

Symanzik effective action for Karsten-Wilczek minimally doubled fermions

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Karsten-Wilczek (KW) fermions are a popular variant of minimally doubled fermions. We construct Symanzik effective action for KW fermions, which is known to break the hypercubic symmetry of the lattice action. In this work we make the two fermionic modes, called tastes, explicit using the point-splitting proposal of Creutz and Misumi and write the KW action in terms of the taste fields. We identify the symmetries of the point-split action and write down the Symanzik effective action up to dimension-5, including the divergent dimension-3, operators for both free and interacting KW fermions.

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I. INTRODUCTION

Chiral perturbation theory (χ PT) [1] is extensively used to explore continuum QCD at low energies and is an essential component in extracting physical observables from lattice simulations. On a discrete spacetime lattice, the first step is construction of Symanzik effective theory [2,3] describing the lattice actions near the continuum limit using higher dimensional operators in orders of lattice spacing a which appears explicitly in the action. Such a chiral Lagrangian is an expansion in the quark masses (like continuum χ PT) and in the lattice spacing. This formalism is known as lattice χ PT. Well-known examples of lattice χ PT are Wilson χ PT [4–6] and Staggered χ PT [7]. In this work, we build the Symanzik effective theory for Minimally doubled fermions as proposed by Karsten and Wilczek [8,9].

According to the Nielsen-Ninomiya no-go theorem [10], fermion actions that have an exact chiral symmetry must have at least two doublers. The doublers are multiple fermionic species on lattice arising due to discretizing spacetime. To have the chiral fermions on the lattice, Domain-wall proposal [11] introduces a fifth dimension while the Ginsparg-Wilson proposal [12] suggests redefinition of chiral symmetry on the lattice as a go-around. Another class of lattice fermions that are explored recently

are minimally doubled fermions. Karsten-Wilczek (KW) fermions are a variant of minimally doubled fermions [13], which breaks the hypercubic symmetry of the lattice action. Another popular variant of minimally doubled fermions (MDF) is Borici-Creutz fermions [14–16]. Numerical investigations have been carried out for different variants of MDF by various groups [17–20]. It is therefore imperative to explore the chiral structure of MDF. In this paper, we focus our study on KW fermions.

In minimally doubled fermions, the fermionic field ψ describes the two doubler fermions, which we call *tastes*. The two tastes residing at separate poles of the KW propagator can be made explicit by *point-splitting* of the field ψ . This procedure of point-splitting is not unique; various suggestions for point-splitting exist and can be found in [21,22], [23], and very recently [24]. In [25], we investigated Tiburzi's point-splitting for KW fermions and attempted to write the Symanzik effective action and the chiral Lagrangian. However, physical and computational problems emerge from the fact that Tiburzi's tastes are infinitely nonlocal. In this paper, we used point-splitting suggested by Creutz-Misumi [21,22].

One distinct advantage of Creutz point-splitting is that in the configuration space the relation between the taste fields, later denoted by (u, d) and originally defined in the momentum space, and the field ψ is fairly intuitive. Furthermore, one can create a taste isospinor from the taste fields and use it to define the point-split action. Besides, the taste fields thus defined are local and hence can be useful for numerical simulations. In Creutz's point-splitting, the mass term for the two tastes is not diagonal in the taste space and, therefore, in the continuum limit the tastes do not have the usual physical flavor states interpretation. We write the lattice action in terms of the isospinor field with the tastes as its components and identify the symmetries of this action. These symmetries

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are used to construct the actions for the Symanzik effective theory. Symanzik effective theory is an expansion in the lattice spacing and hence is crucial in performing the continuum extrapolation. In this work, Symanzik effective theory has been constructed up to dimension-5 operators for Creutz point-split KW fermions. Free and interacting cases have been treated separately and Symanzik effective theories have been constructed for both.

This paper has been organized as follows: in Sec. II, we describe the KW action followed by a review of Creutz's proposal for point-splitting. We use the taste fields for constructing the point-split action and identify its symmetries. In Sec. III we construct the Symanzik effective action for the free KW fermions. Following, in Sec. IV, we study the transformations of the link variables and construct the Symanzik effective action for interacting KW fermions. We conclude the discussion of our work with a summary. In Appendix A, we present some calculations of symmetry transformations of some of the operators as an example. In Appendix B, we present a brief discussion on Tiburzi's point-splitting.

II. CREUTZ POINT-SPLITTING RELATIONS

The free Karsten-Wilczek action on lattice [21], where $\hat{e}_4$ is the unit vector in the time direction, is given in Eq. (10),

$$\begin{aligned} \mathcal{S} = & \frac{1}{2} \sum_{j=1}^3 \sum_x [\bar{\psi}_x \gamma_j \psi_{x+\hat{j}} - \bar{\psi}_{x+\hat{j}} \gamma_j \psi_x] \\ & - \frac{i}{2} \sum_x [\bar{\psi}_x \gamma_4 \psi_{x+\hat{e}_4} + \bar{\psi}_{x+\hat{e}_4} \gamma_4 \psi_x] + m_0 \sum_x \bar{\psi}_x \psi_x \\ & - \frac{i}{2} \sum_{j=1}^3 \sum_x [\bar{\psi}_x \gamma_4 \psi_{x+\hat{j}} + \bar{\psi}_{x+\hat{j}} \gamma_4 \psi_x - 2\bar{\psi}_x \gamma_4 \psi_x]. \end{aligned} \quad (1)$$

The corresponding Dirac operator in the momentum space is

$$D(p) = i \sum_{j=1}^3 \gamma_j \sin(p_j) + i\gamma_4 \left(3 - \sum_{\mu=1}^4 \cos p_\mu \right) \quad (2)$$

and the poles of the propagator $D^{-1}(p)$ are at $\vec{p} = 0$, $p_4 = \pm\pi/2$. The doublers or tastes residing at the poles are separated using Creutz point-splitting relations,

$$u(q) = \frac{1}{2} \left(1 + \sin \left(q_4 + \frac{\pi}{2} \right) \right) \psi \left(q + \frac{\pi}{2} \hat{e}_4 \right), \quad (3)$$

$$d(q) = \frac{1}{2} \Gamma \left(1 - \sin \left(q_4 - \frac{\pi}{2} \right) \right) \psi \left(q - \frac{\pi}{2} \hat{e}_4 \right), \quad (4)$$

where in the above Eq. (4), the matrix Γ is $\Gamma = i\gamma_4\gamma_5$, $\{\gamma_5, \Gamma\} = 0$ and $\Gamma^2 = 1$. The presence of the Γ matrix distinguishes the opposite chiralities of the two tastes. In the configuration space, the Creutz point-splitting relations for KW fermions are realized as

$$u_x = \frac{1}{2} e^{i\frac{\pi}{2}x_4} \left(\psi_x + \frac{i}{2} (\psi_{x-e_4} - \psi_{x+e_4}) \right), \quad (5)$$

$$d_x = \frac{1}{2} \Gamma e^{-i\frac{\pi}{2}x_4} \left(\psi_x - \frac{i}{2} (\psi_{x-e_4} - \psi_{x+e_4}) \right). \quad (6)$$

