

μ -extension of iterated integrals and nested sums in quantum field theory

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The analytic integration of single-scale Feynman integrals emerging in perturbative calculations in quantum field theories can be performed within special classes of functions, which appear as consecutive generalizations of the polylogarithm in form of Kummer-Poincaré iterative integrals over special alphabets and extensions thereof. These are the polylogarithms, Nielsen integrals, iterated integrals over linear denominator terms, cyclotomic letters, letters induced by quadratic forms, and square-root valued letters. These integrals are solutions of first-order factorizing differential equations. They are related to specific nested sums via the Mellin transform and their expansions around $x = 0$. We construct the μ -extensions of these iterated integrals and the associated nested sums. We present closed form solutions or provide algorithms in the case of more involved cases to derive the respective μ -extensions and study the algebras of the μ -extended function spaces. Except for the case of square-root valued alphabets, the μ -extension maps into the same function space polynomially in μ . This is also the case for the associated nested sums. For square-root valued alphabets or sums containing central binomials, the μ -extension leads to higher transcendental functions. In all other cases the μ -extension preserves the Hopf algebra structure implied by the (quasi)shuffle product, by supplementing μ to the ground field.

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I. INTRODUCTION

The construction of q extensions of elementary and special functions, leading to the q calculus, has a long history in mathematics, as has been discussed in detail in Ref. [1]. It started with L. Euler's theory of partitions during the 1740s [2]. Different q analogs have been derived for various special functions, see Refs. [3–12] for surveys. They are applied to quantum mechanical systems modifying the Heisenberg algebra by q deformation [13,14]

$$\hat{x}\hat{p} - q\hat{p}\hat{x} = i, \quad (1)$$

with q the parameter of the extension.¹ Here $\hat{x}$ and $\hat{p}$ denote the position and the momentum operators. The extended functions are related to quantum groups [15,16].² The q extension in quantum field theory is implied by

¹In some extensions the rhs of Eq. (1) is modified by a function of a Hermitian operator.

²In the case of the q extensions the original field theory is obtained in the limit $q \rightarrow 1$.

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$$a_\lambda(k)a_{\lambda'}^\dagger(k') - qa_{\lambda'}^\dagger(k')a_\lambda(k) = g_{\lambda\lambda'}\delta^{(3)}(k - k'), \quad (2)$$

see Refs. [17,18], where $a_\lambda(k)$ and $a_{\lambda'}^\dagger(k')$ are the creation and annihilation operators. In this context also concepts of space-time quantization due to noncommutative geometries are discussed, cf. Refs. [19,20].³ Different kinds of q extensions are used in the literature. In Appendix A we give a brief survey on this.

Another extension consists of the μ deformation. Here the quantity μ is interpreted as a deformation parameter originally introduced by Jannussis [28], see also a more general approach in Ref. [29], in the context of deformations of the canonical Heisenberg algebra. In particular, within the framework of the q -deformed harmonic oscillators with the structure of Lie-admissible algebra [30], given by commutation relations

$$aa^\dagger - a^\dagger a = \varphi_\mu(N + 1) - \varphi_\mu(N), \quad \varphi_\mu(N) = \frac{N}{1 + \mu N},$$

$$\mu \in [0, 1], \quad (3)$$

where N denotes the number operator and μ the extension parameter, in the quantum mechanical case. For $\mu \rightarrow 0$ the

³For early attempts see, e.g., [21–23]. Snyder's approach in Ref. [23] is related, cf. Ref. [24], to the quantum Poincaré group, [15,25,26], see also Ref. [27].

nondeformed theory is obtained. An analogous relation holds for quantum field theories.

The higher functions discussed in the present paper emerge in loop corrections in the non- q or μ -deformed quantum field theories, such as quantum chromodynamics or quantum electrodynamics. One expects that in the case of the μ -extension the μ deformation of these functions is emerging.

The μ deformation has since been employed in various physical contexts, including deformed oscillator systems with quasi-Fibonacci spectrum and generalized Bose-gas models with deformed total number of particles, partition functions, equation of state, etc., which depend on the parameter μ [31,32]. These constructions modify the thermodynamic properties such as r -particle momentum correlation functions and virial coefficients. Here μ can change the interparticle interaction [29,33–35]. It also applies as entropic characteristics of composite boson and interacting systems, including the case of treating these jointly [36–39].

Furthermore, the μ deformation has been applied to diverse condensed systems, including models of galactic dark matter halo structures and induced quantum gravity with μ -deformed Einstein equations and a μ -dependent effective cosmological constant [39–45].

On the algebraic level, they lead to deformed Heisenberg-type algebras with nontrivial operator properties, including extensions exhibiting pseudo-Hermiticity and multiparameter generalizations, such as, e.g., (p, q, μ) deformations, cf. Refs. [45,46].

In the present paper we consider the μ -extension of higher special functions emerging in perturbative quantum field theory. Besides being of mathematical interest, structures of this kind may emerge in perturbative calculations in μ -deformed quantum field theories. Both in q - and μ -extended field theories corresponding extensions

of the polylogarithm have been already considered, cf. Refs. [34,47,48]. These functions form the simplest cases of function spaces studied in the present paper.

