes the commutator of the star product of θ, θ' . This is the W_∞ -algebra in two dimensions [8–11].¹ Notice that this is really the $U(1)$ gauge transformation restricted to a particular Landau level. One can also apply this transformation to the edge modes, it then gives an extension to the current algebra on the edge. This feature, we emphasize, will hold in general. In higher dimensions, the analog of W_∞ will again be the $U(1)$ (and possible non-Abelian) gauge transformations restricted to a particular Landau level. Alternate interpretations such as area-preserving diffeomorphisms will be dimension dependent.

Returning to the general case, we start by showing, in the next section, how the commutator anomaly for the generators of the gauge transformation is related to a two-cocycle [13–15] which is in turn, related to the descent equations for anomalies [15]. We will then use the fact that any lack of gauge invariance is canceled between the explicit variation of the bulk action and the anomaly of the boundary theory to relate the extension of the current algebra, or the commutator anomaly of the edge modes, to the bulk action. The *raison d'être* for this approach is that, while one can work out the extension of the current algebra by standard canonical quantization if an edge action is available, an explicit expression for the edge action has been derived only for certain geometries [4,5,16]. On the other hand, generally for complex manifolds of any even dimension, one can construct an effective action for the bulk dynamics of the system, including fluctuations of the gauge fields and the metric, using arguments based on the Dolbeault index theorem [17,18]. This is briefly reviewed in Sec. III. So by relating the extension of the current algebra to the bulk action, we can derive a general expression for any even dimension. This is carried out in Sec. IV.

¹The nature of the star commutator is not relevant for now, so we do not display it here. It is basically the star commutator for the contravariant symbol, as explained in [12].

The anomaly descent part of our argument applies in general. It requires a bulk topological effective action whose gauge variation reproduces the boundary anomaly, but does not by itself require a complex structure. In the present paper, such bulk actions are constructed using the Dolbeault index, and this specialization assumes that the spatial manifold is complex. Quantum Hall systems on manifolds without a complex structure, such as the four-sphere with a non-Abelian instanton background, should instead be formulated using the appropriate Dirac-type index density. Once the corresponding bulk Chern-Simons action is identified, the same descent procedure determines the two-cocycle and hence the extension of the edge current algebra. A detailed implementation for such geometries is beyond the scope of the present work.

A few other points regarding the situation in arbitrary dimensions are worthy of comment. Primarily, the symmetry of interest is the $U(1)$ gauge transformation of the symplectic potential. This coincides with area-preserving diffeomorphisms in two dimensions because the symplectic two-form is proportional to the geometric area. As mentioned before, the expression of this symmetry in the bulk is the W_∞ -transformation, with the restriction to the edges giving the current algebra [8–11]. In higher dimensions, the corresponding symmetries are again $U(1)$ transformations which preserve the symplectic form. The phase volume is preserved,² but the situation is more restrictive, the two-form itself has to be preserved. So, in higher dimensions, the analogs of the W_∞ transformations do not correspond to an arbitrary volume-preserving transformation, but a restricted set. Second, in higher dimensions we also have the possibility of non-Abelian background fields, and, correspondingly, non-Abelian gauge symmetries. The edge dynamics will show an anomaly for these as well, leading to a commutator anomaly for the corresponding currents. So we have these additional current algebraic structures in higher dimensions, going beyond the familiar W_∞ of the two-dimensional situation. The full theory maintains the symmetry with cancellation of the variation of the bulk action and the anomaly due to the boundary modes. So we again use our arguments to work out the corresponding current algebra in Sec. IV. As mentioned earlier, actions for the edge modes are available for certain special cases even beyond two spatial dimensions. For example, we have previously obtained the action for the edge modes for quantum Hall droplets in $\mathbb{CP}^k$, both with constant and fluctuating Abelian and non-Abelian background gauge fields [4,5,16]. In Sec. IV, we also carry out a direct quantization of these actions to obtain the current algebra. The results are shown to agree with the cocycle derived from the topological arguments.

²Up to a multiplicative constant, the phase volume is the same as the geometrically defined volume for a complex manifold with constant Abelian curvature, including manifolds such as $\mathbb{CP}^k$.

Section V is devoted to a short discussion of the results delineating the differences between the two-dimensional and higher dimensional situations. We give a summary of the main results in Sec. VI. There are also four short Appendices outlining some technical results on deriving the bulk action from the Dolbeault index density, some useful formulas, the edge actions and on the canonical quantization of the edge modes.

II. THE TWO-COCYCLE AND THE COMMUTATOR ANOMALY

In this section we present the key arguments for relating the commutator anomaly to a two-cocycle. This will be done in two parts. In the first part of this section we will consider the commutator anomaly for a field theory in its own right. In the second part we will then consider this field theory as describing the edge dynamics of a theory in one higher dimension, thereby linking the commutator anomaly to a bulk action in the higher dimension.

A. Commutator anomaly in field theory

We start with a brief discussion of anomalies in terms of partition functions or effective actions [13–15]. While the commutator anomaly is discussed in these papers, the argument we present here is somewhat independent and complete in its own right.

Consider a field theory of scalars and/or fermions, generically denoted by Φ , and gauge fields, denoted by A , in an even dimensional spacetime $\mathcal{M}_{2k}$, say of dimension $2k$. The action is denoted by $S(\Phi, A)$ and the partition function $Z(A)$ or effective action for the gauge fields is given by the functional integral of e^{iS} over the matter fields Φ ; i.e.,

$$Z(A) = e^{iS_{\text{eff}}(A)} = \mathcal{N} \int [D\Phi] e^{iS(\Phi, A)}, \quad (3)$$

where $\mathcal{N}$ is a normalization constant. The symmetry transformation of interest will be of the form $\Phi \rightarrow \Phi^g = g\Phi$ for matter fields and $A \rightarrow A^g = gAg^{-1} - dg g^{-1}$ for gauge fields, where $g \in G$. Generically, we take G to be a Lie group with elements of the form $g = e^{it_a \theta^a}$, t_a forming a Hermitian basis of the generators of G . The action $S(\Phi, A)$ has the symmetry $S(\Phi^g, A^g) = S(\Phi, A)$. However, the regularization needed to define the functional integration can render this symmetry anomalous. This can be viewed as due to the nontrivial Jacobian resulting from the change of variables $\Phi \rightarrow \Phi^g$, i.e., from

$$[D\Phi^g] = e^{i\beta(g, A)} [D\Phi]. \quad (4)$$

Thus, correspondingly, we can write

$$\begin{aligned} Z(A^g) &= \mathcal{N} \int [D\Phi] e^{iS(\Phi, A^g)} \\ &= \mathcal{N} \int [D\Phi^g] e^{iS(\Phi^g, A^g)} = \mathcal{N} \int [D\Phi] e^{iS(\Phi, A)} e^{i\beta(g, A)} \\ &= e^{i\beta(g, A)} Z(A). \end{aligned} \quad (5)$$

In going from the first line to the second we make a semantic change of variables for the integration from Φ to Φ^g . Then since $S(\Phi^g, A^g) = S(\Phi, A)$, we can use (4) to get to next equality and the final result. In (5), $\beta(g, A)$ is in the form of an integral over $\mathcal{M}_{2k}$ of polynomials of the fields and their derivatives along with factors of g and $g^{-1}dg$, with the condition $\beta(1, A) = 0$. Typically, $\beta(g, A)$ also involves an integration over a line on the group connecting the identity ($g = 1$) to the finite element g . The key point is that $\beta(g, A)$ cannot be written as the variation under g of any expression S_c which is of the form of an integral of the fields and their derivatives. If this were possible, one could remove the anomaly $\beta(g, A)$ by redefining the action as $S \rightarrow S - S_c$. This would be equivalent to a different choice of regularization.

The group composition law imposes a consistency requirement on $\beta(g, A)$. The action of $gg' \in G$ on the fields may be viewed as the transformation by g' followed by the action of g . In other words $\Phi^{gg'} = gg'\Phi = (\Phi^{g'})^g$, $A^{gg'} = (A^{g'})^g$. Applied to $Z(A)$, we find

$$\begin{aligned} Z(A^{gg'}) &= e^{i\beta(gg', A)} Z(A) \\ &= e^{i\beta(g, A^{g'})} Z(A^{g'}) = e^{i[\beta(g, A^{g'}) + \beta(g', A)]} Z(A). \end{aligned} \quad (6)$$

This shows that the anomaly $\beta(g, A)$ should obey the condition³

$$\beta(gg', A) - \beta(g, A^{g'}) - \beta(g', A) = 0. \quad (7)$$

This is the so-called Wess-Zumino consistency condition [19]. The anomaly $\beta(g, A)$ is defined on the space of fields, but it also has an integration along a line on the group from $g = 1$ to g , so it can be considered as a 1-cochain. The consistency requirement (7) is then a closure condition on $\beta(g, A)$, so it becomes a 1-cocycle.

So far the action and the effective action involve integration over all of $\mathcal{M}_{2k}$. For defining the wave functions, we separate off the time coordinate, taking spacetime to be of the form $V_{2k-1} \times \tau$, where V_{2k-1} denotes the spatial manifold of dimension $(2k - 1)$ and τ denotes a segment of the real line $\mathbb{R}$, corresponding to time. We will take $\tau = [t_i, t_f]$, where t_i denotes the initial time and t_f will

³Since we discuss the effective action here, we allow for integrations by parts with vanishing boundary contributions and the right-hand side of (7) is zero. More generally, as we shall see shortly, the right-hand side will be the integral of a total derivative.

be the final value of t under consideration. The wave function of the theory at time t_f is defined by the functional integral

$$\Psi[\Phi_f, A_f] = \mathcal{N} \int [D\Phi] d\mu(A) e^{iS(\Phi, A)}, \quad (8)$$

where we integrate over all fields obeying the boundary condition that $\Phi(t_f, \vec{x}) = \Phi_f(\vec{x})$, $A(t_f, \vec{x}) = A_f(\vec{x})$, and $\mathcal{N}$ is again some normalization factor. The functional measure for the gauge fields is the invariant measure on the space of gauge orbits. The specific state for which $\Psi[\Phi_f, A_f]$ is the wave function as given in (8) is determined by the initial choice of the state (at time t_i), which we do not display here explicitly since it is not relevant to arguments which follow.

As before, the existence of an anomaly for the action of G will involve the nontrivial transformation of the measure of functional integration. We then get

$$\begin{aligned} \Psi(\Phi_f^g, A_f^g) &= \int [D\Phi^g] d\mu(A^g) e^{iS(\Phi^g, A^g)} \\ &= \int [D\Phi] d\mu(A) e^{iS(\Phi, A)} e^{i\beta(g, A)}. \end{aligned} \quad (9)$$

The group composition law for the transformations now gives

$$\begin{aligned} \Psi((\Phi_f^{g'})^g, (A_f^{g'})^g) &= \int [D\Phi^{g'}] d\mu(A^{g'}) e^{iS(\Phi^{g'}, A^{g'})} e^{i\beta(g, A^{g'})} \\ &= \int [D\Phi] d\mu(A) e^{iS(\Phi, A)} e^{i\beta(g, A^{g'})} e^{i\beta(g', A)} \end{aligned} \quad (10)$$

$$\Psi(\Phi_f^{gg'}, A_f^{gg'}) = \int [D\Phi] d\mu(A) e^{iS(\Phi, A)} e^{i\beta(gg', A)}. \quad (11)$$

These two results should match since $(\Phi_f^{g'})^g = \Phi_f^{gg'}$, $(A_f^{g'})^g = A_f^{gg'}$. Generally, and as we shall see explicitly a little later, the difference of β 's in (10) and (11) is of the form

$$\beta(gg', A) - \beta(g, A^{g'}) - \beta(g', A) = \int d\omega(g, g', A), \quad (12)$$

where $\omega(g, g', A)$ is a 2-cochain. In the case of integration over all time with appropriate vanishing conditions on the boundary of $\mathcal{M}_{2k}$, the right-hand side integrates to zero in agreement with (7) for the effective action. However, in the present case, we are integrating over $V_{2k-1} \times \tau$, so the right-hand side of (12) leads to integrals on the boundary. We will choose the fields to vanish on the spatial boundary, so that, effectively, we can take $\partial V_{2k-1} = \emptyset$, as far as the fields of interest are concerned. On the right-hand side of (12), we are then left with $\oint_{V_{2k-1}} \omega(g, g', A_f)$ at t_f and a similar term

at t_i . The integral at t_f can be taken out of the functional integration since it involves only data at t_f and we can write (11) as

$$\begin{aligned} \Psi(\Phi_f^{gg'}, A_f^{gg'}) &= \int [D\Phi] d\mu(A) e^{iS(\Phi, A)} e^{i\beta(gg', A)} \\ &= e^{i \oint_{V_{2k-1}} \omega(g, g', A_f)} \int [D\Phi] d\mu(A) e^{iS(\Phi, A)} \\ &\quad \times e^{i\beta(g, A^{g'})} e^{i\beta(g', A)} \\ &= e^{i \oint_{V_{2k-1}} \omega(g, g', A_f)} \Psi\left(\left(\Phi_f^{g'}\right)^g, \left(A_f^{g'}\right)^g\right). \end{aligned} \quad (13)$$