To write the free KW action in terms of the point-split tastes u_x and d_x , we reorganize the Eqs. (5) and (6) as

$$\psi_x = e^{-i\frac{\pi}{2}x_4} u_x + e^{i\frac{\pi}{2}x_4} \Gamma d_x, \quad (7)$$

$$\bar{\psi}_x = e^{i\frac{\pi}{2}x_4} \bar{u}_x - e^{-i\frac{\pi}{2}x_4} \bar{d}_x \Gamma, \quad (8)$$

and substitute the reorganized relations in the free KW action of Eq. (1). We further introduce a taste isospin notation Ψ in two-dimensional taste space as

$$\Psi_x = \begin{pmatrix} u_x \\ d_x \end{pmatrix} \quad \text{and} \quad \bar{\Psi}_x = (\bar{u}_x \quad \bar{d}_x). \quad (9)$$

In terms of the taste isospinor Ψ , we get the KW point-split action in configuration space,

$$\begin{aligned} \mathcal{S} = & \frac{1}{2} \sum_{k=1}^3 \sum_x [\bar{\Psi}_x \{ (\gamma_k \otimes \sigma_3) + (-1)^{x_4} (\Gamma \gamma_k \otimes i\sigma_2) \} \Psi_{x+k} - \bar{\Psi}_{x+k} \{ (\gamma_k \otimes \sigma_3) + (-1)^{x_4} (\Gamma \gamma_k \otimes i\sigma_2) \} \Psi_x] \\ & + \frac{1}{2} \sum_x [\bar{\Psi}_x \{ -(\gamma_4 \otimes \sigma_3) - (-1)^{x_4} (\Gamma \gamma_4 \otimes i\sigma_2) \} \Psi_{x+e_4} + \bar{\Psi}_{x+e_4} \{ (\gamma_4 \otimes \sigma_3) - (-1)^{x_4} (\Gamma \gamma_4 \otimes i\sigma_2) \} \Psi_x] \\ & - \frac{i}{2} \sum_{k=1}^3 \sum_x [\bar{\Psi}_x \{ (\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1) \} \Psi_{x+k} + \bar{\Psi}_{x+k} \{ (\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1) \} \Psi_x \\ & - 2\bar{\Psi}_x \{ (\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1) \} \Psi_x] + m_0 \sum_x [\bar{\Psi}_x \{ (1 \otimes \sigma_3) + (-1)^{x_4} (\Gamma \otimes i\sigma_2) \} \Psi_x] \end{aligned} \quad (10)$$

where the tensor product in the action indicates Dirac matrix $\otimes$ taste matrix. In the order of their appearance in the action, the terms can be understood as

$$\mathcal{S} = [\text{kinetic spatial}] + [\text{kinetic temporal}] + [\text{Karsten - Wilczek}] + [\text{mass}]. \quad (11)$$

As an example, let us consider an explicit construction of one of the terms, say the mass term in the last line of the above point-split action Eq. (10),

$$\begin{aligned} m_0 \sum_x \bar{\psi}_x \psi_x &= m_0 \sum_x [e^{i\frac{a}{2}x_4} \bar{u}_x - e^{-i\frac{a}{2}x_4} \bar{d}_x \Gamma] \times [e^{-i\frac{a}{2}x_4} u_x + e^{i\frac{a}{2}x_4} \Gamma d_x] \\ &= m_0 \sum_x [\bar{u}_x u_x + (-1)^{x_4} \bar{u}_x \Gamma d_x - \bar{d}_x d_x - (-1)^{x_4} \bar{d}_x \Gamma u_x] \\ &= m_0 \sum_x \bar{\Psi}_x \{ (1 \otimes \sigma_3) + (-1)^{x_4} (\Gamma \otimes i\sigma_2) \} \Psi_x. \end{aligned} \quad (12)$$

The remaining terms in the point-split action can be constructed in a similar way.

The symmetries of the point-split action Eq. (10) in terms of taste isospin field Ψ_x are given in Table I, Charge conjugation and time reversal individually are not symmetries of the original action but their product is. Charge conjugation and time reversal are broken by the kinetic temporal and Karsten-Wilczek term.

From the symmetry arguments, the taste isospinor Ψ_x can transform under four groups— $U(1)_V$, $U(1)_A$, $SU(2)_V$ and $SU(2)_A$. Out of these only the $U(1)_V$ preserves the taste isospin symmetry. The $SU(2)_V$ has three generators, namely, $(1 \otimes \sigma_1)$, $(1 \otimes \sigma_2)$, and $(1 \otimes \sigma_3)$. Out of these, the Karsten-Wilczek term breaks $(1 \otimes \sigma_2)$ and $(1 \otimes \sigma_3)$ but preserves $(1 \otimes \sigma_1)$. The remaining terms of the free action break all generators of $SU(2)_V$.

The group $U(1)_A$ is generated by $(\gamma_5 \otimes 1)$. This group is broken by all the terms of the action. The $SU(2)_A$ has three generators: $(\gamma_5 \otimes \sigma_1)$, $(\gamma_5 \otimes \sigma_2)$, and $(\gamma_5 \otimes \sigma_3)$. The kinetic spatial and kinetic temporal terms break $(\gamma_5 \otimes \sigma_1)$ and $(\gamma_5 \otimes \sigma_2)$ but preserve the generator $(\gamma_5 \otimes \sigma_3)$. Whereas, the mass term breaks $(\gamma_5 \otimes \sigma_1)$ and $(\gamma_5 \otimes \sigma_3)$ but preserves $(\gamma_5 \otimes \sigma_2)$. On the other hand, the KW term breaks $(\gamma_5 \otimes \sigma_1)$ but preserves $(\gamma_5 \otimes \sigma_2)$ and $(\gamma_5 \otimes \sigma_3)$. Note that in the chiral limit, $m_0 \rightarrow 0$, the entire point-split action is invariant under the generator $(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$. The last line is crucial for the construction of the Symanzik effective theory in the next section.

A list of discrete transformations and the way they act on ψ_x and Ψ_x are given in the Table II. Once the symmetries of the action are made explicit, the next step is to construct the free Symanzik effective action. We can now move to the

TABLE I. Symmetries of point-split free action Eq. (10).

