

Matrix model proposal for quantum gravity and the quantum mechanics of black holes

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We propose a quantum mechanical theory of quantum spaces described by large N noncommutative geometry as a model for quantum gravity. The model admits a fuzzy sphere as a static solution. Over the fuzzy geometry, the quantum mechanics of the fermions is given by a sum of oscillators with equal frequency. The energy state where exactly half of the Fermi sea is filled contains the maximal amount of degeneracy. This state of the fuzzy sphere obeys the mass-radius relation of a Schwarzschild black hole if the fuzzy sphere is identified with the black hole horizon. Moreover, the set of states in the Fermi sea gives precisely the Bekenstein-Hawking entropy. We thus propose that quantum black holes are described by fuzzy spheres with a half-filled Fermi sea in our model. We also consider a system of two fuzzy spheres by embedding them as blocks in the matrix quantum mechanics. When the distance r between the two fuzzy spheres is small, the total energy of the system can be computed using perturbation theory. We show that, in the leading order of large N limit, the interaction energy depends on $-GM_1M_2$ exactly in the manner as in Newton gravity. To reproduce the correct r dependence in the long range, we expect the inclusion of large N corrections and quantum effects will be needed.

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I. INTRODUCTION

A black hole arises as a solution of the classical general relativity. However, a number of properties of a black hole, such as the existence of an area dependent Bekenstein-Hawking entropy [1,2], the black hole information puzzle [3–5], the existence of a black hole singularity [6], the exhibition of holography [7,8], etc., are puzzling and it is believed that only in a theory of quantum gravity can these problems be properly understood and resolved.

A prime candidate to a theory of quantum gravity is string theory. Previously, brane models of a black hole have been considered in nonperturbative string theory, with the Bekenstein-Hawking entropy reproduced successfully from a microstates counting [9,10]. Recently, progress has been achieved for black holes constructed in the AdS/CFT correspondence [11], where the Page curve behavior of the entanglement entropy of the Hawking radiation is remarkably obtained [12–15]. Nevertheless, despite these successes, it is highly desirable to have a direct formulation of the quantum gravity itself and be able

to describe the fundamental degrees of freedom of quantized spacetime and the set of microstates explicitly without resorting to supersymmetry and duality. In this regards, large N matrix model formulation of string theory, such as the Banks-Fischler-Shenker-Susskind (BFSS) matrix model [16] or the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model [17], is particularly appealing since in the matrix model, space/time emerges from the more fundamental quantum mechanical matrix degrees of freedom in the large N limit, and one can study the quantum properties of spacetime using the matrix quantum mechanics.

The main stream approach to a quantum black hole has been a top-down one, where the properties of the black hole are studied using a certain candidate theory of quantum gravity such as those mentioned above. However, so far it has not been possible to study a Schwarzschild black hole this way such that the expected properties of a quantum black hole can be obtained and addressed. In a recent series of studies [18–20], we have initiated a bottom-up approach to a quantum black hole by using the anticipated properties of the quantum black hole as “empirical input” to guide the construction of the fundamental theory of quantum gravity. We took the assumption that quantum gravity can be formulated in terms of a quantum mechanics of fermionic and bosonic degrees of freedom, and a generic quantum mechanical model of black hole was proposed. The degrees

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of freedom of the quantum mechanics is supposed to take some generic form,

$$L = i\psi^\dagger \dot{\psi} + \psi^\dagger h(X)\psi - V(X), \quad (1.1)$$

where $h(X)$ denotes some Yukawa coupling and $V(X)$ the self-interaction. Our goal was to identify basic properties of quantum gravity which are essential to the construction of the theory. Our hope was that the model would be helpful in ways similar to that of the Bohr atomic model to the building of quantum mechanics. We found [19] that if our model admits a constant density of energy eigenstates and if the Fermi sea of the system is filled up to a Fermi energy level that is inversely proportional to the system size, then the Schwarzschild radius of black hole is reproduced for the system. Moreover, the system is in a highly degenerate energy state whose counting of microstates gives precisely the Bekenstein-Hawking entropy. While the result is quite encouraging, the identified conditions are sufficient ones, and it is not clear if all of the assumptions can indeed be implemented consistently in a quantum mechanics. The construction of a consistent quantum mechanical model of spacetime which includes a quantum black hole is the main motivation of this work.

In this work, we propose an explicit large N quantum mechanics of non-Abelian bosonic and fermionic variables as a model of quantum gravity. The use of large N is inspired by the previous success of the BFSS and IKKT matrix models; see, for example, [21,22] for review. There are three bosonic variables X^a in the adjoint of $SU(N)$ which correspond to the three spatial coordinates of a quantized noncommutative space. We note that the emergence of noncommutative geometry is generally expected when space (or spacetime) is quantized in quantum gravity.¹ In addition, there are two fermionic variables which couple minimally to the geometry, and make up the fundamental Hilbert space of quantum gravity. We show that our model of quantized spacetime admits a fuzzy sphere as a static solution. Over the fuzzy geometry, the quantum mechanics of the fermions gives a Fermi sea with uniform energy levels in the large N limit. It is amazing that the energy state where exactly half of the Fermi sea is filled contains the maximal amount of degeneracy and satisfies the mass-radius relation of a Schwarzschild black hole if the fuzzy sphere is identified with the black hole horizon. Moreover, the set of states in the half-filled Fermi sea gives precisely the Bekenstein-Hawking entropy. We therefore propose that quantum black holes are described by fuzzy spheres with a half-filled Fermi sea in our model. Quite amazingly, the quantum mechanics proposed here realizes pretty much all of the ideas outlined earlier in [18–20].

¹The study of quantized spacetime in terms of noncommutative geometry goes back to the study of Snyder [23] and Yang [24].

To show that the fuzzy spheres are gravitating objects, one needs to show that the interaction between them agrees with gravity, e.g., with Newton gravity in the long distance limit. We thus consider a system of two fuzzy spheres by embedding them as blocks in the matrix quantum mechanics. When the distance r between the two fuzzy spheres is small, the total energy of the system can be computed using perturbation theory. We show that, in the leading order of the large N limit, the interaction energy depends on the product of black hole masses and the Newton constant exactly as is expected in Newton gravity. It is possible that the inclusion of large N corrections as well as quantum loop effects could reproduce the correct Newton limit at a large distance. We leave this important problem for further study.