If we consider the gauge transformation on the wave function to be implemented by the action of an operator $U(g)$, we can write

$$\Psi(\Phi_f^g, A_f^g) = U(g) \Psi(\Phi_f, A_f). \quad (14)$$

The relation (13) then implies that the composition law is realized at the level of the operators $U(g)$ as

$$U(gg') = e^{i \oint_{V_{2k-1}} \omega(g, g', A_f)} U(g) U(g'). \quad (15)$$

This shows that we have a projective realization of the action of $U(g)$ on the wave functions. At the level of infinitesimal generators, this rule implies that we have a commutator anomaly. Let θ and θ' denote the group parameters corresponding to g and g' , respectively, with $g = e^{it_a \theta^a}$, $g' = e^{it_a \theta'^a}$. We then have, for small values of the parameters,

$$gg' = e^{it_a(\theta^a + \theta'^a - \frac{1}{2}(\theta \times \theta')^a + \dots)}, \quad (\theta \times \theta')^a = f^{abc} \theta^b \theta'^c. \quad (16)$$

Here, f^{abc} are the structure constants of the Lie algebra defined by $[t^a, t^b] = i f^{abc} t^c$. Also let $G(\theta)$ denote the operator which generates the gauge transformation, so that we can write $U(g) = e^{iG(\theta)}$. $G(\theta)$ is linear in θ^a , so we get

$$\begin{aligned} U(gg') &\approx e^{iG(\theta + \theta' - \frac{1}{2}\theta \times \theta' + \dots)}, \\ U(g)U(g') &= e^{iG(\theta)} e^{iG(\theta')} \approx e^{iG(\theta) + iG(\theta') - \frac{1}{2}[G(\theta), G(\theta')] + \dots}. \end{aligned} \quad (17)$$

It is now easy to see that (15) becomes

$$[G(\theta), G(\theta')] = iG(\theta \times \theta') + 2i \oint_{V_{2k-1}} \omega(\theta, \theta', A_f). \quad (18)$$

This displays $\oint \omega$ as a commutator anomaly. One can also consider the triple combination $g_1 g_2 g_3$. The reduction of this, in consistency with associativity of group transformations, leads to a condition on $\omega(g, g', A_f)$. This is a closure

condition, qualifying $\int \omega(g, g', A_f)$ as a 2-cocycle. A similar result holds at the infinitesimal level for $\omega(\theta, \theta', A_f)$ in (18); it is equivalent to the Jacobi identity.

It is also useful to relate $\omega(\theta, \theta', A_f)$ to the descent equations for anomalies. (See lectures by Jackiw and by Zumino in [14].) Let ω_{2k+1} denote the Chern-Simons term associated to an index density $\mathcal{I}_{2k+2}$ in $2k+2$ dimensions; i.e., $d\omega_{2k+1} = 2\pi\mathcal{I}_{2k+2}$.⁴ ω_{2k+1} is a $(2k+1)$ -form made of gauge fields, spin connections, curvatures, etc. The gauge fields are connections on a principal fiber bundle and gauge transformations can be represented by exterior differentiation, denoted by δ , along the fiber directions. Writing the bundle connection as $\mathcal{A} = gAg^{-1} - dg g^{-1}$, we see that

$$d\mathcal{A} + \mathcal{A}\mathcal{A} = gFg^{-1}$$

$$\delta\mathcal{A} = dv + \mathcal{A}v + v\mathcal{A} = \mathcal{D}v, \quad v = \delta g g^{-1}, \quad (19)$$

keeping in mind that $d\delta + \delta d = 0$. The gauge invariance of $\mathcal{I}_{2k+2}$ tells us that

$$d(\delta\omega_{2k+1}) = -\delta(d\omega_{2k+1}) = -2\pi\delta\mathcal{I}_{2k+2} = 0 \quad (20)$$

so we have a locally defined form $\alpha_{2k}^{(1)}$ such that $\delta\omega_{2k+1} = d\alpha_{2k}^{(1)}$. The superscript on $\alpha_{2k}^{(1)}$ indicates that it has one power of $v = \delta g g^{-1}$. Interpreting v as θ (for small θ), $\int(-\alpha_{2k}^{(1)})$ is the anomaly for the $2k$ -dimensional field theory; i.e.,

$$\beta(g, A) \Big|_{g \approx 1 + i t_a \theta^a} \approx - \int_{\mathcal{M}_{2k}} \alpha_{2k}^{(1)}(\theta, A). \quad (21)$$

Since $\delta^2 = 0$, we also have $d(\delta\alpha_{2k}^{(1)}) = 0$, implying that we can write $\delta\alpha_{2k}^{(1)} = d\omega_{2k-1}^{(2)}$. The differential form $\omega_{2k-1}^{(2)}$ will have two powers of v , antisymmetrized, which may be identified as θ and θ' . The integral of $\omega_{2k-1}^{(2)}$ over the $(2k-1)$ -dimensional spatial manifold is the commutator anomaly, i.e.,

$$\oint_{V_{2k-1}} \omega(\theta, \theta', A_f) = - \oint_{V_{2k-1}} \omega_{2k-1}^{(2)}(\theta, \theta', A_f). \quad (22)$$

Reasoning along similar lines as before, $\delta\omega_{2k-1}^{(2)} = d\omega_{2k-2}^{(3)}$, so that $\oint \omega_{2k-1}^{(2)}$ is δ -closed, i.e., $\delta \oint \omega_{2k-1}^{(2)} = 0$. This implies that $\oint \omega_{2k-1}^{(2)}$ is a 2-cocycle. As mentioned above, at the level of commutators, this is equivalent to the Jacobi identity.

Although we outlined the descent procedure using the exterior derivative δ , one can also identify ω by carrying out infinitesimal transformations and antisymmetrizing. Since $\alpha_{2k}^{(1)}$ has one power of the (infinitesimal) parameter θ , we

may write (21) as $\beta \approx - \int \alpha(\theta)$. The condition (12) then gives

$$\delta_\theta \alpha(\theta', A) - \delta_{\theta'} \alpha(\theta, A) - \alpha(\theta \times \theta', A)$$

$$= -2 \oint_{V_{2k-1}} \omega(\theta, \theta', A_f). \quad (23)$$

This gives a straightforward way to identify ω .

In this section we have given a general argument on how to relate the commutator anomaly in a field theory to the 2-cocycle derived from Chern-Simons term via the descent procedure.

B. Bulk and boundary theories

So far we have considered a field theory in $2k$ -dimensional spacetime in its own right. We now consider the placement of the argument given above in the context of a higher dimensional field theory. Towards this, consider a $(2k+1)$ -dimensional spacetime manifold of the form $\mathcal{M}_{2k+1} = V_{2k} \times \tau$, where the boundary of V_{2k} is the space V_{2k-1} considered above. The bulk action for $\mathcal{M}_{2k+1}$ should be such that its variation under the symmetry transformation produces a boundary term which is canceled by the anomaly of the boundary theory. In other words, there is a cancellation of the anomaly between the bulk and boundary dynamics and the full theory preserves the gauge symmetry. All the considerations of anomalies in this section up to the previous paragraph are to be taken as applying to the boundary theory on $V_{2k-1} \times \tau$.

Since the anomaly of the boundary theory is $-\int \alpha_{2k}^{(1)}$, it is easy to see that the action for the $(2k+1)$ -dimensional theory should be

$$S = \int_{\mathcal{M}_{2k+1}} \omega_{2k+1} + S(\Phi, A), \quad (24)$$

where $S(\Phi, A)$ is the action for the theory considered in (8), which is defined on the manifold $V_{2k-1} \times \tau$. The action (24) is free of anomalies, the anomaly of the boundary theory canceling against the variation of the bulk part of the action.

The cancellation between the gauge variation of the bulk action and the anomaly of the boundary degrees of freedom is also known as the anomaly-inflow mechanism, familiar in particular from the work of Callan and Harvey [20]. In the present application, the bulk quantum Hall Chern-Simons action supplies the inflow, while the chiral edge theory carries the corresponding boundary anomaly. Our use of this mechanism is to infer the boundary current-algebra two-cocycle from the known bulk action, even when an explicit edge action is unavailable.

We are going to apply these ideas to the case of the quantum Hall effect. The effective bulk actions for QHE with integer filling fraction ν on complex manifolds of arbitrary even dimensions were obtained in [17] using the Dolbeault index theorem. The action is in fact the

⁴The factor of 2π ensures that we have the correct normalization for the anomaly.

Chern-Simons term ω_{2k+1} corresponding to the Dolbeault index density $\mathcal{I}_{2k+2}$. The background values for the gauge and gravitational fields in these actions were arbitrary, except for being constrained to a particular topological class.

We also note that the bulk action for the Laughlin-type states with fractional values of ν with Abelian gauge fields were obtained for $(2+1)$ and $(4+1)$ dimensions in [18], as a generalization of the parton picture previously used in two dimensions. The main observation here is that the bulk action is easier to construct and is known for general backgrounds, as we indicate in Appendix A.

By contrast, the situation for the boundary action is more limited. For the case of the integer QHE on $\mathbb{CP}^k$ manifolds, with Abelian and non-Abelian background magnetic fields proportional to the “constant” $U(k)$ curvatures of $\mathbb{CP}^k$, the boundary action was obtained in [4,5] using the coadjoint orbit method. The result is basically a higher dimensional generalization of the 2D chiral bosonic action. Fluctuating gauge fields were also introduced via a gauged version of the coadjoint orbit method in [16,21], leading to an effective action with cancellation of anomalies between the bulk and the boundary. All these results were for the fixed geometric background of $\mathbb{CP}^k$ and in a large- n limit.

We can now see that the arguments presented here lead to a simple and general strategy for obtaining the commutator anomaly or the nontrivial extension of the current algebra associated to the gauge transformations of interest for the chiral edge modes of the quantum Hall droplet. As mentioned above, the bulk actions are known for general backgrounds. So we can obtain the corresponding two-cocycle via the descent procedure mentioned above. The arguments given above then show that the two-cocycle given in (18), and which is constructed via descent from the Chern-Simons form as in (22) or (23), can be identified as the extension term for the current algebra acting on the states of the boundary theory.

The key point is that even if we do not have the general action for the chiral edge modes, we can determine the corresponding current algebra for the edge modes by identifying the two-cocycle from the bulk action, the latter, as mentioned above, being known in full generality. It is the cancellation between the gauge variation of the bulk action and the anomaly of the boundary theory which makes this possible. This is the main point of this paper. We will now work out some specific cases in the next section. We will also show that, in cases where the edge action is known, such as the case of QHE on $\mathbb{CP}^k$ as mentioned above, the extended current algebra obtained by these topological arguments also agrees with the direct quantization of the edge theory.

III. BULK ACTIONS FROM AN INDEX THEOREM

In Appendix A, we have briefly recapitulated the arguments based on the Dolbeault index theorem [22] from which one can obtain the form of the bulk effective action for a quantum Hall droplet of fermions [17]. For simplicity

we will only consider the case of the droplets in the lowest Landau level. In a general dimension, the bulk action for $\nu = 1$ takes the form

$$S^{\text{bulk}} = \int \left[\text{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(A) \right]_{2k+1} + S^{\text{grav}}. \quad (25)$$

The Chern-Simons term $(CS)_{2p+1}(A)$ is defined by

$$\frac{1}{2\pi} d(CS)_{2p+1}(A) = \frac{1}{(p+1)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{p+1} \quad (26)$$

and $\text{td}(T_c K)$ is the Todd class for the complex tangent space, see Appendix A for more details. [The Chern-Simons term is the same as the ω_{2p+1} defined earlier if the index density is just the Chern character, i.e., the purely A -dependent right-hand side of (26).] In the expansion of the Todd class and the CS terms in (25), we should use the differential form of rank $2k+1$ for the action, as indicated by the subscript $2k+1$.