Structurally, there is a difference in the function spaces, for which we are constructing the extension in the following, and those that were mathematically considered in Refs. [3–12]. In the latter case the determining differential or difference equations are usually of low order, while for the functions we are considering the orders are growing with the weight of the functions.

The most simple functions appearing in the solution of the differential equations in quantum field theory are the logarithm $\ln(1-x)$ [49,50], the dilogarithm $\text{Li}_2(x)$ [51,52], and the polylogarithms $\text{Li}_n(x)$ [53,54], see also [55–57]. Here we consider the parameter $x \in [0, 1]$. The next level of extension consists in the Nielsen integrals [58,59], followed by the harmonic polylogarithms [60]. These are special subspaces of the generalized harmonic polylogarithms, which were introduced by E. Kummer, see Refs. [61–69]. Another extension of the harmonic polylogarithms is the cyclotomic harmonic polylogarithms [70] and the iterated integrals inspired by general quadratic forms [71].

In the previous cases the denominator functions are either linear functions or polynomials. Among the iterated integrals, which obey first order factorizing differential equations, there are also those derived by square-root valued letters [72]. In contrast to special functions that obey second order differential equations or can even be obtained as special cases of Gauß' hypergeometric function, the iterative integrals occurring in quantum field theoretic calculations obey higher order differential equations in general. Likewise, the corresponding nested sums obey higher order difference equations. In Table I we summarize these function spaces.

TABLE I. Different classes of nested sums and iterated integrals defined in Secs. III and IV.

Nested sums			
Type	Expression	Alphabet	Refs.
Harmonic sums	$S_{\vec{a}}(N)$	$\mathfrak{S}_H$	[73,74]
Generalized harmonic sums	$S_{\vec{a}}(\vec{b}, N)$	$\mathfrak{S}_{GH}$	[68,69]
Cyclotomic harmonic sums	$S_{\{a_1, b_1, c_1\}, \dots, \{a_l, b_l, c_l\}}(s_1, \dots, s_l)(N)$	$\mathfrak{S}_C$	[70]
Finite binomial sums	$\binom{2n}{n}^{\pm 1}$ occur in the letters of the alphabet	$\mathfrak{S}_B$	[72]
Iterated integrals			
Polylogarithms	$\text{Li}_n(x)$	$\mathfrak{A}_P$	[55–57]
Nielsen integrals	$S_{n,p}(x)$	$\mathfrak{A}_P$	[58,59]
Harmonic polylogarithms	$H_{\vec{a}}(x)$	$\mathfrak{A}_H$	[60]
Generalized harmonic polylogarithms	$G_{\vec{a}}(x)$	$\mathfrak{A}_{KP}$	[61,68,69]
Cyclotomic harmonic polylogarithms	$H_{\overrightarrow{\{a, b\}}}(x)$	$\mathfrak{A}_C$	[70]
Iterative integrals implied by quadratic forms	$H_{\overrightarrow{((k, b_k, c_k), d_k)}}(x)$	$\mathfrak{A}_R$	[71]
Iterative integrals over square-root valued letters	alphabet contains letters with factors of the kind $(ax^2 + bx + c)^{\pm 1/2}$	$\mathfrak{A}_{SQ}$	[72]

The iterated integrals are related to special types of nested sums via a Mellin transform from x space to N space by

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x). \quad (4)$$

In this way one obtains the harmonic sums [73,74], the generalized harmonic sums [68,69], the cyclotomic harmonic sums [70], the sums resulting from the functions in Ref. [71], and nested finite sums containing central binomials $^{2n}_n$ [72]. These sum structures do also occur in the Taylor expansion of the iterated integrals at $x = 0$.

The easiest way to construct the μ analog to a given special function $F(x)$ consists in determining the coefficient function $f[n]$ given by the expansion

$$F(x) = \sum_{n=0}^{\infty} f[n] x^n. \quad (5)$$

Here $f[n]$ has to be known in closed form for each value of n . In some cases $F(x)$ has logarithmic singularities to a finite order at $x = 0$ with the representation

$$F(x) = \sum_{m=0}^M \ln^m(x) \sum_{n=0}^{\infty} f_m[n] x^n. \quad (6)$$

Also cases with an expansion

$$F(x) = \sum_{m=0}^M \ln^m(x) \sum_{n=0}^{\infty} f_m[n] x^{n+\alpha}, \quad \alpha \in \mathbb{R} \quad (7)$$

are admissible. For $\alpha = -1/2$ they occur in the case of square-root valued alphabets, cf. Ref. [72].

In many cases the functions $f_m[n]$ are the solution of difference equations. If a large number of special values of these functions for $n = 0, 1, 2, \dots$ can be determined, one may use guessing algorithms [75–78] to obtain the corresponding difference equation

$$\sum_{l=0}^L p_l(n) f_m[n+l] = 0, \quad (8)$$

where $p_l(n)$ are polynomials in n . If Eq. (8) factorizes to first order, it can be solved by using the algorithms available in the package SIGMA [79,80]. For all the special functions given by iterated integrals defined in Sec. II A we will consider in this paper, the above steps can be carried out. The functions $f_m[n]$ have a closed form sum-product representation, for which one can obtain the μ analog at least in the form of an infinite series. In various cases these series can be summed. For real representations this is the method of choice, if simpler approaches are not available.