1. Spatial rotations	2. Parity	3. Charge conjugation $\times$ time reversal
4. Site reflection	5. $U(1)_V$	

next step and using these symmetries, construct the free Symanzik effective action.

III. FREE SYMANZIK EFFECTIVE ACTION FOR KW FERMIONS

We construct the Symanzik effective action to next-to-leading order (NLO) by considering operators up to dimension-5. The Symanzik action has the following form in terms of lattice spacing a :

$$\mathcal{S}_{\text{Symanzik}} = a^{-1} S_{-1} + S_0 + a S_1 + a^2 S_2 + \cdots \quad (13)$$

where S_k consists of operators of dimension- $(k+4)$. For instance, for dimension-3 operators, $k = -1$, we have the term $a^{-1} S_{-1}$. The operators that go in S_k are invariant under the symmetries of the lattice action, which in this case is the point-split KW action in Eq. (10). Corresponding to l -different operators of dimension- $(k+4)$, we associate with $\mathcal{O}_l^{(k+4)}(x)$ a coefficient $c_l^{(k+4)}$, often referred to as the low-energy constant (LEC),

$$S_k = \sum_l \int d^4x c_l^{(k+4)} \mathcal{O}_l^{(k+4)}. \quad (14)$$

We tabulate the $(k+4)$ -dimension operators up to $k = 1$ in the Tables III–VIII. In these tables, the leftmost column

TABLE II. Symmetry transformations and their actions on fermionic ψ_x and isospinor Ψ_x . The first three transformations are linear while site-reflection is antilinear. Charge conjugation matrix $C = i\gamma_2\gamma_4$, time reversal matrix $T = \gamma_4\gamma_5$ and $\Gamma = i\gamma_4\gamma_5$.

Transformation	Action on ψ_x	Action on Ψ_x
Parity	$\gamma_4 \psi_{-\mathbf{x}, x_4}$	$(\gamma_4 \otimes \sigma_3) \Psi_{-\mathbf{x}, x_4}$
Charge conjugation	$C^{-1} \bar{\psi}_x^T$	$(C^{-1} \Gamma \otimes i\sigma_2) \bar{\Psi}_x^T$
Time reversal	$T \psi_{\mathbf{x}, -x_4}$	$(T \Gamma \otimes \sigma_1) \Psi_{\mathbf{x}, -x_4}$
Site reflection	$T \bar{\psi}_{1-\mathbf{x}, x_4}^T$	$(T \otimes \sigma_3) \bar{\Psi}_{1-\mathbf{x}, x_4}^T$

TABLE III. The rotationally invariant dimension-3 operators and their transformations under various symmetries and a generator of $SU(2)_A$. Invariance of an operator is denoted by $\mathcal{I}$ while those that are not are denoted by $\times$.

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$\bar{\Psi}(1 \otimes 1)\Psi$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(1 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5 \otimes 1)\Psi$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$
$\bar{\Psi}(\gamma_5 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5 \gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$

TABLE IV. The rotationally invariant dimension-4 operators containing a partial derivative.

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$\bar{\Psi}(1 \otimes 1)\partial_4\Psi$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\times$
$\bar{\Psi}(1 \otimes \sigma_i)\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_4 \otimes 1)\partial_4\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(\gamma_4 \otimes \sigma_i)\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_5 \otimes 1)\partial_4\Psi$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$
$\bar{\Psi}(\gamma_5 \otimes \sigma_i)\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_5 \gamma_4 \otimes 1)\partial_4\Psi$	$\mathcal{I}$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_i)\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_k \otimes 1)\partial_k\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(\gamma_k \otimes \sigma_i)\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_5 \gamma_k \otimes 1)\partial_k\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(\gamma_5 \gamma_k \otimes \sigma_i)\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_4 \gamma_k \otimes 1)\partial_k\Psi$	$\times$	$\mathcal{I}$	$\times$	$\mathcal{I}$	$\times$	$\mathcal{I}$
$\bar{\Psi}(\gamma_4 \gamma_k \otimes \sigma_i)\partial_k\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\times, \times, \mathcal{I}$

TABLE V. The rotationally invariant dimension-4 operators proportional to the fermion mass m .

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$m\bar{\Psi}(1 \otimes 1)\Psi$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$m\bar{\Psi}(1 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m\bar{\Psi}(\gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\times$
$m\bar{\Psi}(\gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$m\bar{\Psi}(\gamma_5 \otimes 1)\Psi$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$
$m\bar{\Psi}(\gamma_5 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$m\bar{\Psi}(\gamma_5 \gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$
$m\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$

TABLE VI. The rotationally invariant dimension-5 operators containing two partial derivatives.

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$\bar{\Psi}(1 \otimes 1)\partial_4\partial_4\Psi$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(1 \otimes \sigma_i)\partial_4\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_4 \otimes 1)\partial_4\partial_4\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_4 \otimes \sigma_i)\partial_4\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5 \otimes 1)\partial_4\partial_4\Psi$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$
$\bar{\Psi}(\gamma_5 \otimes \sigma_i)\partial_4\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5\gamma_4 \otimes 1)\partial_4\partial_4\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_i)\partial_4\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(1 \otimes 1)\partial_k\partial_k\Psi$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$\bar{\Psi}(1 \otimes \sigma_i)\partial_k\partial_k\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$\bar{\Psi}(\gamma_4 \otimes 1)\partial_k\partial_k\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\sum_k \bar{\Psi}(\gamma_4 \otimes \sigma_i)\partial_k\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5 \otimes 1)\partial_k\partial_k\Psi$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$
$\bar{\Psi}(\gamma_5 \otimes \sigma_i)\partial_k\partial_k\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5\gamma_4 \otimes 1)\partial_k\partial_k\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_i)\partial_k\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_k \otimes 1)\partial_k\partial_k\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_k \otimes \sigma_i)\partial_k\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_4\gamma_k \otimes 1)\partial_k\partial_k\Psi$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\times$
$\bar{\Psi}(\gamma_4\gamma_k \otimes \sigma_i)\partial_k\partial_k\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$\bar{\Psi}(\gamma_5\gamma_k \otimes 1)\partial_k\partial_k\Psi$	$\mathcal{I}$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\times$
$\bar{\Psi}(\gamma_5\gamma_k \otimes \sigma_i)\partial_k\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$

TABLE VII. The rotationally invariant dimension-5 operators containing a partial derivative and proportional to the fermion mass m . Sum over k is assumed.