The purely gravitational part of the action S^{grav} is defined by

$$S^{\text{grav}} = 2\pi \int \Omega_{2k+1}^{\text{grav}}, \quad d\Omega_{2k+1}^{\text{grav}} = [\text{td}(T_c K)]_{2k+2}, \quad (27)$$

The expression (25) is general, but it is useful to write it out explicitly for some low dimensions. For this, we first note that the CS terms are given by

$$\begin{aligned} (CS)_1 &= i\text{Tr}(A), \\ (CS)_3 &= \frac{i^2}{4\pi} \text{Tr} \left(AF - \frac{1}{3} A^3 \right), \\ (CS)_5 &= \frac{i^3}{3!(2\pi)^2} \text{Tr} \left(AF^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right), \\ (CS)_7 &= \frac{1}{4!(2\pi)^3} \text{Tr} \left(AF^3 - \frac{2}{5} A^3 F^2 - \frac{1}{5} (AFA^2F - A^4F) \right. \\ &\quad \left. - \frac{1}{35} A^7 \right). \end{aligned} \quad (28)$$

The traces involved in the Todd class will depend on the dimension since the curvature of the manifold takes values in the algebra of the holonomy group $U(k)$. In $2+1$ dimensions, for which $k=1$, so $U(k) = U(1)$, we get the bulk action

$$S_{2+1}^{\text{bulk}} = \frac{i^2}{4\pi} \int \left\{ AdA + A d\omega + \frac{1}{6} \omega d\omega \right\}, \quad (29)$$

where ω is the spin connection. The fields (A, ω) are given in an anti-Hermitian basis to be uniform in notation with what we used in earlier sections and in (28). (Thus A in this equation stands for $-i$ times the real gauge field.) The last term, which is

independent of the gauge fields will not be important for the commutator anomaly for the gauge transformations.

In $4 + 1$ dimensions, the bulk action becomes

$$\begin{aligned} S_{4+1}^{\text{bulk}} &= \int \left[(CS)_5 + \frac{c_1}{2} (CS)_3 + \frac{c_1^2 + c_2}{12} (CS)_1 \right] + S^{\text{grav}} \\ &= i^3 \int \left\{ \left[\frac{1}{3!(2\pi)^2} \text{Tr} \left(AF^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right) \right] \right. \\ &\quad \left. + \frac{d\omega^0}{8\pi^2} \text{Tr} \left(AF - \frac{1}{3} A^3 \right) \right. \\ &\quad \left. + \left[\frac{5}{48\pi^2} d\omega^0 d\omega^0 - \frac{1}{96\pi^2} \text{Tr}(\tilde{R} \tilde{R}) \right] \text{Tr} A \right\} + S^{\text{grav}}. \end{aligned} \quad (30)$$

The curvature has the form given in (31), except that $\mathbb{1}$ and t_a are now 3×3 matrices since the holonomy group is $U(3)$.

In all the cases of QHE on arbitrary even spatial dimensions analyzed in [3–5] an Abelian background field of the form $dA = -in\Omega_K$,⁵ where Ω_K is the Kähler two-form on the spatial manifold, was used, in addition to any non-Abelian background fields or fluctuations of the gauge potentials. A high magnetic field (obtainable for large n) is needed to have a large number of states and large occupancy so that one can have a well-defined droplet of fermions. The actions for the edge modes of the droplet were also worked out. It is therefore useful to expand the bulk actions given above around the background value of A by shifting $A \rightarrow A + a$, where $da = \Omega = -in\Omega_K$. This expansion is important in obtaining the two-cocycle in a form which can then be compared to the direct quantization of the edge actions in [4,5,16]. It is easy to check that⁶

$$(CS)_{2p+1}(A + a) = [(e^{ida/2\pi})(CS)(A)]_{2p+1} + (CS)_{2p+1}(a). \quad (33)$$

⁵ Ω is written in anti-Hermitian form in accordance with the convention we have used. However, we use Hermitian Ω_K since that is more conventional in the literature on complex projective spaces.

⁶For the bulk actions, to identify the correct local action densities in terms of Ω , certain integrations by parts are allowed and have been carried out.

Again, we use an anti-Hermitian basis. In $4 + 1$ dimensions, the holonomy group is $U(2)$ since $k = 2$. Thus the curvature is of the form

$$R = d\omega^0 \mathbb{1} + (-it_a)R^a = d\omega^0 \mathbb{1} + \tilde{R}. \quad (31)$$

Here $\mathbb{1}$ is the 2×2 identity matrix and t_a are the generators of $SU(2)$ in the fundamental representation, with $\text{Tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$. We have not displayed the purely gravitational part since it is not relevant for the commutator anomaly.

In $6 + 1$ dimensions, we can similarly work out the bulk action as

$$\begin{aligned} S_{6+1}^{\text{bulk}} &= \int \left[(CS)_7 + \frac{c_1}{2} (CS)_5 + \frac{c_1^2 + c_2}{12} (CS)_3 + \frac{c_1 c_2}{24} (CS)_1 \right] + S^{\text{grav}} \\ &= \int \left\{ (CS)_7 + i \frac{3}{4\pi} d\omega^0 (CS)_5 + i^2 \left[\frac{1}{4\pi^2} (d\omega^0)^2 - \frac{1}{96\pi^2} \text{Tr}(\tilde{R} \tilde{R}) \right] (CS)_3 \right. \\ &\quad \left. + i^3 \left[\frac{3}{64\pi^3} (d\omega^0)^3 - \frac{d\omega^0}{128\pi^3} \text{Tr}(\tilde{R} \tilde{R}) \right] (CS)_1 \right\} + S^{\text{grav}}. \end{aligned} \quad (32)$$

Keeping only the A -dependent terms, where A now is of order n^0 , we find,

$$\begin{aligned} S_{2+1}^{\text{bulk}} &= \frac{i^2}{4\pi} \int [AdA + Ad\omega + 2\Omega A] \\ S_{4+1}^{\text{bulk}} &= \int \left\{ (CS)_5 + \frac{i(\Omega + d\omega^0)}{2\pi} (CS)_3 \right. \\ &\quad \left. + \frac{i^2}{8\pi^2} \left[\Omega^2 + 2\Omega d\omega^0 + \frac{5}{6} d\omega^0 d\omega^0 - \frac{1}{12} \text{Tr}(\tilde{R} \tilde{R}) \right] \right. \\ &\quad \left. \times (CS)_1 \right\}, \end{aligned} \quad (34)$$

$$\begin{aligned} S_{6+1}^{\text{bulk}} &= \int \left\{ (CS)_7 + i \left(\frac{\Omega}{2\pi} + \frac{3d\omega^0}{4\pi} \right) (CS)_5 \right. \\ &\quad \left. + \frac{i^2}{8\pi^2} \left[\Omega^2 + 3\Omega d\omega^0 + 2d\omega^0 d\omega^0 - \frac{1}{12} \text{Tr}(\tilde{R} \tilde{R}) \right] (CS)_3 \right. \\ &\quad \left. + \frac{i^3}{3!(2\pi)^3} \left[\Omega^3 + \frac{9}{2} \Omega^2 d\omega^0 + 6\Omega (d\omega^0)^2 + \frac{9}{4} (d\omega^0)^3 \right. \right. \\ &\quad \left. \left. - \left(\frac{\Omega}{4} + \frac{3d\omega^0}{8} \right) \text{Tr}(\tilde{R} \tilde{R}) \right] (CS)_1 \right\}. \end{aligned} \quad (35)$$

A similar analysis was carried out for some fractional values of the filling fraction ν in [18]. The strategy was to use the so-called parton picture where the fundamental charge carrier, the electron, is viewed as made of several partons of fractional charge. Additional gauge fields are

needed to obtain the binding of the partons to form the electron. Equivalently, these fields lead to constraints which ensure that the observable charge carriers are the electrons. *A priori*, there are additional anomalies due to the gauge symmetries of these additional gauge fields. The cancellation

of these anomalies, needed to ensure that the constraints can be consistently implemented, then lead to the fractional states. The resulting bulk actions were worked out in [18] for $2 + 1$ and $4 + 1$ dimensions for Abelian gauge fields. For filling fraction $\nu = 1/m$, these are given by

$$S_{2+1, \nu=1/m}^{\text{bulk}} = \frac{i^2}{4\pi} \int \left[\frac{AdA}{m} + Ad\omega \right] + \frac{i^2}{4\pi} \int \left(\frac{m}{4} - \frac{1}{12} \right) \omega d\omega$$

$$\xrightarrow{A \rightarrow A+a} \frac{i^2}{4\pi} \int \left[\frac{AdA}{m} + Ad\omega + \frac{2\Omega A}{m} \right] + \dots \quad (36)$$

The result for $4 + 1$ dimensions is

$$S_{4+1, \nu=1/m}^{\text{bulk}} = i^3 \int \left\{ \frac{1}{24\pi^2} \frac{AF^2}{m^2} + \frac{1}{8\pi^2} d\omega^0 \frac{AF}{m} + \left[\frac{5}{48\pi^2} (d\omega^0)^2 - \frac{1}{96\pi^2} \text{Tr}(\tilde{R} \tilde{R}) \right] A \right\} + S^{\text{grav}}$$

$$\xrightarrow{A \rightarrow A+a} i^3 \int \left\{ \frac{1}{24\pi^2} \frac{AF^2}{m^2} + \frac{1}{8\pi^2} \left(\frac{\Omega}{m^2} + \frac{d\omega^0}{m} \right) AF + \frac{1}{8\pi^2} \left(\frac{\Omega^2}{m^2} + 2 \frac{\Omega d\omega^0}{m} + \frac{5}{6} (d\omega^0)^2 - \frac{1}{12} \text{Tr}(\tilde{R} \tilde{R}) \right) A \right\} + \dots \quad (37)$$

The anomaly cancellation for the partons used in [18] to formulate the fractional Hall states is significantly more involved for non-Abelian background gauge fields, so, in (36) and (37) we have given the results only for Abelian gauge fields.

IV. THE COMMUTATOR ANOMALY FOR THE QUANTUM HALL DROPLETS

The commutator anomaly can now be derived in a straightforward way. We are interested in gauge transformations of the gauge field A . Therefore, for now, the A -independent terms, i.e., the purely gravitational terms in the action, are not of interest. Further, since the commutator anomaly involves two gauge transformations (with parameters θ, θ'), we need at least 2 factors of A to get a nonzero contribution. So the terms involving $(CS)_1$ are not important for this purpose. From the gauge variation of the CS terms we then find

$$\int_{\mathcal{M}_{2k+1}} \delta_\theta (CS)_{2k+1} = \int_{\mathcal{M}_{2k}} \alpha_{2k}(\theta, A)$$

$$\alpha_2(\theta, A) = \frac{i^2}{4\pi} \text{Tr}[d\theta A]$$

$$\alpha_4(\theta, A) = \frac{i^3}{24\pi^2} \text{Tr} \left[d\theta \left(AdA + \frac{1}{2} A^3 \right) \right]$$

$$= \frac{i^3}{24\pi^2} \text{Tr} \left[d\theta \left(AF - \frac{1}{2} A^3 \right) \right]$$

$$\alpha_6(\theta, A) = \frac{i^4}{24(2\pi)^3} \text{Tr} \left[d\theta \left(AdAdA - \frac{1}{5} (A^2 dAA - 3AdAA^2 - 4A^3 dA) + \frac{2}{5} A^5 \right) \right]$$

$$= \frac{i^4}{24(2\pi)^3} \text{Tr} \left[d\theta \left(AF^2 - \frac{1}{5} A^3 F - \frac{2}{5} AFA^2 - \frac{1}{5} A^2 FA + \frac{1}{5} A^5 \right) \right]. \quad (38)$$