The plan of the paper is as follows. In Sec. II, we give our proposal of a large N quantum mechanics as a fundamental formulation of quantum gravity in three dimensions. In Sec. III, we show that the fuzzy sphere solution with a half-filled Fermi sea has the desired properties of a quantum black hole, namely with the black hole mass-radius relation and the Bekenstein-Hawking black hole entropy reproduced correctly. In Sec. IV, we discuss the mode stability of our fuzzy sphere solution and show that it is stable with respect to mass preserving perturbations. In Sec. V, we consider a system of two fuzzy spheres and derive their interaction energy in the small separation limit. We discuss how the inclusion of large N corrections as well as quantum loop effects could generate the desired gravitational forces between the quantum black holes. Further discussion is found in Sec. VI.

II. A PROPOSED MODEL OF QUANTUM GRAVITY

Let us consider a model of quantized space where the spatial coordinates become operators X^a ($a = 1, 2, 3$) and are represented by $SU(N)$ matrices. Here N is large and characterizes the dimensions of the Hilbert space of quantum gravity. In addition, we propose to include fermionic degrees of freedom in the fundamental formulation of quantum gravity. In our model proposed below, they are given by the two-component spinors $\psi^\dagger, \psi$ in the adjoint representation of $SU(N)$. Our choice of degrees of freedom is a minimalistic one and is allowed since we do not assume supersymmetry. In fact, it is important to realize that going bottom-up, there is no physical reason to insist on having supersymmetry. We remark that in the usual top-down approach, supersymmetry is often needed for the reason of consistency of formulation, e.g., for string/M theory in 10 or 11 dimensions. However, supersymmetry is irrelevant for the well-definedness and consistency of quantum mechanics as there is no issue of renormalizability here. Also, the usual advantages of supersymmetric quantum mechanics, such as exact computation of quantities such as index or partition function, does not appeal to us since here we are after the construction of a physical theory

rather than a theory that is exact or calculable. Therefore we will not insist on having supersymmetry. This allows for a much wider choice of terms in the Lagrangian which are forbidden otherwise.

We propose to consider the following quantum mechanical model of spacetime, with the Lagrangian:

$$L = \text{tr} \left[\frac{1}{2M_0} \dot{X}^{a2} + \frac{M_P}{N^2} ([X^a, X^b]^2 + 4X^{a2}) + i\psi^\dagger \dot{\psi} - a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi \right] - a_3 r_X M_P \quad (2.1)$$

and the Hamiltonian

$$H = \text{tr} \left[\frac{M_0}{2} P^{a2} - \frac{M_P}{N^2} ([X^a, X^b]^2 + 4X^{a2}) + a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi \right] + a_3 r_X M_P, \quad (2.2)$$

where $P^a = \dot{X}^a/M_0$ is the conjugate momentum for the bosonic degrees of freedom. The matrices $X^a, \psi, \psi^\dagger$ are dimensionless $N \times N$ traceless Hermitian matrices, meaning that the model (2.1) is supposed to describe the dynamics with respect to the center of mass frame. σ^a are the Pauli matrices. We note that, unlike the IKKT or BFSS matrix model where the maximal supersymmetry fixes the field content and the action uniquely, our model is nonsupersymmetric and there is a negative mass term. In (2.1), the quartic term is fixed to have a coefficient 4 times that of the mass term. This can always be achieved with a rescaling of the variables X^a . Here $M_0 = a_0^2 M_P$ and the Lagrangian is specified by a mass scale M_P (Planck mass) that sits in front of the bosonic potential, whose relation with the Newton constant G will be specified below. The field ψ is normalized so that the fermionic kinetic term has a unit coefficient. We have adopted a definite N^2 dependence in the bosonic potential and the Yukawa coupling term. The bosonic kinetic term and the Yukawa term are then specified up to a choice of the coefficients a_0 and a_2 . In addition, we have included a topological action term specified by a_3 . Here r_X is the rank of the matrix $\Gamma := [X^a, X^b]^2$. The r_X term does not affect the equation of motion, but measures the energy of space due to noncommutativity. It is $r_X = 0$ for Abelian configurations (including the Minkowski vacuum $X^a = 0$) and $r_X = N$ for the fuzzy sphere solution. We note that the coefficients a_0, a_2, a_3 are dimensionless. They are not fixed by symmetry argument, but will be determined below “phenomenologically” by requiring the properties of the fuzzy sphere solutions match up with that expected of the quantum Schwarzschild black holes. In the large N limit, our model is equivalent to some nonsupersymmetric version of the IKKT-like instantonic model in four dimensions by a large N reduction [25].

The Lagrangian (2.1) has an $SO(3)$ rotational invariance where X^i transforms as a vector and ψ in the spinor representation. The theory is also invariant under the global $SU(N)$ transformation,

$$\begin{aligned} X^a &\rightarrow UX^a U^\dagger, & \psi &\rightarrow U\psi U^\dagger, \\ \psi^\dagger &\rightarrow U\psi^\dagger U^\dagger, & UU^\dagger &= 1. \end{aligned} \quad (2.3)$$

We remark that in the SUSY BFSS matrix model, the $SU(N)$ symmetry is usually gauged in order to close the SUSY algebra. In [26], it was shown that the global BFSS matrix quantum mechanics, although it contains extra nonsinglet states that do not come in supersymmetry multiplets, is nevertheless also consistent. In our case, it is also possible to gauge the $SU(N)$ and impose a singlet Gauss law constraint. We do not consider this possibility here but simply noting that the fuzzy sphere solution is also a solution of the gauged model and our analysis below remains the same. We remark that our proposal is similar in philosophy to that of the M(atrix) theory [21]. However, we do not assume supersymmetry, and so we can write down our model directly for three space dimensions, and we can include a mass term. We note that a mass term has been considered in the IKKT matrix model where interesting classical solutions such as fuzzy spacetime [27] and expanding universe [28–31] have been obtained. However, the possible connection of a nonsupersymmetric large N quantum mechanics with black hole is new.