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$m \bar{\Psi}(1 \otimes 1)\partial_4\Psi$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\times$
$m \bar{\Psi}(1 \otimes \sigma_i)\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$m \bar{\Psi}(\gamma_4 \otimes 1)\partial_4\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_4 \otimes \sigma_i)\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m \bar{\Psi}(\gamma_5 \otimes 1)\partial_4\Psi$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_5 \otimes \sigma_i)\partial_4\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \times, \times$	$\times, \times, \mathcal{I}$
$m \bar{\Psi}(\gamma_5\gamma_4 \otimes 1)\partial_4\Psi$	$\mathcal{I}$	$\times$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_i)\partial_4\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \times, \times$	$\times, \times, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m \bar{\Psi}(\gamma_k \otimes 1)\partial_k\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_k \otimes \sigma_i)\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m \bar{\Psi}(\gamma_5\gamma_k \otimes 1)\partial_k\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_5\gamma_k \otimes \sigma_i)\partial_k\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m \bar{\Psi}(\gamma_4\gamma_k \otimes 1)\partial_k\Psi$	$\times$	$\mathcal{I}$	$\times$	$\mathcal{I}$	$\times$	$\mathcal{I}$
$m \bar{\Psi}(\gamma_4\gamma_k \otimes \sigma_i)\partial_k\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\times, \times, \mathcal{I}$

TABLE VIII. The rotationally invariant dimension-5 operators proportional to m^2 .

Operators	$(\gamma_5 \otimes \sigma_3)$ of $SU(2)_A$	Parity	Time reversal	Charge conjugation	Charge $\times$ time	Site reflection
$m^2 \bar{\Psi}(1 \otimes 1)\Psi$	$\times$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\mathcal{I}$
$m^2 \bar{\Psi}(1 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \mathcal{I}$
$m^2 \bar{\Psi}(\gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\mathcal{I}$	$\times$	$\times$	$\mathcal{I}$	$\times$
$m^2 \bar{\Psi}(\gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\times, \times, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$
$m^2 \bar{\Psi}(\gamma_5 \otimes 1)\Psi$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$	$\times$
$m^2 \bar{\Psi}(\gamma_5 \otimes \sigma_i)\Psi$	$\mathcal{I}, \mathcal{I}, \times$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\times, \times, \times$	$\mathcal{I}, \times, \times$	$\mathcal{I}, \mathcal{I}, \times$
$m^2 \bar{\Psi}(\gamma_5 \gamma_4 \otimes 1)\Psi$	$\mathcal{I}$	$\times$	$\times$	$\times$	$\mathcal{I}$	$\times$
$m^2 \bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_i)\Psi$	$\times, \times, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \mathcal{I}$	$\times, \mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{I}, \times$

lists the rotationally invariant operators of appropriate dimensions. The following columns show the invariance property or lack of it for various symmetry transformations, except for the second column which discusses the invariance property under the generator $(\gamma_5 \otimes \sigma_3)$ of the group $SU(2)_A$ in the taste space. The $\mathcal{I}$ entries in a row symbolize invariance of the operator under such transformations and the $\times$ entries symbolize the breaking of the invariance. The rows with single entries of $\mathcal{I}$ or $\times$ for every symmetry transformation correspond to operators with unit matrix in the taste space while the rows with three entries correspond to operators having one of the three $\sigma_1, \sigma_2, \sigma_3$ in the taste space.

None of the tables here contain dimension-2 operators since no local operators of dimension-2 that are invariant under rotations and gauge transformations can be constructed. Hence the dimension-3 operators are the LO operators in the Symanzik action Eq. (14).

In Table III, the transformation properties of dimension-3 operators are listed. From the table, as an example, we consider the fourth row which shows the invariance properties of the operator $\bar{\Psi}(\gamma_4 \otimes \sigma_i)\Psi$. We check the transformation of this term under parity, for instance. Because this term corresponds to the taste matrix σ_i , the

parity invariance holds only for σ_3 hence $\mathcal{I}$. Whereas for the two other sigma matrices, parity invariance is broken, hence $\times$. Because parity invariance is the symmetry of the lattice action, operators entering the Symanzik action are required to satisfy parity invariance as well. Hence, the operators $\bar{\Psi}(\gamma_4 \otimes \sigma_i)\Psi$ with $i = 1, 2$ cannot be included in the Symanzik action. In Appendix A, we provide details for calculating the symmetry properties of the operator taking a dimension-4 operator, appearing in Table IV as an example. From the dimension-3 operators given in Table III, based on the discussion above, the operators that satisfy all the symmetries of the lattice action qualify for the Symanzik action. They are

$$i\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi. \quad (15)$$

Tables IV and V list all rotationally invariant dimension-4 operators with single derivatives (∂_k, ∂_4) and one power of the quark mass m , respectively.

From a similar reasoning to the one provided in the selection of the dimension-3 operator, the terms that enter the Symanzik action in NLO are, where sum over k is understood,

$$\begin{aligned}
& \bar{\Psi}(\gamma_4 \otimes 1)\partial_4\Psi, \quad \bar{\Psi}(\gamma_4 \otimes \sigma_3)\partial_4\Psi, \quad i\bar{\Psi}(\gamma_5 \otimes \sigma_1)\partial_4\Psi, \quad \bar{\Psi}(\gamma_k \otimes 1)\partial_k\Psi, \\
& \bar{\Psi}(\gamma_k \otimes \sigma_3)\partial_k\Psi, \quad m\bar{\Psi}(1 \otimes 1)\Psi, \quad m\bar{\Psi}(1 \otimes \sigma_3)\Psi, \quad im\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \\
& im\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi, \quad m\bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi, \quad m\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_2)\Psi.
\end{aligned} \quad (16)$$

Similarly, the operators coming from Tables VI, VII, and VIII that enter the NNLO of the Symanzik action are

$$\begin{aligned}
& i\bar{\Psi}(\gamma_4 \otimes 1)\partial_4\partial_4\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes \sigma_3)\partial_4\partial_4\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)\partial_4\partial_4\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes 1)\partial_k\partial_k\Psi, \\
& i\bar{\Psi}(\gamma_4 \otimes \sigma_3)\partial_k\partial_k\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)\partial_k\partial_k\Psi, \quad i\bar{\Psi}(\gamma_k \otimes 1)\partial_4\partial_k\Psi, \quad i\bar{\Psi}(\gamma_k \otimes \sigma_3)\partial_4\partial_k\Psi, \\
& m\bar{\Psi}(\gamma_4 \otimes 1)\partial_4\Psi, \quad m\bar{\Psi}(\gamma_4 \otimes \sigma_3)\partial_4\Psi, \quad im\bar{\Psi}(\gamma_5 \otimes \sigma_1)\partial_4\Psi, \quad m\bar{\Psi}(\gamma_k \otimes 1)\partial_k\Psi, \\
& m\bar{\Psi}(\gamma_k \otimes \sigma_3)\partial_k\Psi, \quad im\bar{\Psi}(1 \otimes 1)\partial_4\Psi, \quad im\bar{\Psi}(1 \otimes \sigma_3)\partial_4\Psi, \quad im\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_2)\partial_4\Psi, \\
& im\bar{\Psi}(\gamma_5 \gamma_k \otimes \sigma_2)\partial_k\Psi, \quad m^2\bar{\Psi}(1 \otimes 1)\Psi, \quad m^2\bar{\Psi}(1 \otimes \sigma_3)\Psi, \quad im^2\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \\
& im^2\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi, \quad m^2\bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi, \quad m^2\bar{\Psi}(\gamma_5 \gamma_4 \otimes \sigma_2)\Psi.
\end{aligned} \quad (17)$$

Note that the lists of operators given here do not contain only mutually independent operators. Redundancy is potentially present in these terms; however, in an effort to construct lattice chiral perturbation theory, having redundant operators poses no major hurdles [7].