Carrying out a second variation of α_2 , we find

$$\delta_\theta \alpha_2(\theta', A) - \delta_{\theta'} \alpha_2(\theta, A) = -\frac{i^2}{4\pi} [\text{Tr}(d\theta' d\theta - d\theta d\theta')] - \frac{i^2}{4\pi} \text{Tr}([[\theta, d\theta'] + [d\theta, \theta']] A)$$

$$= -\alpha_2([\theta, \theta'], A) + \frac{i^2}{4\pi} \text{Tr}(d\theta d\theta' - d\theta' d\theta)$$

$$= \alpha_2(\theta \times \theta', A) + d \left[\frac{i^2}{4\pi} \text{Tr}(\theta d\theta' - \theta' d\theta) \right]. \quad (39)$$

Comparing this with (23), we see that the term in the square brackets in this equation gives $-2\omega(\theta, \theta', A)$. [The argument A here is A_f if we compare this with (23).] Similarly, carrying out the second variations we find

$$\begin{aligned} \delta_\theta \alpha_4(\theta', A) - \delta_{\theta'} \alpha_4(\theta, A) - \alpha_4(\theta \times \theta', A) &= d \left[\frac{i^3}{24\pi^2} \text{Tr}[(\theta d\theta' - \theta' d\theta) dA] \right] \\ \delta_\theta \alpha_6(\theta', A) - \delta_{\theta'} \alpha_6(\theta, A) - \alpha_6(\theta \times \theta', A) &= d \left[\frac{1}{24(2\pi)^3} \text{Tr} \left((d\theta d\theta' - d\theta' d\theta) \left(AdA + \frac{3}{5} A^3 \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{5} (d\theta Ad\theta' A^2 - d\theta' Ad\theta A^2) \right) \right]. \end{aligned} \quad (40)$$

We can thus identify the corresponding 2-cocycles as

$$\begin{aligned} 2\omega_1(\theta, \theta', A) &= -\frac{i^2}{4\pi} \text{Tr}(\theta d\theta' - \theta' d\theta) \\ 2\omega_3(\theta, \theta', A) &= -\frac{i^3}{24\pi^2} \text{Tr}[(\theta d\theta' - \theta' d\theta) dA] \\ 2\omega_5(\theta, \theta', A) &= -\frac{1}{24(2\pi)^3} \text{Tr} \left[(d\theta d\theta' - d\theta' d\theta) \left(AdA + \frac{3}{5} A^3 \right) + \frac{1}{5} (d\theta Ad\theta' A^2 - d\theta' Ad\theta A^2) \right], \end{aligned} \quad (41)$$

where $\theta = i(\theta^0 + t_a \theta^a)$, and θ^0, θ^a are the Abelian and non-Abelian real-valued transformation parameters. [For simplicity we have set $A_f \equiv A$ in (41)].

Based on arguments given in Sec. II, the anomalous current algebra for the boundary theory defined on $\mathcal{M}_{2k}$ takes the form

$$[G(\theta), G(\theta')] = iG(\theta \times \theta') + 2i \oint_{V_{2k-1}} \omega_{2k-1}(\theta, \theta', A), \quad (42)$$

where $G(\theta)$ is the generator of gauge transformations for the edge modes.

In the next section we will give the anomalous current algebras for the edge modes in an explicit form for different dimensions *based solely on our knowledge of the bulk effective actions*. We will then compare them with the current algebras obtained from the edge effective actions for QHE on $\mathbb{CP}^k$ we derived in [4,5,16].

Before we proceed though, a comment about the given expressions for α_{2k} is appropriate. Clearly the definition of $\alpha_{2k}^{(1)}$ via $\delta\omega_{2k+1} = d\alpha_{2k}^{(1)}$ specifies the form of $\alpha_{2k}^{(1)}$ only up to a total derivative of the form $d\tilde{\alpha}_{2k-1}^{(1)}$. Being a total derivative, the contribution of this term resides on the boundary and so this may be viewed as an additional term in the definition of the generator of the gauge transformations for the edge modes, i.e., $G(\theta) \rightarrow G(\theta) + \tilde{\alpha}_{2k-1}^{(1)}$. Since $G(\theta)$ is in general a composite operator, regularization is needed and its renormalized form can have a regularization dependence which is precisely of this form. Thus in choosing a specific form for $\alpha_{2k}^{(1)}$, we are choosing a particular regularization. This can change the form of $\omega_{2k-1}(\theta, \theta', A)$ in a

cohomologically trivial way (see lectures by Jackiw in [14]). Of course, the part of the commutator anomaly which is cohomologically significant is unaltered by the ambiguity due to $\tilde{\alpha}_{2k-1}^{(1)}$.

A. 2+1 dimensions

Comparison of these results with (42) shows that the generator of the gauge transformations on the wave functions of the 1+1 dimensional field theory (relevant to Hall droplets in 2+1 dimensions) should obey the anomalous algebra

$$[G(\theta), G(\theta')] = iG(\theta \times \theta') + \frac{i}{4\pi} \oint \text{Tr}(\theta d\theta' - \theta' d\theta), \quad (43)$$

where $\theta = i(\theta^0 + t_a \theta^a)$. The generator of gauge transformations on the matter fields on the boundary will correspond to the time component of a suitable current, $J_0(\theta) = \oint [\theta^0 J_0(\varphi) + \theta^a J_0^a(\varphi)]$, where $J_0(\varphi), J_0^a(\varphi)$ are the Abelian and non-Abelian currents and φ denotes the angular coordinate along the edge of the droplet. In this case, with $\text{Tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$, the algebra in (43) reduces to

$$[J_0(\varphi), J_0(\varphi')] = -\frac{i}{2\pi} \partial_\varphi \delta(\varphi - \varphi'), \quad (44)$$

$$\begin{aligned} [J_0^a(\varphi), J_0^b(\varphi')] &= i f^{abc} J_0^c(\varphi) \delta(\varphi - \varphi') \\ &\quad - \frac{i}{4\pi} \delta^{ab} \partial_\varphi \delta(\varphi - \varphi'). \end{aligned} \quad (45)$$

Although here we display the non-Abelian current algebra as well to show the nature of the central extension, in the

case of Hall effect in two spatial dimensions the gauge field of interest is Abelian, specifically, the electromagnetic field, and therefore only the algebra (44) is relevant.

The result (44) is obtained from the two-cocycle derived from the bulk action. We can now compare with direct calculation from the edge dynamics. It is known that the edge modes for the $(1+1)$ -dimensional boundary theory can be described by a chiral scalar field Φ [23], with action

$$S = -\frac{1}{4\pi} \int dt d\varphi \left[\dot{\Phi} \frac{\partial \Phi}{\partial \varphi} + v_F \left(\frac{\partial \Phi}{\partial \varphi} \right)^2 \right], \quad (46)$$

where v_F is the Fermi velocity at the edge of the droplet. [The action (46) can also be derived using the coadjoint orbit method, see Appendix C.] The canonical quantization of this theory leads to the commutation rule

$$[\Phi(\varphi), \Phi(\varphi')] = i\pi[\Theta(\varphi - \varphi') - \Theta(\varphi' - \varphi)], \quad (47)$$

where $\Theta(\varphi - \varphi')$ is the step function. These commutation rules also show that

$$\begin{aligned} [-(1/2\pi)\partial_\varphi \Phi(\varphi), \Phi(\varphi')] &= -i\delta(\varphi - \varphi'), \\ [-(1/2\pi)\partial_\varphi \Phi(\varphi), -(1/2\pi)\partial_{\varphi'} \Phi(\varphi')] &= -\frac{i}{2\pi} \partial_\varphi \delta(\varphi - \varphi'). \end{aligned} \quad (48)$$

We see that we can therefore identify the canonical generator of the gauge transformations as

$$J_0(\theta^0) = -\frac{1}{2\pi} \int d\varphi \theta^0 \partial_\varphi \Phi. \quad (49)$$

It is easily verified that the gauge transformation of Φ is obtained as $e^{iJ_0(\theta^0)} \Phi e^{-iJ_0(\theta^0)} = \Phi + \theta^0$. The second relation in (48) shows that the canonical quantization of the boundary theory reproduces the commutator anomaly in (44).

Turning to the fractional Hall effect of $\nu = 1/m$ with the action given in (36), we see that we can easily modify our considerations above to get the anomalous algebra

$$[G(\theta^0), G(\theta^{0'})] \equiv [J_0(\theta^0), J_0(\theta^{0'})] = -\frac{i}{2\pi m} \oint \theta^0 \partial \theta^{0'}. \quad (50)$$

This corresponds to an edge action of the form

$$S = -\frac{1}{4\pi m} \int dt d\varphi \left[\dot{\Phi} \frac{\partial \Phi}{\partial \varphi} + v_F \left(\frac{\partial \Phi}{\partial \varphi} \right)^2 \right]. \quad (51)$$

B. 4 + 1 dimensions

For the $(3+1)$ -dimensional boundary theory corresponding to a droplet in quantum Hall effect in $4+1$ dimensions, the generators of the gauge transformations obey the anomalous algebra

$$\begin{aligned} [G(\theta), G(\theta')] &= iG(\theta \times \theta') \\ &= -\frac{1}{8\pi^2} \oint \text{Tr} \left[(\theta d\theta' - \theta' d\theta) \left(\frac{dA}{3} + \Omega + d\omega^0 \right) \right]. \end{aligned} \quad (52)$$

This result follows from (34) and (41), i.e., just from the bulk action. This is the general result for a complex spatial manifold, and we have kept terms of all orders in n for the anomalous term on the right-hand side.

In the case of the spatial manifold being $\mathbb{CP}^k$, there is a direct calculation of the edge action at large n using the method of coadjoint orbits [4,5], as summarized in Appendix C. So we will now consider the simplification of the anomaly term in (52) for $\mathbb{CP}^2$ since that will provide a point of comparison with the quantization of the edge modes. In general, for $\mathbb{CP}^k$, the term proportional to Ω on the right-hand side of (52) will be the leading term at large n , since $\Omega = -in\Omega_K$. Here Ω_K is the Fubini-Study Kähler form

$$\Omega_K = i \left[\frac{dz^i d\bar{z}^i}{(1 + \bar{z} \cdot z)} - \frac{\bar{z} \cdot dz \cdot d\bar{z}}{(1 + \bar{z} \cdot z)^2} \right]. \quad (53)$$

The complex coordinates of $\mathbb{CP}^2$ are z^i , $i = 1, 2$, which can be related to real local coordinates by $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$. Elementary algebraic relations from Appendix B then lead to the result

$$\begin{aligned} &-\frac{1}{8\pi^2} \oint \text{Tr}[(\theta d\theta' - \theta' d\theta)\Omega] \\ &= \frac{in}{4\pi^2} \oint dS_3 \frac{r_D^2}{(1 + r_D^2)^3} \text{Tr}(\theta \mathcal{L} \theta' - \theta' \mathcal{L} \theta), \end{aligned} \quad (54)$$

where $\mathcal{L}$ denotes the angular derivative

$$\mathcal{L} = -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} \partial_\nu = i \left(z \cdot \frac{\partial}{\partial \bar{z}} - \bar{z} \cdot \frac{\partial}{\partial z} \right) \equiv \frac{\partial}{\partial \varphi}, \quad (55)$$

where φ is the common phase for all z^i . We have considered a spherical droplet of radius r_D for simplicity. The integration in (54), indicated by dS_3 , is over the three-sphere which is the boundary of the droplet. Further, as is clear from (55), $\mathcal{L}$ is the angular derivative on the surface along the direction Poisson conjugate to the radial direction $\hat{x}_\mu$.

