

Tripartite entanglement in $e^+e^- \rightarrow t\bar{t}Z$

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(Received 4 July 2026; accepted 7 October 2026; published 23 October 2026)

Multipartite entanglement is a uniquely quantum form of correlation that captures collective properties of a composite quantum state beyond those encoded in its bipartite subsystems. We investigate this phenomenon in the process $e^+e^- \rightarrow t\bar{t}Z$ at a future lepton collider, where the final-state spins span the tripartite Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. Starting from the standard model helicity amplitudes, we reconstruct the full 12×12 spin density matrix and characterize its entanglement structure through one-to-one negativities, one-to-other negativities, and the genuine multipartite negativity, evaluated at three increasingly inclusive levels of phase-space integration. Pairwise entanglement is generally suppressed relative to the collective (one-to-other) and the genuine multipartite entanglement, and all measures decrease as more kinematic information is integrated out. Assuming quantum tomography in the fully leptonic decay channel at $\sqrt{s} = 1$ TeV, we find that collective entanglement should be accessible at a realistic high-luminosity polarized lepton collider. By contrast, certifying genuine multipartite entanglement is more challenging, with only limited sensitivity projected for a specific polarization benchmark within the luminosity expected at the International Linear Collider. The study establishes $e^+e^- \rightarrow t\bar{t}Z$ as an attractive laboratory for probing multipartite entanglement in high-energy collisions and provides a general mixed state framework that applies to any tripartite spin system.

DOI: [10.1103/188d-nfqp](https://doi.org/10.1103/188d-nfqp)

I. INTRODUCTION

Polarization and spin correlations are powerful probes of the dynamics of high-energy collisions, playing a central role in both precision measurements and searches for physics beyond the standard model. The interpretation of these observables through the lens of quantum information science has recently opened a new perspective on these studies, for instance, probing whether a system is entangled. Most analyses to date have focused on bipartite systems, including $t\bar{t}$ [1–41], $\tau^+\tau^-$ [42–50], and electro-weak gauge boson final states [51–79], where the quantum state is classified simply as separable or entangled. The theoretical framework for studying tripartite entanglement and nonlocal correlations in particle physics processes was

introduced in Refs. [80,81]. Tripartite quantum correlations have subsequently been explored in a few collider studies [82–85]. Nevertheless, a comprehensive characterization of experimentally accessible tripartite spin systems, particularly in the presence of realistic mixed quantum states, remains limited.

Extending the study of quantum entanglement from bipartite to multipartite systems opens a qualitatively new frontier. Unlike bipartite states, multipartite quantum states admit a hierarchy of inequivalent separability classes. A tripartite state may be fully separable, bi-separable, or genuinely multipartite entangled (GME), and its quantum correlations cannot, in general, be reconstructed from one-body observables or pairwise correlations alone [80,81,86–89]. Remarkably, genuine multipartite entanglement may persist even when all two-body entanglement measures vanish. Understanding how such collective quantum correlations arise in collider processes, and whether they survive realistic phase-space averaging and experimental reconstruction, constitutes an important open question at the interface of quantum information science and high-energy physics.

From the perspective of collider phenomenology, multipartite correlations represent a natural extension of the

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study of spin observables. While polarization and pairwise spin correlations have become standard tools in new physics searches [90–103], the corresponding multipartite quantum correlations remain largely unexplored. Quantum tomography provides access to the complete spin density matrix, making it possible to move beyond one-body and two-body observables, probing the full structure of multipartite quantum states. This opens the possibility of identifying novel observables that are inaccessible to traditional analyses and may provide complementary information on the dynamics of particle interactions.

In this work, we perform a systematic study of tripartite spin entanglement in $e^+e^- \rightarrow t\bar{t}Z$ at future lepton colliders. The final state contains three spin-carrying particles and spans the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, allowing direct access to pairwise, collective, and genuinely multipartite quantum correlations. We construct the full spin density matrix and characterize its separability structure through a hierarchy of entanglement measures applicable to mixed states: (1) one-to-one negativities, which probe pairwise entanglement between subsystems [87], (2) one-to-other negativities, which quantify the entanglement between a single particle and the remaining pair, and (3) the genuine multipartite negativity (GMN) [88], which detects correlations that cannot be reduced to any bi-separable description. Unlike entanglement diagnostics that rely on pure state constructions, the measures employed here remain valid for arbitrary mixed states, providing a natural framework for realistic collider measurements. These observables resolve the full hierarchy of separability classes accessible to the $t\bar{t}Z$ spin state. We find that the dominant quantum correlations are collective rather than pairwise. Pairwise entanglement is already suppressed prior to phase-space integration, while one-to-other entanglement and genuine multipartite entanglement remain sizeable across broad regions of phase space. We identify fiducial kinematic regions that maximize the retained multipartite entanglement and estimate the integrated luminosity required to probe these effects at future lepton colliders.

The prospects for studying these effects are particularly promising, since $e^+e^- \rightarrow t\bar{t}Z$ will be accessible at future high-energy lepton colliders. The process naturally produces a tripartite spin system and can be reconstructed with high precision at facilities operating above the $t\bar{t}Z$ threshold, including the high-energy stages of the International Linear Collider (ILC) and the Compact Linear Collider [104,105]. Their clean experimental environment, polarized beams, and excellent event reconstruction capabilities provide the ingredients required for detailed studies of multipartite spin states. Similar opportunities arise at the proposed multi-TeV muon collider through the analogous process $\mu^+\mu^- \rightarrow t\bar{t}Z$ [106]. In this context, multipartite correlations complement the established top quark precision program by providing a window into the collective quantum structure of the full spin density matrix.

This paper is organized as follows. In Sec. II, we set up the kinematics and construct the spin density matrix for $e^+e^- \rightarrow t\bar{t}Z$ from helicity amplitudes. In Sec. III, we introduce the entanglement measures employed in this work and review the separability structure of tripartite quantum systems. Section IV presents the entanglement properties of the differential and fully inclusive spin states. In Sec. V, we analyze partially inclusive states and assess the prospects for observing multipartite entanglement at a future lepton collider. Our conclusions are presented in Sec. VI. The details of the quantum tomography reconstruction for the $t\bar{t}Z$ process are provided in Appendix A.

II. HELICITY AMPLITUDES AND SPIN QUANTUM STATES

In this section we set up the kinematics of $e^+e^- \rightarrow t\bar{t}Z$, compute the tree-level helicity amplitudes, and construct the spin density matrix of the final state in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ (dimension $2 \times 2 \times 3 = 12$). The resulting density matrix is the central object on which all of the entanglement measures of Sec. III will be evaluated.

A. Kinematics

We consider the process

$$e^-(k_1, \lambda_1) + e^+(k_2, \lambda_2) \rightarrow t(p_1, h_1) + \bar{t}(p_2, h_2) + Z(p_3, h_3) \quad (2.1)$$

at a future lepton collider with center-of-mass energy $\sqrt{s}$. The four-momenta k_1, k_2, p_1, p_2 , and p_3 are assigned to the electron, positron, top, antitop, and Z-boson, respectively, in the lab frame. Similarly, $\lambda_1, \lambda_2, h_1, h_2$, and h_3 are the helicities of these particles. The spatial z -axis is chosen along the incoming electron beam, $\hat{z} \equiv \hat{k}_1$. Neglecting the electron mass, one can write $k_1^\mu = \frac{\sqrt{s}}{2}(1, 0, 0, 1)$ and $k_2^\mu = \frac{\sqrt{s}}{2}(1, 0, 0, -1)$. We consider the idealized case with no initial-state radiation. In this limit, the lab frame coincides with the center-of-mass frame.

Using the rotational symmetry around the beam axis, we fix the x -axis such that the $\hat{x} - \hat{z}$ plane is aligned with the production plane of Z and the $t\bar{t}$ subsystem and the x -component of the Z 's momentum is positive (see the left panel of Fig. 1). Introducing the polar angle θ_Z for the Z 's momentum, one can write $p_3^\mu = (E_3, q \sin \theta_Z, 0, q \cos \theta_Z)$ with $E_3 = \sqrt{m_Z^2 + q^2}$. In the lab frame, the $t\bar{t}$ subsystem is back-to-back with the Z -boson. The momentum of the $t\bar{t}$ subsystem is $p_{t\bar{t}}^\mu = (p_1 + p_2)^\mu = (E_{t\bar{t}}, -q \sin \theta_Z, 0, -q \cos \theta_Z)$ with $E_{t\bar{t}} = \sqrt{m_{t\bar{t}}^2 + q^2}$, where $m_{t\bar{t}}^2 = (p_1 + p_2)^2$ is the $t\bar{t}$ invariant mass.

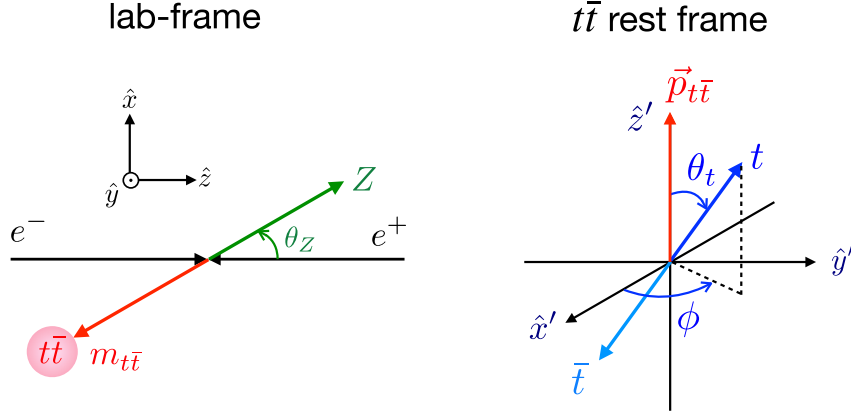


FIG. 1. Definition of the kinematical variables. Left: lab-frame view defining $m_{t\bar{t}}$ and θ_Z . Right: $t\bar{t}$ rest-frame view defining θ_t and ϕ .

The conservation of the energy, $\sqrt{s} = E_3 + E_{t\bar{t}}$, leads to $q = \frac{1}{2}\sqrt{s - 2(m_{t\bar{t}}^2 + m_Z^2)} + \frac{1}{s}(m_{t\bar{t}}^2 - m_Z^2)^2$.

The kinematics of t and $\bar{t}$ are determined as follows. We first describe the t and $\bar{t}$ momenta at the $t\bar{t}$ rest frame as $p'_1 = [\frac{m_{t\bar{t}}}{2}, p'(\sin\theta_t\cos\phi, \sin\theta_t\sin\phi, \cos\theta_t)]$ and $p'_2 = [\frac{m_{t\bar{t}}}{2}, -p'(\sin\theta_t\cos\phi, \sin\theta_t\sin\phi, \cos\theta_t)]$ with $p' = \sqrt{\frac{m_{t\bar{t}}^2}{4} - m_t^2}$. Here, θ_t and ϕ are the polar and azimuthal angles of the top momentum, where the polar axis is taken to be the boost direction of the $t\bar{t}$ subsystem (see the right panel of Fig. 1). The lab-frame momenta, p_1 and p_2 , are obtained by Lorentz-transforming p'_1 and p'_2 to the lab frame.

With the above parametrization, the lab-frame momenta p_1 , p_2 , and p_3 are completely fixed (up to the cylindrical symmetry of the collider) by the following four parameters:

- (i) $m_{t\bar{t}}$: $t\bar{t}$ invariant mass, $m_{t\bar{t}} \in [2m_t, \sqrt{s} - m_Z]$;
- (ii) θ_Z : polar angle of Z in the lab frame, $\cos\theta_Z \in [-1, 1]$;
- (iii) θ_t : polar angle of t in the $t\bar{t}$ rest frame, $\cos\theta_t \in [-1, 1]$; and
- (iv) ϕ : azimuthal angle of t in the $t\bar{t}$ rest frame, $\phi \in [0, 2\pi)$.

In this parametrization, CP acts on the final state as $t \leftrightarrow \bar{t}$ together with $\vec{p}_i \rightarrow -\vec{p}_i$ for every final-state momentum. This reverses $\hat{p}_Z$ in the lab frame, which sends $\phi_Z \rightarrow \phi_Z + \pi$. Reusing the cylindrical symmetry of the collider to reset $\phi_Z = 0$ (a rotation by π about $\hat{z}$), CP acts on the four variables as

$$CP: (m_{t\bar{t}}, \cos\theta_Z, \cos\theta_t, \phi) \rightarrow (m_{t\bar{t}}, -\cos\theta_Z, -\cos\theta_t, -\phi). \quad (2.2)$$

The sign flips of $\cos\theta_t$ and ϕ are most easily checked by decomposing CP into its two factors. We treat CP as an active transformation acting only on the final state: the unpolarized initial state is itself CP -symmetric, so the lab axes $(\hat{x}, \hat{y}, \hat{z})$ are kept fixed throughout. Under C , $\vec{p}_i$ is replaced by $\vec{p}_i = -\vec{p}_i$ in the $t\bar{t}$ rest frame, contributing $\cos\theta_t \rightarrow -\cos\theta_t$ and $\phi \rightarrow \phi + \pi$. Under P (with the accompanying rotation), only the final-state $\hat{p}_Z$ flips in the lab (not $\hat{z}$), hence, $\hat{z}' = -\hat{p}_Z$ and $\hat{y}' \propto \hat{z} \times \hat{p}_Z$ each change sign, while $\hat{x}' = \hat{y}' \times \hat{z}'$ picks up two such minus signs and is unchanged. Projecting onto the new triad then contributes an extra $\phi \rightarrow \pi - \phi$. Composing the two gives (2.2). For unpolarized beams the tree-level process is CP -invariant, so any CP -even observable is left invariant by (2.2) and any CP -conjugate pair, e.g., $(N_{tZ}, N_{\bar{t}Z})$, satisfies $O_{tZ}(\cos\theta_Z, \cos\theta_t, \phi) = O_{\bar{t}Z}(-\cos\theta_Z, -\cos\theta_t, -\phi)$.

B. Helicity amplitudes

To obtain the spin quantum state of the final state, we compute the helicity amplitudes of $e^+e^- \rightarrow t\bar{t}Z$ at the leading order. There are five types of tree-level diagrams, labeled by $I = a, \dots, e$, shown in Fig. 2. The amplitudes for fixed momenta and helicities are given as the sum of five diagrams,

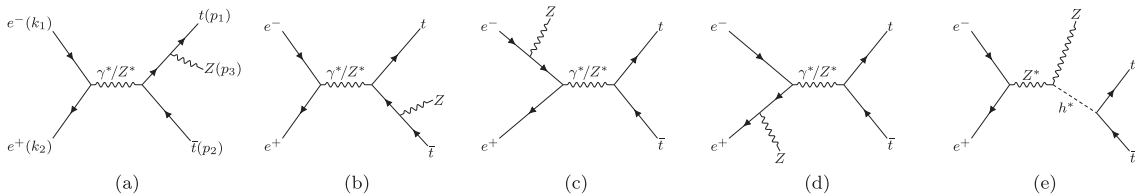


FIG. 2. Tree-level Feynman diagrams contributing to $e^+e^- \rightarrow t\bar{t}Z$ in the standard model. Diagrams (a) and (b): Z radiation from t and $\bar{t}$. Diagrams (c) and (d): Z radiation from e^- and e^+ . Diagram (e): Higgsstrahlung $e^+e^- \rightarrow Zh^* \rightarrow Zt\bar{t}$.

$$\mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} = \sum_{I=a, \dots, e} \mathcal{M}^{(I)}[\bar{u}(k_1, \lambda_1), v(k_2, \lambda_2), u(p_1, h_1), \bar{v}(p_2, h_2), \epsilon(p_3, h_3)], \quad (2.3)$$

where $\mathcal{M}^{(I)}$ is the contribution from the diagram I , which is a function of four Dirac spinors corresponding to e^- , e^+ , t , and $\bar{t}$ and the polarization vector ϵ^μ of the Z -boson. The fermion and antifermion spinors $u(p, \lambda)$ and $v(p, \lambda)$ with the helicity $\lambda = \pm$ and the momentum $p^\mu = (E, \vec{p})$ with $E = \sqrt{m^2 + |\vec{p}|^2}$ and $\vec{p} = |\vec{p}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ are given by [107–109]¹

$$\begin{aligned} u(p, \lambda) &= \begin{pmatrix} \omega_{-\lambda}(p) \chi_\lambda(p) \\ \omega_\lambda(p) \chi_\lambda(p) \end{pmatrix}, & v(p, \lambda) &= \begin{pmatrix} -\lambda \omega_\lambda(p) \chi_{-\lambda}(p) \\ \lambda \omega_{-\lambda}(p) \chi_{-\lambda}(p) \end{pmatrix}, & \omega_\lambda(p) &= \sqrt{E + \lambda |\vec{p}|}, \\ \chi_+ &= \begin{pmatrix} e^{-i\phi/2} \cos \frac{\theta}{2} \\ e^{i\phi/2} \sin \frac{\theta}{2} \end{pmatrix}, & \chi_- &= \begin{pmatrix} -e^{-i\phi/2} \sin \frac{\theta}{2} \\ e^{i\phi/2} \cos \frac{\theta}{2} \end{pmatrix}. \end{aligned} \quad (2.4)$$

When $\theta = 0$ and π , the phase ambiguity in $\chi_\pm$ is fixed by setting $\phi = 0$. For the Z -boson momentum $p_3^\mu = (E_3, q \sin \theta_Z, 0, q \cos \theta_Z)$, the polarization vectors corresponding to helicities $h_3 \in \{+1, 0, -1\}$ are given by

$$\begin{aligned} \epsilon^\mu(p_3, \pm) &= \pm \frac{1}{\sqrt{2}} (0, -\cos \theta_Z, \pm i, \sin \theta_Z), \\ \epsilon^\mu(p_3, 0) &= \left(\frac{q}{m_Z}, \frac{E_3}{m_Z} (\sin \theta_Z, 0, \cos \theta_Z) \right). \end{aligned} \quad (2.5)$$

These polarization vectors satisfy $p_3 \cdot \epsilon(p_3, h_3) = 0$ and the standard orthonormality relations. We explicitly evaluate the helicity amplitudes in Eq. (2.3) with these spinors and polarization vectors. Since the electron mass is neglected, chirality conservation at the $e^+ e^- \gamma^* / Z^*$ vertex implies helicity conservation along the fermion line, so the amplitudes vanish unless $\lambda_1 = -\lambda_2$,

$$\mathcal{M}_{h_1, h_2, h_3}^{++} = \mathcal{M}_{h_1, h_2, h_3}^{--} = 0. \quad (2.6)$$

In the high-energy regime relevant here, the neutral-current contribution is naturally described in terms of the chiral couplings g_L^f and g_R^f (or, equivalently, g_V^f and g_A^f). For $q^2 \gg m_Z^2$, the γ^* and Z^* propagators have the same leading $1/q^2$ scaling, while their different coupling structures remain imprinted in the helicity amplitudes. In addition, the observed on-shell Z can be radiated from the fermion lines, and the Higgsstrahlung topology $e^+ e^- \rightarrow Zh^* \rightarrow Zt\bar{t}$ provides a contribution with a distinct coupling structure. The spin density matrix, therefore, receives coherent contributions from these different electroweak topologies.