IV. INTERACTING KW ACTION

In the interacting case, the only but important difference is the introduction of gluon fields through covariant derivatives and gauge action. In terms of gluon link variables $U(x, x + \mu)$ the KW action becomes

$$\begin{aligned} \mathcal{S} = & \frac{1}{2} \sum_{k=1}^3 \sum_x [\bar{\psi}_x \gamma_k U(x, x+k) \psi_{x+\hat{k}} - \bar{\psi}_{x+\hat{k}} \gamma_k U^\dagger(x, x+k) \psi_x] - \frac{i}{2} \sum_x [\bar{\psi}_x \gamma_4 U(x, x+\hat{e}_4) \psi_{x+\hat{e}_4} + \bar{\psi}_{x+\hat{e}_4} \gamma_4 U^\dagger(x, x+\hat{e}_4) \psi_x] \\ & + m_0 \sum_x \bar{\psi}_x \psi_x - \frac{i}{2} \sum_{k=1}^3 \sum_x [\bar{\psi}_x \gamma_4 U(x, x+k) \psi_{x+\hat{k}} + \bar{\psi}_{x+\hat{k}} \gamma_4 U^\dagger(x, x+k) \psi_x - 2\bar{\psi}_x \gamma_4 \psi_x]. \end{aligned} \quad (18)$$

The realization of point-splitting also gets altered accordingly with the addition of link variables. To achieve a gauge-invariant point-split KW action, we define our point-splitting,

$$u_x = \frac{1}{2} e^{i\frac{\pi}{2}x_4} \left[\psi_x + \frac{i}{2} (U(x, x_b) \psi_{x_b} - U(x, x_f) \psi_{x_f}) \right], \quad (19)$$

$$d_x = \frac{1}{2} \Gamma e^{-i\frac{\pi}{2}x_4} \left[\psi_x - \frac{i}{2} (U(x, x_b) \psi_{x_b} - U(x, x_f) \psi_{x_f}) \right], \quad (20)$$

where $x_b = x - e_4$ and $x_f = x + e_4$. However, Eq. (19) and Eq. (20) can be inverted to write ψ_x in terms of u_x and d_x and the relation connecting ψ_x and u_x , d_x remains unchanged as in Eq. (7) and Eq. (8). Defining taste isospinor Ψ as before in Eq. (9) we get the point-split interacting KW action in Eq. (21). The order of appearance of various terms in the interacting action remains the same as stated in Eq. (11):

$$\begin{aligned} \mathcal{S} = & \frac{1}{2} \sum_{k=1}^3 \sum_x [\bar{\Psi}_x \{(\gamma_k \otimes \sigma_3) + (-1)^{x_4} (\Gamma \gamma_k \otimes i\sigma_2)\} U(x, x+k) \Psi_{x+k} - \bar{\Psi}_{x+k} \{(\gamma_k \otimes \sigma_3) + (-1)^{x_4} (\Gamma \gamma_k \otimes i\sigma_2)\} U^\dagger(x, x+k) \Psi_x] \\ & + \frac{1}{2} \sum_x [\bar{\Psi}_x \{-(\gamma_4 \otimes \sigma_3) - (-1)^{x_4} (\Gamma \gamma_4 \otimes i\sigma_2)\} U(x, x+\hat{e}_4) \Psi_{x+\hat{e}_4} + \bar{\Psi}_{x+\hat{e}_4} \{(\gamma_4 \otimes \sigma_3) - (-1)^{x_4} (\Gamma \gamma_4 \otimes i\sigma_2)\} U^\dagger(x, x+\hat{e}_4) \Psi_x] \\ & - \frac{i}{2} \sum_{k=1}^3 \sum_x [\bar{\Psi}_x \{(\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1)\} U(x, x+k) \Psi_{x+k} + \bar{\Psi}_{x+k} \{(\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1)\} U^\dagger(x, x+k) \Psi_x \\ & - 2\bar{\Psi}_x \{(\gamma_4 \otimes 1) - (-1)^{x_4} (\Gamma \gamma_4 \otimes \sigma_1)\} \Psi_x] + m_0 \sum_x \bar{\Psi}_x \{(1 \otimes \sigma_3) + (-1)^{x_4} (\Gamma \otimes i\sigma_2)\} \Psi_x. \end{aligned} \quad (21)$$

The next step to construct the interacting Symanzik action is to identify the symmetries of the interacting lattice action in Eq. (21). For this we need to know how the link variables transform under the discrete symmetry transformations. Various discrete symmetries and their action on the link variables is summarized in Table IX.

Combining the transformations of the fermionic fields given in Table II with those of the link variables given in Table IX, we conclude that the symmetries of the interacting point-split action are identical to those of the free point-split action in Eq. (10). Since no dimension-3 operator involves a link variable, the operators that qualify for the

Symanzik action at dimension-3 remain unchanged. For readability we rewrite those operators,

$$i\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi. \quad (22)$$

All dimension-3 operators diverge in the continuum limit as they are proportional to the inverse lattice spacing i.e., a^{-1} . Hence care needs to be taken to absorb these divergences. A proper renormalization scheme for the KW Symanzik action must absorb all the dimension-3 operators by renormalizing appropriate quantities that couple to operators of higher dimensions. Quantities to

TABLE IX. Symmetry transformations and their actions on gluon fields $A_\mu(z) = A_\mu^B(z)\tau^B$ and the link variables $U(x, x + \hat{\mu})$ [26]. Here too, the first three transformations are linear while site-reflection is anti-linear. Action on spatial and temporal parts is shown separately.