III. FUZZY SPHERE SOLUTION WITH FERMI SEA

The classical equation of motion for a bosonic matrix configuration is given by

$$-\frac{1}{M_0} \ddot{X}^a + \frac{4M_P}{N^2} ([X^b, [X^a, X^b]] + 2X^a) = 0, \quad \psi = 0. \quad (3.1)$$

For a static configuration, this becomes

$$[[X^a, X^b], X^b] = 2X^a. \quad (3.2)$$

This can be solved by the spin $j = (N-1)/2$ representation of $SU(2)$, which are given by $2j+1 = N$ dimensional matrices satisfying

$$[X^a, X^b] = i\epsilon_{abc} X^c. \quad (3.3)$$

Because of the Casimir relation

$$\sum_a X^{a2} = \frac{N^2 - 1}{4} \mathbf{1}, \quad (3.4)$$

the configurations (3.3) and (3.4) define a fuzzy sphere. Since $r_X = N$, the classical fuzzy sphere solution carries an

energy

$$E_B = (2a_3 - 1) \frac{M_P N}{2} \quad (3.5)$$

in the leading order limit of large N .

Next, let us analyze the Yukawa coupling term,

$$H_F = a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi. \quad (3.6)$$

Over the fuzzy sphere geometry, the matrix $K := \sigma^a X^a$ satisfies $K^2 + K - \frac{N^2-1}{4} \mathbf{1} = 0$ and so it has eigenvalues $(N-1)/2$ or $-(N+1)/2$. Since it is traceless, therefore $N+1$ of the eigenvalues are positive and $N-1$ of them are the negative ones. Let us from now on consider the leading large N limit. In the leading order of large N limit, K has the eigendecomposition

$$K_{(m\alpha)(n\beta)} = \frac{N}{2} \sum_{p=1}^N \left(\mathcal{U}_{m\alpha}^p \mathcal{U}_{n\beta}^{p\dagger} - \mathcal{V}_{m\alpha}^p \mathcal{V}_{n\beta}^{p\dagger} \right). \quad (3.7)$$

Here $\mathcal{U}_{n\beta}^p, \mathcal{V}_{n\beta}^p, p = 1, \dots, N$ are eigenvectors of K with positive and negative eigenvalue. Introduce the fermionic oscillators

$$\xi_k^p := \mathcal{U}_{n\beta}^{p\dagger} \psi_{nk\beta}, \quad \chi_k^{p\dagger} := \mathcal{V}_{n\beta}^{p\dagger} \psi_{nk\beta} \quad (3.8)$$

and H_F reads

$$H_F = \frac{a_2 M_P}{2N} \sum_{p,k=1}^N \left(\xi_k^{p\dagger} \xi_k^p - \chi_k^p \chi_k^{p\dagger} \right). \quad (3.9)$$

Here the Grassmanian variables ξ_k^p and χ_k^p resemble the oscillators b and d of the Dirac theory of electrons.

The theory (2.2) can be canonically quantized. For the bosonic variables, we impose the commutation relation,

$$[X_{mk}^a, P_{nl}^b] = i\delta^{ab} \delta_{mn} \delta_{kl}. \quad (3.10)$$

As for the fermionic variables, the conjugate momentum $\pi = i\psi^\dagger$ give rises to the anticommutation relation

$$\{\psi_{m\alpha}^\dagger, \psi_{nl}\} = \delta_{mn} \delta_{kl} \delta_{\alpha\beta}. \quad (3.11)$$

This gives equivalently

$$\{\xi_k^p, \xi_l^{q\dagger}\} = \delta^{pq} \delta_{kl}, \quad \{\chi_k^p, \chi_l^{q\dagger}\} = \delta^{pq} \delta_{kl}. \quad (3.12)$$

The quantized fermion Hamiltonian can be obtained from (3.9) with the subtraction of a constant,

$$H_F = \frac{a_2 M_P}{2N} \sum_{p,k=1}^N \left(\xi_k^{p\dagger} \xi_k^p + \chi_k^{p\dagger} \chi_k^p \right), \quad (3.13)$$

such that there is no zero point energy for the oscillators. This may also be obtained with a prescription of normal ordering defined on the oscillators ξ_k^p and χ_k^p , $:\chi_k^p \chi_k^{p\dagger} := -\chi_k^{p\dagger} \chi_k^p$, etc. The quantized Hamiltonian H_F has the eigenstates

$$\left| \Psi_{k_1 \dots k_r, l_1 \dots l_s}^{p_1 \dots p_r, q_1 \dots q_s} \right\rangle := \xi_{k_1}^{p_1\dagger} \dots \xi_{k_r}^{p_r\dagger} \chi_{l_1}^{q_1\dagger} \dots \chi_{l_s}^{q_s\dagger} |0\rangle, \quad (3.14)$$

where $|0\rangle$ is the fermionic Fock vacuum defined by

$$\xi_k^p |0\rangle = \chi_l^q |0\rangle = 0, \quad \forall p, q, k, l \quad (3.15)$$

and the eigenvalues

$$E_F = \frac{a_2 M_P}{2N} (N^2 + n), \quad n := r + s - N^2, \quad (3.16)$$

where $n = -N^2, \dots, N^2$ specifies the energy level within the Fermi sea. The lowest level $n = -N^2$ corresponds to an empty Fermi sea, while the highest level $n = N^2$ corresponds to a completely filled Fermi sea. We remark that one may label the oscillators with the collective indices $a := (p, k)$ and the fermionic Hamiltonian can be written as

$$H_F = \frac{a_2 M_P}{2N} \sum_{a=1}^{N^2} \xi_a^\dagger \xi_a + \chi_a^\dagger \chi_a. \quad (3.17)$$

Physically, (3.17) may be given an interpretation that the fuzzy sphere has been divided into N^2 cells with unit cell area of $\Delta A = 4\pi l_p^2$ of the Planck size. Each cell, whose location is labeled by $a = (p, k)$, is populated by a pair of oscillators ξ_a, χ_a which describes the quantum fluctuations over the fuzzy sphere. This is strikingly similar to the description of “partons” in the holographic picture of black hole suggested by Susskind [8]. It is quite amusing that this conjectured feature of holography get a concrete realization in our quantum mechanics of spacetime. In the next section, we propose to identify the quantum Schwarzschild black hole with the fuzzy sphere solution with a half-filled Fermi sea $n = 0$ in our model.

Before we move on, it is necessary to discuss the stability of the fuzzy sphere solution. In general relativity, it is known that the Schwarzschild metric and the Kerr metric is stable against mode perturbations that do not change the mass M and angular momentum J of these vacuum solutions. This statement of mode stability was originally proven [32,33] for perturbations that can be written as a superposition of spherical harmonic modes. Recently, it has been shown that [34] the Schwarzschild metric is stable against general linear perturbations that does not

necessarily need to be written as a finite sum of spherical harmonic modes. However, the linear stability of the Kerr metric remains an open problem. The full nonlinear stability of a vacuum solution in general relativity is a hard problem and has only been established for the Minkowski metric [35]. For us here, we will be satisfied with a mode analysis and show that our fuzzy sphere solution is stable against mode perturbations that do not change the energy of the solution.