To facilitate comparison with the edge action derived in [4,5], and summarized in Appendix C, we note that the states of the lowest Landau level for $\mathbb{CP}^2$ form a symmetric

representation of $SU(3)$ of rank n , which may be decomposed into a series of $SU(2)$ representations. In [4,5], we considered a droplet corresponding to the first M $SU(2)$ representations to be occupied. In this case $r_D^2 = M/n$ for large n , M , with $n \gg M$. We can then simplify the right-hand side of (54) to get

$$-\frac{1}{8\pi^2} \oint \text{Tr}[(\theta d\theta' - \theta' d\theta)\Omega] = -i \frac{M}{2\pi^2} \oint dS_3 (\theta^0 \mathcal{L}\theta^{0'}) \quad (56)$$

This is for the Abelian case. In the non-Abelian case with $\theta = it_a \theta^a$, $\theta' = it_a \theta'^a$, we get

$$\begin{aligned} -\frac{1}{8\pi^2} \oint \text{Tr}[(\theta d\theta' - \theta' d\theta)\Omega] &= i \frac{M}{2\pi^2} \oint dS_3 \text{Tr}(\theta \mathcal{L}\theta') \\ &= -i \frac{M}{4\pi^2} \oint dS_3 (\theta^a \mathcal{L}\theta'^a) \end{aligned} \quad (57)$$

The remaining terms in (52) are subleading in $1/n$. Their simplification is more straightforward and gives

$$\begin{aligned} -\frac{1}{8\pi^2} \oint \text{Tr}[(\theta d\theta' - \theta' d\theta) \left(\frac{dA}{3} + d\omega^0 \right)] \\ = -\frac{1}{16\pi^2} \oint dS^{\nu\alpha\beta} \text{Tr} \left[(\theta \partial_\nu \theta' - \theta' \partial_\nu \theta) \left(\frac{1}{3} f_{\alpha\beta} + R_{\alpha\beta}^0 \right) \right], \end{aligned} \quad (58)$$

where

$$dS^{\nu\alpha\beta} = r_D^3 \hat{x}_\mu \epsilon^{\mu\nu\alpha\beta} dS_3. \quad (59)$$

Also $f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $R_{\alpha\beta}^0 = \partial_\alpha \omega_\beta^0 - \partial_\beta \omega_\alpha^0$ is the $U(1)$ component of the curvature.

While all these terms (56)–(58) can contribute to the commutator anomaly, the leading term at large n is given by (56) and (57). Although no explicit factor of $1/n$ is shown in (58), we can see that it is down by a power of n because of the factor r_D^3 in $dS^{\nu\alpha\beta}$. One factor r_D is used to convert ∂_ν to an angular derivative, while the remaining $r_D^2 = M/n$. We emphasize that despite the simplifications we have carried out, the result (58) is still what is derived from the bulk action. We will show that the quantization of the edge action as derived in [4,5,16], and discussed in Appendix C, will reproduce the result (56) and (57), so in effect verifying the large n limit of (52). Since the quantization is similar for all $\mathbb{CP}^k$, we defer the details of this to the next section.

Finally, turning to the fractional Hall effect of $\nu = 1/m$, by comparison of the actions (34) and (37), we can write down the anomalous commutation rules (for the Abelian gauge transformations) as

$$\begin{aligned} [G(\theta^0), G(\theta^{0'})] &\equiv [J_0(\theta^0), J_0(\theta^{0'})] \\ &= -i \frac{M}{2\pi^2 m^2} \oint dS_3 (\theta^0 \mathcal{L}\theta^{0'}) \\ &\quad + \frac{1}{8\pi^2} \oint dS^{\nu\alpha\beta} (\theta^0 \partial_\nu \theta^{0'}) \left(\frac{f_{\alpha\beta}^0}{3m^2} + \frac{R_{\alpha\beta}^0}{m} \right). \end{aligned} \quad (60)$$

The leading term again shows that the action for the edge modes should be

$$S = -\frac{M}{4\pi^2 m^2} \int dt dS_3 [\dot{\Phi} \mathcal{L}\Phi + v_F (\mathcal{L}\Phi)^2]. \quad (61)$$

C. 6+1 and higher dimensions

Turning to 6+1 dimensions, we see from (34) that one can get contributions from $(CS)_7$, $(CS)_5$, and $(CS)_3$. While we can simplify the results along lines similar to what we did above, various expressions become progressively more cumbersome as we go up in dimensions. There is, however, significant simplification in the large n limit. The dominant term is from $(CS)_3$ with a product of Ω 's, since Ω is taken to be of order n . From (33), we see that the dominant term in $2k+1$ dimensions is

$$S_{2k+1} = \frac{1}{(k-1)!} \int \left(\frac{i\Omega}{2\pi} \right)^{k-1} (CS)_3. \quad (62)$$

Using (18) and (41) we find that the algebra of the generators of the gauge transformations works out to be

$$\begin{aligned} [G(\theta), G(\theta')] &- iG(\theta \times \theta') \\ &= -\frac{i}{2(2\pi)^k} \oint \sqrt{\det i\Omega} \hat{x}_\mu (i\Omega^{-1})^{\mu\nu} \text{Tr}(\theta \partial_\nu \theta' - \theta' \partial_\nu \theta), \end{aligned} \quad (63)$$

where we have used the relation (B1) to carry out the simplification.

Specifically for the case of $\mathbb{CP}^k$, $\Omega = -in\Omega_K$, with $\sqrt{\det \Omega_K} = 2^k / (1+r^2)^{k+1}$ and this expression can be simplified as

$$\begin{aligned} [G(\theta), G(\theta')] &- iG(\theta \times \theta') = i \frac{M^{k-1}}{2\pi^k} \oint \text{Tr}(\theta \mathcal{L}\theta') \\ &= -i \frac{M^{k-1}}{2\pi^k} \oint \theta^0 (\mathcal{L}\theta^{0'}). \end{aligned} \quad (64)$$

The second line applies to the Abelian case. Since $\mathbb{CP}^k = SU(k+1)/U(k)$, the states of the lowest Landau level for a symmetric rank- n representation of $SU(k+1)$ which consists of a number of $SU(k)$ representations, of ranks from zero to n . As explained in Appendix C, the

droplet is taken to have all the low rank $SU(k)$ representations, from rank = 0 to rank = M , to be occupied [4,5].

The effective action for the edge modes for an Abelian quantum Hall droplet on $\mathbb{CP}^k$ was derived in [4] as

$$S = -\frac{1}{4\pi^k} M^{k-1} \int [\Phi \mathcal{L} \Phi + v_F (\mathcal{L} \Phi)^2]. \quad (65)$$

The canonical quantization of this theory is worked out in Appendix D and gives the commutation rule

$$(M^{k-1}/2\pi^k) \oint_y [(\mathcal{L} \theta^0) \Phi(y), \Phi(x)] = -i\theta^0(x). \quad (66)$$

This shows that we can identify the operator

$$J_0(\theta^0) = -(M^{k-1}/2\pi^k) \oint \theta^0 \mathcal{L} \Phi \quad (67)$$

as the generator of the Abelian gauge transformations $\Phi \rightarrow \Phi + \theta^0$. From the canonical commutation rule (66), we also see that

$$[J_0(\theta^0), J_0(\theta^{0'})] = -i \frac{M^{k-1}}{2\pi^k} \oint \theta^0 \mathcal{L} \theta^{0'}. \quad (68)$$

This result is in complete agreement with the commutator anomaly obtained from topological arguments in (64).

As mentioned before, it is possible to have non-Abelian background gauge fields in spatial dimensions >2 . There is a corresponding commutator anomaly for the non-Abelian gauge transformations as indicated in (57) for $\mathbb{CP}^2$ and in (64) for the general $\mathbb{CP}^k$. It is possible to compare these results also with the direct quantization of the edge modes. The action for the edge modes for $\mathbb{CP}^k$ was obtained by direct calculation using coadjoint orbits in [5]. We found that the edge action is a higher dimensional chiral Wess-Zumino-Witten (WZW) action gauged in a left-right symmetric way with respect to the background non-Abelian gauge field, which is of order n^0 and proportional to the $SU(k)$ spin connection of the manifold. The $U(1)$ background field is, as before, $\tilde{F} = \Omega = -in\Omega_K$.⁷ This construction was further generalized in [16] to include fluctuating gauge fields. This produced a higher dimensional chiral WZW action gauged in a left-right asymmetric way, where $A^L = A + \bar{A}$, $A^R = \bar{A}$, with A , $\bar{A}$ being the fluctuating and fixed background gauge fields, respectively. The salient features of this effective action are recalled in Appendix C. Here we quote that it is of the form

$$S = \frac{N}{2nN'} \left[- \int dt d\mu \frac{\partial \rho_0}{\partial r^2} \text{Tr}[(V^{-1} D_0 V + v_F V^{-1} \hat{\mathcal{L}} V)(V^{-1} \hat{\mathcal{L}} V)] + \Gamma_{\text{WZ}} \right],$$

$$\Gamma_{\text{WZ}} = \frac{k}{2\pi} \int \rho_0 \text{Tr} \left[-d(A^L dV V^{-1} + A^R V^{-1} dV - A^L V A^R V^{-1}) + \frac{1}{3} (V^{-1} dV)^3 \right] \wedge \left(\frac{\Omega_K}{2\pi} \right)^{k-1}, \quad (69)$$

$$\begin{aligned} \hat{\mathcal{L}} V &= -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} (\partial_\nu V + A_\nu^L V - V A_\nu^R) \\ &= \mathcal{L} V - 2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} (A_\nu^L V - V A_\nu^R). \end{aligned} \quad (70)$$

By virtue of the wedge product with $(\Omega_K)^{k-1}$, this is a Kähler-Chern-Simons theory [24], a generalization of the WZW model to higher dimensions for a Kähler manifold. Since $\mathbb{CP}^k = SU(k+1)/U(k)$, the one-particle LLL wave functions $\Psi(u)$ are functions of u , transforming as an irreducible representation of $SU(k+1)$, of dimension N under the left action of $SU(k+1)$ on u , and as a representation of dimension N' under the right action of $SU(k) \subset SU(k+1)$ on u . The field V is a unitary $(N' \times N')$ matrix. We consider the droplet as made of all $SU(k)$ representations [contained in the $SU(k+1)$ representation corresponding to $\Psi(u)$] to be filled up to rank M . The density of the droplet is ρ_0 , which, at large n , is a spherically symmetric step function,

$$\rho_0 = \Theta(1 - (nr^2/M)), \quad r^2 = \bar{z} \cdot z. \quad (71)$$

Since the action is of the first order in time derivatives, and because of the nature of the WZ term, there are some

subtleties in the canonical quantization. We will give some of the technical details in Appendix D. The result is that we can define a current

$$J(\theta) = \frac{M^{k-1}}{2\pi^k} \oint dS_{2k-1} \text{Tr}[\theta(\mathcal{L} V - V A_{\mathcal{L}}^R) V^{-1}], \quad (72)$$

where $A_{\mathcal{L}}^R = -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} A_\nu^R$ is the angular component of A^R corresponding to the direction $\mathcal{L}$. This current generates left gauge transformations on V ,

$$[J(\theta), V] = -i\theta V. \quad (73)$$

The algebra of this current is then

$$[J(\theta), J(\theta')] = iJ(\theta \times \theta') + i \frac{M^{k-1}}{2\pi^k} \oint dS_{2k-1} \text{Tr}(\theta \mathcal{L} \theta'). \quad (74)$$

⁷Since we use an anti-Hermitian basis for gauge fields, A in this paper is iA of [16]. Also, to avoid confusion with the generator of gauge transformations, we use V in place of G used for the field variable in [5,16].

This is a generalized Kac-Moody algebra. The WZW action in (69) is the usual one promoted to higher dimensions by use of the factor $(\Omega_K/2\pi)^{k-1}$. Effectively we have a set of $(1+1)$ -dimensional chiral theories on the edge, the spatial direction being the angle Poisson-conjugate to the radius of the droplet. Therefore the appearance of the Kac-Moody algebra as in (74) is basically reproducing, apart from the level number, (43) in $(2+1)$ D and is entirely as expected. It is also easy to understand the level number in (74). As shown in Appendix C, it is, again as expected, given by the number of chiral modes on the edge of the droplet.

The result (74) is in complete agreement with the first line of (64). Thus we do have a derivation of the anomalous algebra of the non-Abelian gauge transformations for the edge modes from topological arguments as in (64) and a direct verification from the edge action in (74). We also point out that actions similar to (69) were derived in [25] for fermion droplets in a general phase space. The results are in agreement for the commutator anomalies given in (68) and (74).

In arriving at the result (74), we have used the action (69), including the gauge fields A^L, A^R . The extension to the algebra turns out to be independent of the gauge fields. This has to do with the fact that in deriving the edge actions we neglected terms of lower order in $1/n$. The inclusion of such terms will lead to gauge-field dependent extensions to the algebra similar to the terms in (58).

V. W_∞ AND OTHER SYMMETRIES

We now turn to the question of how our results may be viewed as a generalization of the familiar W_∞ symmetry. Recall that a high value for the background magnetic field inhibits inter-Landau level transitions, and therefore in quantum Hall systems one can consider dynamics restricted to a specific Landau level, most often the lowest one. The states of the single particle Hilbert space are given by the eigenstates of a suitable gauged Laplace operator, for which the background magnetic field will have a component which is Abelian and proportional to the $U(1)$ component of the curvature for a general complex manifold. This is the term that can be scaled with n to define a suitable large n limit. If the manifold is Kähler, the Abelian magnetic field will be proportional to the Kähler form Ω_K . Generally, there can be additional non-Abelian fields as well.