The same method can also be applied to calculate the Mellin inversion of these iterated integrals $F(x)$, which are given by nested sum-product structures. In this paper we will derive both the μ analogs to the different classes of iterated integrals and nested sums.

The paper is organized as follows. In Sec. II we define the iterated integrals and nested sums for which μ -extensions will be derived and discuss their (quasi)shuffle algebras. In Sec. III we introduce the iterated integrals for which we are going to derive the μ -extension. Likewise, we discuss the different finite nested sums appearing in this context in Sec. IV. In Sec. V we summarize the basic operations needed to derive the μ -extensions of the functions discussed in Secs. III and IV. The μ -extensions of the nested sums are calculated in Sec. VI and the ones of the iterated integrals in Sec. VII either in complete form or by devising algorithms by which one can obtain the extension in individual cases. Section VIII contains the conclusions. In Appendix A we summarize a number of variants of q extensions, which have been proposed in the literature. In Appendixes B and C we describe a series of technical aspects.

II. BASIC DEFINITIONS

In the following we will first define a series of properties of iterated integrals and nested sums and some basic operations which are used to get the μ -extension of these classes of functions.

A. Iterated integrals and nested sums

Given the alphabet $\mathfrak{A}$

$$\mathfrak{A} = \{f_1(x), \dots, f_m(x)\}, \quad (9)$$

where $f_i(x)$ are continuous functions in $x \in]0, 1[$, which are not higher transcendental functions. An iterated integral $G_{a_1, \dots, a_m}(x)$ over $\mathfrak{A}$ is defined by

$$G_{a_1, \dots, a_m}(x) = \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 f_{a_2}(y_2) \dots \times \int_0^{y_{m-1}} dy_m f_{a_m}(y_m). \quad (10)$$

Furthermore, we demand that all integrals contributing to (10) are indefinite, with an ordered sequence of variables $\{x, y_1, \dots, y_m\}$. Iterated integrals are the solution of a differential operator that factorizes at first order, with

$$\frac{d}{dx} G_{a_1, \dots, a_m}(x) = f_{a_1}(x) G_{a_2, \dots, a_m}(x). \quad (11)$$

Likewise, we consider the alphabet $\mathfrak{S}$

$$\mathfrak{S} = \{s_1(k), \dots, s_m(k)\}, \quad k \in \mathbb{N}. \quad (12)$$

Here, $s_l(k)$ are sum-product structures. The nested finite sums are defined by

$$S_{a,b_1,\dots,b_m}(N) = \sum_{k=1}^N s_a(k) S_{b_1,\dots,b_m}(k), \quad (13)$$

where the letters a, b_i label elements out of $\mathfrak{S}$.

B. Shuffle and quasishuffle algebras

The notion of alphabets $\mathfrak{S}$ and $\mathfrak{A}$ used below is essential for studying the products of iterated integrals and nested sums, deriving representations within the respective shuffle and quasishuffle algebras [68,74,81,82].

The products of two iterated integrals $G_{a_1,\dots,a_l}(x)$ and $G_{b_1,\dots,b_m}(x)$ are the sum of all iterated integrals of weight $l+m$ over all shuffles

$$\{c_1, \dots, c_{l+m}\} = \{a_1, \dots, a_l\} \sqcup \{b_1, \dots, b_m\}. \quad (14)$$

Here $\sqcup$ denotes the shuffle product. The indices a_i, b_j refer to the alphabet $\mathfrak{A}_F$ of functions to be iterated. Here the order of letters of the set $\{a_1, \dots, a_l\}$ and $\{b_1, \dots, b_m\}$ is preserved, and all other combinations are allowed. One obtains

$$G_{a_1,\dots,a_l}(x) \cdot G_{b_1,\dots,b_m}(x) = \sum_{C \in \{c_1,\dots,c_{l+m}\}} G_C(x). \quad (15)$$

The nested sums obey quasishuffle products, since there are additional trace terms, which vanish in the case of functions because they are of measure zero. We illustrate the quasishuffle relations for the product of generalized harmonic sums, which are most important in the following. The corresponding alphabet is given in Eq. (75). The product of the sums $S_{a_1,\dots,a_l}(c_1, \dots, c_l; N)$ and $S_{b_1,\dots,b_m}(d_1, \dots, d_m; N)$ is obtained by

$$\begin{aligned} S_{a_1,\dots,a_l}(c_1, \dots, c_l; N) \cdot S_{b_1,\dots,b_m}(d_1, \dots, d_m; N) &= \sum_{n=1}^N \frac{c_1^n}{n^{a_1}} S_{a_2,\dots,a_l}(c_2, \dots, c_l; n) \cdot S_{b_1,\dots,b_m}(d_1, \dots, d_m; n) \\ &+ \sum_{n=1}^N \frac{d_1^n}{n^{b_1}} S_{a_1,\dots,a_l}(c_1, \dots, c_l; n) \cdot S_{b_2,\dots,b_m}(d_2, \dots, d_m; n) \\ &- \sum_{n=1}^N \frac{(c_1 \cdot d_1)^n}{n^{a_1+b_1}} S_{a_2,\dots,a_l}(c_2, \dots, c_l; n) \cdot S_{b_2,\dots,b_m}(d_2, \dots, d_m; n) \end{aligned} \quad (16)$$

iteratively, cf. Refs. [68,69]. The application of Eq. (16) leads to a linear combination of generalized harmonic sums with integer coefficients, cf. Ref. [69], with additional terms beyond the shuffle contributions.