For linearized analysis, we look at the quadratic potential of the fluctuations δX^a . Over a general classical solution X^a , we have

$$U = \frac{M_P}{N^2} V, \quad V = -\text{tr}(F_{ab}^2 + 2[X^a, X^b][\delta X^a, \delta X^b] + 4\delta X^a \delta X^a), \quad (3.18)$$

where

$$F_{ab} := [X^a, \delta X^b] - [X^b, \delta X^a] \quad (3.19)$$

is the field strength. The diagonalization of V is standard. It is convenient to introduce the derivative operator L^a whose action on a matrix f is given by $L^a f = [X^a, f]$; in terms of which we have

$$F_{ab} = -i\epsilon_{abc}(\epsilon \cdot L)_{cd}\delta X^d, \quad (3.20)$$

where ϵ_a is a vector matrices with components $(\epsilon_a)_{bc} := -i\epsilon_{abc}$. So far, this is general. For X^a given by the fuzzy sphere (3.3), (3.4), the second term in V of (3.18) can also be expressed in terms of F_{ab} and the potential takes the compact form

$$V = 2\text{tr}(\delta X^a N_{ab} \delta X^b), \quad \text{where} \quad N_{ab} := ((\epsilon \cdot L)^2 - (\epsilon \cdot L) - 2)_{ab}. \quad (3.21)$$

As a result, the equation of motion for the linearized fluctuation is given by

$$\delta \ddot{X}^a + \frac{4a_0^2 M_P^2}{N^2} N_{ab} \delta X^b = 0, \quad (3.22)$$

and the study of the mode stability of the fluctuations reduces to the eigenvalue problem of N_{ab} :

$$N_{ab} \phi^b = \lambda \phi^a. \quad (3.23)$$

The operator N_{ab} is diagonalized by the vector harmonics. To construct them, note that ϵ_a is an angular momentum of angular momentum quantum number $\ell_\epsilon = 1$ since

$$[\epsilon_a, \epsilon_b] = i\epsilon_{abc}\epsilon_c, \quad \epsilon_a^2 = 2. \quad (3.24)$$

An eigenbasis is given by $|\ell_\epsilon, m_\epsilon\rangle$, $m_\epsilon = -1, 0, +1$:

$$\epsilon^2 |\ell_\epsilon, m_\epsilon\rangle = 2|\ell_\epsilon, m_\epsilon\rangle, \quad \epsilon_z |\ell_\epsilon, m_\epsilon\rangle = m_\epsilon |\ell_\epsilon, m_\epsilon\rangle. \quad (3.25)$$

Note also that, for the fuzzy sphere background, L^a is an angular momentum operator. An eigenbasis is given by $|\ell, m_z\rangle$,

$$L^2 |\ell, m_z\rangle = \ell(\ell+1) |\ell, m_z\rangle, \quad L_z |\ell, m_z\rangle = m_z |\ell, m_z\rangle \quad (3.26)$$

for $m_z = -\ell, \dots, \ell$ and $0 \leq \ell \leq N-1$. Here the angular momentum quantum number ℓ is cut off by N due to the noncommutativity of the fuzzy sphere. The spherical harmonics $\hat{Y}_{m_z}^\ell$ is now obtained in the matrix representation as $(\hat{Y}_{m_z}^\ell)_{n_1 n_2} = \langle n_1 n_2 | \ell, m_z \rangle$. The eigenvalue problem of the operator $\epsilon \cdot L$ can now be easily solved with the help of the total angular momentum operator $J^a := \epsilon^a + L^a$. (J^2, J_z) is diagonalized by

$$J^2 |J, M_J\rangle = J(J+1) |J, M_J\rangle, \quad J_z |J, M_J\rangle = m_J |J, M_J\rangle, \quad (3.27)$$

and

$$J = 1 \quad \text{for } \ell = 0 \quad \text{and} \quad J = \ell - 1, \ell, \ell + 1 \quad \text{for } \ell \geq 1. \quad (3.28)$$

As a result, $\epsilon \cdot L = \frac{1}{2}(J(J+1) - \ell(\ell+1) - 2)$ and λ has the eigenvalues,

$$\epsilon \cdot L = \begin{cases} 0 \\ -(\ell+1) \\ -1 \\ \ell \end{cases}, \quad \lambda = \begin{cases} -2 & \text{for } \ell = 0, J = 1 \\ \ell(\ell+3) & \text{for } \ell \geq 1, J = \ell - 1 \\ 0 & \text{for } \ell \geq 1, J = \ell \\ (\ell+1)(\ell-2) & \text{for } \ell \geq 1, J = \ell + 1. \end{cases} \quad (3.29)$$

For given ℓ and J as given by (3.28), the corresponding eigenstate can be constructed by using the Clebsch–Gordan coefficient $C_{\ell m_z 1 p}^{J M_J}$,

$$|J, M_J\rangle = \sum_{m_z=-\ell}^{\ell} \sum_{p=0, \pm 1} C_{\ell m_z 1 p}^{J M_J} |\ell m_z\rangle \otimes |1 p\rangle, \quad (3.30)$$

where here we have used $p = 0, \pm 1$ to denote the polarization. The eigenfunction ϕ^a of (3.23) can then be constructed with the matrix elements,

$$(\phi^a)_{n_1 n_2} = (\langle n_1 n_2 | \otimes \langle a |) | J M_J \rangle, \quad (3.31)$$

where $|n_1 n_2\rangle$ are the basis states of the matrix space and $|a\rangle$ are the basis states of the three-dimensional vector space. As a result, we obtain the eigenvector ϕ^a as given by the vector spherical harmonics,

$$\phi^a = \sum_{m_z=-\ell}^{\ell} \sum_{p=0,\pm 1} C_{\ell m_z 1 p}^{J M_J} e_a^p \hat{Y}_{m_z}^{\ell} := (\hat{Y}_{\ell}^{J M_J})_a, \quad (3.32)$$

where $e_a^p := \langle a | 1 p \rangle$ is the a th component of the polarization vector e^p . Note that the eigenvalue (3.29) is independent of M_J and so each of the eigenvalues in (3.29) has a degeneracy of $2J + 1$. In total, we have $3N^2$ eigenmodes in (3.29). Not all of these modes are admissible fluctuations, however.