C. Spin quantum states

For fixed external momenta and initial polarization (λ_1, λ_2) , the unnormalized final spin state in the Hilbert

¹Here, (θ, ϕ) denote the polar and azimuthal angles of $\vec{p}$ in the lab frame, and should not be confused with θ_Z , θ_t , and ϕ introduced in Sec. II A.

space $H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ is given by

$$|\tilde{\psi}_{\lambda_1, \lambda_2}\rangle = \sum_{h_1, h_2, h_3} \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} |h_1, h_2, h_3\rangle. \quad (2.7)$$

The corresponding unnormalized 12×12 density matrix is

$$R_{(h_1, h_2, h_3), (h'_1, h'_2, h'_3)}^{\lambda_1, \lambda_2} = \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} [\mathcal{M}_{h'_1, h'_2, h'_3}^{\lambda_1, \lambda_2}]^*. \quad (2.8)$$

The normalized density matrix is then obtained as $\rho^{\lambda_1, \lambda_2} = R^{\lambda_1, \lambda_2} / \text{Tr}[R^{\lambda_1, \lambda_2}]$.

In a realistic experimental situation, the incoming beams are not 100% polarized. We introduce the polarization parameters $\mathcal{P}_{e^\pm} \in [-1, 1]$ to describe the e^- and e^+ beams as [25,40]

$$\rho_{e^\pm} = \frac{1}{2} (1 + \mathcal{P}_{e^\pm}) |+\rangle\langle+|_{e^\pm} + \frac{1}{2} (1 - \mathcal{P}_{e^\pm}) |-\rangle\langle-|_{e^\pm}. \quad (2.9)$$

For imperfectly polarized beams with $\mathcal{P}_{e^-}$ and $\mathcal{P}_{e^+}$, the final-state density matrix is in general mixed and is obtained as the cross-section-weighted average over initial helicities. The unnormalized density matrix in this case is given by

$$\begin{aligned} R[\mathcal{P}_{e^-}, \mathcal{P}_{e^+}] &= \sum_{\lambda_1, \lambda_2} P_{\lambda_1, \lambda_2} R^{\lambda_1, \lambda_2}, \\ P_{\lambda_1, \lambda_2} &= \frac{1}{4} (1 + \lambda_1 \mathcal{P}_{e^-}) (1 + \lambda_2 \mathcal{P}_{e^+}), \end{aligned} \quad (2.10)$$

where P_{λ_1, λ_2} is the joint probability of having initial helicities (λ_1, λ_2) . Since the SM amplitudes vanish for $\lambda_1 = \lambda_2$, only the $(+, -)$ and $(-, +)$ helicity configurations contribute. Throughout Sec. IV we use the unpolarized configuration $\mathcal{P}_{e^-} = \mathcal{P}_{e^+} = 0$, for which

$$\rho^{\text{unpol}} = \frac{R[0,0]}{\text{Tr} R[0,0]}, \quad R[0,0] = \frac{1}{4}[R^{+-} + R^{-+}]. \quad (2.11)$$

In Sec. V, where we estimate the prospects for an entanglement observation at a future e^+e^- collider, we additionally consider two polarized benchmarks, Pol+ with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (+0.8, -0.3)$ and Pol- with $(-0.8, +0.3)$ —the corresponding $\rho[\mathcal{P}_{e^-}, \mathcal{P}_{e^+}]$ is obtained directly from Eq. (2.10).

The diagonal elements of the R -matrix correspond to the differential cross section

$$\frac{d\sigma_{h_1, h_2, h_3}^{\lambda_1, \lambda_2}}{d\Phi} = \frac{1}{2s} R_{(h_1, h_2, h_3), (h_1, h_2, h_3)}^{\lambda_1, \lambda_2}. \quad (2.12)$$

In an experimental setup, the density matrix ρ must be averaged over the events contained in some selected phase-space region Σ , which leads to

$$\rho_{\Sigma}^{\lambda_1, \lambda_2} = \frac{R_{\Sigma}^{\lambda_1, \lambda_2}}{\text{Tr}[R_{\Sigma}^{\lambda_1, \lambda_2}]}, \quad R_{\Sigma}^{\lambda_1, \lambda_2} = \int_{\Sigma} d\sigma^{\lambda_1, \lambda_2} \rho^{\lambda_1, \lambda_2} = \int_{\Sigma} \frac{d\Phi}{2s} R^{\lambda_1, \lambda_2}, \quad (2.13)$$

where $d\sigma^{\lambda_1, \lambda_2} = \sum_{h_1, h_2, h_3} d\sigma_{h_1, h_2, h_3}^{\lambda_1, \lambda_2}$ and the full phase-space integration is

$$\int_{\text{full}} d\Phi = \frac{1}{1024\pi^4} \int_{(m_{t\bar{t}}^{\min})^2}^{(m_{t\bar{t}}^{\max})^2} dm_{t\bar{t}}^2 \kappa_{tZ}^{\frac{1}{2}} \kappa_{t\bar{t}}^{\frac{1}{2}} \int_{-1}^1 d\cos\theta_Z \times \int_{-1}^1 d\cos\theta_t \int_0^{2\pi} d\phi \quad (2.14)$$

with $m_{t\bar{t}}^{\min} = 2m_t$, $m_{t\bar{t}}^{\max} = \sqrt{s} - m_Z$, $\kappa_{tZ} = \lambda(s, m_{t\bar{t}}^2, m_Z^2)/s^2$, and $\kappa_{t\bar{t}} = 1 - 4m_t^2/m_{t\bar{t}}^2$, where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the Källén function.

The averaging procedure generally makes the state mixed due to the entanglement between the spin and the momentum degrees of freedom. Note that for each event the spin quantization axes are taken along the directions of the momentum of each particle (i.e., the helicity basis). This means that in the phase-space integration, we sum density matrices defined with different spin quantization axes. In the literature, the result of such an average is called the “fictitious quantum state” [6, 14]. It is this object on which all the entanglement measures introduced in Sec. III will be evaluated. For unpolarized beams the fictitious final state is, therefore, given by

$$\rho_{\Sigma}^{\text{unpol}} = \frac{R_{\Sigma}[0,0]}{\text{Tr} R_{\Sigma}[0,0]}, \quad R_{\Sigma}[0,0] = \frac{1}{4}[R_{\Sigma}^{+-} + R_{\Sigma}^{-+}]. \quad (2.15)$$

The polarized analog $\rho_{\Sigma}[\mathcal{P}_{e^-}, \mathcal{P}_{e^+}]$ used in Sec. V is obtained by normalizing $R_{\Sigma}[\mathcal{P}_{e^-}, \mathcal{P}_{e^+}] = \sum_{\lambda_1, \lambda_2} P_{\lambda_1, \lambda_2} \times R_{\Sigma}^{\lambda_1, \lambda_2}$, with the joint probabilities P_{λ_1, λ_2} given in Eq. (2.10).

III. ENTANGLEMENT MEASURES

The spin density matrix ρ (and ρ_{Σ}) constructed in the previous sections describes a tripartite spin state of the $t\bar{t}Z$ system in the Hilbert space $\mathcal{H} = \mathbb{C}_t^2 \otimes \mathbb{C}_{\bar{t}}^2 \otimes \mathbb{C}_Z^3$, which has dimension 12. In this section, we introduce several entanglement measures to characterize the multipartite entanglement structure of the state.

A. Separability and LOCC monotonicity

One of the defining properties of quantum entanglement is nonseparability. For a bipartite mixed state ρ_{AB} acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the state is called *separable* if it admits a convex decomposition,

$$\rho_{\text{sep}}^{AB} = \sum_i q_i \rho_i^A \otimes \rho_i^B, \quad \rho_i^A \in \mathcal{S}(\mathcal{H}_A), \quad \rho_i^B \in \mathcal{S}(\mathcal{H}_B), \quad (3.1)$$

with $q_i \geq 0$ and $\sum_i q_i = 1$, where $\mathcal{S}(\mathcal{H})$ is the set of all density matrices of $\mathcal{H}$. The set of entangled states is defined as the complement of the set of separable states. Based on this notion of separability, we consider an entanglement measure, a non-negative function $E(\rho) \geq 0$, which vanishes for all separable states. If $E(\rho) > 0$ for every entangled state, i.e., every nonseparable state, the measure E is called *faithful*.

For a tripartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, the structure of separability becomes richer, since several inequivalent notions of separability may arise. First, the state ρ is said to be *fully separable* if it admits a convex decomposition

$$\rho^{\text{fs}} = \sum_i q_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C, \quad (3.2)$$

with $q_i \geq 0$, $\sum_i q_i = 1$, and $\rho_i^I \in \mathcal{S}(\mathcal{H}_I)$ for $I = A, B, C$. Fully separable states contain only classical correlations among the three subsystems, and all entanglement measures vanish for such states.

Beyond fully separable states, there is a set of states that are separable for a given bipartition $M|\bar{M}$. A state is called *M-separable* if it admits a decomposition

$$\rho^{M|\bar{M}} = \sum_i q_i \rho_i^M \otimes \rho_i^{\bar{M}}, \quad (3.3)$$

with $q_i \geq 0$ and $\sum_i q_i = 1$. In the tripartite system, there are three types of such states, $\rho^{A|BC}$, $\rho^{B|AC}$, and $\rho^{C|AB}$.

Below we will define entanglement measures that can probe $M|\bar{M}$ entanglement.

A third notion of separability is bi-separability. A tripartite state ρ is called *bi-separable* if it can be expressed as a convex mixture of states that are separable across different bipartitions,

$$\rho^{\text{bs}} = q_1 \rho^{A|BC} + q_2 \rho^{B|AC} + q_3 \rho^{C|AB}, \quad (3.4)$$

with $q_i \geq 0$ and $q_1 + q_2 + q_3 = 1$. A general mixed state may be bi-separable even if it is non- M -separable for all cuts, $M = A, B$, and C . A state that is not bi-separable is called GME. Figure 3 illustrates the different separability classes in the tripartite quantum system.

Another fundamental property of entanglement is the monotonicity under the local operations and classical communication (LOCC). Under the LOCC protocol, distant observers can perform unitary transformations and measurements only on their own subsystem and exchange only classical information. Since LOCC cannot generate entanglement, no nonclassical correlations can be created through such operations. An entanglement measure E is called an entanglement monotone if it is nonincreasing under LOCC,

$$E(\Lambda_{\text{LOCC}}(\rho)) \leq E(\rho), \quad (3.5)$$

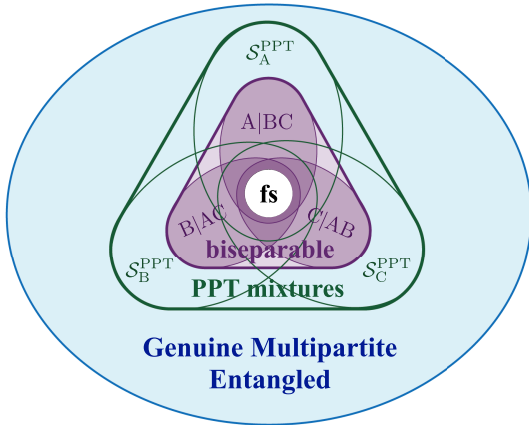


FIG. 3. Separability classification of tripartite quantum states (cf. [110]). The outer ellipse represents the set of all tripartite states. The small disk in the center, labeled “fs,” is the set of fully separable states. The purple rounded triangle is the set of bi-separable states, given by the convex hull of the three M -separable subsets $S^{M|\bar{M}}$ ($M = A, B, C$), indicated by $A|BC$, $B|AC$, and $C|AB$. States outside the purple region are GME. The green rounded triangle is the set of PPT mixtures, the convex hull of the three single-cut PPT subsets S_M^{PPT} . Every bi-separable state is a PPT mixture, but the converse may fail: the region between the purple and green triangles consists of bound-entangled PPT mixtures, which exist whenever the local Hilbert-space dimensions are larger than 2×3 for at least one cut.

where $\Lambda_{\text{LOCC}}(\rho)$ denotes the post-LOCC state of ρ . This property gives the magnitude of E an operational meaning: it imposes a partial ordering on quantum states according to their entanglement content and identifies maximally entangled states as those achieving the largest possible value of E within a given Hilbert space.

Finally, an entanglement measure E is said to be *convex* if it does not increase under probabilistic mixing,

$$E\left(\sum_i q_i \rho_i\right) \leq \sum_i q_i E(\rho_i). \quad (3.6)$$

Convexity reflects the intuitive fact that classical uncertainty cannot increase quantum entanglement on average.

B. Partial transpose and Peres-Horodecki criterion

While the definitions of separability introduced in Sec. III A are conceptually fundamental, they are generally not practical for determining whether a given mixed state is entangled. In particular, checking separability requires establishing the existence of a convex decomposition of the form of Eqs. (3.1)–(3.4), which is in general a highly nontrivial problem. For this reason, operational entanglement criteria that can be efficiently evaluated are of central importance. One of the most widely used criteria is the Peres-Horodecki criterion [111,112], based on the partial transpose operation.

For a state ρ acting on $\mathbb{C}^{d_M} \otimes \mathbb{C}^{d_{\bar{M}}}$, the partial transpose with respect to subsystem M is defined by

$$(\rho^{T_M})_{i_M i_{\bar{M}}, j_M j_{\bar{M}}} = \rho_{j_M i_{\bar{M}}, i_M j_{\bar{M}}}. \quad (3.7)$$

If a state ρ is M -separable, its partial transpose takes the form

$$\rho^{T_M} = \sum_i q_i (\rho_i^M)^T \otimes \rho_i^{\bar{M}}, \quad (3.8)$$

with $q_i \geq 0$ and $\sum_i q_i = 1$. Since $(\rho_i^M)^T$ is a positive semidefinite Hermitian matrix with a unit trace, the same holds for ρ^{T_M} . Hence, the presence of at least one negative eigenvalue in ρ^{T_M} immediately rules out M -separability and certifies entanglement across the bipartition $M|\bar{M}$ [111,112]. A state whose partial transpose is positive semidefinite is called *positive-partial-transpose* (PPT), while a state with at least one negative eigenvalue in ρ^{T_M} is called *negative-partial-transpose* (NPT).

Whether the converse statement holds depends on the dimensions of the bipartite system. For 2×2 and 2×3 systems, the Peres-Horodecki criterion is both necessary and sufficient for separability, implying that every entangled state is necessarily NPT. For larger systems, however, there exist entangled states whose partial transpose remains positive semidefinite. Such states are known as PPT-entangled or bound-entangled states.