While the Hilbert space is defined by the eigenstates of the Laplacian, the states of the LLL also correspond to the states obtained by quantization of a suitable symplectic two-form. This is because the projection to the LLL identifies the spatial manifold as the phase space for the dynamics. For a Kähler manifold, the Abelian magnetic field, i.e., symplectic form is given by

$$\Omega_{\text{symp}} = n\Omega_K, \quad (75)$$

where Ω_K is the Kähler two-form. This result applies in all dimensions, the case of the plane is obtained as the large radius limit of the two-sphere, with $n = 2Br^2$, for magnetic field B . For a droplet of fermions, the low energy excitations obviously correspond to unitary transformations of the single particle Hilbert space (of the given Landau level). Thus, fundamentally, these are the quantum transformations of interest. The classical analog of these are canonical transformations preserving the symplectic structure.

Notice that Ω_{symp} is also the magnetic field, so the symplectic potential $\mathcal{A}$ (defined by $\Omega_{\text{symp}} = d\mathcal{A}$) is also the gauge potential for the electromagnetic field. Thus canonical transformations which preserve Ω_{symp} can also be viewed as $U(1)$ gauge transformations.

There is one more identification we can obtain in two dimensions. In this case, Ω_{symp} is proportional to the area element, so the canonical transformations can also be viewed as area-preserving diffeomorphisms. These are the W_∞ transformations, whose quantum version is given by the unitary transformations of the single particle Hilbert space. Since the droplet of fermions is incompressible due to the fermionic nature of the particles, these transformations are also naturally interpreted as incompressible flows of a quantum Hall droplet.

In two dimensions, we can therefore conclude that the algebra of W_∞ transformations on the edge modes has an extension given by the commutator anomaly for the $U(1)$ gauge transformations.

This is not exactly the state of affairs obtained in higher dimensions. If the background field is strictly Abelian, then we can use the symplectic form in (75). Canonical transformations are then realized as gauge transformations of the gauge potential. The commutator anomaly is still obtained for these transformations acting on the edge states; however the connection to diffeomorphisms is more subtle. The natural analog of area-preserving diffeomorphisms in higher dimensions would be volume-preserving diffeomorphisms. However, there are volume-preserving transformations which do not preserve Ω_{symp} . In fact any diffeomorphism with unit Jacobian will preserve the volume. But these are not acceptable for the dynamics of the droplet, the acceptable ones must preserve the symplectic two-form Ω_{symp} . Of course, the acceptable transformations which preserve Ω_{symp} will also preserve the phase volume $(\Omega_{\text{symp}})^k$, which is the geometrically defined volume for a Kähler manifold, up to a multiplicative constant. But these are not the most general volume preserving diffeomorphisms. Thus, while we do have an anomalous algebra of transformations on the edge modes, similar to the 2D case, with the anomaly as derived in the previous sections, the transformations are more stringent than the simple analog of area-preserving diffeomorphisms.

There are also additional features in higher dimensions. One can have non-Abelian background fields. In these

cases, the single particle Hilbert space has a basis of wave functions with an additional index for the non-Abelian symmetry. (In other words, they are sections of a suitable vector bundle, not a line bundle.) For a proper single particle phase space description, Ω_{symp} should have additional terms (and additional degrees of freedom) compared to what is given in (75). (This could arise from the action for particles carrying non-Abelian charge, as in [26].) Nevertheless, we do have the non-Abelian gauge transformations as a symmetry of the full theory, with an algebra for the generators on edge modes carrying a commutator anomaly as derived earlier.

We conclude that the Abelian and non-Abelian gauge transformations restricted to a particular Landau level are the suitable higher dimensional generalizations of the W_∞ symmetries in two dimensions. But these transformations do not have a simple geometric interpretation of preserving area or volume, or leading to incompressible flows. They are actually more stringent symmetries. The anomalous algebra of such transformations on the edge modes is the natural generalization based on the descent equations of anomalies.

Although we have focused on commutator anomalies for gauge transformations, we note that the bulk actions also have purely gravitational terms as indicated in (27) or like the $\omega d\omega$ -term in (29). They can have associated two-cocycles which can be viewed as the commutator anomaly for the generators of the local Lorentz transformations. For example, in the two-dimensional case, one can have an anomaly for spatial rotations. As is well known, one can also trade the anomaly of Lorentz transformations for a diffeomorphism anomaly [27]. In particular, by adding an appropriate local counterterm, the gravitational anomaly may be represented as a local Lorentz anomaly or as a diffeomorphism anomaly. So it may even be appropriate to consider combinations of these transformations with the gauge transformations, since the latter are related to area-preserving diffeomorphisms. Clearly there is more to be explored here, we leave this for future work.

VI. SUMMARY

We have identified the Abelian and non-Abelian gauge transformations restricted to a Landau level as the appropriate higher-dimensional analogs of the W_∞ transformations familiar from the two-dimensional quantum Hall effect. In Sec. II, we gave general arguments for relating the two-cocycle which gives the extension of the algebra for the generators of gauge transformations to the well-known anomaly descent equations. In effect, this utilizes the bulk-boundary anomaly inflow. Specifically, we obtained the extended algebra of these transformations, i.e., the higher dimensional version of the W_∞ -algebra, on the edge of a quantum Hall droplet. Explicit formulas for the extended algebras are given in Sec. IV for $2+1$, $4+1$, and $6+1$ dimensions. We emphasize that the key point

here is that the construction requires only the bulk topological action and therefore applies even when a complete microscopic edge action is not available. For quantum Hall droplets on $\mathbb{CP}^k$, where explicit edge actions are known, we also showed that the resulting extensions agree with direct canonical quantization of the edge modes. This also provides, for cases where it is possible, an independent verification of the general arguments deriving the extended algebra from the bulk action.

A key conclusion of our analysis is that the higher dimensional analogs of W_∞ transformations correspond to transformations which preserve the symplectic two-form. In two dimensions, for dimensional reasons, we could consider them as related to area-preserving diffeomorphisms. But, in general, there is an important distinction compared to the two-dimensional case since the invariance of the symplectic two-form is a stronger requirement than the preservation of the volume form. Consequently, the higher-dimensional analog of W_∞ is not the full algebra of volume-preserving diffeomorphisms, although the transformations do preserve the volume. Further, higher-dimensional systems admit non-Abelian background fields and corresponding anomalous non-Abelian edge current algebras. They have also been derived in Sec. IV.

In the complex geometries studied here, the bulk action is obtained from the Dolbeault index density. However, the anomaly-inflow and descent argument itself is more general. For each context, it is really a question of what the appropriate index density is. For example, for quantum Hall systems on manifolds without a complex structure, we should expect the relevant index density to be the one for the Dirac complex; its associated Chern-Simons action should thus play the role of the bulk action leading to the corresponding two-cocycle. While not discussed in the present paper, extending the explicit constructions to such manifolds, and incorporating the gauge and gravitational transformations into a unified description, are natural directions for future work.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. National Science Foundation Grants No. PHY-2412479 and No. PHY-2412480.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: BULK ACTION FROM THE DOLBEAULT INDEX THEOREM

We start with a brief review of construction of the bulk effective action for $\nu = 1$ for all dimensions using an appropriate index theorem [17]. The spatial manifold of

interest, i.e., V_{2k} discussed in the last section, will be a complex manifold K of complex dimension k , which includes, as a special case, the complex projective spaces $\mathbb{CP}^k$.

For a $2k$ -dimensional real manifold, the spin connections and curvatures take values in the Lie algebra of the holonomy group $SO(2k)$, corresponding to the group of local frame rotations at a point. For a complex manifold, coordinate transformations which preserve the complex structure are holomorphic transformations. This restricts the holonomy group to $U(k) \subset SO(2k)$. Correspondingly, the frame fields can be taken to be holomorphic and antiholomorphic one-forms [i.e., of the (1,0) and (0,1) type] which are combinations of the real ones given by the complex structure. The tangent space also has similar combinations which lead to a complex tangent space $T_c K$. The curvature of the manifold takes values in the Lie algebra of $U(k)$. For defining the Landau levels on V_{2k} , we take the magnetic fields to be proportional to the curvatures. It can be Abelian, corresponding to the $U(1)$ generator of $U(k)$, or non-Abelian, taking values in the Lie algebra of $SU(k) \subset U(k)$. The single-particle wave functions of the LLL obey the holomorphicity condition

$$\bar{D}_i \Psi = 0. \quad (\text{A1})$$

Thus the wave functions Ψ are part of the Dolbeault complex, in fact, the twisted Dolbeault complex since

we have background gauge fields as well. The number of normalizable solutions to (A1) is therefore given by the index theorem for the twisted Dolbeault complex as

$$\text{Index}(\bar{D}) = \int_K \text{td}(T_c K) \wedge \text{ch}(V), \quad (\text{A2})$$

where $\text{td}(T_c K)$ is the Todd class on the complex tangent space of K and $\text{ch}(V)$ is the Chern character of the relevant vector bundle [22]. For any vector bundle with curvature $\mathcal{F}$, the Chern classes c_i are defined by

$$\det \left(1 + \frac{i\mathcal{F}}{2\pi} t \right) = \sum_i c_i t^i. \quad (\text{A3})$$

The Todd class may also be represented, via the splitting principle, in terms of a generating function as

$$\text{td} = \prod_i \frac{x_i}{1 - e^{-x_i}}, \quad (\text{A4})$$

where x_i represent the “eigenvalues” of the curvature in a suitable canonical form [22]. This will be the diagonal form for a complex basis (as in $T_c K$) or the canonical antisymmetric form for real antisymmetric $i\mathcal{F}$. Rewriting (A4) using traces of powers of curvatures, the Todd class can be expanded out as

$$\text{td} = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_1^2 + c_2) + \frac{1}{24} c_1 c_2 + \frac{1}{720} (-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) + \dots \quad (\text{A5})$$

The Chern classes involved in this expansion, for low dimensions, can be easily worked out from (A3) and are given as follows.

$$\begin{aligned} c_1(T_c K) &= \text{tr} \frac{iR}{2\pi}, \\ c_2(T_c K) &= \frac{1}{2} \left[\left(\text{tr} \frac{iR}{2\pi} \right)^2 - \text{tr} \left(\frac{iR}{2\pi} \right)^2 \right], \\ c_3(T_c K) &= \frac{1}{3!} \left[\left(\text{tr} \frac{iR}{2\pi} \right)^3 - 3 \text{tr} \frac{iR}{2\pi} \text{tr} \left(\frac{iR}{2\pi} \right)^2 + 2 \text{tr} \left(\frac{iR}{2\pi} \right)^3 \right], \\ c_4(T_c K) &= \frac{1}{4!} \left[\left(\text{tr} \frac{iR}{2\pi} \right)^4 - 6 \left(\text{tr} \frac{iR}{2\pi} \right)^2 \text{tr} \left(\frac{iR}{2\pi} \right)^2 + 8 \text{tr} \frac{iR}{2\pi} \text{tr} \left(\frac{iR}{2\pi} \right)^3 + 3 \text{tr} \left(\frac{iR}{2\pi} \right)^2 \text{tr} \left(\frac{iR}{2\pi} \right)^2 - 6 \text{tr} \left(\frac{iR}{2\pi} \right)^4 \right]. \end{aligned} \quad (\text{A6})$$

The vector bundle V relevant to the Chern character in (A2), is defined by the internal gauge symmetry structure of the fermions fields. The fermion wave functions are sections of this bundle. The Chern character can be written out as

$$\text{ch}(V) = \text{tr}(e^{i\mathcal{F}/2\pi}) = \dim V + \text{Tr} \frac{i\mathcal{F}}{2\pi} + \frac{1}{2!} \text{Tr} \frac{i\mathcal{F} \wedge i\mathcal{F}}{(2\pi)^2} + \dots, \quad (\text{A7})$$

where $\dim V$ is the dimension of the bundle V . Here $\mathcal{F}$ is the gauge field strength F . (If spin is included, $\mathcal{F}$ will also include the curvature of the spin bundle.)