Various explicit examples in the case of harmonic sums are given in Refs. [74,82]. One example is

$$S_b(N) \cdot S_{a_1,\dots,a_l}(N) = S_{b,a_1,\dots,a_l}(N) + S_{a_1,b,a_2,\dots,a_l}(N) + \dots + S_{a_1,\dots,a_l,b}(N) - S_{b \wedge a_1, a_2, \dots, a_l}(N) - \dots - S_{a_1, a_2, \dots, b \wedge a_l}(N), \quad (17)$$

with

$$\begin{aligned} S_{b,a_1,\dots,a_l}(N) + S_{a_1,b,a_2,\dots,a_l}(N) + \dots + S_{a_1,\dots,a_l,b}(N) \\ = S_b(N) \sqcup S_{a_1,\dots,a_l}(N) \end{aligned} \quad (18)$$

and

$$a \wedge b = (|a| + |b|) \text{sign}(a) \text{sign}(b). \quad (19)$$

The numbers of algebraically independent iterative integrals and nested sums spanning the respective function space can be calculated by Witt-formulas [83,84] or by counting the number of Lyndon words [85–87], see also Ref. [82].

III. CLASSES OF ITERATED INTEGRALS

We consider the following classes of iterated integrals: the polylogarithms [55–57], the Nielsen integrals [58,59], harmonic polylogarithms [60], generalized harmonic polylogarithms, also called Kummer-Poincaré integrals [61–69], cyclotomic iterated integrals [70], iterated integrals implied by quadratic forms [71], and iterated integrals containing square-root valued letters [72].

A. Polylogarithms

Polylogarithms belong to the iterated integrals over the alphabet

$$\mathfrak{A}_p = \left\{ \frac{1}{x}, \frac{1}{1-x} \right\}. \quad (20)$$

The second letter only appears as the innermost letter, on which the first letter is iterated, see also Eq. (47).

For $x \in [-1, 1]$ the series representation

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^n}{k^n}, \quad n \geq 2, \quad (21)$$

holds. Eq. (21) is also the expansion around $x = 0$, according to Eq. (5). Furthermore, one has

$$x \frac{d}{dx} \text{Li}_n(x) = \text{Li}_{n-1}(x), \quad (22)$$

with

$$\text{Li}_1(x) = -\ln(1-x), \quad (23)$$

$$\text{Li}_0(x) = \frac{x}{1-x}. \quad (24)$$

For $n < 1$ the polylogarithms are rational functions. For $n < 2$ the polylogarithm diverges for $x = 1$.

B. Nielsen integrals

Nielsen integrals $S_{n,p}(x)$ [58,59] are all iterated integrals over the alphabet (20) and cover the classical polylogarithms

$$S_{n-1,1}(x) = \text{Li}_n(x). \quad (25)$$

They are defined by

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \ln^{n-1}(z) \ln^p(1-zx) \frac{dz}{z}, \quad n, p \geq 1, \quad (26)$$

see also Eq. (47). The integral representation (26) shows that $x = 1$ is a branch point both for polylogarithms and Nielsen integrals, at which a cut $[1, \infty)$ starts. In the case of the Nielsen integrals one first iterates the second letter in

$\mathfrak{A}_p$ p times, and then the first letter only. Allowing the iteration of both letters in arbitrary order leads to a subclass of the harmonic polylogarithms, cf. Sec. III C.

The hierarchy of Nielsen integrals is implied by

$$x \frac{d}{dx} S_{n,p}(x) = S_{n-1,p}(x), \quad (27)$$

$$S_{n,p}(x) = \int_0^x S_{n-1,p}(y) \frac{dy}{y}. \quad (28)$$

The differential relation allows also to define $S_{n,p}(x)$ for $n < 1$, unlike Eq. (26). Later on also the Nielsen integrals

$$S_{0,p}(x) = \frac{(-1)^p}{p!} \ln^p(1-x), \quad (29)$$

$$S_{-1,p}(x) = \frac{(-1)^{p-1}}{(p-1)!} \frac{x}{1-x} \ln^{p-1}(1-x), \quad (30)$$

contribute. They are diverging at $x = 1$.

The expansion of Nielsen integrals around $x = 0$ can be obtained in closed form,

$$S_{n,2}(x) = \sum_{k=2}^{\infty} S_1(k-1) \frac{x^k}{k^{n+1}}, \quad (31)$$

$$S_{n,3}(x) = \frac{1}{2!} \sum_{k=3}^{\infty} [S_1^2(k-1) - S_2(k-1)] \frac{x^k}{k^{n+1}}, \quad (32)$$

$$S_{n,4}(x) = \frac{1}{3!} \sum_{k=4}^{\infty} [S_1^3(k-1) - 3S_1(k-1)S_2(k-1) + 2S_3(k-1)] \frac{x^k}{k^{n+1}}, \quad (33)$$

for the lowest indices p . Here $S_l(k-1)$ denote the single harmonic sums.