Transformation	$A_\mu(z)$	$U(x, x + \hat{\mu})$
Parity (spatial)	$A_j(z) \xrightarrow{P} -A_j(-\mathbf{z}, z_4)$	$U(n, n + \hat{j}) \xrightarrow{P} U^\dagger((- \mathbf{n} - \hat{j}, n_4), (-\mathbf{n}, n_4))$
Parity (temporal)	$A_4(z) \xrightarrow{P} A_4(-\mathbf{z}, z_4)$	$U(n, n + \hat{e}_4) \xrightarrow{P} U((- \mathbf{n}, n_4), (-\mathbf{n}, n_4 + 1))$
Charge conjugation	$A_\mu(z) \xrightarrow{C} -A_\mu^T(z)$	$U(n, n + \hat{\mu}) \xrightarrow{C} U^*(n, n + \hat{\mu})$
Time reversal (spatial)	$A_j(z) \xrightarrow{T} A_j(\mathbf{z}, -z_4)$	$U(n, n + \hat{j}) \xrightarrow{T} U((\mathbf{n}, -n_4), (\mathbf{n} + \hat{j}, -n_4))$
Time reversal (temporal)	$A_4(z) \xrightarrow{T} -A_4(\mathbf{z}, -z_4)$	$U(n, n + \hat{e}_4) \xrightarrow{T} U^\dagger((\mathbf{n}, -n_4 - 1), (\mathbf{n}, -n_4))$
Site reflection (spatial)	$A_j(z) \xrightarrow{SR} -A_j^*(\mathbf{1} - \mathbf{z}, -z_4)$	$U(n, n + \hat{j}) \xrightarrow{SR} U^T((\mathbf{1} - \mathbf{n} - \hat{j}, n_4), (\mathbf{1} - \mathbf{n}, n_4))$
Site reflection (temporal)	$A_4(z) \xrightarrow{SR} A_4^*(\mathbf{1} - \mathbf{z}, -z_4)$	$U(n, n + \hat{e}_4) \xrightarrow{SR} U^*((\mathbf{1} - \mathbf{n}, n_4), (\mathbf{1} - \mathbf{n}, n_4 + 1))$

be renormalized may contain wave function, fermionic speed of light, and gluonic speed of light [27].

For dimension-4 and dimension-5 operators that are proportional to m and m^2 respectively, the operators that qualify for the Symanzik action remain unchanged. Since the rotational and discrete transformations of $\partial\Psi$ and are

identical with those of $D\Psi$, the operators that qualify for the Symanzik action in dimension-4 and dimension-5 are obtained by replacing ∂ with D . Additionally, in dimension-4 we will get two purely gluonic terms that are invariant under the same set of symmetries. At the NLO, thus we get the operators,

$$\begin{aligned}
&\bar{\Psi}(\gamma_4 \otimes 1)D_4\Psi, \quad \bar{\Psi}(\gamma_4 \otimes \sigma_3)D_4\Psi, \quad i\bar{\Psi}(\gamma_5 \otimes \sigma_1)D_4\Psi, \quad \bar{\Psi}(\gamma_k \otimes 1)D_k\Psi, \\
&\bar{\Psi}(\gamma_k \otimes \sigma_3)D_k\Psi, \quad m\bar{\Psi}(1 \otimes 1)\Psi, \quad m\bar{\Psi}(1 \otimes \sigma_3)\Psi, \quad im\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \\
&im\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi, \quad m\bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi, \quad m\bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_2)\Psi, \quad F_{0k}F_{0k}, \quad F_{jk}F_{jk}.
\end{aligned} \tag{23}$$

At the NNLO, i.e. dimension-5, the operators entering the Symanzik action are

$$\begin{aligned}
&i\bar{\Psi}(\gamma_4 \otimes 1)D_4D_4\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes \sigma_3)D_4D_4\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)D_4D_4\Psi \\
&i\bar{\Psi}(\gamma_4 \otimes 1)D_kD_k\Psi, \quad i\bar{\Psi}(\gamma_4 \otimes \sigma_3)D_kD_k\Psi, \quad \bar{\Psi}(\gamma_5 \otimes \sigma_1)D_kD_k\Psi \\
&i\bar{\Psi}(\gamma_k \otimes 1)D_4D_k\Psi, \quad i\bar{\Psi}(\gamma_k \otimes \sigma_3)D_4D_k\Psi, \quad m\bar{\Psi}(\gamma_4 \otimes 1)D_4\Psi \\
&m\bar{\Psi}(\gamma_4 \otimes \sigma_3)D_4\Psi, \quad im\bar{\Psi}(\gamma_5 \otimes \sigma_1)D_4\Psi, \quad m\bar{\Psi}(\gamma_k \otimes 1)D_k\Psi \\
&m\bar{\Psi}(\gamma_k \otimes \sigma_3)D_k\Psi, \quad im\bar{\Psi}(1 \otimes 1)D_4\Psi, \quad im\bar{\Psi}(1 \otimes \sigma_3)D_4\Psi \\
&im\bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_2)D_4\Psi, \quad im\bar{\Psi}(\gamma_5\gamma_k \otimes \sigma_2)D_k\Psi, \quad m^2\bar{\Psi}(1 \otimes 1)\Psi \\
&m^2\bar{\Psi}(1 \otimes \sigma_3)\Psi, \quad im^2\bar{\Psi}(\gamma_4 \otimes 1)\Psi, \quad im^2\bar{\Psi}(\gamma_4 \otimes \sigma_3)\Psi \\
&m^2\bar{\Psi}(\gamma_5 \otimes \sigma_1)\Psi, \quad m^2\bar{\Psi}(\gamma_5\gamma_4 \otimes \sigma_2)\Psi.
\end{aligned} \tag{24}$$

As in the case of free Symanzik action, redundancy in operators has not been addressed here.

V. DISCUSSION

In this paper we addressed the construction of Symanzik effective theory of the Karsten-Wilczek variant of minimally doubled fermions. First we considered the free KW lattice action and its doublers u_x and d_x at the poles of the fermion propagator are separated using the Creutz-Misumi

point-splitting relations. Using these two doublers or tastes we form a isospin doublet Ψ and then rewrite the free KW lattice action. This form of the lattice action was convenient in identifying the taste space symmetries. The invariance under discrete and taste space symmetries determined the selection of the continuum operators up to dimension-5 for the Symanzik effective action. Subsequently, we construct the Symanzik action for the interacting KW fermions. Once we determine the transformation properties of the link variables and continuum gauge fields under discrete

transformations, we select the dimension-4 and dimension-5 operators in a way similar to the free case. In the interacting case, the dimension-3 operators remain unchanged and two purely gluonic operators appear at dimension-4. The first step in constructing lattice χ PT for MDF is writing down the Symanzik action. Once the Symanzik action is found the chiral Lagrangian can be constructed using the method of spurion analysis [5–7]. Spurion analysis does not require the knowledge of the numerical values of the low-energy constants in the Symanzik effective action. The fixing of the coefficients of the operators in the chiral Lagrangian comes from lattice simulations.