Finally, we note that the integral in the index theorem (A2) is to be interpreted in the standard way, namely, in the product of the Todd class and the Chern character, we pick the set of terms which give a $2k$ -form and integrate it over the manifold.

The argument for the construction of the bulk action in [17] is the following. The index gives the degeneracy of the LLL. Therefore the charge density J_0 , for fermions of unit charge, can be identified with the index density for the $\nu = 1$ state. In terms of the effective action, we can thus write

$$\frac{\delta S^{\text{bulk}}}{\delta A_0} = J_0 = \text{Index density.} \quad (\text{A8})$$

This shows that the leading term of the effective action may be taken as a Chern-Simons term $CS(A)$ whose variational derivative with respect to A_0 will give the index density. In other words, we can “integrate up” the relation (A8), and appropriately covariantize to obtain the topological part of the bulk effective action S . (In principle, there can be terms arising from dipole and higher multipole terms in J_0 . These do not affect the degeneracy, but can contribute to the action. However, because they involve multipole terms in J_0 , they will lead to higher derivatives of the fields and will be metric dependent as well. The action given by the index density will be the topological term and is also the leading term in the sense of a derivative expansion.)

This argument does not determine the purely gravitational terms in the effective action. If we have a droplet with edge modes, they can generate a gravitational anomaly as well. The purely gravitational terms in S should be determined by the gravitational anomaly via the descent method used for anomalies. For more details on this, see [17].

The conclusion is that the bulk effective action for $\nu = 1$ higher dimensional QHE is of the form

$$\begin{aligned} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \dots \alpha_{2k-1} \alpha_{2k}} M_{\alpha_1 \alpha_2} M_{\alpha_3 \alpha_4} \dots M_{\alpha_{2k-1} \alpha_{2k}} &= 2^k k! \sqrt{\det M}, \\ \epsilon^{\mu\nu \alpha_3 \alpha_4 \dots \alpha_{2k-1} \alpha_{2k}} M_{\alpha_3 \alpha_4} \dots M_{\alpha_{2k-1} \alpha_{2k}} &= 2^k k! \left[-\frac{1}{2k} \right] (M^{-1})^{\mu\nu} \sqrt{\det M}. \end{aligned} \quad (\text{B1})$$

We have used these relations in Secs. IV B and IV C in text, setting $M_{\mu\nu} = i\Omega_{\mu\nu}$. Specifically for the manifold $\mathbb{CP}^k$, the Kähler form is the Fubini-Study form given in (53). This obeys the integral formula

$$\int_{\mathbb{CP}^k} \left(\frac{\Omega_K}{2\pi} \right)^k = 1. \quad (\text{B2})$$

$$\begin{aligned} S^{\text{bulk}} &= \int \left[\text{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(A) \right]_{2k+1} \\ &+ 2\pi \int \Omega_{2k+1}^{\text{grav}}. \end{aligned} \quad (\text{A9})$$

Here $(CS)_{2p+1}(A)$ is the Chern-Simons term associated with just the gauge part and is defined by

$$\frac{1}{2\pi} d(CS)_{2p+1} = \frac{1}{(p+1)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{p+1}. \quad (\text{A10})$$

In terms of ω_{2p+1} , we have $(CS)_{2p+1} = \omega_{2p+1}$, with $d\omega_{2p+1}$ as the purely gauge field dependent part of the index density. Again, as in the case of the index theorem, one should expand the terms in the square brackets in (A9) in powers of curvatures and F and pick out the terms corresponding to the $(2k+1)$ -form. This is indicated by the subscript $2k+1$ for the square brackets.

The purely gravitational contribution is the last term in (A9). It is the analog of the Chern-Simons term for the Todd class defined by

$$d\Omega_{2k+1}^{\text{grav}} = [\text{td}(T_c K)]_{2k+2}. \quad (\text{A11})$$

The argument for the term S^{grav} is that the gravitational anomaly for the $2k$ -dimensional boundary field theory is obtained from the $(2k+2)$ -dimensional index density by the standard descent procedure.

APPENDIX B: SOME USEFUL FORMULAS

In this appendix, we collect some formulas useful for simplifications carried out in text. In general, for an antisymmetric matrix $M_{\mu\nu}$, we have

The metric on $\mathbb{CP}^k$ is the Fubini-Study metric given by

$$ds^2 = \left[\frac{dz^i d\bar{z}^i}{(1 + \bar{z} \cdot z)} - \frac{\bar{z} \cdot dz \cdot d\bar{z}}{(1 + \bar{z} \cdot z)^2} \right]. \quad (\text{B3})$$

It is the symmetric product of the differentials in (B3), there is no wedge product. In terms of real coordinates $x^\mu, \mu = 1, \dots, 2k$, defined by $z^i = x^{2i-1} + ix^{2i}$,

$\bar{z}^i = x^{2i-1} - ix^{2i}$, $i = 1, 2, \dots, k$, the volume element corresponding to this metric is

$$d\mu = \frac{k!}{\pi^k} \frac{d^{2k}x}{(1+r^2)^{k+1}}, \quad \int_{\mathbb{CP}^k} d\mu = \int_{\mathbb{CP}^k} \left(\frac{\Omega_K}{2\pi}\right)^k = 1, \quad (\text{B4})$$

where $r^2 = x^\mu x_\mu$. This is the volume element which appears in the first term of (69). For the case of $\mathbb{CP}^2$, these relations give

$$\epsilon^{\mu\nu\alpha\beta}(\Omega_K)_{\mu\nu}(\Omega_K)_{\alpha\beta} = 8\sqrt{\det \Omega_K} = \frac{32}{(1+r^2)^3}, \quad r^2 = x^\mu x_\mu$$

$$\epsilon^{\mu\nu\alpha\beta}(\Omega_K)_{\mu\nu} = -2\sqrt{\det \Omega_K}(\Omega_K^{-1})^{\alpha\beta}. \quad (\text{B5})$$

APPENDIX C: EFFECTIVE EDGE ACTIONS FOR QHE ON $\mathbb{CP}^k$

The effective edge actions for $\nu = 1$ QHE on $\mathbb{CP}^k$, both with constant Abelian and non-Abelian gauge fields were derived in [4,5] using a coadjoint orbit construction. Here we summarize the results.

1. Abelian case

The states of the lowest Landau level for $\mathbb{CP}^k$ form a symmetric representation of $SU(k+1)$ of rank n , which may be decomposed into a series of $SU(k)$ representations. For simplicity we have considered a spherical droplet by having the first M $SU(k)$ representations occupied. To obtain such a droplet as the multifermion ground state we used an appropriate spherically symmetric confining potential. The density of the droplet, in the large n limit, then has the form

$$\rho_0 = \Theta(1 - (nr^2/M)), \quad r^2 = \bar{z} \cdot z. \quad (\text{C1})$$

This defines a spherical droplet of radius r_D , where $r_D^2 = M/n$ for large n and M , with $n \gg M$.

For an Abelian constant magnetic field of the form $\bar{F} = -in\Omega_K$, where Ω_K is the Fubini-Study Kähler form defined in (53), we found that, in the large n -limit, the quantum Hall dynamics is confined to the boundary of the droplet and is described by a chiral action of the form

$$S = -\frac{1}{4\pi^k} M^{k-1} \int dt dS^{2k-1} [\dot{\Phi} \mathcal{L} \Phi + v_F (\mathcal{L} \Phi)^2], \quad (\text{C2})$$

where Φ is a scalar field, v_F is the Fermi velocity (which is related to the gradient of the confining potential energy on the edge of the droplet) and $\mathcal{L}$ defines a particular tangential direction along the boundary of the droplet. In particular,

$$\mathcal{L} = -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} \partial_\nu = i \left(z \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right). \quad (\text{C3})$$

In two dimensions, with $\mathbb{CP}^1 \sim S^2$, $\mathcal{L} = \frac{\partial}{\partial \varphi}$ is the derivative tangential to the circular edge boundary and the equation of motion derived from (C2) shows the chirality of the edge modes. In higher dimensions, there are several tangential directions along the boundary droplet. However, the theory selects a particular tangential direction $\mathcal{L}$ for the propagation of the edge modes. This is Poisson conjugate to the radial direction $\hat{x}^\mu$, as indicated in (C3). Effectively, the higher-dimensional edge action is a collection of $(1+1)$ -dimensional chiral edge modes. The number of such modes is indicated by the overall level number in (C2). In particular, since $S^{2k-1}/S^1 = \mathbb{CP}^{k-1}$, separating out the S^1 direction (corresponding to $\mathcal{L}$), we get a $\mathbb{CP}^{k-1}$ -worth of $(1+1)$ -dimensional theories [4,5]. The volume of $\mathbb{CP}^{k-1}$ as given by the integral over dS_{2k-1} , after factoring out S^1 , is $\pi^{k-1}/(k-1)!$. The edge modes which are the last occupied $SU(k)$ states form a rank M symmetric representation, which is of dimension

$$\frac{(M+k-1)!}{(M!(k-1)!)} \approx \frac{M^{k-1}}{(k-1)!} = \frac{\pi^{k-1}}{(k-1)!} \left[\frac{M^{k-1}}{\pi^{k-1}} \right] \quad (\text{C4})$$

for large M . Since the factor $\pi^{k-1}/(k-1)!$ is already accounted for by dS_{2k-1} in (C2), we see that the level number is correctly given by the number of chiral modes of the edge of the droplet.

2. Non-Abelian case

The formulation of QHE on $\mathbb{CP}^k$ allows the possibility of non-Abelian background magnetic fields. In particular, one can choose “constant” magnetic fields proportional to the $U(k)$ curvature components. This means that, in addition to the background $U(1)$ magnetic field which is given by Ω , we choose the $SU(k)$ gauge potential $\bar{A}$ proportional to the spin connection of $\mathbb{CP}^k$ [5]. In this case, the single-particle lowest Landau level wave functions are sections of a $U(k)$ bundle on $\mathbb{CP}^k$ carrying appropriate $U(k)$ charges. In particular, since $\mathbb{CP}^k = SU(k+1)/U(k)$ can be parametrized by a group element $u \in SU(k+1)$, the one-particle LLL wave functions $\Psi(u)$ are functions of u . They transform as an irreducible representation of $SU(k+1)$, of dimension N under the left action of $SU(k+1)$ on u , and as a representation of dimension N' under the right action of $SU(k) \subset SU(k+1)$ on u . [In the Abelian case, $N' = 1$, and the wave functions are singlets under the right action of $SU(k)$.]

The $SU(k+1)$ representation of the LLL states can be viewed as made of a set of $SU(k)$ representations. All such $SU(k)$ representations up to rank M are taken to be filled to form the droplet. The density of the droplet is ρ_0 , which, at large n , is again a spherically symmetric step-function as in (C1). For the large n simplification of the action, we can again use $N/N' \rightarrow n^k/k!$.

We found in [5] that the effective action at large n is also a chiral boundary action which has the form of a generalized, higher dimensional Wess-Zumino-Witten action given by

$$S = -\frac{N}{2nN'} \int dt d\mu \frac{\partial \rho_0}{\partial r^2} \text{tr}[(V^{-1} \dot{V} + v_F V^{-1} \hat{\mathcal{L}} V) V^\dagger \hat{\mathcal{L}} V] \\ + \frac{Nk}{4\pi nN'} \int \rho_0 \left[-d(i\bar{A} dV V^{-1} + i\bar{A} V^{-1} dV) \right. \\ \left. + \frac{1}{3} (V^{-1} dV)^3 \right] \wedge \left(\frac{\Omega_K}{2\pi} \right)^{k-1}, \quad (\text{C5})$$

where the field V is a unitary ($N' \times N'$) matrix⁸ and

$$\hat{\mathcal{L}} = -(\Omega_K^{-1})^{\mu\nu} 2r_D \hat{x}_\mu D_\nu \quad (\text{C6})$$

$\hat{\mathcal{L}}$ is the component of the covariant derivative $D = \partial + [\bar{A}, \cdot]$ perpendicular to the radial direction, along a special tangential direction on the droplet boundary. This is the gauged version of the operator $\mathcal{L}$ defined in (C3). The density of the droplet is the spherically symmetric ρ_0 given in (C1).