The expansion coefficients in Eqs. (31)–(33) are related to Stirling numbers of the first kind [59,88,89]. They can be written as modified Faa di Bruno-Ramanujan determinants [74,90,91]

$$\begin{bmatrix} k \\ p \end{bmatrix} = \frac{1}{(p-1)!} \begin{vmatrix} S_1(k-1) & 1 & 0 & 0 & 0 & \dots & 0 \\ S_2(k-1) & S_1(k-1) & 2 & 0 & 0 & \dots & 0 \\ S_3(k-1) & S_2(k-1) & S_1(k-1) & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{p-1}(k-1) & \dots & \dots & \dots & \dots & S_1(k-1) \end{vmatrix}. \quad (34)$$

One obtains

$$S_{n,p}(x) = \sum_{k=p}^{\infty} \left[\begin{matrix} k \\ p \end{matrix} \right] \frac{x^k}{k^{n+1}}, \quad (35)$$

providing the representation of Eq. (5). Note that in the hierarchy equations (27), (28) the harmonic sums structures, cf. Sec. IV A, are not affected, which is important for the μ -extension in Sec. VII B.

C. Harmonic polylogarithms

The harmonic polylogarithms $H_{\vec{a}}(x)$ are iterated integrals (10) over the alphabet [60],

$$\mathfrak{A}_H = \{f_0(x), f_1(x), f_{-1}(x)\} \equiv \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right\}, \quad (36)$$

with

$$H_{b,\vec{a}}(x) = \int_0^x dy f_b(y) H_{\vec{a}}(y) \quad (37)$$

and

$$\frac{d}{dx} H_{b,\vec{a}}(x) = f_b(x) H_{\vec{a}}(x). \quad (38)$$

With longer and longer alphabets the expansions around $x = 0$ lead to more and more involved explicit structures. The simplest harmonic polylogarithms are

$$H_0(x) = \ln(x), \quad (39)$$

$$H_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad (40)$$

$$H_{-1}(x) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}. \quad (41)$$

The general series-representation of a specific harmonic polylogarithm can be obtained by the function GETMOMENT of the package HARMONICSUMS [60,69,70,72–74,82,92–103].⁴ The expansion coefficients $f[n]$ of all harmonic polylogarithms around $x = 0$ are polynomials of nested harmonic sums and rational function of n , including factors of $(-1)^n$. In the calculations performed in the present paper it is necessary to perform some of the infinite sums first as finite sums to an upper quantifier N by the package SIGMA [79,80] and afterward to perform the limit $N \rightarrow \infty$ using the routines SIGMAREDUCE and SIGMALIMIT.⁵

We consider an example at weight $w = 6$,

$$H_{-1,0,1,0,1,-1}(x) = \sum_{n=1}^{\infty} \left\{ \left[-\frac{1}{n^6} + \frac{S_5(n)}{n} - \frac{S_{-2,-3}(n)}{n} - \frac{S_{-4,-1}(n)}{n} + \frac{S_{-2,2,-1}(n)}{n} \right] (-1)^n + \frac{S_{-1}(n)}{n^5} + \frac{S_{-3}(n)}{n^3} - \frac{S_{2,-1}(n)}{n^3} \right\} x^n. \quad (42)$$

One obtains this result by

$$\text{GetMoment}[H[-1, 0, 1, 0, 1, -1, x], x, n]. \quad (43)$$

Differentiating Eq. (42) for x

```
HarmonicSumsSum[D[GetMoment[H[-1, 0, 1, 0, 1, -1, x], x, n]*x^n, x], {n, 1, Infinity}]
// TransformToSSums // SinfToH // VariableToArgument
```

one obtains

$$\frac{H_{0,1,0,1,-1}(x)}{1+x} \quad (44)$$

as expected, with

$$\frac{H_{0,1,0,1,-1}(x)}{1+x} = \sum_{n=1}^{\infty} (-1)^n [-S_5(n) + S_{-4,-1}(n) + S_{-2,-3}(n) - S_{-2,2,1}(n)] x^n. \quad (45)$$

⁴If not noted otherwise all commands quoted in the following are those of the package HARMONICSUMS.

⁵Commands in SIGMA and HARMONICSUMS may require one to newly define the sum type, which can be obtained by TOHARMONICSUMSSUM prior to TRANSFORMTOSSUMS.

The classical polylogarithms and the Nielsen integrals form subsets of the harmonic polylogarithms with

$$\text{Li}_n(x) = \text{H}_{\underbrace{0, \dots, 0}_{n-1}, 1}(x), \quad (46)$$

$$S_{n,p}(x) = \text{H}_{\underbrace{0, \dots, 0}_n, \underbrace{1, \dots, 1}_p}(x), \quad (47)$$

from which one may also obtain the relations Eqs. (31), (33), (35). However, both the polylogarithms and Nielsen integrals do not form shuffle algebras, since their products are in general not contained in these subsets. The harmonic polylogarithms form a shuffle algebra, cf. Sec. II B.