The perturbation is unstable if the eigenvalue is negative, i.e., for the modes $\ell = 0, J = 1$ and $\ell = 1, J = 2$. The mode $\ell = 1, J = 2$ corresponds to a scaling $\delta X^a \propto X^a$ of the fuzzy sphere. This changes the energy of the fuzzy sphere. The mode $\ell = 0, J = 1$ corresponds to a shift of an overall $U(1)$ factor, which is not allowed since X^a is traceless. Fluctuations represented by the other modes, energy preserving or not, are stable. Therefore we find that the single fuzzy sphere solution is stable against linearized energy preserving perturbations.

Finally, we note that upon quantization, the quadratic fluctuation modes become oscillators with frequency determined by λ in (3.29). We adopt a definition of the quantized bosonic Hamiltonian by subtracting a constant such that the fuzzy sphere has vanishing zero point energy. In other words, the ground state of the single fuzzy sphere has a vanishing zero point energy for both the bosonic as well as the fermionic oscillations.

IV. QUANTUM BLACK HOLE AS FUZZY SPHERE WITH A HALF-FILLED FERMI SEA

Consider the fuzzy sphere solution with a half-filled Fermi sea ($n = 0$). The total energy of the fuzzy sphere system is given by

$$E = \frac{\gamma N M_P}{2}, \quad \text{where } \gamma := a_2 + 2a_3 - 1. \quad (4.1)$$

To compare with the energy of the Schwarzschild black hole, let us introduce dimensional coordinates $Y^a = 2l_P X^a$ for some length scale l_P ; the fuzzy sphere solution becomes

$$[Y^a, Y^b] = \frac{2iR}{\sqrt{N^2 - 1}} \epsilon_{abc} Y^c, \quad \sum_a Y^{a2} = R^2 \mathbf{1}, \quad (4.2)$$

where $R^2 = (N^2 - 1)l_P^2$. This describes a fuzzy sphere of radius $R = Nl_P$ in the large N limit. The energy (4.1) can then be put in the form

$$E = \frac{R}{2G} \quad (4.3)$$

if we identify M_P and l_P with the Newton constant G as

$$G = \frac{l_P}{\gamma M_P}. \quad (4.4)$$

The relation (4.3) is precisely the Schwarzschild mass-radius relation if the fuzzy sphere is identified with the black hole horizon and the total energy of the system is identified with the mass M of the black hole,

$$E = M. \quad (4.5)$$

We note that the relation (4.5) holds if we assume the principle of equivalence of internal (nongravitational) energy and gravitational mass. In the next section, we will consider multiple fuzzy spheres configuration in our model and show that the interaction energy depends of the product $GM_1 M_2$ precisely as in Newton's gravity if the gravitational mass is given by (4.5). This gives a simple explanation of the equivalence principle from the quantum mechanics.

Next let us consider the microstates counting. The level n eigenvalue has a degeneracy of

$$\Omega_n = \binom{2N^2}{N^2 + n}, \quad (4.6)$$

which corresponds to the ways to fill exactly $N^2 + n$ of the oscillator levels. For the $n = 0$ energy state, we have

$$\Omega_0 = 2^{2N^2} \quad (4.7)$$

in the leading order of large N . These microstates of the system at the energy (4.3) give rise to the entropy $S = \log_2 \Omega_0$:

$$S = 2N^2. \quad (4.8)$$

This is precisely the Bekenstein-Hawking entropy of a Schwarzschild black hole if

$$l_P = \sqrt{\frac{2G}{\pi}}. \quad (4.9)$$

As a result of (4.4), M_P is given by

$$M_P = \frac{1}{\gamma} \sqrt{\frac{2}{\pi G}}. \quad (4.10)$$

It is remarkable that our quantum mechanical theory (2.2) admits a solution that matches precisely the desired properties of a quantum black hole. We thus propose that a

quantum black hole is described in our theory by a fuzzy sphere geometry with a half-filled Fermi sea. We note that the matching works for any coefficients a_0, a_2, a_3 as long as $\gamma > 0$. This arbitrariness of the action can be fixed by considering the rotating Kerr black hole [36]. We note that the state $n = 0$ is singled out by the fact that Ω_n is maximized at $n = 0$. It is amazing that a quantum black hole in our description is characterized to be one, given the mass of the black hole is fixed, having the maximal allowed amount of microstates. In the next section, we provide further justification by showing that the fuzzy spheres interact with each other with a GM_1M_2 dependence that is characteristic of Newton gravity.

Finally, let us comment on the Bekenstein entropy bound [37] which set the maximal amount of entropy that can be contained within a region of space with radius R and energy E :

$$S \leq 2\pi RE. \quad (4.11)$$

Although counterexamples of this bound is known, it has been proven in quantum field theory [38]. Presumably it also holds in a consistent theory of quantum gravity. Let us check it against our theory. Indeed the bound is saturated for the $n = 0$ state of the fuzzy sphere. For positive n , the inequality is satisfied since $S(n) < S(0) = 2\pi RE_0 < 2\pi RE_n$. However, it is easy to see that the bound is violated for negative n . Physically, it means the fuzzy spheres in the $n = 0$ state (black hole) and the positive n more energetic state (presumably represent stars and compact objects) are allowed in the theory, while those states with negative n should be excluded. It is possible that these states becomes unstable in the higher order perturbation theory. It would be interesting to understand this better.

V. MULTIPLE BLACK HOLES

The reproduction of the static properties of a quantum black hole supports the conjecture that the fuzzy sphere solution does describe a quantum black hole and that the theory (2.2) is a creditable proposal. Below we perform a preliminary analysis on the interaction of fuzzy spheres and show that some characteristic features of Newton gravity can be reproduced.