This effort, however, does not come without challenges. An important hurdle associated with Creutz point-splitting is the interpretation of tastes in the continuum limit, but it is still possible to progress and construct a lattice chiral perturbation theory with the Creutz point-splitting. The alternative suggestions made, of late, in [24] where the tastes have a natural flavor interpretation in the continuum limit, also seem to be an interesting direction to explore.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: TRANSFORMATIONS OF $\bar{\Psi}_x(\gamma_k \otimes \sigma_i)\partial_k \Psi_x$

In this appendix, for the purpose of demonstration we choose a particular dimension-4 operator for the free KW case and show some of its transformation properties listed in Table IV. From Table II, under parity transformation,

$$x \xrightarrow{P} (-\mathbf{x}, x_4), \quad \partial_\mu \xrightarrow{P} (-\partial_\mu, \partial_4) \quad \text{and} \quad \Psi_x \xrightarrow{P} (\gamma_4 \otimes \sigma_3) \Psi_{-\mathbf{x}, x_4}. \quad (\text{A1})$$

Integrating this operator under parity,

$$\begin{aligned} & \int d^4x \bar{\Psi}_x(\gamma_k \otimes \sigma_i)\partial_k \Psi_x \\ & \xrightarrow{P} \int d^4x \bar{\Psi}_{-\mathbf{x}, x_4}(\gamma_4 \otimes \sigma_3)(\gamma_k \otimes \sigma_i)(\gamma_4 \otimes \sigma_3)(-\partial_k) \Psi_{-\mathbf{x}, x_4} \\ & = \int d^4x \bar{\Psi}_x(\gamma_k \otimes \sigma_3 \sigma_i \sigma_3) \partial_k \Psi_x. \end{aligned} \quad (\text{A2})$$

The only operator which is invariant under parity corresponds to $i = 3$ i.e. $\bar{\Psi}_x(\gamma_k \otimes \sigma_i)\partial_k \Psi_x$.

Under site reflection, referring to the fourth line of Table II,

$$\Psi_x \xrightarrow{SR} (T \otimes \sigma_3) \bar{\Psi}_{1-\mathbf{x}, x_4}^T. \quad (\text{A3})$$

Again, integrating this operator under site reflection, we get

$$\begin{aligned} & \int d^4x \Psi_{1-\mathbf{x}, x_4}^T (T \otimes \sigma_3) (\gamma_k^* \otimes \sigma_i^*) (T \otimes \sigma_3) (-\partial_k) \bar{\Psi}_{1-\mathbf{x}, x_4}^T \\ & = - \int d^4x \Psi_x^T (T \gamma_k^T T \otimes \sigma_3 \sigma_i^T \sigma_3) \partial_k \bar{\Psi}_x^T \\ & = \int d^4x \Psi_x^T (\gamma_k^T \otimes \sigma_3 \sigma_i^T \sigma_3) \partial_k \bar{\Psi}_x^T \\ & = - \int d^4x \partial_k \bar{\Psi}_x^T (\gamma_k \otimes \sigma_3 \sigma_i \sigma_3) \Psi_x^T \\ & = \int d^4x \bar{\Psi}_x(\gamma_k \otimes \sigma_3 \sigma_i \sigma_3) \partial_k \Psi_x. \end{aligned} \quad (\text{A4})$$

Here too, the only operator that is invariant under site reflection is for $\sigma_i = \sigma_3$.

For transformation under the generator $(\gamma_5 \otimes \sigma_3)$ of the group $SU(2)_A$, we expand in small parameter ϵ ,

$$\begin{aligned} & \int d^4x \bar{\Psi}_x [(1 \otimes 1) + i\epsilon(\gamma_5 \otimes \sigma_3)] (\gamma_k \otimes \sigma_i) \\ & \times [(1 \otimes 1) + i\epsilon(\gamma_5 \otimes \sigma_3)] \partial_k \Psi_x \\ & = \int d^4x \bar{\Psi}_x [(\gamma_k \otimes \sigma_i) + i\epsilon\{(\gamma_5 \otimes \sigma_3), (\gamma_k \otimes \sigma_i)\}] \partial_k \Psi_x + \dots \\ & = \int d^4x \bar{\Psi}_x [(\gamma_k \otimes \sigma_i) + i\epsilon(\gamma_5 \gamma_k \otimes [\sigma_3, \sigma_i])] \partial_k \Psi_x. \end{aligned} \quad (\text{A5})$$

The operator $\bar{\Psi}_x(\gamma_k \otimes \sigma_i)\partial_k \Psi_x$ is invariant under the generator $(\gamma_5 \otimes \sigma_3)$ of the group $SU(2)_A$ for $i = 3$.

APPENDIX B: TIBURZI'S POINT-SPLITTING

We commented before that the way point-splitting is achieved for minimally doubled fermions is not unique. Tiburzi [23] used a different form of the KW action than ours given in Eq. (1) and consequently his point-splitting is different than Creutz-Misumi's [21]. The difference in the action is in the temporal part of the kinetic term,

$$\begin{aligned} \mathcal{S} = \sum_x & \left[\sum_{\mu=1}^4 [\bar{\psi}(x)(\gamma_\mu - i\gamma_4(1 - \delta_{\mu 4}))\psi(x + \hat{\mu}) \right. \\ & \left. - \bar{\psi}(x + \hat{\mu})(\gamma_\mu + i\gamma_4(1 - \delta_{\mu 4}))\psi(x)] \right. \\ & \left. + \bar{\psi}(x)(m_0 + 3i\gamma_4)\psi(x) \right]. \end{aligned} \quad (\text{B1})$$

The point-splitting suggested by Tiburzi is based on dividing the Brillouin zone into two parts,

$$\begin{aligned}\psi(k)|_{k_\mu \in \mathcal{B}} &= \psi^{(1)}(k), \\ \psi(k)|_{k_\mu \notin \mathcal{B}} &= \gamma_4 \gamma_5 \psi^{(2)}(T_{\pi 4} k),\end{aligned}\quad (\text{B2})$$

where $\psi^{(1)}$ and $\psi^{(2)}$ represent the two taste fields in the momentum space. The shifted momentum is abbreviated as $T_{\pi 4} k_\mu = \{\mathbf{k}, (k_4 + \pi) \bmod 2\pi\}$. Thus the division of the Brillouin zone is based only on the temporal component of the momentum. In this manner, two halves of the Brillouin zone are attributed to the two tastes. Similar to Eq. (9), an isospinor is constructed containing the two momentum space taste fields,

$$\Psi(k) = \begin{pmatrix} \psi^{(1)}(k) \\ \psi^{(2)}(k) \end{pmatrix}. \quad (\text{B3})$$