D. Generalized harmonic polylogarithms

Generalized harmonic polylogarithms, also called Kummer-Poincaré iterated integrals, Refs. [61–69], are iterated integrals over the alphabet

$$\mathfrak{A}_{KP} = \left\{ \frac{1}{x-c} \right\}, \quad c \in \mathbb{C}. \quad (48)$$

They are given by

$$G_{b,\vec{a}}(x) = \int_0^x dy f_b(y) G_{\vec{a}}(y), \quad f_c(x) \in \mathfrak{A}_{KP}. \quad (49)$$

Since $G_{b,\vec{a}}(x)$ may have poles in the integration regions, these have to be removed.⁶ The harmonic polylogarithms form a subset of the generalized harmonic polylogarithms. In the following we will use the symbol H also for G .

The expansion of generalized harmonic polylogarithms around $x = 0$ can be carried out in the same way as in Sec. III C. We illustrate this in the example of a weight $w = 4$ function $H_{-1/2,2,-1,2/3}(x)$. The command GetMoment [$H[-1/2, 2, -1, 2/3, x], x, n$] yields

$$H_{-1/2,2,-1,2/3}(x) = \sum_{n=1}^{\infty} \left\{ -\frac{\left(\frac{3}{2}\right)^n}{n^4} + \frac{(-1)^n S_1\left(-\frac{3}{2}, n\right)}{n^3} + \frac{2^{-n} S_2(3, n)}{n^2} + \frac{(-2)^n S_3\left(-\frac{3}{4}, n\right)}{n} - \frac{2^{-n} S_{1,1}\left(-2, -\frac{3}{2}, n\right)}{n^2} - \frac{(-2)^n S_{1,2}\left(-\frac{1}{4}, 3, n\right)}{n} - \frac{(-2)^n S_{2,1}\left(\frac{1}{2}, -\frac{3}{2}, n\right)}{n} + \frac{(-2)^n S_{1,1,1}\left(-\frac{1}{4}, -2, -\frac{3}{2}, n\right)}{n} \right\} x^n. \quad (50)$$

We check now the differentiation relation, based on Eq. (49). By applying the same computational steps as in the last section one obtains

$$\frac{d}{dx} H_{-1/2,2,-1,2/3}(x) = \sum_{n=1}^{\infty} 2(-2)^n \left[-S_3\left(-\frac{3}{4}, n\right) - S_{1,1,1}\left(-\frac{1}{4}, -2, -\frac{3}{2}, n\right) + S_{1,2}\left(-\frac{1}{4}, 3, n\right) + S_{2,1}\left(\frac{1}{2}, -\frac{3}{2}, n\right) \right] x^n. \quad (51)$$

The summation yields

$$\begin{aligned} \frac{d}{dx} H_{-1/2,2,-1,2/3}(x) &= \frac{2}{1+2x} H_{2/x, -1/x, 2/(3x)}(1) \\ &= \frac{1}{\frac{1}{2}+x} H_{2,-1,2/3}(x). \end{aligned} \quad (52)$$

Here one is advised to work term by term. The last step in Eq. (52) is performed by the command GLVariableToArgument

E. Cyclotomic harmonic polylogarithms

The cyclotomic harmonic polylogarithms [70] are iterative integrals over the alphabet

$$\mathfrak{A}_C = \left\{ \frac{p_{i,j}(x)}{C_i(x)} \right\}, \quad (53)$$

where $C_i(x)$ is the i th cyclotomic polynomial

$$\begin{aligned} &\{1-x, 1+x, 1+x+x^2, 1+x^2, \\ &1+x+x^2+x^3+x^4, 1-x+x^2, \dots\} \end{aligned} \quad (54)$$

and

$$p_{i,j} \in \{1, 1, \{1, x\}, \{1, x\}, \{1, x, x^2, x^3\}, \{1, x\}, \dots\}. \quad (55)$$

The cyclotomic polynomials are the real factors of

$$(1-x^n), \quad n \in \mathbb{N}, \quad n \geq 1, \quad (56)$$

counting the respective new polynomials factors.

⁶This is possible using the command GLRemovePole[f(x) x_0] of the package HARMONICSUMS, with x_0 the pole position.

This class of iterated integrals is already very involved. Individual expansions around $x = 0$ can be obtained by computer algebraic codes, however. To obtain the closed form solution of $f[n]$ one guesses the corresponding difference equation by using the algorithm of Refs. [75–78]. One obtains a first-order factorizing difference equation that can be solved by the package SIGMA [79,80] to obtain the corresponding closed form representation. This will involve cyclotomic harmonic sums.