Let us consider a block diagonal configuration of the following form:

$$X^a = \left(\begin{array}{c|c} X_1^a & 0 \\ \hline 0 & X_2^a \end{array} \right), \quad (5.1)$$

where X_i^a is of dimensions $N_i \times N_i$, $i = 1, 2$ and $N = N_1 + N_2$. We consider large N_i such that the ratios $\alpha_i := N_i/N$ are fixed. The matrices X^a are traceless, but X_i^a are generally not. Let us introduce the coordinates,

$$x_i^a := \frac{1}{N_i} \text{tr} X_i^a, \quad (5.2)$$

which satisfies

$$N_1 x_1^a + N_2 x_2^a = 0, \quad (5.3)$$

due to the tracelessness of X^a . We note that, since the mass of the isolated black hole is given by

$$M_i = \frac{\gamma N_i M_P}{2}, \quad (5.4)$$

the relation (5.3) allows x_i^a to be interpreted as the location of the black holes with respect to the center of mass of the system. As a result, X_i^a can be decomposed into its trace and the traceless part X_i^{0a} as

$$X_i^a = x_i^a + X_i^{0a} \quad (5.5)$$

and the Hamiltonian of the theory is given by

$$H = H_1 + H_2 + H_{12}, \quad (5.6)$$

plus kinetic terms for X_i^{0a} and x_i^a . Here

$$H_i := H_i^0 + H_i', \quad i = 1, 2, \quad (5.7)$$

where the term

$$H_i^0 := \alpha_i^2 \text{tr}_i \left[- \left(\frac{M_P}{N_i^2} [X_i^{0a}, X_i^{0b}]^2 + \frac{4M_P}{N_i^2} X_i^{0a2} \right) + \frac{a_2 M_P}{N_i^2} : \psi^\dagger \sigma^a X_i^{0a} \psi : \right] + a_3 r_{X_i} M_P \quad (5.8)$$

is obtained from the non-Abelian part X_i^{0a} , while the fermionic term

$$H_i' = \frac{a_2 M_P}{N^2} \sum_{m,n=1}^{N_i} : \psi_{mn}^\dagger \sigma^a x_i^a \psi_{mn} : \quad (5.9)$$

and the bosonic term

$$H_{12} := - \frac{4M_P}{N^2} (N_1 x_1^2 + N_2 x_2^2) \quad (5.10)$$

are obtained from the Abelian $U(1)$ part x_i^a of the configuration. In the above, the trace tr_i is taken over the $N_i \times N_i$ subspace. Finally, the terms H_i', H_{12} represent interaction between the two blocks and depend on their separation distance (in dimensionless matrix units),

$$\Delta x := \sqrt{(x_1^a - x_2^a)^2}. \quad (5.11)$$

This comes because (5.3) implies that $x_1^a = \alpha_2(x_1^a - x_2^a)$, etc. As a result, the fermionic interaction term H'_1 is proportional to Δx , and the bosonic interaction term is proportional to Δx^2 ,

$$H_{12} = -\frac{4M_P}{N}\alpha_1\alpha_2\Delta x^2. \quad (5.12)$$

In general, solutions to the equation of motion of (5.6) are not static. Nevertheless, one can consider static configuration and use it to derive the static potential between the blocks. More general velocity dependent forces can also be derived by considering time dependent configuration. Therefore let us consider the equation of motion in the static limit. Note that H_1^0 is identical to the Hamiltonian for a single black hole except for a replacement of $M_P \rightarrow \alpha_1^2 M_P$. As a result, we obtain the same equation of motion and X_1^{0a} is solved by fuzzy sphere with radius

$$R_1 = N_1 l_P. \quad (5.13)$$

The Hamiltonian H_1^0 admits the eigenstates

$$|\Psi_{k_1 \dots k_r, q_1 \dots q_s}^{p_1 \dots p_r, q_1 \dots q_s}\rangle := \xi_{k_1}^{p_1 \dagger} \dots \xi_{k_r}^{p_r \dagger} \chi_{l_1}^{q_1 \dagger} \dots \chi_{l_s}^{q_s \dagger} |0\rangle, \quad (5.14)$$

where ξ_k^p, χ_l^p are defined as in (3.8) using the eigendecomposition (3.7) for $\sigma^a X_i^{0a}$ and the indices m, n range over the block of X_i . Consider the set of states where half of the oscillators over the fuzzy sphere are excited (i.e., $r + s = N_1^2$), the fuzzy sphere has the energy

$$H_1^0 = \frac{M_P N_1}{2}(a_2 + 2\alpha_1 a_3 - \alpha_1^2). \quad (5.15)$$

The coarse graining of this ensemble of microstates gives the Bekenstein-Hawking entropy $S_1 = 2N_1^2 = A_1/4G$.

Next, let us consider the interaction term H_1^a . For small $\Delta x/N$, we can consider H_1^a to be a perturbation and use the perturbation theory for degeneracy eigenstates to determine its correction to the energy. This requires the knowledge of the matrix elements $\langle i | H_1^a | j \rangle$ and its eigenvalues λ_p . Here $|i\rangle, i = 1, \dots, \Omega_0 := 2^{2N_1^2}$ are the unperturbed states (5.14) with $r + s = N_1^2$. This is, however, not only impossible to do as it involves too large a number of states. Physically, since the set of microstates is not observed, it is more meaningful to consider the average corrections over the set of microstates:

$$\langle H_1^a \rangle := \sum_{|i\rangle \in \mathbb{V}_0} p_i \langle i | H_1^a | i \rangle, \quad (5.16)$$

where p_i is the probability of occurrence of the state $|i\rangle$. For an isolated system, we can take the probability to be equal for each of the states and, hence, $p_i = 1/\Omega_0$. This corresponds to a microcanonical ensemble. We show in the Appendix that this ensemble average is given by

$$\langle H_1^a \rangle = \frac{a_2 M_P}{N^2} \text{tr}(\sigma^a X_1^{0a} \sigma^b X_1^b) = 0. \quad (5.17)$$

As a result, the total energy of the two fuzzy spheres system is given by

$$E = \text{const} - \frac{4a_2 M_P}{N} \alpha_1 \alpha_2 \Delta x^2, \quad (5.18)$$

where the constant term is independent of Δx^2 . The second term is a correction to the first term in the leading ordering of small $\Delta x^2/N^2$. In general, one should include the higher order perturbation as well as the interaction term from the off diagonal blocks of (5.1) and quantum loop effects [39]. By summing over all of these contributions, one can expect the final result for the energy of the two fuzzy spheres system to take the form

$$E = \text{const} - \gamma N M_P \alpha_1 \alpha_2 f\left(\frac{\Delta x}{N}\right), \quad (5.19)$$

where f is some function of $\Delta x/N$. It takes the form $f(x) = 1 + \frac{4a_2}{\gamma} x^2 + O(x^3)$ for small x in our tree level computation. Define the coordinate distance between the two black holes as $r = \Delta x l_P$, and define the interaction energy as $V(r) := E(r) - E(r = \infty)$. We have $V(r) = -\frac{GM_1 M_2}{r} g(r)$, where $M_i = \gamma N_i M_P / 2$ is the mass of the black hole in isolation, and $g(r) := \frac{4\gamma}{R} f(\frac{r}{R})$. Note that the factor $GM_1 M_2$ appears exactly the way one expects for Newton gravity. In order to reproduce Newton gravity