The action S in (C5) is a higher dimensional generalization of a Wess-Zumino-Witten action, vectorially gauged with respect to the time independent background gauge field $\bar{A}$. The equations of motion further show that the propagation is along a particular direction and the theory is chiral. The first term on the right-hand side in (C5) is evidently on the boundary of the droplet since the derivative of ρ_0 is a δ -function. The second term is a WZW-type term written as an integral over a manifold of dimension equal to $(2k+1)$, corresponding to the droplet and time. The usual 3-form in the integrand of the WZW-term, $(V^{-1} dV)^3$, has now been augmented to the appropriate $(2k+1)$ -form $(V^{-1} dV)^3 \wedge \Omega_K^{k-1}$. Since the WZW-term is the integral of a locally exact form [5], the whole action S is an edge action.

Further, as in the Abelian case, because the propagation is along a particular tangential direction, namely the Poisson conjugate to the radial direction, the higher dimensional edge action (C5) can be considered as a collection of $(1+1)$ -dimensional non-Abelian chiral modes. Since $N/N' \rightarrow n^k/k!$ for large n , one can easily show that the multiplicity and the level number are the same as in the Abelian case. [The action (C5) coincides with (C2) when $N' = 1$].

Once we allow for fluctuating (dynamical) gauge fields, the chiral action (C5) becomes anomalous. It was shown in [16] that the effective action in the presence of fluctuating gauge fields is of the form

⁸In earlier papers, this field was denoted by G . We have changed the notation here to avoid confusion, as G is used to denote a different quantity in the present paper.

$$S = -\frac{N}{2nN'} \int dt d\mu \frac{\partial \rho_0}{\partial r^2} \text{tr}[(V^{-1} D_0 V + v_F V^{-1} \hat{\mathcal{L}} V) V^{-1} \hat{\mathcal{L}} V] \\ + \frac{Nk}{4\pi nN'} \int \rho_0 \left[-d(A^L dV V^\dagger + A^R V^\dagger dV - A^L V A^R V^{-1}) \right. \\ \left. + \frac{1}{3} (V^{-1} dV)^3 \right] \wedge \left(\frac{\Omega_K}{2\pi} \right)^{k-1}, \quad (\text{C7})$$

where $A^L = A + \bar{A}$, $A^R = \bar{A}$, with $A, \bar{A}$ being the fluctuating and fixed background gauge field, respectively. The derivative operator $\hat{\mathcal{L}}$, as before, denotes the component of the generalized covariant derivative $D_\mu V = \partial_\mu V + A_\mu^L V - V A_\mu^R$ along the tangential direction Poisson conjugate to the radius, i.e.,

$$\hat{\mathcal{L}} V = -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} (\partial_\nu V + A_\nu^L V - V A_\nu^R). \quad (\text{C8})$$

The effective edge action (C7) is a higher dimensional chiral Wess-Zumino-Witten action gauged in a left-right asymmetric way. Gauge transformations correspond to left transformations on V , which would be, for an invariant action, compensated by the transformation of the fluctuating field A . The action (C7) is, however, not gauge-invariant; its noninvariance under gauge transformations is cancelled by the change in the bulk topological action [16].

APPENDIX D: CANONICAL QUANTIZATION OF THE EDGE MODES

The quantum dynamics of the edge modes can be obtained using standard methods of geometric quantization. We will start with the Abelian case where the kinetic term in the action is of the form

$$S = C \int dt \oint \Phi \mathcal{L} \Phi, \quad (\text{D1})$$

where C is a constant. We consider an interval of time $[t_i, t_f]$. The variation of the action takes the form

$$\delta S = -2C \int dt \oint \delta \Phi (\mathcal{L} \Phi) + C \oint \mathcal{L} \Phi \delta \Phi \Big|_{t_i}^{t_f}. \quad (\text{D2})$$

This identifies the canonical (or symplectic) one-form and two-form for the theory as

$$\mathcal{A} = C \oint \mathcal{L} \Phi \delta \Phi, \quad \Omega_{\text{symp}} = \delta \mathcal{A} = C \oint (\mathcal{L} \delta \Phi) \delta \Phi, \quad (\text{D3})$$

where δ denotes the exterior derivative on the space of fields.

Consider the vector field $V_\theta = \oint (\theta^0 \delta / \delta \Phi)$. The interior contraction of this with Ω_{symp} is

$$V_\theta] \Omega_{\text{symp}} = 2C \oint (\mathcal{L}\theta^0) \delta\Phi = -\delta[-2C \oint (\mathcal{L}\theta^0) \Phi]. \quad (\text{D4})$$

This identifies $J_0(\theta^0) = -2C \oint (\mathcal{L}\theta^0) \Phi$ as the function on the phase space which, via Poisson brackets, implements the transformation given by V_θ . Using $\{f, h\} = -V_f] \delta h$, we can work out the Poisson brackets to get

$$\{-2C \oint (\mathcal{L}\theta^0) \Phi, \Phi(x)\} = -V_\theta] \delta\Phi(x) = -\theta^0(x). \quad (\text{D5})$$

We can now obtain the quantum commutation rules

$$\begin{aligned} [J_0(\theta^0), \Phi(x)] &= -i\theta^0(x), \\ [J_0(\theta^0), J_0(\theta^0)] &= i(-V_\theta] \delta J_0(\theta^0)) = i2C \oint \theta^0(\mathcal{L}\theta^0). \end{aligned} \quad (\text{D6})$$

This leads to the results in (48) and (68) in text, with $C = -M^{k-1}/(4\pi^k)$.

In the case of the non-Abelian edge modes with the action (69), we can follow a similar strategy of varying the action and carrying out an integration by parts for the terms with the time derivative of the field variations. The boundary term at t_f will give the canonical one-form $\mathcal{A}$. For the action (69), we find

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4,$$

$$\mathcal{A}_1 = \lambda \int d\mu \partial_\mu \rho_0 (\Omega_K^{-1})^{\mu\nu} \text{Tr}(V^{-1} \delta V V^{-1} D_\nu V), \quad (\text{D7})$$

$$\mathcal{A}_2 = -\frac{\lambda k}{2\pi} \int \rho_0 d(\text{Tr}[\delta V V^{-1} A^L + A^R V^{-1} \delta V]) \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}, \quad (\text{D8})$$

$$\mathcal{A}_3 = \frac{\lambda k}{2\pi} \int \rho_0 d(\text{Tr}[\delta V V^{-1} dV V^{-1}]) \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}, \quad (\text{D9})$$

$$\mathcal{A}_4 = -\frac{\lambda k}{2\pi} \int \rho_0 \text{Tr}[d(\delta V V^{-1}) dV V^{-1}] \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}, \quad (\text{D10})$$

where $\lambda = N/(2nN')$. A couple of comments are in order. $\mathcal{A}_1$ arises from the kinetic term in (69), $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are due to Γ_{WZ} . The variation of Γ_{WZ} yields a total derivative. The time-derivative part of it integrates to a boundary term at t_f with no further time derivatives. This is the term $\mathcal{A}_4$ in (D7). The appearance of this term is tied to the fact that we cannot express Γ_{WZ} entirely in terms of $2k$ -dimensional spacetime integral of a local density. An extra coordinate is needed, which, in the present case is the radial coordinate of the droplet. The expression for $\delta\mathcal{A}_4$ will produce a total derivative and so, while $\mathcal{A}_4$ cannot be written as an

expression on the boundary of the droplet, its contribution to Ω_{symp} is defined on the boundary. There are also terms in $\delta\Gamma_{\text{WZ}}$ which are on the boundary of the droplet, which, upon a further integration over time, gives the additional terms $\mathcal{A}_2$, $\mathcal{A}_3$. We will shortly convert these terms to integrals on the boundary of the droplet, but, for now, we have displayed them as integrals of total (spatial) derivatives. Using the expression for ρ_0 from (71), we can further simplify all these terms. For $\mathcal{A}_1$ we get

$$\begin{aligned} \mathcal{A}_1 &= \frac{\lambda k!}{\pi^k} \int dS_{2k-1} dr \frac{r^{2k-1}}{(1+r^2)^{k+1}} \left[-\delta\left(\frac{M}{n} - r^2\right) \right] \\ &\quad \times 2x_\mu (\Omega_K^{-1})^{\mu\nu} \text{Tr}(V^{-1} \delta V V^{-1} D_\nu V) \\ &\approx \frac{\lambda k!}{2\pi^k} \frac{M^{k-1}}{n^{k-1}} \int dS_{2k-1} \text{Tr}(V^{-1} \delta V V^{-1} \hat{\mathcal{L}} V), \end{aligned} \quad (\text{D11})$$

where $\hat{\mathcal{L}}V$, as given in (C8), is the gauged version of the derivative $\mathcal{L}$ given in (55). Upon taking a second variation, we find

$$\Omega_1 \equiv \delta\mathcal{A}_1 = \frac{\lambda k!}{2\pi^k} \frac{M^{k-1}}{n^{k-1}} \int dS_{2k-1} \text{Tr}(v^2 V^{-1} \hat{\mathcal{L}} V - v(\hat{\mathcal{L}} v)), \quad (\text{D12})$$

where $v = V^{-1} \delta V$ and $\hat{\mathcal{L}}v = \mathcal{L}v + A^R v - vA^R$. Notice that $v = V^{-1} \delta V$ is invariant under group translations on the left of V and undergoes conjugation for the right translations. So $\hat{\mathcal{L}}v$ is the correct gauge covariant derivative of v .

For the remaining terms (D8)–(D10) it is easier to take the second variation before simplifying. We find

$$\begin{aligned} \delta\mathcal{A}_2 &= -\frac{\lambda k}{2\pi} \int \rho_0 d[\text{Tr}(v^2 (V^{-1} A^L V - A^R))] \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}, \\ \delta\mathcal{A}_3 &= \frac{\lambda k}{2\pi} \int \rho_0 d[\text{Tr}(-v dv + v^2 V^{-1} dV)] \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}, \\ \delta\mathcal{A}_4 &= -\frac{\lambda k}{2\pi} \int \rho_0 d[\text{Tr}(-v dv + 2v^2 V^{-1} dV)] \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1}. \end{aligned} \quad (\text{D13})$$

Combining these terms we get

$$\begin{aligned} \delta(\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) &= -\frac{\lambda k}{2\pi} \int \rho_0 d(\text{Tr}[v^2 V^{-1} dV]) \\ &\quad \wedge \left(\frac{\Omega_K}{2\pi}\right)^{k-1} \\ &= -\frac{\lambda k!}{2\pi^k} \frac{M^{k-1}}{n^{k-1}} \int dS_{2k-1} \text{Tr}(v^2 V^{-1} \hat{\mathcal{L}} V). \end{aligned} \quad (\text{D14})$$

We now add this to (D12) to get

$$\begin{aligned}\Omega_{\text{symp}} &= -\frac{\lambda k! M^{k-1}}{2\pi^k n^{k-1}} \int dS_{2k-1} \text{Tr}(v(\hat{\mathcal{L}}v)) \\ &\approx -\frac{M^{k-1}}{4\pi^k} \int dS_{2k-1} \text{Tr}(v(\hat{\mathcal{L}}v)).\end{aligned}\quad (\text{D15})$$

Once again we can consider left translations on V , with the vector field V_θ such that $V_\theta \rfloor \delta V = \theta V$. The contraction of this with Ω_{symp} gives

$$V_\theta \rfloor \Omega_{\text{symp}} = -\delta \left[\frac{M^{k-1}}{2\pi^k} \int dS_{2k-1} \text{Tr}(\theta(\mathcal{L}V - VA_\mathcal{L}^R)V^{-1}) \right], \quad (\text{D16})$$

where $A_\mathcal{L}^R = -2r_D \hat{x}_\mu (\Omega_K^{-1})^{\mu\nu} A_\nu^R$. We therefore define the generator of the transformation as

$$J(\theta) = \frac{M^{k-1}}{2\pi^k} \int dS_{2k-1} \text{Tr}(\theta(\mathcal{L}V - VA_\mathcal{L}^R)V^{-1}). \quad (\text{D17})$$

Working out the Poisson brackets and upgrading to the commutation rules, we find the algebra

$$\begin{aligned}[J(\theta), V] &= -i\theta V, \\ [J(\theta), J(\theta')] &= iJ(\theta \times \theta') + i\frac{M^{k-1}}{2\pi^k} \int dS_{2k-1} \text{Tr}(\theta(\mathcal{L}\theta')).\end{aligned}\quad (\text{D18})$$

These are the results (73) and (74) quoted in text.

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