Let us illustrate this in the following example. We consider the real-valued cyclotomic harmonic polylogarithm

$$H_{\{6,1\},\{1,0\},\{4,0\}}(x) \equiv \int_0^x dx_1 \frac{x_1}{1-x_1+x_1^2} \int_0^{x_1} dx_2 \frac{1}{-1+x_2} \int_0^{x_2} dx_3 \frac{1}{1+x_3^2}, \quad (57)$$

which contains letters of cyclotomy 6, cyclotomy 1, and cyclotomy 4. To obtain a closed-form series expansion $f[n]$ we have to change to a complex-valued representation⁷

$$H_{\{6,1\},\{1,0\},\{4,0\}}(x) = -\frac{1}{2(1+\sqrt{3}-1)} \left[i \left[G\left(\frac{1}{(-1)^{1/3}-\tau}, \frac{1}{1-\tau}, \frac{1}{i-\tau}; x\right) + G\left(\frac{1}{(-1)^{1/3}-\tau}, \frac{1}{1-\tau}, \frac{1}{i+\tau}; x\right) \right] \right. \\ \left. - (-1)^{5/6} \left[G\left(\frac{1}{(-1)^{2/3}+\tau}, \frac{1}{1-\tau}, \frac{1}{i-\tau}; x\right) + G\left(\frac{1}{(-1)^{2/3}+\tau}, \frac{1}{1-\tau}, \frac{1}{i+\tau}; x\right) \right] \right]. \quad (58)$$

Here G denotes an iterative integral, giving the list of letters, which we express by the symbolic integration variable τ in all individual letters.

The commands GLToH and GetMoment yield the desired result

$$f[n] = \frac{1}{2(1+\sqrt[3]{-1})n^3} \left\{ (-i)^{1+n} + i^{1+n} - (-1)^{5/6}(-i)^n + (-1)^{5/6}i^n + (i+\sqrt[6]{-1})(2-\sqrt[3]{-1}+2(-1)^{2/3})nS_1(-i, n) \right. \\ - ((i+\sqrt[6]{-1})(2-\sqrt[3]{-1}+2(-1)^{2/3})nS_1(i, n)) + n^2(-1)^{\frac{1}{6}(5+2n)}S_2(-\sqrt[6]{-1}, n) \\ - n^2(-1)^{\frac{1}{6}(5+2n)}S_2(\sqrt[6]{-1}, n) + in^2e^{-\frac{1}{3}in\pi}S_2(-(-1)^{5/6}, n) - in^2e^{-\frac{1}{3}in\pi}S_2((-1)^{5/6}, n) \\ - in^2e^{-\frac{1}{3}in\pi}S_{1,1}(\sqrt[3]{-1}, -i, n) + in^2e^{-\frac{1}{3}in\pi}S_{1,1}(\sqrt[3]{-1}, i, n) - n^2(-1)^{\frac{1}{6}(5+2n)} \\ \left. S_{1,1}(-(-1)^{2/3}, -i, n) + n^2(-1)^{\frac{1}{6}(5+2n)}S_{1,1}(-(-1)^{2/3}, i, n) \right\}, \quad (59)$$

given by generalized sums with complex-valued weights. The first expansion terms read

$$H_{\{6,1\},\{1,0\},\{4,0\}}(x) = \frac{x^4}{8} + \frac{x^5}{6} + \frac{x^6}{12} - \frac{x^7}{35} - \frac{5x^8}{72} - \frac{73x^9}{2835} + \frac{29x^{10}}{700} + O(x^{11}). \quad (60)$$

The derivative of Eq. (57) is given by

$$\frac{x}{1-x+x^2} H_{\{1,0\},\{4,0\}}(x) = \sum_{n=1}^{\infty} x^n \frac{1}{2(1+\sqrt{3}-1)} \\ = x \left\{ [-S_2(-\sqrt[6]{-1}, n) - S_{1,1}(-(-1)^{2/3}, i, n) + S_2(\sqrt[6]{-1}, n) + S_{1,1}(-(-1)^{2/3}, -i, n)](-1)^{n/3} \right. \\ + [-S_2((-1)^{5/6}, n) - S_{1,1}(\sqrt[3]{-1}, -i, n) + S_2(-(-1)^{5/6}, n) + S_{1,1}(\sqrt[3]{-1}, i, n)] \\ \left. \times (-1)^n(-\sqrt[3]{-1})^{2n} \right\} \sqrt[6]{-1} \\ = \frac{-ix}{2(1-x+x^2)} [-S_2(ix, \infty) - S_{1,1}(x, -i, \infty) + S_2(-ix, \infty) + S_{1,1}(x, i, \infty)]. \quad (61)$$

F. Iterated integrals implied by quadratic forms

The alphabet $\mathfrak{A}_R$ for these integrals, cf. Ref. [71], is given by

$$\mathfrak{A}_R = \mathfrak{A}_{KP} \cup \left\{ \frac{1}{x^2 + b_i x + c_i}, \frac{x}{x^2 + b_i x + c_i} \right\}, \quad b_i, c_i \in \mathbb{R}, 4c_i > b_i^2. \quad (62)$$

The notation for the index set is illustrated by

$$H_{((1,b_1,c_1),d_1)}(x) = \int_0^x dy \frac{y^{d_1}}{y^2 + b_1 y + c_1}, \quad d_i \in \{0, 1\}. \quad (63)$$