C. Negativity

Our primary variable to characterize the entanglement structure of the $t\bar{t}Z$ state is the *negativity*. For a state ρ acting on $\mathbb{C}^{d_M} \otimes \mathbb{C}^{d_{\bar{M}}}$, the negativity is defined by [87]

$$N_M(\rho) = \frac{\|\rho^{T_M}\|_1 - 1}{2}, \quad (3.9)$$

where $\|\mathcal{O}\|_1 = \text{Tr}\sqrt{\mathcal{O}^\dagger \mathcal{O}}$ is the trace norm or the sum of the singular values of $\mathcal{O}$. Since $\text{Tr}\rho^{T_M} = 1$, N_M can also be computed in terms of the eigenvalues λ_i of ρ^{T_M} as

$$N_M(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2} = \sum_{\lambda_i < 0} |\lambda_i|. \quad (3.10)$$

By construction, it is clear that N_M is non-negative, invariant under local basis changes, and positive if ρ is NPT with respect to M . Since all M -separable states are PPT with respect to M , $N_M(\rho) = 0$ for all M -separable states. For 2×2 and 2×3 systems, every entangled state is necessarily NPT, and the negativity is, therefore, a faithful entanglement measure. For larger quantum systems, however, positivity of the partial transpose is no longer sufficient for separability, since PPT-entangled states may exist. Thus, the negativity is not a faithful entanglement measure in general.

The negativity has several other desired properties: it is convex, nonincreasing under LOCC, and symmetric under $M \leftrightarrow \bar{M}$. The general upper bound on N_M for a $d_M \times d_{\bar{M}}$ system is

$$N_M(\rho) \leq \frac{d_{\min} - 1}{2}, \quad d_{\min} \equiv \min(d_M, d_{\bar{M}}). \quad (3.11)$$

The upper bound is saturated by a maximally entangled state of the form $\frac{1}{\sqrt{d_{\min}}} \sum_{i=1}^{d_{\min}} |i\rangle_M |i\rangle_{\bar{M}}$.

Using the negativity, we measure two types of bipartite entanglement in the $t\bar{t}Z$ system: one-to-one entanglement and one-to-other entanglement.

D. One-to-one entanglement

For a tripartite quantum state ρ describing a system of ABC , it is natural to ask how strongly two individual subsystems are entangled with each other. We call this type of entanglement *one-to-one entanglement*. To measure the entanglement between A and B , one first traces out subsystem C and constructs the *reduced density matrix* to describe the AB subsystem

$$\rho_{AB} \equiv \text{Tr}_C[\rho]. \quad (3.12)$$

The entanglement between A and B can then be quantified by the negativity $N_A(\rho_{AB}) = N_B(\rho_{AB})$.

For a state ρ of the $t\bar{t}Z$ spin system, one can define three reduced density matrices describing different pairs,

$$\begin{aligned} \rho_{t\bar{t}} &= \text{Tr}_Z[\rho] \quad (2 \times 2), & \rho_{tZ} &= \text{Tr}_{\bar{t}}[\rho] \quad (2 \times 3), \\ \rho_{\bar{t}Z} &= \text{Tr}_t[\rho] \quad (2 \times 3). \end{aligned} \quad (3.13)$$

The parenthetical labels denote the Hilbert-space dimensions of the retained subsystems after the partial trace. Correspondingly, we define the *one-to-one negativities* as

$$N_{t\bar{t}} \equiv N_t(\rho_{t\bar{t}}), \quad N_{tZ} \equiv N_Z(\rho_{tZ}), \quad N_{\bar{t}Z} \equiv N_Z(\rho_{\bar{t}Z}). \quad (3.14)$$

From Eq. (3.11), these quantities have the same range,

$$\begin{aligned} (\text{separable}) \quad 0 &\leq N_{t\bar{t}}, \quad N_{tZ}, \\ N_{\bar{t}Z} &\leq \frac{1}{2} \quad (\text{maximally entangled}). \end{aligned} \quad (3.15)$$

Since the Peres-Horodecki criterion is a necessary and sufficient condition for entanglement in the 2×2 and 2×3 systems, these one-to-one negativities are faithful entanglement measures: they vanish if and only if the corresponding reduced state is separable.

E. One-to-other entanglement

Another natural question for the $t\bar{t}Z$ state is whether each particle is entangled with the rest of the system. This type of entanglement is called *one-to-other entanglement*. Taking the partial transpose with respect to the single subsystem Z , t , and $\bar{t}$, we define the following *one-to-other negativities*:

$$\begin{aligned} N(Z|t\bar{t}) &\equiv N_Z(\rho) \quad (3 \times 4), & N(t|\bar{t}Z) &\equiv N_t(\rho) \quad (2 \times 6), \\ N(\bar{t}|tZ) &\equiv N_{\bar{t}}(\rho) \quad (2 \times 6). \end{aligned} \quad (3.16)$$

The parentheses denote the dimensions of the two subsystems defined by the partition. Hence, the corresponding negativities are not faithful entanglement measures. The upper bound Eq. (3.11) gives the following ranges:

$$\begin{aligned} (\text{PPT}) \quad 0 &\leq N(Z|t\bar{t}) \leq 1, \\ 0 &\leq N(t|\bar{t}Z), N(\bar{t}|tZ) \leq \frac{1}{2}. \end{aligned} \quad (\text{maximally entangled}) \quad (3.17)$$

Since all M -separable states are PPT with respect to the bipartition $M|\bar{M}$, a nonvanishing negativity, $N_M(\rho) > 0$, guarantees non- M -separability, certifying one-to-other entanglement for the corresponding bipartition.

F. Genuine multipartite negativity

One of the central questions for the $t\bar{t}Z$ spin state is whether it exhibits quantum correlations that are genuinely

tripartite in nature. The GME is defined as non-bi-separability [see Eq. (3.4)]. Even if all one-to-other negativities are nonzero, the state can be bi-separable possessing no GME.

In Ref. [88], Jungnitsch, Moroder, and Gühne (JMG) introduced a GME measure called the GMN, which is applicable and calculable for arbitrary mixed states. To understand this measure, we first define *PPT mixtures*, which are the states that admit the decomposition

$$\rho^{\text{PPT-mix}} = p_A \rho_A^{\text{PPT}} + p_B \rho_B^{\text{PPT}} + p_C \rho_C^{\text{PPT}}, \quad (3.18)$$

with $p_M \geq 0$ and $\sum_M p_M = 1$ ($M = A, B, C$). In this decomposition, $\rho_M^{\text{PPT}} \in \mathcal{S}_M^{\text{PPT}}$, where $\mathcal{S}_M^{\text{PPT}}$ is the set of all states that are PPT with respect to M . Since M -separable states are necessarily PPT with respect to M , $\rho^{M|\bar{M}} \in \mathcal{S}_M^{\text{PPT}}$. Thus, all bi-separable states in Eq. (3.4) belong to the set of all PPT mixtures $\mathcal{S}^{\text{PPT-mix}}$. The idea of JMG is to construct a tight witness that can detect non-PPT mixtures, $\rho \notin \mathcal{S}^{\text{PPT-mix}}$. Such a state cannot be bi-separable and is, therefore, GME (see Fig. 3).

The construction of Ref. [88] is based on a particular class of entanglement witnesses called *fully decomposable witnesses*. A Hermitian operator $\mathcal{W}$ is called fully decomposable if, for every bipartition $M = A, B, C$, it admits a decomposition

$$\mathcal{W} = P_M + Q_M^{T_M}, \quad 0 \leq P_M, \quad 0 \leq Q_M \leq 1, \quad (3.19)$$

for some Hermitian operators P_M and Q_M . The pairs (P_M, Q_M) are allowed to be different for different M but the operator $\mathcal{W}$ on the left-hand side is common to all M .

For any state $\rho_M^{\text{PPT}} \in \mathcal{S}_M^{\text{PPT}}$ and any $\mathcal{W}$ admitting the M -decomposition (3.19), we have

$$\text{Tr}[\rho_M^{\text{PPT}} \mathcal{W}] = \text{Tr}[\rho_M^{\text{PPT}} P_M] + \text{Tr}[(\rho_M^{\text{PPT}})^{T_M} Q_M] \geq 0, \quad (3.20)$$

because the operators in both traces are positive semi-definite. A fully decomposable $\mathcal{W}$ satisfies (3.20) for *every* bipartition M and, therefore, its expectation value on an arbitrary PPT mixture is also non-negative,

$$\begin{aligned} \text{Tr}[\rho^{\text{PPT-mix}} \mathcal{W}] &= p_A \text{Tr}[\rho_A^{\text{PPT}} \mathcal{W}] + p_B \text{Tr}[\rho_B^{\text{PPT}} \mathcal{W}] \\ &\quad + p_C \text{Tr}[\rho_C^{\text{PPT}} \mathcal{W}] \geq 0, \end{aligned} \quad (3.21)$$

where each term is bounded by applying (3.20) with the M -decomposition of $\mathcal{W}$ matching that term. Hence, finding a fully decomposable $\mathcal{W}$ with $\text{Tr}[\rho \mathcal{W}] < 0$ certifies $\rho \notin \mathcal{S}^{\text{PPT-mix}}$, which in turn implies that ρ is GME.

To quantify how strongly a state violates the PPT mixture condition, we minimize $\text{Tr}[\rho \mathcal{W}]$ over the set of all fully decomposable witnesses,

$$\tilde{\mathcal{N}}_G(\rho) \equiv \min_{\{P_M, Q_M\}} \text{Tr}[\rho \mathcal{W}], \quad \text{subject to (3.19) for all } M. \quad (3.22)$$

The optimization variables are the three Hermitian operator pairs (P_M, Q_M) for $M = A, B, C$. Note that $\mathcal{W} = 0$ (with $P_M = Q_M = 0$) is always feasible, so $\tilde{\mathcal{N}}_G(\rho) \leq 0$ for any ρ . The GMN is then defined as

$$\mathcal{N}_G(\rho) \equiv -\tilde{\mathcal{N}}_G(\rho) \geq 0. \quad (3.23)$$

By construction, $\mathcal{N}_G(\rho) > 0$ certifies $\rho \notin \mathcal{S}^{\text{PPT-mix}}$. Remarkably, the converse statement also holds,

$$\rho \notin \mathcal{S}^{\text{PPT-mix}} \Leftrightarrow \mathcal{N}_G(\rho) > 0. \quad (3.24)$$

Indeed, for any $\rho \notin \mathcal{S}^{\text{PPT-mix}}$, the hyperplane separation theorem applied to the closed convex set $\mathcal{S}^{\text{PPT-mix}}$ guarantees the existence of a Hermitian operator $\mathcal{W}$ satisfying $\text{Tr}[\rho \mathcal{W}] < 0$ and $\text{Tr}[\sigma \mathcal{W}] \geq 0$ for all $\sigma \in \mathcal{S}^{\text{PPT-mix}}$. For each M , any state PPT with respect to that single cut belongs trivially to $\mathcal{S}^{\text{PPT-mix}}$, so $\mathcal{S}_M^{\text{PPT}} \subseteq \mathcal{S}^{\text{PPT-mix}}$, and the same operator $\mathcal{W}$ satisfies $\text{Tr}[\sigma \mathcal{W}] \geq 0$ for all $\sigma \in \mathcal{S}_M^{\text{PPT}}$. The Horodecki characterization of decomposable witnesses [112] states that any Hermitian operator non-negative on $\mathcal{S}_M^{\text{PPT}}$ admits a decomposition $P_M + Q_M^{T_M}$ with $P_M, Q_M \geq 0$. Applying this characterization separately for each $M = A, B, C$ then provides three (in general different) decomposition pairs (P_M, Q_M) of the same $\mathcal{W}$, one per cut, showing that $\mathcal{W}$ is fully decomposable. By a positive rescaling of $\mathcal{W}$, one can always make the largest eigenvalue of Q_M less than or equal to 1 for every M simultaneously, ensuring $0 \leq Q_M \leq 1$, while preserving the sign of $\text{Tr}[\rho \mathcal{W}]$.

A few remarks are in order. First, the set of PPT mixtures contains the set of bi-separable states, but the converse may not hold due to the existence of PPT-entangled states (in dimensions $d_M \times d_{\bar{M}}$ larger than 2×3). A state with $\mathcal{N}_G(\rho) = 0$ is, therefore, a PPT mixture but is not guaranteed to be bi-separable, and the GMN provides only a sufficient (but not necessary) condition for GME. Second, the optimization in (3.22) is a semidefinite program and can be solved efficiently with standard numerical routines [88, 113]. Third, the GMN inherits several desirable properties from the underlying witness construction [88, 113]: it is non-negative, convex, nonincreasing under LOCC, and invariant under local unitary transformations.

Finally, the GMN is bounded from above by the smallest one-to-other negativity,

$$\mathcal{N}_G(\rho) \leq \min_M N(M|\bar{M}). \quad (3.25)$$

The argument is immediate from the definition: for any fully decomposable witness $\mathcal{W} = P_M + Q_M^{T_M}$, the constraints $P_M \geq 0$ and $0 \leq Q_M \leq 1$ imply

$$\text{Tr}[\rho\mathcal{W}] = \text{Tr}[\rho P_M] + \text{Tr}[\rho^{T_M} Q_M], \quad (3.26)$$

where the first term is non-negative because $P_M \geq 0$ and $\rho \geq 0$, and the second is bounded below by $-N_M(\rho)$ because $0 \leq Q_M \leq 1$ and the most negative value of $\text{Tr}[\rho^{T_M} Q_M]$ is $\sum_{\lambda_i < 0} \lambda_i = -N_M(\rho)$, attained when Q_M projects onto the negative eigenspace of ρ^{T_M} . Minimizing the left-hand side over all fully decomposable $\mathcal{W}$ gives $\mathcal{N}_G(\rho) \leq N_M(\rho)$ for each M separately and, therefore, (3.25). The bound (3.25) is generally strict for mixed states but is saturated on pure states, $\mathcal{N}_G(|\psi\rangle\langle\psi|) = \min_M N_M(|\psi\rangle\langle\psi|)$ [88]. Combining (3.25) with the universal bound on the negativity (3.11), we obtain the state-independent upper bound

$$\mathcal{N}_G(\rho) \leq \frac{d_{\min} - 1}{2}, \quad d_{\min} \equiv \min(d_A, d_B, d_C). \quad (3.27)$$

This universal bound is saturated for Greenberger-Horne-Zeilinger (GHZ)-type maximally entangled states.

Applying these state-dependent and -independent bounds to the $t\bar{t}Z$ system, we have

$$\mathcal{N}_G(\rho) \leq \min[N(Z|t\bar{t}), N(t|\bar{t}Z), N(\bar{t}|tZ)], \quad (3.28)$$

and

$$(\text{PPT-mixture}) 0 \leq \mathcal{N}_G(\rho) \leq \frac{1}{2}(\text{maximally GME}), \quad (3.29)$$

respectively.

IV. ENTANGLEMENT IN $t\bar{t}Z$

We now evaluate the entanglement measures of Sec. III on the spin quantum states ρ and ρ_Σ defined in Sec. II C for $e^+e^- \rightarrow t\bar{t}Z$ in the standard model. Throughout this section we focus on unpolarized e^+e^- beams. The impact of beam polarization is discussed in Sec. V. The numerical results are organized along two complementary levels of phase-space (non)integration, which differ in how much of the kinematic information has been traced out:

- (i) *Differential* (a phase-space point). For a fixed choice of $(m_{t\bar{t}}, \cos\theta_Z, \cos\theta_t, \phi)$ the state ρ is a cross-section-weighted sum of two pure states, $\rho \propto R^{+-} + R^{-+}$, originating from the $+-$ and $-+$ initial helicity configurations. At this level, ρ is a *genuine* quantum state and the entanglement measures are invariant under local unitary transformations and, therefore, independent of the choice of spin quantization axes. We use this representation to explore the rich structure of the $t\bar{t}Z$ spin state across the four-dimensional phase space.
- (ii) *Fully inclusive*. All four phase-space variables are integrated against the squared matrix element, leaving the spin density matrix ρ_Σ as a function of $\sqrt{s}$ only (for fixed beam polarization). This is the most

inclusive measurement, corresponding to a measurement that sums over all $t\bar{t}Z$ kinematic configurations. The averaging mixes density matrices defined with different spin quantization axes, so ρ_Σ is a *fictitious* quantum state in the sense of Eq. (2.13). Hence, the resulting entanglement measures depend on the choice of basis. Throughout this work we adopt the helicity basis.

A. Entanglement in differential states

We start by discussing the entanglement at a fixed phase-space point. The differential state ρ depends on four kinematic variables $(m_{t\bar{t}}, \cos\theta_Z, \cos\theta_t, \phi)$. Throughout this subsection we set $\sqrt{s} = 1$ TeV and examine two representative parameter planes:

- (i) The $(m_{t\bar{t}}, \cos\theta_Z)$ plane, with the remaining variables fixed to $\cos\theta_t = \phi = 0$.
- (ii) The $(\cos\theta_Z, \phi)$ plane, with $m_{t\bar{t}} = 600$ GeV and $\cos\theta_t = 0$ fixed.