$$V(r) = -\frac{GM_1 M_2}{r} \quad (5.20)$$

in the large distance limit $r/R = \Delta x/N \gg 1$, the function f has to be such that $g = 1 + O(1/r)$. In order to check whether Newton gravity arises in the large distance limit, it is important to devise a method to reliably compute the potential $V(r)$ between the fuzzy spheres. We note that in our model a classical force arises between the fuzzy sphere due to the mass term. It is also possible to consider a variant of our model by considering a Chern-Simons term instead of a mass term, see, e.g., [40–42] for a reduced matrix model with a Chern-Simons term. The model also admits fuzzy spheres as a solution but in this case there is no classical force between them and the Newton gravity would have to emerge from the loop, similar to the situation in the BFSS matrix model [21, 39]. We leave this important issue for further analysis. More generally, it is important to understand whether and how tensorial Einstein gravity may emerge in the low energy approximation in our model.

VI. DISCUSSION

In this paper we have proposed a model of quantum space and gravity as a large N quantum mechanics of non-Abelian bosonic and fermionic coordinates. The quantum mechanics has a static solution whose bosonic part is given by a fuzzy sphere. Over the fuzzy sphere geometry, the fermionic part of the theory is given by a collection of fermionic oscillators all with the same frequency. The energy state where exactly half of the Fermi sea is filled contains the maximal amount of degeneracy. This half-filled fuzzy sphere observes the mass-radius relation of a Schwarzschild black hole if the fuzzy sphere radius is identified with the horizon size. Moreover, the coarse graining of the set of quantum states in the Fermi sea gives precisely the Bekenstein-Hawking entropy. As a result, we propose that a quantum black hole is described by a half-filled fuzzy sphere in our model.

We have also considered interaction between these fuzzy spheres by including the quantum mechanical perturbation arisen from a separation of them in the matrix space. We find that the interaction energy between the fuzzy spheres has a dependence on the masses and Newton constant exactly as in the Newton gravity. We conjecture that, when the full large N resummation of all of the higher order quantum effects including the off-diagonal block terms is performed, the exact result will capture the correct gravitational interaction between the static quantum black holes, with the Newton gravity reproduced in the large distance limit. The obtained modified gravity may be relevant for the dark matter and dark energy problem.

In this paper we have considered static solutions. It would be interesting to consider a nonstatic solution in order to capture the relativistic effects. Our proposed quantum mechanics is for quantum space. A rotational black hole should be included in our model. It is interesting to understand whether and how matter would arise in our quantum mechanics; in particular, how gauge interaction can be incorporated and how a charged black hole can be obtained. It is also interesting to consider a solution with nontrivial fermionic background so as to arrive at a flat solution.

We note that the change of coordinates $x^\mu \rightarrow x'^\mu(x)$ in general relativity is replaced in our description by the unitary transformation

$$X^i \rightarrow UX^iU^{-1}. \quad (6.1)$$

It is important to understand how the diffeomorphism symmetry of general relativity emerges from the quantum mechanics in some classical limit, and how Einstein gravity emerges and get modified in our theory. Einstein has believed that a geodesic equation should not be taken as an independent assumption but derived from the field equations for empty space. This program can be implemented using our quantum mechanics. For example, one

may consider the motion of a light probe block in the presence of a heavy one ($N_1 \ll N_2$). In some limit, we may ignore the backreaction of the light block on the heavy one and take the heavy block as a background. It is possible that one may encode the effect of the heavy block on the probe in terms of some effective metric and effective energy-momentum tensor, and the motion of the probe in the background metric in terms of some effective geodesic motion.

In the general analysis of Almheiri-Marolf-Polchinski-Sully [43] concerning firewalls and complementarity, it was concluded that the following three statements cannot all be true: (i) Hawking radiation is in a pure state, (ii) the information carried by the radiation is emitted from the region near the horizon, with low energy effective field theory valid beyond some microscopic distance from the horizon, and (iii) the infalling observer encounters nothing unusual at the horizon. Our study of the two fuzzy sphere systems shows that nothing singular occurs as the distance between them decreases to zero. Taking one of them as a probe, it means no firewall and no black hole singularity. On the other hand, since the black hole horizon is described by a noncommutative fuzzy geometry in our model, the existence of a low energy effective theory is questionable due to UV/IR mixing [44] (see also UV/IR mixing in fuzzy sphere [45]) and modified causality and nonlocality of noncommutative field theory [46]. It would be interesting to study these points further. In our description, the fuzzy sphere is dynamical in general and it will be interesting to develop some kind of effective membrane description for the horizon, and to compare it with the classical membrane paradigm [47].

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APPENDIX: ENSEMBLE AVERAGE OF FIRST ORDER PERTURBATION

In this appendix, we use perturbation theory to compute the average correction to the energy of the fuzzy spheres system due to the fermionic term of the Hamiltonian.

In general, consider adjoint fermions ψ_{mn} of $SU(N)$ and the Hamiltonian,

$$h_0 =: \psi^\dagger \sigma^a X^{0a} \psi :, \quad (A1)$$

where X^{0a} is given by a fuzzy sphere. In the large N limit, the matrix $\sigma^a X^{0a}$ admits the eigenvectors $\mathcal{U}^p, \mathcal{V}^p$ that correspond to positive and negative eigenvalues $N/2, -N/2$, respectively. One has the eigendecomposition

$$(\sigma^a X^{0a})_{(ma)(n\beta)} = \frac{N}{2} \sum_{p=1}^N (\mathcal{U}_{ma}^p \mathcal{U}_{n\beta}^{p\dagger} - \mathcal{V}_{ma}^p \mathcal{V}_{n\beta}^{p\dagger}). \quad (\text{A2})$$

The eigenvectors $\mathcal{U}^p, \mathcal{V}^p$ satisfy the orthonormality condition

$$\mathcal{U}_{ma}^{p\dagger} \mathcal{U}_{ma}^q = \mathcal{V}_{ma}^{p\dagger} \mathcal{V}_{ma}^q = \delta^{pq}, \quad \mathcal{U}_{ma}^{p\dagger} \mathcal{V}_{ma}^q = 0 \quad (\text{A3})$$

and the completeness relation

$$\mathcal{U}_{ma}^p \mathcal{U}_{n\beta}^{p\dagger} + \mathcal{V}_{ma}^p \mathcal{V}_{n\beta}^{p\dagger} = \delta_{mn} \delta_{\alpha\beta}. \quad (\text{A4})$$