The Karsten-Wilczek action in the momentum space can be written using this isospinor as

$$\begin{aligned}\mathcal{S} = \int_{\mathcal{B}} \frac{d^4 k}{(2\pi)^4} \bar{\Psi}(k) &\left[\sum_{\mu} i(\gamma_{\mu} \otimes 1) \sin k_{\mu} \right. \\ &\left. - i(\gamma_4 \otimes \tau_3) \sum_j (\cos k_j - 1) \right] \Psi(k)\end{aligned}\quad (\text{B4})$$

where the tensor product is between the spinor space and the taste space, as before. In the configuration space the field associated with the two tastes is the infinitely time-smearred,

$$\delta_{\mathcal{B}} \psi_x = \int_{\mathcal{B}} \frac{dk_4}{2\pi} \sum_{y_4} e^{ik_4(x_4 - y_4)} \psi_{\mathbf{x}, y_4}, \quad (\text{B5})$$

$$\delta_{\bar{\mathcal{B}}} \bar{\psi}_x = \int_{\bar{\mathcal{B}}} \frac{dk_4}{2\pi} \sum_{y_4} e^{ik_4(x_4 - y_4)} \bar{\psi}_{\mathbf{x}, y_4}. \quad (\text{B6})$$

Since the taste fields involve summation over all lattice points in the temporal direction, they are infinitely non-local. When performing lattice computations, such non-localities may increase the computational costs and can cause issues with the interpretation of the hadrons created from such nonlocal quark fields. Additionally, problems are expected while extrapolating the lattice computation results to infinite spacetime. Hence, we choose to use Creutz-Misumi's point-splitting over Tiburzi's as the nonlocality in Creutz point-splitting is finite.

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- [1] J. Gasser and H. Leutwyler, Chiral perturbation theory to one loop, *Ann. Phys. (N.Y.)* **158**, 142 (1984).
 - [2] K. Symanzik, Continuum limit and improved action in lattice theories. 1. Principles and ϕ^4 theory, *Nucl. Phys. B* **226**, 187 (1983).
 - [3] K. Symanzik, Continuum limit and improved action in lattice theories. 2. O(N) nonlinear sigma model in perturbation theory, *Nucl. Phys. B* **226**, 205 (1983).
 - [4] O. Bar, G. Rupak, and N. Shores, Chiral perturbation theory at $\mathcal{O}(a^2)$ for lattice QCD, *Phys. Rev. D* **70**, 034508 (2004).
 - [5] G. Rupak and N. Shores, Chiral perturbation theory for the Wilson lattice action, *Phys. Rev. D* **66**, 054503 (2002).
 - [6] S. R. Sharpe and R. L. Singleton, Jr, Spontaneous flavor and parity breaking with Wilson fermions, *Phys. Rev. D* **58**, 074501 (1998).
 - [7] W.-J. Lee and S. R. Sharpe, Partial flavor symmetry restoration for chiral staggered fermions, *Phys. Rev. D* **60**, 114503 (1999).
 - [8] L. H. Karsten, Lattice fermions in Euclidean space-time, *Phys. Lett. B* **104**, 315 (1981).
 - [9] F. Wilczek, Lattice fermions, *Phys. Rev. Lett.* **59**, 2397 (1987).
 - [10] H. B. Nielsen and M. Ninomiya, No-go theorem for regularizing chiral fermions, Technical Report No. RL-81-052, 1981, [10.1016/0370-2693\(81\)91026-1](https://arxiv.org/abs/10.1016/0370-2693(81)91026-1).
 - [11] D. B. Kaplan, A method for simulating chiral fermions on the lattice, *Phys. Lett. B* **288**, 342 (1992).
 - [12] P. H. Ginsparg and K. G. Wilson, A remnant of chiral symmetry on the lattice, *Phys. Rev. D* **25**, 2649 (1982).
 - [13] M. Creutz and T. Misumi, Classification of minimally doubled fermions, *Phys. Rev. D* **82**, 074502 (2010).
 - [14] A. Borici, Creutz fermions on an orthogonal lattice, *Phys. Rev. D* **78**, 074504 (2008).
 - [15] A. Borici, Minimally doubled fermion revival, *Proc. Sci. LATTICE2008* (2008) 231 [arXiv:0812.0092].
 - [16] M. Creutz, Four-dimensional graphene and chiral fermions, *J. High Energy Phys.* **04** (2008) 017.
 - [17] A. Kishore, S. Basak, and D. Chakrabarti, Eigenspectra of minimally doubled fermions, *Proc. Sci. LATTICE2024* (2025) 355 [arXiv:2501.10336].
 - [18] S. Dürr and J. H. Weber, Dispersion relation and spectral range of Karsten-Wilczek and Borici-Creutz fermions, *Phys. Rev. D* **102**, 014516 (2020); *Phys. Rev. D* **108**, 059903(E) (2023).
 - [19] J. H. Weber, S. Capitani, and H. Wittig, Numerical studies of minimally doubled fermions, *Proc. Sci. LATTICE2013* (2014) 122 [arXiv:1312.0488].
 - [20] S. Basak, D. Chakrabarti, and J. Goswami, Mixed action with Borici-Creutz fermions on a staggered sea, *Phys. Rev. D* **96**, 074505 (2017).

- [21] M. Creutz, Minimal doubling and point splitting, *Proc. Sci. LATTICE2010* (**2010**) 078 [[arXiv:1009.3154](#)].
- [22] M. Creutz, T. Kimura, and T. Misumi, Index theorem and overlap formalism with naive and minimally doubled fermions, *J. High Energy Phys.* **12** (2010) 041.
- [23] B.C. Tiburzi, Chiral lattice fermions, minimal doubling, and the axial anomaly, *Phys. Rev. D* **82**, 034511 (2010).
- [24] J. H. Weber, Spin-taste representation of minimally doubled fermions from first principles: Karsten-Wilczek fermions, *Phys. Rev. D* **112**, 014509 (2025).
- [25] K. S. Shukre, D. Chakrabarti, and S. Basak, Chiral Lagrangian for Karsten-Wilczek minimally doubled fermion, *Proc. Sci. LATTICE2024* (**2025**) 357 [[arXiv:2411.14848](#)].
- [26] A. Lahiri, P. B. Pal, and M. Shokova, Discrete symmetry transformations on non-Abelian gauge fields, *Phys. Educator* **05**, 2350007 (2023).
- [27] S. Capitani, M. Creutz, J. Weber, and H. Wittig, Renormalization of minimally doubled fermions, *J. High Energy Phys.* **09** (2010) 027.