We first discuss the purity of the full and the reduced spin states, which provides a simple measure of their mixedness. For a state ρ of a d -dimensional quantum system, the purity is defined as $\gamma(\rho) \equiv \text{Tr}(\rho^2)$ with

$$(\text{maximally mixed}) \frac{1}{d} \leq \gamma \leq 1 \quad (\text{pure}). \quad (4.1)$$

The lower bound $\frac{1}{d}$ is saturated only by the maximally mixed state, $\rho = \frac{1}{d}\mathbb{1}$, for which all entanglement measures vanish. While purity does not, in general, quantify entanglement, it provides useful insight into the correlation structure of the state: a strong reduction in the purity of a reduced density matrix signals substantial correlations between the traced and retained subsystems. As we will see below, the regions with the lowest reduced-state purities also exhibit the largest multipartite entanglement.

Figure 4 shows the purity of the differential $t\bar{t}Z$ spin state, $\gamma(\rho)$, and the three reduced purities $\gamma_{t\bar{t}} = \text{Tr}(\rho_{t\bar{t}}^2)$, $\gamma_{tZ} = \text{Tr}(\rho_{tZ}^2)$, and $\gamma_{\bar{t}Z} = \text{Tr}(\rho_{\bar{t}Z}^2)$ in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane. The lower color-bar limits correspond to the maximally mixed values $\frac{1}{12}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{6}$. The full state purity $\gamma(\rho)$ is uniformly high, $\sim 0.65 - 0.70$. The only source of mixing here is the cross-section-weighted sum over the two initial helicities $+-$ and $-+$, whose amplitudes have comparable magnitudes across the plane, leaving the mixture close to an even superposition rather than dominated by one helicity. For fully polarized beams, $(\mathcal{P}_e^-, \mathcal{P}_e^+) = (\mp 1, \pm 1)$, this sum collapses to a single term and the differential state is exactly pure, $\gamma(\rho) = 1$. The reduced purities are significantly smaller, reflecting the correlations between the traced and retained subsystems. The most pronounced reduction occurs in $\gamma_{t\bar{t}}$, which reaches $\simeq 0.30 - 0.35$ around $\cos\theta_Z \simeq 0$ and $m_{t\bar{t}} \simeq 400 - 500$ GeV. This indicates strong correlations between the $t\bar{t}$ pair and the Z -boson, such that tracing out the latter leaves a highly mixed $t\bar{t}$ state. The γ_{tZ}

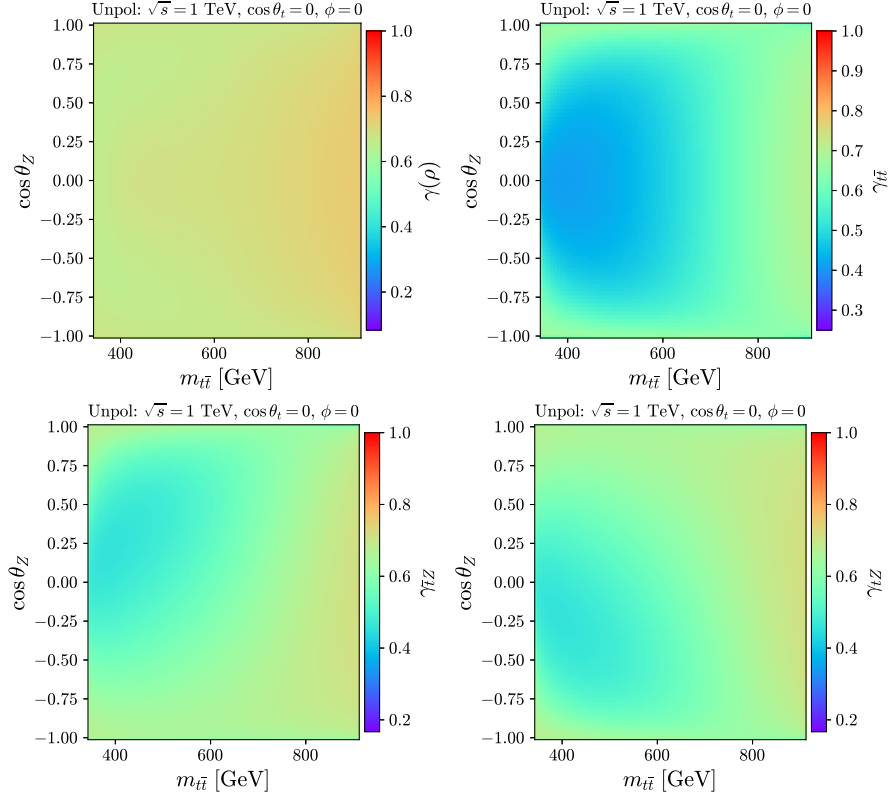


FIG. 4. Purity of the differential $t\bar{t}Z$ spin state at $\sqrt{s} = 1$ TeV for $\cos\theta_t = \phi = 0$, shown in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane for unpolarized beams. The panels display the purity of the full state, $\gamma(\rho) = \text{Tr}(\rho^2)$ (top left), and the purities of the reduced states $\gamma_{t\bar{t}}$ (top right), γ_{tZ} (bottom left), and $\gamma_{t\bar{t}Z}$ (bottom right), obtained by tracing out the Z , $\bar{t}$, and t degrees of freedom, respectively. The lower limits of the color scales correspond to the maximally mixed values $\frac{1}{12}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{6}$ for the respective states.

and γ_{tZ} purities display shallower minima of approximately 0.4 in different regions of phase space. The two are CP -conjugate, $\gamma_{tZ}(\cos\theta_Z) = \gamma_{tZ}(-\cos\theta_Z)$, while $\gamma_{t\bar{t}}$ is CP -even and is, therefore, symmetric under $\cos\theta_Z \rightarrow -\cos\theta_Z$, in line with (2.2).

Figure 5 shows the three one-to-one negativities $N_{t\bar{t}}$, N_{tZ} , and $N_{t\bar{t}Z}$. The dashed contour in the $N_{t\bar{t}}$ panel marks $N_{t\bar{t}} = 10^{-3}$, the threshold below which $N_{t\bar{t}}$ is treated as effectively vanishing. $N_{t\bar{t}}$ is nearly zero throughout most of phase space, becoming nonzero only near the soft- Z boundary, $m_{t\bar{t}} \rightarrow \sqrt{s} - m_Z \simeq 909$ GeV, and for $|\cos\theta_Z| \rightarrow 1$. Both are limits in which the Z effectively decouples from the $t\bar{t}$ pair: at the upper $m_{t\bar{t}}$ boundary the Z momentum vanishes, and in the collinear limit the initial-state radiation diagrams [Figs. 2(c) and 2(d)] dominate, with the matrix element approximately factorizing into a universal $e \rightarrow eZ$ splitting times the on-shell $e^+e^- \rightarrow t\bar{t}$ amplitude in the high-energy limit. In both cases $\rho \rightarrow \rho_{t\bar{t}} \otimes \rho_Z$, and the $t\bar{t}$ reduced state roughly recovers the standard $e^+e^- \rightarrow t\bar{t}$ entanglement [40]. This behavior is closely related to the strong $t\bar{t}$ - Z correlations in the purity analysis: over most of phase space, tracing out the Z leaves a highly mixed $t\bar{t}$ state, suppressing the entanglement in $\rho_{t\bar{t}}$. Only in regions where the $t\bar{t}$ - Z correlations weaken does

the reduced $t\bar{t}$ state retain enough coherence to exhibit nonzero $t\bar{t}$ entanglement. In contrast, N_{tZ} and $N_{t\bar{t}Z}$ are nonzero over large regions of phase space, reaching comparable maxima of approximately 0.25, and are exact CP mirror images of each other under Eq. (2.2). The maxima remain below the algebraic bound $\frac{1}{2}$ because the corresponding reduced states are substantially mixed by their correlations with the third spin.

Figure 6 shows the three one-to-other negativities $N(Z|t\bar{t})$, $N(t|\bar{t}Z)$, and $N(\bar{t}|tZ)$. All three are positive over essentially the entire $(m_{t\bar{t}}, \cos\theta_Z)$ plane, signaling robust entanglement across every bipartition. $N(Z|t\bar{t})$ reaches $\sim 0.5 - 0.6$ in the central region $\cos\theta_Z \simeq 0$, $m_{t\bar{t}} \simeq 500 - 700$ GeV (about half of its algebraic maximum), while $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$ reach $\simeq 0.40 - 0.45$ in broad central regions, approaching their algebraic bound $\frac{1}{2}$.² As in Fig. 5, $N(Z|t\bar{t})$ is CP -even and $N(t|\bar{t}Z) \leftrightarrow N(\bar{t}|tZ)$ under (2.2). Combined with the $N_{t\bar{t}} \simeq 0$, this pattern points to a GHZ-like entanglement structure, in which each particle is

²Recall that the algebraic upper bounds are $N(Z|t\bar{t}) \leq 1$ for the 3×4 partition and $N(t|\bar{t}Z), N(\bar{t}|tZ) \leq \frac{1}{2}$ for the two 2×6 partitions, as shown in Eq. (3.17).

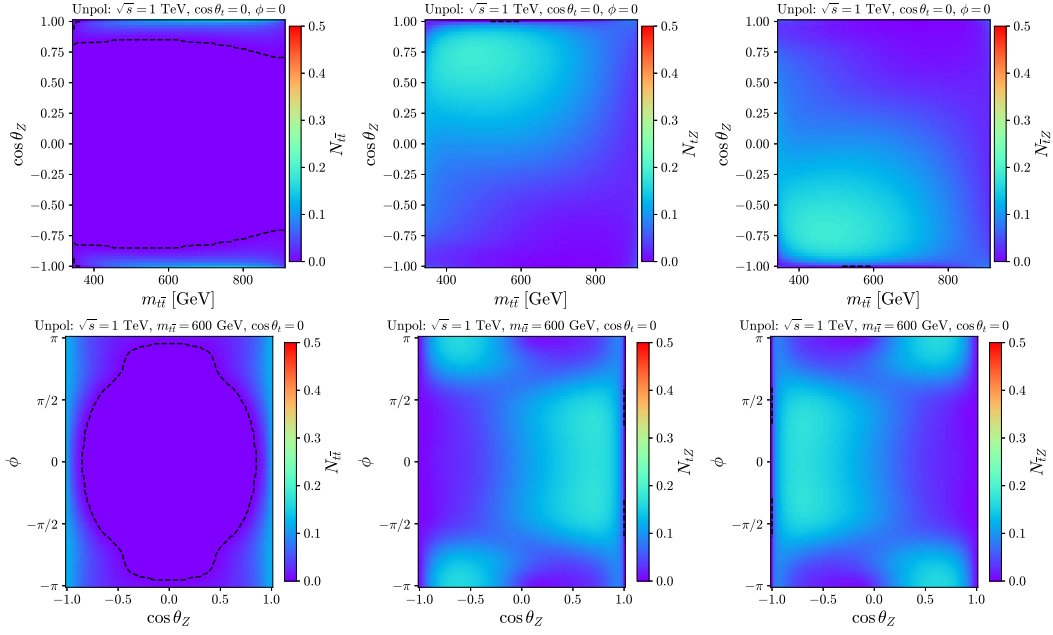


FIG. 5. One-to-one negativities of the differential $t\bar{t}Z$ spin state at $\sqrt{s} = 1$ TeV for unpolarized beams: $N_{t\bar{t}}$ (left), N_{tZ} (middle), and $N_{\bar{t}Z}$ (right). Top row: $(m_{t\bar{t}}, \cos \theta_Z)$ plane at fixed $\cos \theta_t = \phi = 0$. Bottom row: $(\cos \theta_Z, \phi)$ plane at fixed $m_{t\bar{t}} = 600$ GeV and $\cos \theta_t = 0$. The dashed black contours mark the boundary at $N_{t\bar{t}} = 10^{-3}$, below which $N_{t\bar{t}}$ is effectively vanishing. For all panels, the color bar saturates at the algebraic upper bound $N = 1/2$.

strongly entangled with the remaining pair while the reduced $t\bar{t}$ state remains nearly separable.

Figure 7 shows the genuine multipartite negativity $\mathcal{N}_G$. It is positive throughout most of the central region of phase space, reaching values of approximately 0.40, around 80%

of the algebraic GHZ bound $(d_{\min} - 1)/2 = \frac{1}{2}$ for the $2 \times 2 \times 3$ Hilbert space. This unambiguously certifies genuine tripartite entanglement in the differential $t\bar{t}Z$ spin state over a large fraction of phase space—a basis-independent statement. The largest values occur in the same kinematic region

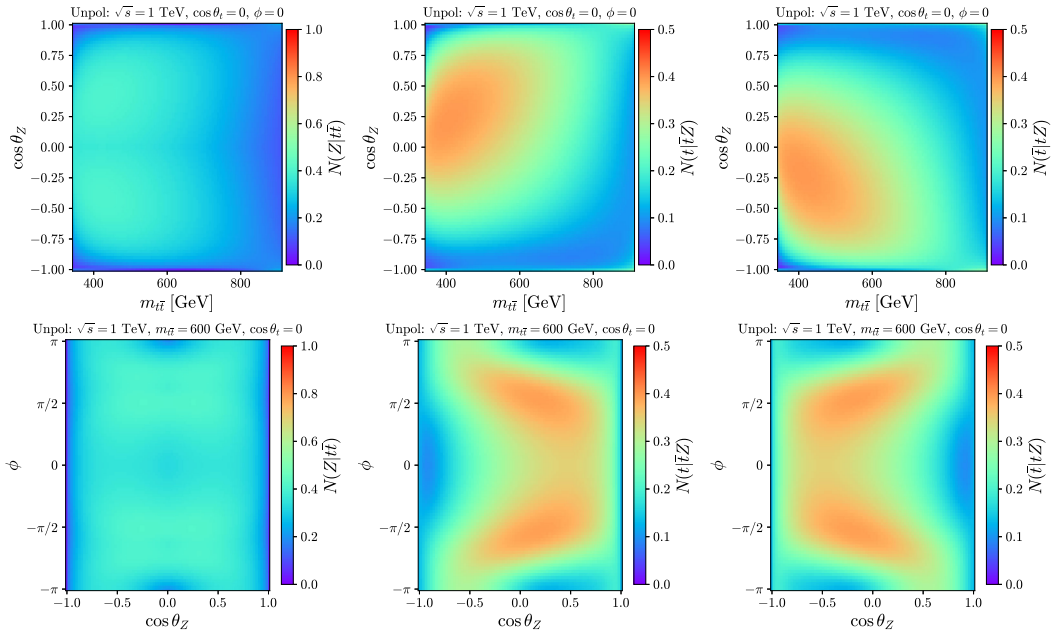


FIG. 6. One-to-other negativities of the differential $t\bar{t}Z$ spin state at $\sqrt{s} = 1$ TeV for unpolarized beams: $N(Z|t\bar{t})$ (left), $N(t|\bar{t}Z)$ (middle), and $N(\bar{t}|tZ)$ (right). Top row: $(m_{t\bar{t}}, \cos \theta_Z)$ plane at fixed $\cos \theta_t = \phi = 0$. Bottom row: $(\cos \theta_Z, \phi)$ plane at fixed $m_{t\bar{t}} = 600$ GeV and $\cos \theta_t = 0$. The color bar saturates at the algebraic upper bound, 1 for the 3×4 partition $N(Z|t\bar{t})$ and $\frac{1}{2}$ for the 2×6 partitions $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$.

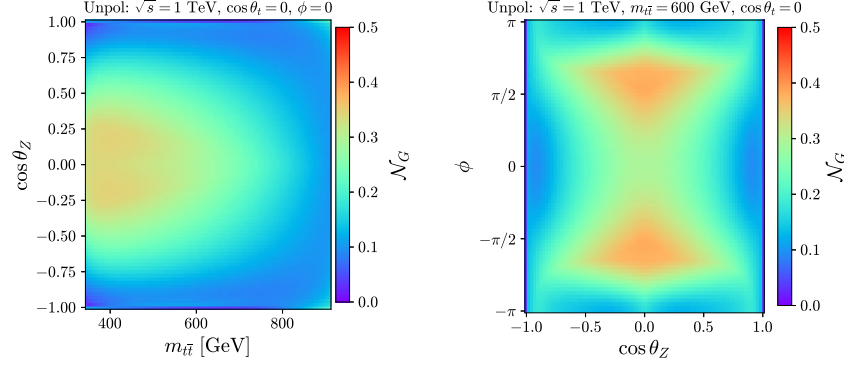


FIG. 7. Genuine multipartite negativity $\mathcal{N}_G$ of the differential $t\bar{t}Z$ spin state at $\sqrt{s} = 1$ TeV for unpolarized beams. Left: $(m_{t\bar{t}}, \cos \theta_Z)$ plane at fixed $\cos \theta_t = \phi = 0$. Right: $(\cos \theta_Z, \phi)$ plane at fixed $m_{t\bar{t}} = 600$ GeV and $\cos \theta_t = 0$. The color bar saturates at the GHZ upper bound $(d_{\min} - 1)/2 = \frac{1}{2}$ for the $2 \times 2 \times 3$ Hilbert space.

where the reduced-state purities are lowest and the one-to-other negativities are largest, confirming that the dominant correlations in the differential state are collective rather than pairwise. The state-dependent bound in Eq. (3.25), $\mathcal{N}_G(\rho) \leq \min_M N_M(\rho)$, is respected, but close to saturation. In the region where $\mathcal{N}_G$ reaches its maximum, the smallest one-to-other negativity is approximately $0.40 - 0.45$, only slightly larger than $\mathcal{N}_G$. For each initial helicity channel the differential state is pure, and for a pure state one has exactly $N(t|\bar{t}Z) = \frac{1}{2} \sqrt{1 - |\vec{P}_t|^2}$, where $\vec{P}_t$ is the top polarization vector at that phase-space point. The bound is, therefore, saturated only for an unpolarized top. The chiral electroweak couplings generate a sizeable top polarization in $e^+e^- \rightarrow t\bar{t}Z$, which keeps $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$, and through Eq. (3.25) also $\mathcal{N}_G$, below $\frac{1}{2}$, while the incoherent sum over the two helicity channels for unpolarized beams lowers them further. $\mathcal{N}_G$ is CP -even, and the displayed slices are correspondingly symmetric under Eq. (2.2).