Introducing the fermionic oscillators,

$$\xi_k^p := \mathcal{U}_{n\beta}^{p\dagger} \psi_{nk\beta}, \quad \chi_k^{p\dagger} := \mathcal{V}_{n\beta}^{p\dagger} \psi_{nk\beta}, \quad (\text{A5})$$

then, the normal ordered h_0 reads

$$h_0 = \frac{N}{2} \sum_{p,k=1}^N (\xi_k^{p\dagger} \xi_k^p + \chi_k^{p\dagger} \chi_k^p). \quad (\text{A6})$$

As a result, h_0 has the eigenstates

$$|\Psi_{k_1 \dots k_r l_1 \dots l_s}^{p_1 \dots p_r q_1 \dots q_s}\rangle := \xi_{k_1}^{p_1\dagger} \dots \xi_{k_r}^{p_r\dagger} \chi_{l_1}^{q_1\dagger} \dots \chi_{l_s}^{q_s\dagger} |0\rangle, \quad (\text{A7})$$

where $|0\rangle$ is the Fock vacuum and the eigenvalues

$$h_0 = n + N^2, \quad (\text{A8})$$

where $n := r + s - N^2 = -N^2, \dots, 0, \dots, N^2$ is the energy level within the Fermi sea. The energy level n has a degeneracy of $\Omega_n = \binom{2N^2}{n+N^2}$. Denoting this set of states by

$$\mathbb{V}_n := \{|i\rangle, \quad i = 1, \dots, \Omega_n\}, \quad (\text{A9})$$

where $|i\rangle$ are those states (A7) with $r + s = n + N^2$. In the main body of the text, we have identified the set Ω_0 of states with the microstates of the Schwarzschild black hole. Now let us consider the Hamiltonian $h = h_0 + h'$ with an operator h' of the form

$$h' := \sum_{m,n=1}^N : \psi_{mn}^\dagger \tilde{h}' \psi_{mn} :. \quad (\text{A10})$$

We are interested in the change of energy of the fuzzy spheres system.

For generality, we will treat h' as a perturbation. Obviously, each of the microstates $|i\rangle \in \mathbb{V}_0$ will receive a different correction. Suppose the individual microstate is not observed, e.g., as in the case of a black hole, then rather than trying to determine the splitting for each individual state, it is more meaningful to compute some kind of average correction over all microstates of the level $n = 0$.

As the system is in isolation, we have a microcanonical ensemble where probability of occurrence is equally likely for all of the microstates, i.e., $p_i = 1/\Omega_0$. As a result, what is observed physically is the ensemble average,

$$\langle h' \rangle := \frac{1}{\Omega_0} \sum_{|i\rangle \in \mathbb{V}_0} \langle i | h' | i \rangle. \quad (\text{A11})$$

In the main text of the paper, we have claimed that $\langle h' \rangle = 0$ for the displaced fuzzy spheres system. To see this, let us use

$$\psi_{mna} = \mathcal{U}_{ma}^p \xi_n^p + \mathcal{V}_{ma}^p \chi_n^{p\dagger} \quad (\text{A12})$$

to rewrite h' as

$$h' = P^{pq} \xi_k^{p\dagger} \xi_k^q - Q^{pq} \chi_k^{p\dagger} \chi_k^q + T^{pq} \xi_k^{p\dagger} \chi_k^{q\dagger} + \text{H.c.}, \quad (\text{A13})$$

where the matrices P^{pq} etc. are given by

$$\begin{aligned} P^{pq} &:= \mathcal{U}_{ma}^{p\dagger} \mathcal{U}_{m\beta}^q \tilde{h}'_{\alpha\beta}, & Q^{pq} &:= \mathcal{V}_{ma}^{p\dagger} \mathcal{V}_{m\beta}^q \tilde{h}'_{\alpha\beta}, \\ T^{pq} &:= \mathcal{U}_{ma}^{p\dagger} \mathcal{V}_{m\beta}^q \tilde{h}'_{\alpha\beta}. \end{aligned} \quad (\text{A14})$$

It is easy to see that the T terms do not contribute to the matrix element. Also, we have the matrix elements

$$\langle i | \xi_k^{p\dagger} \xi_k^q | i \rangle = n_i^\xi \delta^{pq}, \quad \langle i | \chi_k^{p\dagger} \chi_k^q | i \rangle = n_i^\chi \delta^{pq}, \quad (\text{A15})$$

where

$$n_i^\xi = N \binom{N^2 - 1}{r} \binom{N^2}{s}, \quad n_i^\chi = N \binom{N^2 - 1}{s} \binom{N^2}{r} \quad (\text{A16})$$

for a state $|i\rangle$ of the form (A7). Therefore

$$\langle i | h' | i \rangle = n_i^\xi \text{tr} P - n_i^\chi \text{tr} Q. \quad (\text{A17})$$

Using $\sum_r \binom{N^2-1}{r} \binom{N^2}{N^2-r} = \Omega_0/2$, the sum over all states of form (A7) with a fixed level $n = 0$ is given by

$$\sum_{|i\rangle \in \mathbb{V}_0} \langle i | h' | i \rangle = \frac{N}{2} \Omega_0 \text{tr}(P - Q). \quad (\text{A18})$$

Using the definitions (A14) and (A2), we obtain

$$\langle h' \rangle = \text{tr}(K \tilde{h}'), \quad (\text{A19})$$

where $K = \sigma^a X_0^a$ is the kernel for the unperturbed Hamiltonian h_0 . We emphasize that the result (A19) holds true, in general, and is independent of the form of $\tilde{h}'$. In this paper, we consider a perturbation with the kernel

$$\tilde{h}' = \sigma^a x^a \quad (\text{A20})$$

due to a separation of the fuzzy spheres. As a result, $\langle h' \rangle = 0$ since $\text{tr}(X^{0a}) = 0$. We note that this result is also expected

since the perturbation (A20) takes the form of a spin in an external magnetic field $\vec{x}$. As the fuzzy sphere on which the microstates are defined is isotropic, the magnetic energy averaged over the set of microstates is zero.

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