A coherent picture emerges from Figs. 4–7. The differential $t\bar{t}Z$ spin state exhibits a clear hierarchy of quantum correlations. Pairwise entanglement is generally suppressed, especially within the $t\bar{t}$ subsystem, while one-to-other entanglement remains large throughout most of phase space. The positivity of $\mathcal{N}_G$ across broad kinematic regions further demonstrates that these correlations are genuinely multipartite. Overall, the dominant quantum structure of the differential state is collective rather than pairwise.

B. Entanglement in fully inclusive states

We now turn to the fully inclusive state, obtained by integrating over the complete $t\bar{t}Z$ phase space. The resulting density matrix $\rho_\Sigma^{\text{unpol}}$ is a fictitious quantum state (see Sec. II C). The left panel of Fig. 8 shows the purity of the full state and of the three two-body reduced states. All purities depend only mildly on $\sqrt{s}$. The equality $\gamma_{tZ} = \gamma_{\bar{t}Z}$ follows from CP invariance after full phase-space

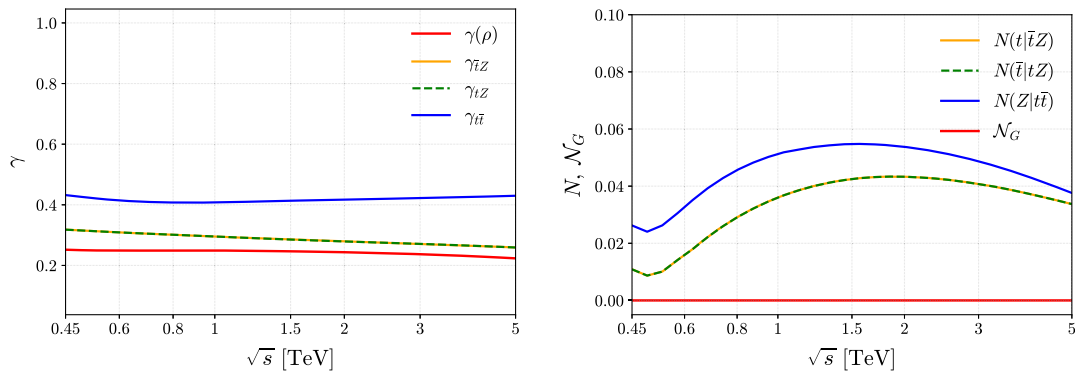


FIG. 8. Entanglement measures of the fully integrated fictitious $t\bar{t}Z$ spin state $\rho_\Sigma^{\text{unpol}}$ in the helicity basis, as functions of the center-of-mass energy $\sqrt{s}$ for unpolarized beams. Left: purity of the full state, $\gamma(\rho_\Sigma)$ (red), and those of the three reduced states, γ_{tZ} (orange), $\gamma_{\bar{t}Z}$ (green dashed), and $\gamma_{t\bar{t}}$ (blue). The curves γ_{tZ} and $\gamma_{\bar{t}Z}$ coincide as a consequence of CP invariance. Right: one-to-other negativities $N(t|\bar{t}Z)$ (orange), $N(\bar{t}|tZ)$ (green dashed), $N(Z|t\bar{t})$ (blue), and the genuine multipartite negativity $\mathcal{N}_G$ (red). The one-to-one negativities $N_{t\bar{t}}$, N_{tZ} , and $N_{\bar{t}Z}$ vanish identically across the whole range and are not shown.

integration. The full state is the most mixed, with $\gamma(\rho_\Sigma) \simeq 0.22 - 0.25$, while the reduced states are less mixed, with $\gamma_{t\bar{t}} \simeq 0.41 - 0.43$ and $\gamma_{tZ} = \gamma_{\bar{t}Z} \simeq 0.26 - 0.32$.

For the one-to-one negativities, we find

$$N_{t\bar{t}} = N_{tZ} = N_{\bar{t}Z} = 0 \quad \text{for all } \sqrt{s}. \quad (4.2)$$

Since the corresponding reduced states act on 2×2 and 2×3 Hilbert spaces, the Peres-Horodecki criterion is both necessary and sufficient for separability. Hence, the vanishing negativities imply that the $t\bar{t}$, tZ , and $\bar{t}Z$ systems are all separable once the full phase space is integrated over.

The right panel of Fig. 8 shows the one-to-other negativities and the genuine multipartite negativity. The one-to-other negativities remain small but nonzero over the entire energy range. By CP invariance, $N(t|\bar{t}Z) = N(\bar{t}|tZ)$, reaching a maximum of approximately 0.043 around $\sqrt{s} \sim 2$ TeV, while $N(Z|t\bar{t})$ peaks at approximately 0.055 around $\sqrt{s} \sim 1.5$ TeV. Thus, full phase-space integration removes all pairwise entanglement but does not erase all collective entanglement across single-particle cuts. For the GMN, however, we obtain $\mathcal{N}_G = 0$ for all $\sqrt{s}$. The fully inclusive $\rho_\Sigma^{\text{unpol}}$ is, therefore, a PPT mixture. Since PPT-entangled states may exist for all three bipartitions, $\mathcal{N}_G = 0$ does not imply bi-separability. Rather, it shows that genuine multipartite entanglement is not detected by the PPT mixture criterion.

V. PROSPECTS FOR ENTANGLEMENT OBSERVATION AT A FUTURE e^+e^- COLLIDER

In the previous section, we showed that the differential $t\bar{t}Z$ spin state exhibits a rich entanglement structure, including strong one-to-other entanglement and genuine multipartite entanglement over broad regions of phase space. These correlations are highly sensitive to phase-space averaging. After full integration, the one-to-one negativities vanish, the surviving one-to-other negativities are strongly suppressed, and the GMN no longer certifies genuine multipartite entanglement. A realistic collider

study cannot reconstruct the density matrix at a single phase-space point. The reconstruction of the spin density matrix from experimental data, known as *quantum state tomography*, requires a sufficiently large sample of $t\bar{t}Z$ events from which the decay angular distributions of the t , $\bar{t}$, and Z can be measured in their respective rest frames. The details of this procedure are presented in Appendix A. The central question is, therefore, whether one can identify kinematic regions that retain a substantial fraction of the differential entanglement while still providing enough events for its reconstruction and reliable certification.

In Sec. VA, we analyze the partially inclusive spin state ρ_Σ obtained by integrating over $\cos\theta_t$ and ϕ , and use the resulting entanglement maps in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane to define fiducial kinematic regions in which the most informative measures remain large. In Sec. VB, we then translate the fiducial entanglement values into the statistical significance and integrated luminosity needed to establish each measure as nonzero. Throughout the section we fix $\sqrt{s} = 1$ TeV, near the maximum of the unpolarized $\sigma(e^+e^- \rightarrow t\bar{t}Z)$, and consider three benchmark beam polarization configurations,

	Unpol	Pol+	Pol−
$(\mathcal{P}_{e^-}, \mathcal{P}_{e^+})$	(0, 0)	(+0.8, −0.3)	(−0.8, +0.3)

(5.1)

We focus on measuring the three one-to-other negativities and the genuine multipartite negativity, as they are the most informative quantities characterizing the tripartite entanglement structure and the most promising to access experimentally, owing to their relatively large values in the differential state.

A. Entanglement in partially inclusive states

In this subsection we analyze the partially inclusive spin state ρ_Σ obtained by integrating ρ over $\cos\theta_t$ and ϕ with the matrix-element-squared weight. This is the relevant state for an analysis in which the kinematic information used to

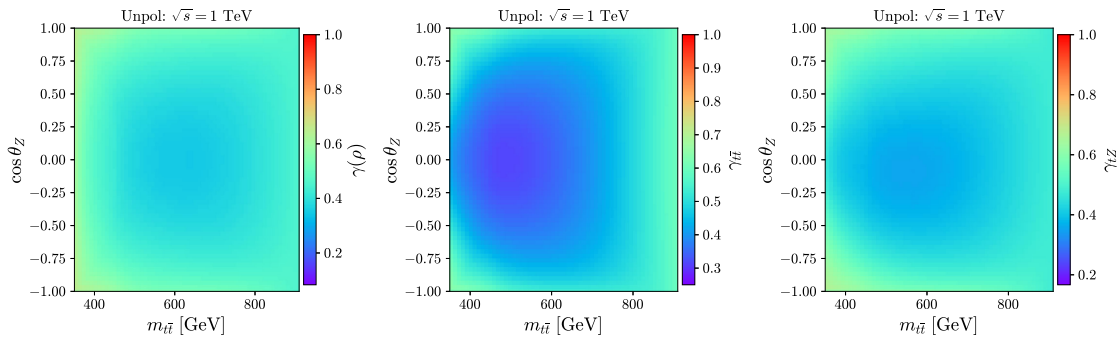


FIG. 9. Purity of the partially inclusive fictitious $t\bar{t}Z$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV for unpolarized beams, in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane after integration over $\cos\theta_t$ and ϕ in the helicity basis. Left: purity of the full state, $\gamma(\rho_\Sigma)$. Middle: purity of the $t\bar{t}$ reduced state, $\gamma_{t\bar{t}}$. Right: purity of the tZ reduced state, γ_{tZ} , which equals $\gamma_{\bar{t}Z}$ by CP invariance.

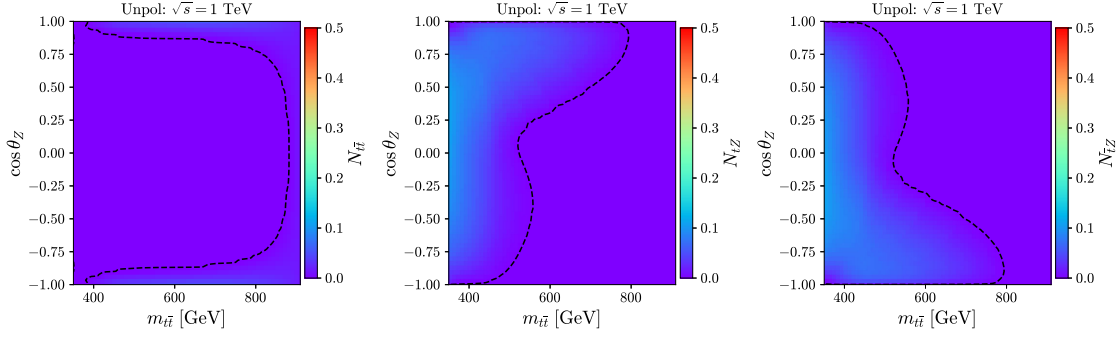


FIG. 10. One-to-one negativities of the partially inclusive $t\bar{t}$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV for unpolarized beams, in the $(m_{t\bar{t}}, \cos \theta_Z)$ plane after integration over $\cos \theta_t$ and ϕ in the helicity basis: $N_{t\bar{t}}$ (left), N_{tZ} (middle), and $N_{t\bar{Z}}$ (right). The dashed black contours mark the threshold $N = 10^{-3}$, below which each measure is treated as effectively vanishing. The color bar saturates at the algebraic upper bound $\frac{1}{2}$.

reconstruct ρ is the $t\bar{t}$ invariant mass and the polar angle of the Z in the lab frame, while the orientation of t in the $t\bar{t}$ rest frame is summed over. We work in the helicity basis throughout. We start with the unpolarized configuration. The results for the Pol+ and Pol− benchmarks, which are qualitatively similar, are collected in Appendix B.

Figure 9 shows the purity of the full state $\gamma(\rho_\Sigma)$ (left) and those of the reduced states $\gamma_{t\bar{t}}$ (middle) and $\gamma_{tZ} = \gamma_{t\bar{Z}}$ (right). The exact equality $\gamma_{tZ} = \gamma_{t\bar{Z}}$ holds point-by-point across the plane due to the CP invariance of the neutral interaction. All three purities are markedly smaller than in the differential case, reflecting the dilution of the spin information by the phase-space averaging. $\gamma(\rho_\Sigma)$ drops to ~ 0.35 in the central region $\cos \theta_Z \simeq 0$, $m_{t\bar{t}} \simeq 600$ GeV and rises toward ~ 0.6 near the corners of the plane, where the available solid angle in $(\cos \theta_t, \phi)$ shrinks and the integration mixes a narrower set of orientations. $\gamma_{t\bar{t}}$ follows the same overall shape with a minimum of ~ 0.25 , close to the lower bound $\frac{1}{4}$, signaling that the $t\bar{t}$ reduced state is nearly maximally mixed in this region. $\gamma_{tZ} = \gamma_{t\bar{Z}}$ takes intermediate values $\sim 0.35 - 0.55$.

Figure 10 shows the three one-to-one negativities $N_{t\bar{t}}$ (left), N_{tZ} (middle), and $N_{t\bar{Z}}$ (right) for unpolarized beams.

The corresponding results for Pol+ and Pol− are presented in Fig. 15 and display a similar behavior. The dashed black contours mark the threshold $N = 10^{-3}$, below which each measure is treated as effectively vanishing. We can see the relation $N_{tZ}(m_{t\bar{t}}, \cos \theta_Z) = N_{t\bar{Z}}(m_{t\bar{t}}, -\cos \theta_Z)$, anticipated by the CP invariance of the interaction. Compared with the differential results of Sec. IV A, all three one-to-one negativities are strongly depleted by phase-space averaging. The $t\bar{t}$ negativity survives only in a narrow region near the soft- Z and collinear boundaries, where the $t\bar{t}$ subsystem approximately decouples from the Z -boson and recovers the familiar entanglement structure of $e^+e^- \rightarrow t\bar{t}$. The negativities N_{tZ} and $N_{t\bar{Z}}$ remain nonzero only at low invariant masses, $m_{t\bar{t}} \lesssim 550$ GeV, with peak values below 0.10. Pairwise entanglement is, therefore, absent over a large fraction of the $(m_{t\bar{t}}, \cos \theta_Z)$ plane, with all three two-body reduced states classified as separable by the negativity criterion.

Figure 11 shows the three one-to-other negativities $N(Z|t\bar{t})$ (left), $N(t|\bar{t}Z)$ (middle), and $N(\bar{t}|tZ)$ (right) for unpolarized beams. The corresponding results for Pol+ and Pol− are shown in Fig. 16 and display essentially the same shape and magnitude. In contrast to the one-to-one

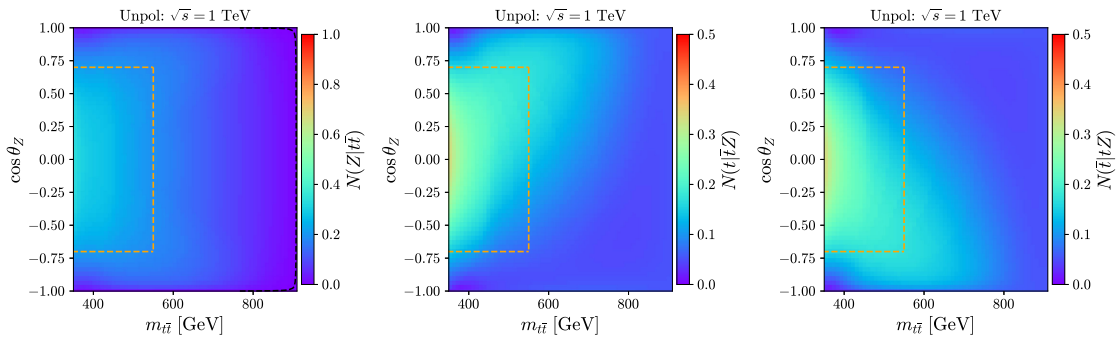


FIG. 11. One-to-other negativities of the partially inclusive $t\bar{t}Z$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV for unpolarized beams, in the $(m_{t\bar{t}}, \cos \theta_Z)$ plane after integration over $\cos \theta_t$ and ϕ in the helicity basis: $N(Z|t\bar{t})$ (left), $N(t|\bar{t}Z)$ (middle), and $N(\bar{t}|tZ)$ (right). The orange dashed lines mark the fiducial region $\Sigma_{1/2}$ defined in Eq. (5.2). The color bar saturates at the algebraic upper bound, 1 for $N(Z|t\bar{t})$ and $\frac{1}{2}$ for $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$.

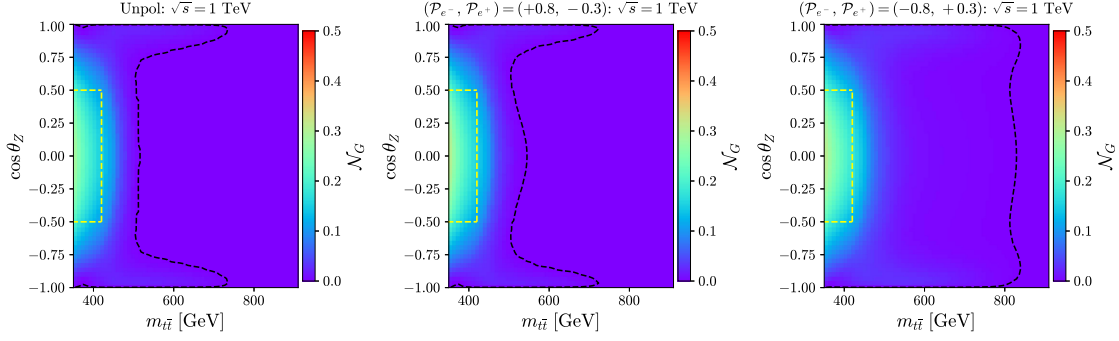


FIG. 12. Genuine multipartite negativity $\mathcal{N}_G$ of the partially inclusive $t\bar{t}Z$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV, in the $(m_{t\bar{t}}, \cos \theta_Z)$ plane in the helicity basis, for the three beam polarization configurations: Unpol (left), Pol+ (middle), and Pol- (right). The dashed black contours mark the threshold $\mathcal{N}_G = 10^{-3}$, below which the GMN is treated as effectively vanishing, and the yellow dashed contour marks the fiducial region Σ_{GME} defined in Eq. (5.3).

negativities, all three measures are positive throughout an extended region of the plane. $N(Z|t\bar{t})$ remains of order 0.1–0.3 over the entire central region of $(m_{t\bar{t}}, \cos \theta_Z)$, signaling that even after the angular integration the Z helicity is nontrivially entangled with the $t\bar{t}$ pair taken as a whole. $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$ are mirror images of each other under $\cos \theta_Z \rightarrow -\cos \theta_Z$ (again by CP) and peak near the $t\bar{t}$ threshold, $m_{t\bar{t}} \simeq 2m_t$, where they reach $\sim 0.3 - 0.4$, close to their algebraic maximum $\frac{1}{2}$. For $m_{t\bar{t}} \gtrsim 700$ GeV both negativities fall rapidly and become very small. The hierarchy among the three one-to-other measures suggests defining a fiducial region in which all three negativities are simultaneously large. A simple rectangular choice is

$$\Sigma_{1|2}: |\cos \theta_Z| < 0.7 \quad \text{and} \quad m_{t\bar{t}} < 550 \text{ GeV}, \quad (5.2)$$

indicated by the orange dashed contour in Fig. 11. We will use $\Sigma_{1|2}$ in Sec. VB as the fiducial region for the one-to-other entanglement observation.

Figure 12 shows the genuine multipartite negativity $\mathcal{N}_G$ of ρ_Σ for the three beam configurations. The pattern is essentially the same in all three: $\mathcal{N}_G > 0$ in a contiguous region near threshold, $m_{t\bar{t}} \lesssim 500$ GeV, at central $\cos \theta_Z$, where it reaches peak values ~ 0.25 —halfway to the GHZ-like upper bound $\frac{1}{2}$ and certifying GME of ρ_Σ in this region. Narrow strips of $\mathcal{N}_G > 0$ also appear near the collinear- Z edges $|\cos \theta_Z| \rightarrow 1$, paralleling the $N_{t\bar{t}} > 0$ regions of Fig. 10. For Pol-, the positive $\mathcal{N}_G$ region extends to $m_{t\bar{t}} \simeq 700\text{--}850$ GeV. Outside these regions $\mathcal{N}_G$ vanishes within numerical precision, so the corresponding ρ_Σ is a PPT mixture. Because PPT mixtures in 2×6 subsystems can still be entangled, this does not strictly exclude bi-separability, but it does mean that the GMN measure can no longer certify GME from the data. The GMN profile is more sharply localized in $(m_{t\bar{t}}, \cos \theta_Z)$ than the one-to-other negativities, so a smaller fiducial region is appropriate,

$$\Sigma_{\text{GME}}: |\cos \theta_Z| < 0.5 \quad \text{and} \quad m_{t\bar{t}} < 420 \text{ GeV}, \quad (5.3)$$

shown as the yellow dashed contour in Fig. 12.

B. Feasibility for observing tripartite entanglement

The fiducial regions $\Sigma_{1|2}$ and Σ_{GME} defined in Sec. VA were tailored to maximize the entanglement signal at the price of a restricted phase-space volume. In this subsection we quantify the resulting trade-off between signal strength and event rate, and translate it into the integrated luminosity required to observe each entanglement measure at a future e^+e^- collider.

Figure 13 shows the leading-order standard model cross section $\sigma(e^+e^- \rightarrow t\bar{t}Z)$ as a function of the center-of-mass energy $\sqrt{s}$, for the three beam polarization configurations. The cross section rises rapidly above the kinematic threshold $\sqrt{s} \gtrsim 2m_t + m_Z \simeq 0.44$ TeV, reaches a maximum around $\sqrt{s} \simeq 0.8\text{--}0.9$ TeV, with peak values of approximately 9, 5, and 3 fb for Pol-, Unpol, and Pol+, respectively, and then

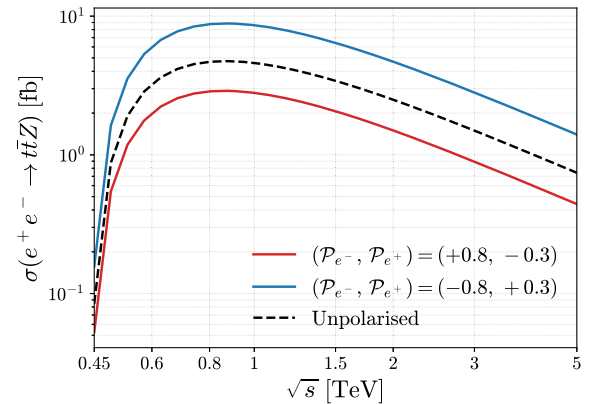


FIG. 13. Tree-level standard model cross section for $e^+e^- \rightarrow t\bar{t}Z$ as a function of the center-of-mass energy $\sqrt{s}$, for the three beam polarization configurations Pol+ with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (+0.8, -0.3)$ (red), Pol- with $(-0.8, +0.3)$ (blue), and unpolarized (black dashed).

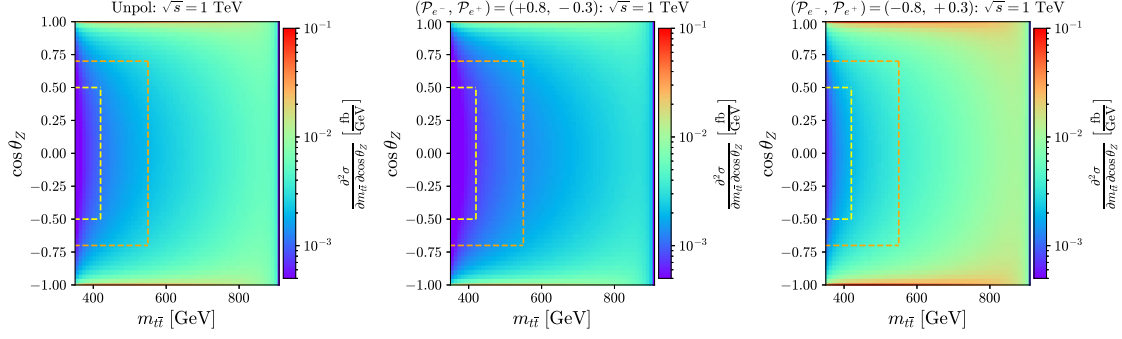


FIG. 14. Doubly differential cross section $\partial^2 \sigma / (\partial m_{t\bar{t}} \partial \cos \theta_Z)$ in fb/GeV at $\sqrt{s} = 1$ TeV, in the $(m_{t\bar{t}}, \cos \theta_Z)$ plane for the three beam configurations: Unpol (left), Pol+ (middle), and Pol– (right). The orange and yellow dashed contours mark the fiducial regions $\Sigma_{1|2}$ and Σ_{GME} defined in Eqs. (5.2) and (5.3).

decreases at higher energies. The enhancement of the Pol– configuration originates from the chiral electroweak couplings of the electron, which favor left-handed electrons and right-handed positrons in the dominant production amplitudes. As a result, Pol– configuration offers the largest event sample for a given integrated luminosity and is the natural choice when statistics dominates the measurement. In the following, we fix $\sqrt{s} = 1$ TeV for each beam polarization scenario. This energy lies close to the maximum of the corresponding $e^+e^- \rightarrow t\bar{t}Z$ production cross section and is a benchmark energy for future high-energy lepton colliders.

Figure 14 shows the doubly differential cross section $\partial^2 \sigma / (\partial m_{t\bar{t}} \partial \cos \theta_Z)$ in fb/GeV at $\sqrt{s} = 1$ TeV. In all three polarizations the events are concentrated near $|\cos \theta_Z| \rightarrow 1$ (collinear Z emission) and at intermediate $m_{t\bar{t}}$, in marked contrast to the entanglement maps of Sec. VA, which favor the central region $|\cos \theta_Z| \lesssim 0.5$ and low $m_{t\bar{t}}$. The fiducial regions $\Sigma_{1|2}$ and Σ_{GME} , therefore, sit in the kinematically suppressed corner of the cross section, and the price for retaining a large fiducial entanglement is a small fiducial event rate.

The relevant numbers are summarized in Table I. For the unpolarized beam, the wider fiducial region $\Sigma_{1|2}$ retains about 10% of the total $\sigma(e^+e^- \rightarrow t\bar{t}Z)$ at $\sqrt{s} = 1$ TeV ($\sigma_{1|2} \simeq 0.41$ fb), while the smaller Σ_{GME} retains only about 1% ($\sigma_{\text{GME}} \simeq 0.054$ fb). Both fiducial cross sections scale

TABLE I. Fiducial cross sections $\sigma_{1|2}$ and σ_{GME} (in fb) for the regions $\Sigma_{1|2}$ and Σ_{GME} , defined in Eqs. (5.2) and (5.3), at $\sqrt{s} = 1$ TeV. We also report the corresponding fiducial values of $N(Z|t\bar{t})$ and $N(t|\bar{t}Z)$ in $\Sigma_{1|2}$, and of $\mathcal{N}_G$ in Σ_{GME} , for the three beam configurations. By the CP symmetry of the neutral-current production, we have $N(\bar{t}|tZ) = N(t|\bar{t}Z)$ in $\Sigma_{1|2}$.

	$\sigma_{1 2}$ [fb]	σ_{GME} [fb]	$N(Z t\bar{t})$	$N(t \bar{t}Z)$	$\mathcal{N}_G$
Unpol	0.41	0.054	0.199	0.136	0.156
Pol+	0.27	0.036	0.216	0.150	0.169
Pol–	0.74	0.099	0.244	0.170	0.181

by a factor of ~ 2.7 from Pol+ to Pol–, mirroring the rate hierarchy of Fig. 13. The fiducial entanglement values, in contrast, depend only weakly on the polarization: $N(Z|t\bar{t}) \simeq 0.20 - 0.24$, $N(t|\bar{t}Z) \simeq 0.14 - 0.17$, and $\mathcal{N}_G \simeq 0.16 - 0.18$, with the strongest signals consistently for Pol– across all three measures. Pol– is, therefore, unambiguously the preferred running condition for the feasibility study below.

It is instructive to ask how much further the idealized limit of fully polarized beams, $(\mathcal{P}_{e-}, \mathcal{P}_{e+}) = (\mp 1, \pm 1)$, could improve on these benchmarks. By Eq. (2.10), the fiducial state is a two-component mixture of the pure helicity channels,

$$\rho_{\Sigma}[\mathcal{P}_{e-}, \mathcal{P}_{e+}] = w_{-+} \rho_{\Sigma}^{-+} + w_{+-} \rho_{\Sigma}^{+-},$$

$$w_{\lambda_1 \lambda_2} = \frac{P_{\lambda_1 \lambda_2} \sigma_{\Sigma}^{\lambda_1 \lambda_2}}{P_{-+} \sigma_{\Sigma}^{-+} + P_{+-} \sigma_{\Sigma}^{+-}}, \quad (5.4)$$

where $\rho_{\Sigma}^{\lambda_1 \lambda_2}$ and $\sigma_{\Sigma}^{\lambda_1 \lambda_2}$ denote the fiducial state and cross section of the channel (λ_1, λ_2) , reached for $(\mathcal{P}_{e-}, \mathcal{P}_{e+}) = (\lambda_1, \lambda_2)$. At $\sqrt{s} = 1$ TeV, the pure-channel cross sections in the fiducial regions are $\sigma_{1|2}^{-+} \simeq 1.24$ fb and $\sigma_{1|2}^{+-} \simeq 0.39$ fb, and $\sigma_{\text{GME}}^{-+} \simeq 0.17$ fb and $\sigma_{\text{GME}}^{+-} \simeq 0.05$ fb. The ratio $\sigma^{-+}/\sigma^{+-} \simeq 3.2$ reflects the chiral electron couplings noted above. The effective weight of the dominant channel in Eq. (5.4) is then $w_{-+} \simeq 0.76$ (Unpol), $w_{+-} \simeq 0.84$ (Pol+), and $w_{-+} \simeq 0.98$ (Pol–): the preferred benchmark Pol– is already within 2% of the pure $(-, +)$ channel. Because the negativities are convex and continuous functions of the state, moving from Pol– to the $(-1, +1)$ polarization can shift them only at the percent level, and the entanglement structure at $(-1, +1)$ is essentially that of Pol–. The main effect of full polarization is instead on the rate: $\sigma_{\Sigma}(-1, +1) = \sigma_{\Sigma}^{-+}$ exceeds the Pol– value by a factor $\simeq 1.7$, which lowers the luminosity requirements derived below by the same factor without altering the conclusions. Although the residual admixture is larger for Pol+, the

TABLE II. Naive significances σ_v for the observation of each entanglement measure at $\mathcal{L} = 8 \text{ ab}^{-1}$, together with the integrated luminosities $\mathcal{L}_{1\sigma}$ and $\mathcal{L}_{2\sigma}$ (in ab^{-1}) required to reach 1σ and 2σ significance, for the three beam configurations. The one-to-other negativities $N(Z|t\bar{t})$ and $N(t|\bar{t}Z)$ are evaluated inside $\Sigma_{1|2}$, and the GMN $\mathcal{N}_G$ is evaluated inside Σ_{GME} . All entries assume $c_{\text{sys}} = 1$.

significance, luminosities	$N(Z t\bar{t})$			$N(t \bar{t}Z)$			$\mathcal{N}_G$		
	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$
Unpol	1.1	6.2	24.8	1.5	3.3	13.2	0.65	19.1	76.4
Pol+	1.0	7.9	31.7	1.4	4.1	16.4	0.57	24.6	98.4
Pol−	1.9	2.3	9.1	2.6	1.2	4.7	1.01	7.8	31.1

$(+1, -1)$ limit is of little practical interest, being disfavored both in rate and in entanglement.

To assess the feasibility of the entanglement observation, we consider quantum tomography in the fully leptonic decay channel ($t \rightarrow b\ell^+\nu$, $\bar{t} \rightarrow \bar{b}\ell^-\bar{\nu}$, and $Z \rightarrow \ell^+\ell^-$, where $\ell = e, \mu, \tau$), which provides a clean final state with two charged leptons from the Z decay and two additional charged leptons from the leptonic top decays. The corresponding branching ratio is $\text{BR} \simeq (0.33)^2 \cdot 0.10 \simeq 0.010$. For the τ modes, we assume truth-level knowledge of the τ directions. For a fiducial cross section σ_Σ and an integrated luminosity $\mathcal{L}$, the expected number of fully leptonic events inside the fiducial region Σ is

$$n_\Sigma = \sigma_\Sigma \cdot \text{BR} \cdot \mathcal{L}. \quad (5.5)$$

To obtain an order-of-magnitude estimate of the statistical requirements for entanglement certification, we adopt a simple scaling model for the uncertainty on an entanglement variable $v \in \{N(Z|t\bar{t}), N(t|\bar{t}Z), N(\bar{t}|tZ), \mathcal{N}_G\}$. For a sample of n_Σ events, we estimate

$$\delta_v = c_{\text{sys}} \cdot \frac{v_{\text{max}}}{\sqrt{n_\Sigma}}, \quad (5.6)$$

where v_{max} is the algebraic upper bound on v , namely, $v_{\text{max}} = 1$ for $N(Z|t\bar{t})$ (the 3×4 partition) and $v_{\text{max}} = \frac{1}{2}$ for $N(t|\bar{t}Z)$, $N(\bar{t}|tZ)$, and $\mathcal{N}_G$. The coefficient $c_{\text{sys}} \geq 1$ parametrizes departures from the naive scaling $v_{\text{max}}/\sqrt{n_\Sigma}$, including reconstruction inefficiencies, finite detector resolution, uncertainties in the quantum state reconstruction procedure, and any backgrounds. Throughout this section we adopt the optimistic benchmark $c_{\text{sys}} = 1$ in order to isolate the impact of event statistics. The resulting luminosity requirements should be interpreted as indicative statistical targets rather than as the outcome of a complete detector-level tomography analysis. A realistic experimental study would generally increase δ_v by a factor of a few, with the corresponding luminosity requirements scaling as c_{sys}^2 . The naive significance at which a nonzero value of v can be established in the fiducial region is

$$\sigma_v \equiv \frac{v_\Sigma}{\delta_v} = \frac{v_\Sigma}{v_{\text{max}}} \sqrt{\sigma_\Sigma \cdot \text{BR} \cdot \mathcal{L}}, \quad (5.7)$$

and the luminosity required to reach a target significance σ_{tgt} is

$$\mathcal{L}_{\sigma_{\text{tgt}}} = \left(\frac{\sigma_{\text{tgt}} v_{\text{max}}}{v_\Sigma} \right)^2 \frac{1}{\sigma_\Sigma \cdot \text{BR}}. \quad (5.8)$$

Because $v_{\text{max}}/v_\Sigma \sim 3 - 5$ for the $t\bar{t}Z$ system, σ_Σ is sub-fb, and $\text{BR} \simeq 10^{-2}$, the required luminosities naturally fall in the multi- ab^{-1} regime, making the observation of the most robust entanglement observables a plausible target for a future high-luminosity lepton collider.

Table II summarizes the naive significances σ_v expected at a benchmark integrated luminosity $\mathcal{L} = 8 \text{ ab}^{-1}$, together with the integrated luminosities $\mathcal{L}_{1\sigma}$ and $\mathcal{L}_{2\sigma}$ required to reach 1σ and 2σ significance through Eq. (5.8). By construction, $\mathcal{L}_{2\sigma} = 4\mathcal{L}_{1\sigma}$, since $\sigma_v \propto \sqrt{\mathcal{L}}$. Three clear messages emerge.

First, the easiest measure to certify is $N(t|\bar{t}Z)$ [equivalently $N(\bar{t}|tZ)$, identical by CP]. This is the result of a fortunate combination: the fiducial value $\sim 0.14 - 0.17$ is a sizeable fraction of the upper bound $v_{\text{max}} = \frac{1}{2}$, and the fiducial region $\Sigma_{1|2}$ used to evaluate it has the largest cross section among the two fiducial choices. Already with $\mathcal{L} \simeq 3.3 \text{ ab}^{-1}$ (Unpol) or 1.2 ab^{-1} (Pol−) the measure can be established as nonzero at the 1σ level, and 2σ certification requires only 13.2 and 4.7 ab^{-1} , respectively. These luminosities are comparable to those envisioned for the ILC at $\sqrt{s} = 1 \text{ TeV}$.

Second, $N(Z|t\bar{t})$ is intrinsically harder to certify than $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$ despite its larger fiducial value ($\sim 0.20 - 0.24$), because the relevant algebraic maximum is $v_{\text{max}} = 1$ rather than $\frac{1}{2}$. The ratio v_Σ/v_{max} that drives σ_v is, therefore, smaller, and the luminosity needed to reach a given significance is correspondingly larger: 24.8 ab^{-1} at 2σ for Unpol, dropping to 9.1 ab^{-1} for Pol−.

Third, certifying genuine multipartite entanglement through $\mathcal{N}_G$ remains the most demanding goal. The fiducial value of $\mathcal{N}_G$ ($\sim 0.16 - 0.18$) is comparable to that of $N(t|\bar{t}Z)$, so the dominant penalty for the GMN comes

from the fiducial cross section in Σ_{GME} , which is $\sim 7 - 8$ times smaller than in $\Sigma_{1|2}$. The resulting luminosities are 19 and 8 ab^{-1} for 1σ in Unpol and Pol–, and 76 and 31 ab^{-1} for 2σ . With the optimistic $c_{\text{sys}} = 1$ assumed here, a 1σ GME certification at $\sqrt{s} = 1$ TeV for Pol– is achievable within the projected ILC luminosity of $\sim 8 \text{ ab}^{-1}$ [114]—the 2σ target for Pol– requires a factor of ~ 4 more than this baseline. In the idealized fully polarized limit $(-1, +1)$, the factor $\simeq 1.7$ gain in σ_{GME} discussed above would reduce these to $\simeq 4.6$ and 18 ab^{-1} , respectively, while the 2σ GME certification would still lie beyond the baseline ILC luminosity. More optimization for the fiducial cuts beyond the $(m_{\bar{t}\bar{t}}, \cos\theta_Z)$ plane, optimization for the quantization axes (beyond the helicity basis) in reconstructing fictitious quantum states, a more refined GME observable, or inclusion of hadronic decay channels would all be valuable directions for further study.

VI. CONCLUSIONS

We have presented a systematic study of the tripartite entanglement structure of the spin state produced in $e^+e^- \rightarrow t\bar{t}Z$. The final-state spin Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ has dimension 12. Starting from the tree-level standard model helicity amplitudes for all five contributing diagrams, we constructed the 12×12 density matrix as a function of the four independent phase-space variables $(m_{\bar{t}\bar{t}}, \cos\theta_Z, \cos\theta_t, \phi)$ and evaluated on it a hierarchy of negativity-based entanglement measures that remain valid for mixed states: the three *one-to-one* negativities $N_{\bar{t}\bar{t}}$, N_{tZ} , and $N_{\bar{t}Z}$, the three *one-to-other* negativities $N(Z|\bar{t}\bar{t})$, $N(t|\bar{t}Z)$, and $N(\bar{t}|tZ)$, and the GMN $\mathcal{N}_G$ of Refs. [88,113]. These quantities were studied at three increasingly inclusive levels: differential (a fixed phase-space point), partially inclusive (integrated over $\cos\theta_t$ and ϕ in the helicity basis), and fully inclusive (integrated over all four kinematic variables).

At the differential level the state ρ is nearly pure [$\gamma(\rho) \simeq 0.65 - 0.70$] and genuinely tripartite entangled over a large fraction of phase space, with $\mathcal{N}_G \sim 0.40$ in the central region. Its entanglement pattern is GHZ-like: the one-to-other negativities $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$ nearly saturate their algebraic maxima of $\frac{1}{2}$, while the pairwise negativities remain small or vanish over most of phase space. Hence, each particle is largely entangled with the remaining two, but each pair carries little extractable entanglement.

At the fully inclusive level the state ρ_Σ becomes nearly maximally mixed [$\gamma(\rho_\Sigma) \simeq 0.22 - 0.25$, only slightly above the $\frac{1}{12}$ floor]: all one-to-one negativities vanish identically, $\mathcal{N}_G = 0$ for all $\sqrt{s}$, and only the one-to-other negativities survive with small but nonzero values [$N(Z|\bar{t}\bar{t}) \lesssim 0.06$ and $N(t|\bar{t}Z) = N(\bar{t}|tZ) \lesssim 0.04$ around $\sqrt{s} \simeq 1.5 - 2$ TeV]. Thus, the genuine tripartite quantum correlations of $e^+e^- \rightarrow t\bar{t}Z$

live in the differential states rather than in the inclusive ones.

Partial integration over $(\cos\theta_t, \phi)$ at fixed $(m_{\bar{t}\bar{t}}, \cos\theta_Z)$ preserves much of the differential entanglement in the threshold region. $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$ peak at $\sim 0.3 - 0.4$ near $m_{\bar{t}\bar{t}} \simeq 2m_t$, $N(Z|\bar{t}\bar{t})$ remains of order $0.1 - 0.3$ over a broad central region, and $\mathcal{N}_G$ reaches ~ 0.25 at low $m_{\bar{t}\bar{t}}$ and central $\cos\theta_Z$. The one-to-one negativities, by contrast, remain comparatively small in this region. Motivated by these observations, we defined the fiducial regions $\Sigma_{1|2}$ [Eq. (5.2)] and Σ_{GME} [Eq. (5.3)] for the one-to-other and GMN measurements, respectively.

Assuming quantum tomography in the fully leptonic decay channel of $t\bar{t}Z$ at $\sqrt{s} = 1$ TeV with the most optimistic statistical assumption $c_{\text{sys}} = 1$, our naive luminosity estimates for 2σ certification are as follows:

- (i) $N(t|\bar{t}Z)$ [equivalently $N(\bar{t}|tZ)$ by CP]: $\mathcal{L} \simeq 13 \text{ ab}^{-1}$ (Unpol) or 5 ab^{-1} (Pol–),
- (ii) $N(Z|\bar{t}\bar{t})$: $\mathcal{L} \simeq 25 \text{ ab}^{-1}$ (Unpol) or 9 ab^{-1} (Pol–), and
- (iii) $\mathcal{N}_G$: $\mathcal{L} \simeq 76 \text{ ab}^{-1}$ (Unpol) or 31 ab^{-1} (Pol–).

The first two targets are accessible within the projected high-luminosity polarized program at the ILC [114]. The GMN target at 1σ for Pol– is comparable to the projected ILC luminosity at $\sqrt{s} = 1$ TeV, while the 2σ target requires a factor of ~ 4 more than this baseline. Beam polarization gives a $\sim 2 - 3$ luminosity-saving factor across all three measures. The GME prospects could be improved by developing refined GME observables (the GMN considered here is a sufficient but not necessary certificate of GME), optimizing the fiducial kinematic regions beyond the $(m_{\bar{t}\bar{t}}, \cos\theta_Z)$ plane, optimizing the quantization axes in reconstructing fictitious quantum states, or including hadronic channels in the quantum state tomography [10,20,22]. Another promising future direction is to extend the present quantum information program of the $t\bar{t}Z$ final state beyond entanglement to other genuinely quantum correlations and resources, such as tripartite Bell nonlocality [81] and “magic” (nonstabilizerness) [19].

Beyond the standard model, the multipartite spin density matrix studied here is a natural probe of new physics in the $t\bar{t}Z$ interaction (anomalous neutral-current vertices, CP -violating contributions, or new heavy mediators), which can leave a distinctive imprint on the entanglement maps presented in Sec. IV and Sec. VA. The framework developed in this paper, combining helicity amplitudes, multi-level phase-space integration, and a hierarchy of negativity-based entanglement measures, applies to any tripartite spin system. Other natural collider targets include $e^+e^-/pp \rightarrow \tau^+\tau^-Z$, W^+W^-Z , and $h \rightarrow \tau^+\tau^-Z$. Furthermore, the present analysis can be naturally extended to $pp \rightarrow t\bar{t}Z$ at the LHC and the FCC-hh. The numerical results and methodology presented here lay out a quantitative quantum information

program for the study of multipartite quantum correlations at current and next-generation colliders.

ACKNOWLEDGMENTS

We thank Błażej Rozwoda for contributions in the early stages of this project. K. S. thanks Hai-Chau Nguyen, Ties-Albrecht Ohst, and Gilberto Tetlamatzi-Xolocotzi for helpful discussions on the computation of genuine multipartite entanglement measures. K. S. also thanks Yuki Nakasone for producing the schematic illustration in Fig. 3. The work of D. G. is supported in part by US Department of Energy Grant No. DE-SC 0016013. Some computing for this project was performed at the High Performance Computing Center at Oklahoma State University, supported in part through the National Science Foundation Grant No. OAC-1531128.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: QUANTUM STATE TOMOGRAPHY

In Sec. II we described how the $t\bar{t}Z$ spin density matrix is constructed from the tree-level helicity amplitudes, and the results presented in Sec. IV and Sec. V are obtained directly with this formalism. Experimentally, however, the density matrix must instead be reconstructed from the measured angular distributions of the t , $\bar{t}$, and Z decay products through a quantum state tomography procedure. In this appendix we present this procedure for the full 12×12 density matrix of the $t\bar{t}Z$ spin state.

1. Spherical tensor representation for spin-1/2 and spin-1

First let us consider the density matrix for a spin-1/2 particle

$$\rho_{1/2} = \frac{1}{2}(\mathbb{I}_2 + \vec{b} \cdot \vec{\sigma}), \quad (\text{A1})$$

where $\vec{b} = (b_1, b_2, b_3)$ is the polarization vector and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Define the spin-1/2 spherical tensor operators t_M^L [115,116],

$$t_{\pm 1}^1 = \mp \frac{1}{\sqrt{6}}(\sigma_1 \pm i\sigma_2), \quad \text{and} \quad t_0^1 = \frac{1}{\sqrt{3}}\sigma_3. \quad (\text{A2})$$

They provide an orthonormal basis and satisfy

$$t_{-M}^L = (-1)^M (t_M^L)^\dagger, \quad \text{Tr}\{t_M^L (t_{M'}^L)^\dagger\} = \frac{2}{3} \delta_{LL'} \delta_{MM'}. \quad (\text{A3})$$

Explicitly,

$$t_1^1 = -\sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_{-1}^1 = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$t_0^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A4})$$

The spin-1/2 density matrix can then be written as

$$\rho_{1/2} = \frac{1}{2}(\mathbb{I}_2 + A_{11}t_1^1 + A_{1-1}t_{-1}^1 + A_{10}t_0^1)$$

$$= \frac{1}{2}(\mathbb{I}_2 + A_{LM}t_M^L), \quad (\text{A5})$$

where the multipole parameters, A_{LM} with $L = 1$ and $-1 \leq M \leq 1$, satisfy $A_{L-M} = (-1)^M A_{LM}^*$ and are related to the components of the polarization vector by

$$A_{11} = -\sqrt{3} \left(\frac{b_1 - ib_2}{\sqrt{2}} \right), \quad A_{1-1} = \sqrt{3} \left(\frac{b_1 + ib_2}{\sqrt{2}} \right),$$

$$A_{10} = \sqrt{3}b_3. \quad (\text{A6})$$

The density matrix for a spin-1 particle can be expressed in an analogous way,

$$\rho_1 = \frac{1}{3}(\mathbb{I}_3 + A_{LM}T_M^L), \quad (\text{A7})$$

where $L = 1, 2$ and $-L \leq M \leq L$. The spherical tensor operators, T_M^L , in terms of spin-1 component operators, J_x , J_y , and J_z , are given by [52]

$$T_{\pm 1}^1 = \mp \frac{\sqrt{3}}{2}(J_x \pm iJ_y), \quad T_0^1 = \sqrt{\frac{3}{2}}J_z,$$

$$T_{\pm 2}^2 = \frac{2}{\sqrt{3}}(T_{\pm 1}^1)^2, \quad T_{\pm 1}^2 = \sqrt{\frac{3}{2}}(T_{\pm 1}^1 T_0^1 + T_0^1 T_{\pm 1}^1),$$

$$T_0^2 = \frac{\sqrt{2}}{3}(T_1^1 T_{-1}^1 + T_{-1}^1 T_1^1 + 2(T_0^1)^2). \quad (\text{A8})$$

As in the spin-1/2 case, they form an orthonormal basis satisfying

$$T_{-M}^L = (-1)^M (T_M^L)^\dagger, \quad \text{Tr}\{T_M^L (T_{M'}^L)^\dagger\} = 3\delta_{LL'} \delta_{MM'}. \quad (\text{A9})$$

Explicitly,

$$T_{-1}^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_0^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_1^1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A10})$$

$$T_{-2}^2 = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_{-1}^2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_0^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A11})$$

$$T_1^2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2^2 = \sqrt{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A12})$$

2. Quantum state tomography for $t\bar{t}Z$

The $t\bar{t}Z$ quantum state is represented by a 12×12 density matrix, which can be parametrized in terms of the spherical tensor basis introduced in the previous section. Using Eqs. (A5) and (A7), the $t\bar{t}Z$ density matrix reads³

$$\begin{aligned} \rho = & \frac{1}{12} (\mathbb{I}_{12} + A_{L_1 M_1}^1 t_{M_1}^{L_1} \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 + A_{L_2 M_2}^2 t_{M_2}^{L_2} \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 + A_{L_3 M_3}^3 \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes T_{M_3}^{L_3} + C_{L_1 M_1 L_2 M_2}^{12} t_{M_1}^{L_1} \otimes t_{M_2}^{L_2} \otimes \mathbb{I}_3 \\ & + C_{L_1 M_1 L_3 M_3}^{13} t_{M_1}^{L_1} \otimes \mathbb{I}_2 \otimes T_{M_3}^{L_3} + C_{L_2 M_2 L_3 M_3}^{23} t_{M_2}^{L_2} \otimes T_{M_3}^{L_3} + C_{L_1 M_1 L_2 M_2 L_3 M_3}^{123} t_{M_1}^{L_1} \otimes t_{M_2}^{L_2} \otimes T_{M_3}^{L_3}), \end{aligned} \quad (\text{A13})$$

where $L_{1,2} = 1$, $-L_{1,2} \leq M_{1,2} \leq L_{1,2}$, $L_3 = 1, 2$, and $-L_3 \leq M_3 \leq L_3$. Here, $A_{L_i M_i}^i$ are the polarizations of particle i with $i = 1$ for the top quark, $i = 2$ for the antitop quark, and $i = 3$ for the Z-boson, $C_{L_i M_i L_j M_j}^{ij}$ are the two-particle spin correlations between particles i and j , and $C_{L_1 M_1 L_2 M_2 L_3 M_3}^{123}$ are the three-particle spin correlations.

The spin information of the top quarks and Z-boson is imprinted in the angular distributions of their decay products. Consider the fully leptonic final state $t(\rightarrow b\ell_1^+\nu_\ell)\bar{t}(\rightarrow \bar{b}\ell_2^-\bar{\nu}_\ell)Z(\rightarrow \ell_3^+\ell_3^-)$. The decay matrices for the top and antitop quarks are given by [118]

$$\Gamma_{1,2} = \begin{pmatrix} \frac{1+\beta_{1,2}\cos\theta_{1,2}}{2} & \frac{\beta_{1,2}\sin\theta_{1,2}}{2}e^{i\varphi_{1,2}} \\ \frac{\beta_{1,2}\sin\theta_{1,2}}{2}e^{-i\varphi_{1,2}} & \frac{1-\beta_{1,2}\cos\theta_{1,2}}{2} \end{pmatrix}, \quad (\text{A14})$$

where $(\theta_{1,2}, \varphi_{1,2})$ are the polar and azimuthal angles of the charged lepton $\ell_{1,2}$ in their mother rest frame, and $\beta_{1,2}$ are their spin analyzing powers. For top quark decays to charged leptons, $\beta_1 = +1$ for ℓ_1^+ and $\beta_2 = -1$ for ℓ_2^- [119]. For the Z-boson decaying into two charged leptons, the decay matrix reads

$$\begin{aligned} \Gamma_3 = & \frac{1}{4} \\ & \times \begin{pmatrix} 1+\delta_3+(1-3\delta_3)\cos^2\theta_3-2\eta_3\cos\theta_3 & \frac{1}{\sqrt{2}}[(1-3\delta_3)\sin 2\theta_3-2\eta_3\sin\theta_3]e^{-i\varphi_3} & (1-3\delta_3)(1-\cos^2\theta_3)e^{-2i\varphi_3} \\ \frac{1}{\sqrt{2}}[(1-3\delta_3)\sin 2\theta_3-2\eta_3\sin\theta_3]e^{-i\varphi_3} & 4\delta_3+2(1-3\delta_3)\sin^2\theta_3 & -\frac{1}{\sqrt{2}}[(1-3\delta_3)\sin 2\theta_3+2\eta_3\sin\theta_3]e^{-i\varphi_3} \\ (1-3\delta_3)(1-\cos^2\theta_3)e^{-2i\varphi_3} & -\frac{1}{\sqrt{2}}[(1-3\delta_3)\sin 2\theta_3+2\eta_3\sin\theta_3]e^{-i\varphi_3} & 1+\delta_3+(1-3\delta_3)\cos^2\theta_3+2\eta_3\cos\theta_3 \end{pmatrix}, \end{aligned} \quad (\text{A15})$$

with (θ_3, φ_3) denoting the angles of the negatively charged lepton, ℓ_3^- , in the Z rest frame. The parameters η_3 and δ_3 are the spin analyzing powers of ℓ_3^- analogous to β in the top quark decay. Neglecting the charged lepton mass, they are given by [70,79]

³We parametrize the density matrix using the spherical tensor basis, which is conventionally used for the Z-boson quantum tomography. An equivalent parametrization in the “Cartesian” basis is given in Ref. [117].

$$\eta_3 = \frac{C_L^2 - C_R^2}{C_L^2 + C_R^2}, \quad \delta_3 = 0, \quad (\text{A16})$$

where $C_{L,R}$ are the left and right chiral couplings of the Z-boson to charged leptons.

We define a basis to compute the polar and azimuthal angles of the charged leptons in their mother rest frames. We start by boosting all particles to the $t\bar{t}Z$ rest frame. The procedure to measure the angles of particle A's decay product is as follows: particle A is brought along the z direction by rotations around the z-axis by ϕ , and around the y-axis by θ . The same rotations are applied to its daughter. The daughter is then boosted to A's rest frame. The $\{\hat{x}, \hat{y}, \hat{z}\}$ basis for particle A is then defined as follows:

- (i) $\hat{z} = \hat{p}_A$,
- (ii) $\hat{x} = \hat{p}_B \times \hat{p}_C$, and
- (iii) $\hat{y} = \hat{z} \times \hat{x}$.

with p_B and p_C in the $t\bar{t}Z$ rest frame and p_A after the described rotations.

The normalized differential distribution for the charged leptons, ℓ_1^+ , ℓ_2^- , and ℓ_3^- , is then given by

$$\begin{aligned} \frac{1}{\sigma} \frac{d\sigma}{d\Omega_1 d\Omega_2 d\Omega_3} &= \left(\frac{2}{4\pi}\right)^2 \left(\frac{3}{4\pi}\right) \text{Tr}\{\rho(\Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3)^T\} \\ &= \frac{1}{(4\pi)^3} [1 + A_{L_1 M_1}^1 B_{L_1}^1 Y_{L_1}^{M_1}(\Omega_1) + A_{L_1 M_1}^2 B_{L_2}^2 Y_{L_2}^{M_2}(\Omega_2) \\ &\quad + A_{L_3 M_3}^3 B_{L_3}^3 Y_{L_3}^{M_3}(\Omega_3) + C_{L_1 M_1 L_2 M_2}^{12} B_{L_1}^1 B_{L_2}^2 Y_{L_1}^{M_1}(\Omega_1) Y_{L_2}^{M_2}(\Omega_2) \\ &\quad + C_{L_1 M_1 L_3 M_3}^{13} B_{L_1}^1 B_{L_3}^3 Y_{L_1}^{M_1}(\Omega_1) Y_{L_3}^{M_3}(\Omega_3) + C_{L_2 M_2 L_3 M_3}^{23} B_{L_2}^2 B_{L_3}^3 Y_{L_2}^{M_2}(\Omega_2) Y_{L_3}^{M_3}(\Omega_3) \\ &\quad + C_{L_1 M_1 L_2 M_2 L_3 M_3}^{123} B_{L_1}^1 B_{L_2}^2 B_{L_3}^3 Y_{L_1}^{M_1}(\Omega_1) Y_{L_2}^{M_2}(\Omega_2) Y_{L_3}^{M_3}(\Omega_3)], \end{aligned} \quad (\text{A17})$$

where $L_{1,2} = 1$, $-L_{1,2} \leq M_{1,2} \leq L_{1,2}$, $L_3 = 1, 2$, and $-L_3 \leq M_3 \leq L_3$, and $B_1^{1,2} = \frac{2\sqrt{\pi}}{3}\beta_{1,2}$, $B_1^3 = -\sqrt{2\pi}\eta_3$, and $B_2^3 = \sqrt{2\pi/5}(1 - 3\delta_3)$. The A and C parameters can be computed from the differential distribution using the orthogonality of the spherical harmonics,

$$\int \frac{1}{\sigma} \frac{d\sigma}{d\Omega_1 d\Omega_2 d\Omega_3} Y_{L_i}^{M_i}(\Omega_i)^* d\Omega_1 d\Omega_2 d\Omega_3 = \frac{B_{L_i}^i}{4\pi} A_{L_i M_i}^i, \quad i = 1, 2, 3, \quad (\text{A18})$$

$$\int \frac{1}{\sigma} \frac{d\sigma}{d\Omega_1 d\Omega_2 d\Omega_3} Y_{L_i}^{M_i}(\Omega_i)^* Y_{L_j}^{M_j}(\Omega_j)^* d\Omega_1 d\Omega_2 d\Omega_3 = \frac{B_{L_i}^i}{4\pi} \frac{B_{L_j}^j}{4\pi} C_{L_i M_i L_j M_j}^{ij}, \quad i \neq j, \quad (\text{A19})$$

$$\int \frac{1}{\sigma} \frac{d\sigma}{d\Omega_1 d\Omega_2 d\Omega_3} Y_{L_1}^{M_1}(\Omega_1)^* Y_{L_2}^{M_2}(\Omega_2)^* Y_{L_3}^{M_3}(\Omega_3)^* d\Omega_1 d\Omega_2 d\Omega_3 = \frac{B_{L_1}^1}{4\pi} \frac{B_{L_2}^2}{4\pi} \frac{B_{L_3}^3}{4\pi} C_{L_1 M_1 L_2 M_2 L_3 M_3}^{123}. \quad (\text{A20})$$

APPENDIX B: ENTANGLEMENT FROM POLARIZED BEAM COLLISIONS

In the main text, we evaluated the partially inclusive entanglement maps in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane for unpolarized beams (see Figs. 10 and 11), and we claimed in Sec. VA and Sec. VB that the one-to-one and one-to-other negativities depend only weakly on the choice of beam polarization. For completeness, we collect in this appendix the corresponding maps for the two polarized configurations used in Sec. V,

$$\text{Pol} + : (\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (+0.8, -0.3), \quad \text{Pol} - : (\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (-0.8, +0.3), \quad (\text{B1})$$

to back up this claim. All plots are evaluated on the partially inclusive fictitious state $\rho_\Sigma(m_{t\bar{t}}, \cos\theta_Z)$ in the helicity basis at $\sqrt{s} = 1$ TeV.

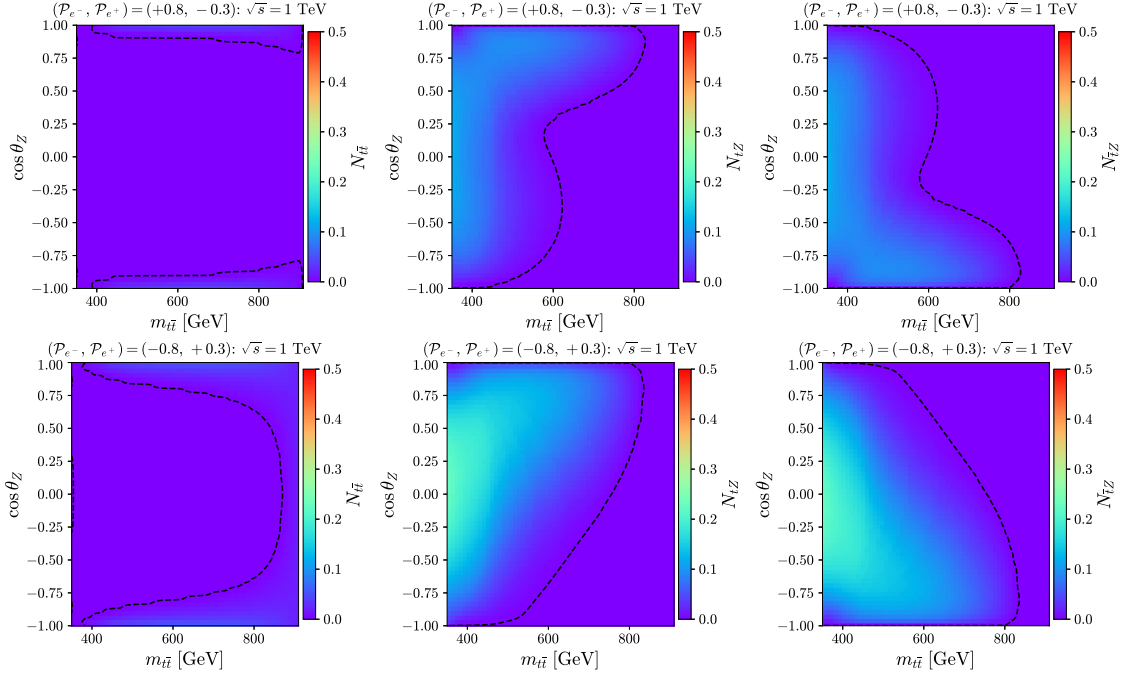


FIG. 15. One-to-one negativities of the partially inclusive $t\bar{t}Z$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV for polarized beams, in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane in the helicity basis: $N_{t\bar{t}}$ (left), N_{tZ} (middle), and $N_{t\bar{t}Z}$ (right). Upper row: Pol+ with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (+0.8, -0.3)$. Lower row: Pol- with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (-0.8, +0.3)$. The dashed black contours mark the threshold $N = 10^{-3}$, below which each measure is treated as effectively vanishing. The color bar saturates at the algebraic upper bound $\frac{1}{2}$.

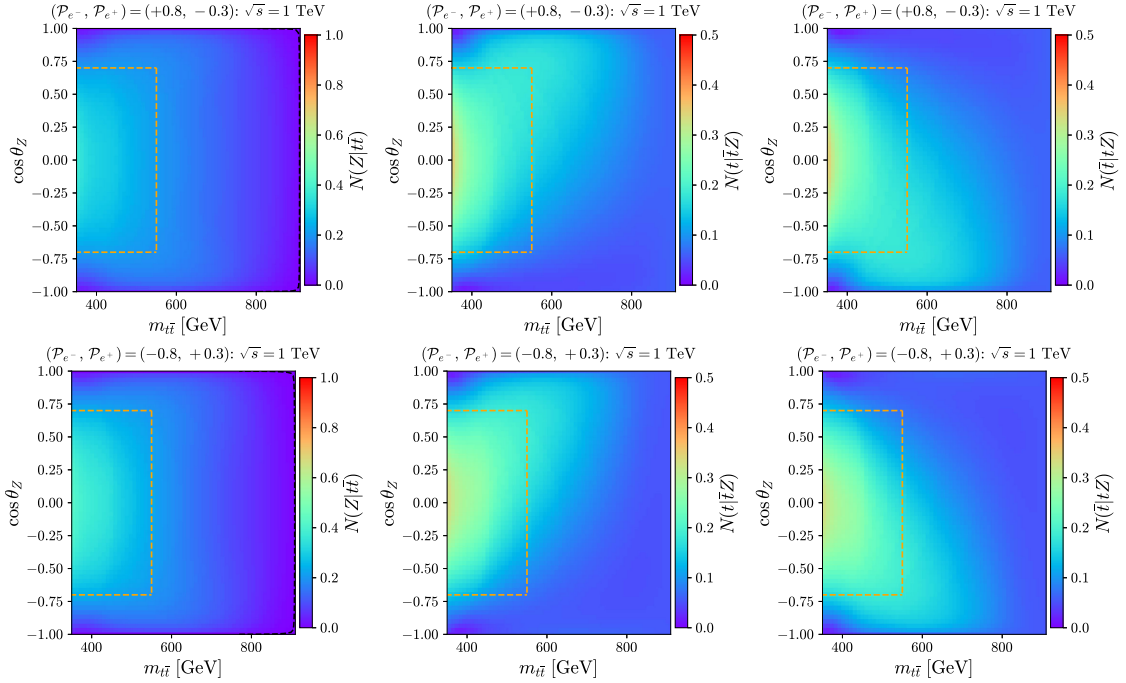


FIG. 16. One-to-other negativities of the partially inclusive $t\bar{t}Z$ spin state ρ_Σ at $\sqrt{s} = 1$ TeV for polarized beams, in the $(m_{t\bar{t}}, \cos\theta_Z)$ plane in the helicity basis: $N(Z|t\bar{t})$ (left), $N(t|\bar{t}Z)$ (middle), and $N(\bar{t}|tZ)$ (right). Upper row: Pol+ with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (+0.8, -0.3)$. Lower row: Pol- with $(\mathcal{P}_{e^-}, \mathcal{P}_{e^+}) = (-0.8, +0.3)$. The color bar saturates at the algebraic upper bound, 1 for $N(Z|t\bar{t})$ and $\frac{1}{2}$ for $N(t|\bar{t}Z)$ and $N(\bar{t}|tZ)$.

The one-to-one negativities $N_{t\bar{t}}$, N_{tZ} , and $N_{\bar{t}Z}$ are shown in Fig. 15, and the one-to-other negativities $N(Z|t\bar{t})$, $N(t|\bar{t}Z)$, and $N(\bar{t}|tZ)$ in Fig. 16. In each figure the upper row corresponds to Pol+ and the lower row to Pol−. A direct comparison with the unpolarized maps in Figs. 10 and 11 shows that the position, shape, and peak heights of the entanglement-positive regions are essentially

unchanged: the polarization acts mainly on the production rate, not on the spin-correlation structure of the partially inclusive state. This justifies the choice of fiducial regions $\Sigma_{1|2}$ and Σ_{GME} from the unpolarized maps in Sec. VA, and supports the conclusion in Sec. VB that the polarization enters the feasibility study almost exclusively through the fiducial cross section.

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