We consider the example $H_{((1,1,2),0),((0,1,-1),0),((0,1,1),0)}(x)$, for which $f[n]$ shall be derived. The original integrals shall be reformulated in terms of general iterative integrals first before one obtains by the methods described before the analytic representation in Eq. (64). By applying the commands

```
GetMoment [GL [ { 1 / (VarGL^2 + VarGL - 2) , 1 / (VarGL + 1) , 1 / (VarGL - 1) } , x] // GLToL //
LToH, x, n] // ReduceToBasis
```

one derives

$$\begin{aligned} H_{((1,1,-2),0),((0,1,-1),0),((0,1,1),0)}(x) &= \int_0^x dy_1 \frac{1}{y_1^2 + y_1 - 2} \int_0^{y_1} dy_2 \frac{1}{y_2 + 1} \int_0^{y_2} dy_3 \frac{1}{y_3 - 1} \\ &= \sum_{n=1}^{\infty} \frac{1}{3n} \left[\frac{1}{2} S_2(n) - \frac{1}{2} S_{-1}(n)^2 - \left(-\frac{1}{2}\right)^n S_2(-2, n) + \left(-\frac{1}{2}\right)^n S_{1,1}(2, -1, n) \right] x^n. \end{aligned} \quad (64)$$

Finally, we check the differentiation relation for this example. We apply to the derivative

$$T_1 = \sum_{n=1}^{\infty} f[n] n x^{n-1}, \quad (65)$$

with $f[n]$ from Eq. (64), the commands

```
{TransformToSSums // SinfToH // VariableToArgument}.
```

This results in the expected result

$$T_1 = \frac{d}{dx} H_{((1,1,-2),0),((0,1,-1),0),((0,1,1),0)}(x) = -\frac{H_{-1,1}(x)}{x^2 + x - 2}. \quad (66)$$

G. Iterated integrals over alphabets containing square-root valued letters

Nested sums containing central binomials have been studied in Ref. [72]. They are related to iterated integrals with square-root valued letters, for which various classes and many examples were given in [72], Eqs. (3.30–3.61), for the letters of the square-root valued letters of the alphabet $\mathfrak{A}_{SQ}$. We first derive the function $f[n]$ in these cases. Let us consider the following iterated integral

$$T_2 = \int_0^x dy_1 \frac{1}{\sqrt{y_1(1-y_1)}} \int_0^{y_1} dy_2 \frac{1}{1+y_2} \int_0^{y_2} dy_3 \frac{1}{\sqrt{1-y_3}}. \quad (67)$$

By applying the command

```
GLToS [T_2]
```

one obtains

$$\begin{aligned}
 T_2 = \frac{1}{\sqrt{x}} & \left\{ \frac{x^3}{5} + \frac{x^5}{16} + \sum_{o_1=6}^{\infty} \frac{8}{(2o_1-1)^3} \left\{ 2^{-2o_1} \binom{2o_1}{o_1} \left[1 - \left(\left(2 + S_1(o_1) + \sum_{i_1=1}^{o_1} \frac{(-1)^{i_1} 2^{2i_1}}{\binom{2i_1}{i_1} i_1} \right. \right. \right. \right. \\
 & \left. \left. \left. + \sum_{i_1=1}^{o_1} \frac{(-1)^{i_1} 2^{2i_1} \sum_{i_2=1}^{i_1} \frac{(-1)^{i_2} 2^{-2i_2} \binom{2i_2}{i_2}}{-1+2i_2}}{\binom{2i_1}{i_1} i_1} - 2 \sum_{i_1=1}^{o_1} \frac{1}{-1+2i_1} \right) o_1 \right] + 2 \left(2 + S_1(o_1) \right. \right. \\
 & \left. \left. + \sum_{i_1=1}^{o_1} \frac{(-1)^{i_1} 2^{2i_1}}{\binom{2i_1}{i_1} i_1} + \sum_{i_1=1}^{o_1} \frac{(-1)^{i_1} 2^{2i_1} \sum_{i_2=1}^{i_1} \frac{(-1)^{i_2} 2^{-2i_2} \binom{2i_2}{i_2}}{-1+2i_2}}{\binom{2i_1}{i_1} i_1} - 2 \sum_{i_1=1}^{o_1} \frac{1}{-1+2i_1} \right) o_1^2 \right] \right. \\
 & \left. - (-1)^{o_1} \left[1 + \sum_{i_1=1}^{o_1} \frac{(-1)^{i_1} 2^{-2i_1} \binom{2i_1}{i_1}}{-1+2i_1} \right] (-1+2o_1) \right\} x^{o_1} \Bigg\}, \tag{68}
 \end{aligned}$$

from which $f[o_1]$ can be obtained up to a polynomial in x and the overall weight by $1/\sqrt{x}$. The reverse command to transform Eq. (68) into Eq. (67) is

$$\text{ComputeGeneratingFunction}[f[n], x\{n, 6, \text{Infinity}\}] \tag{69}$$

for the main contribution and adding the remaining terms.

We check the differential relation, upon which the hierarchy of square-root valued iterated integrals is built. With the command GLTOS applied to the derivative Eq. (68) we obtain the representation

$$\frac{d}{dx} T_2 = \frac{x^2}{2} + \sum_{n=3}^{\infty} x^n \left[\frac{2^{1-2n} \binom{2n}{n}}{n(-1+2n)} - \frac{1}{n} 2(-1)^n \sum_{i_1=1}^n \frac{(-\frac{1}{4})^{i_1} \binom{2i_1}{i_1}}{-1+2i_1} \right]. \tag{70}$$

Using ComputeGeneratingFunction results in

$$\frac{d}{dx} T_2 = \frac{2}{\sqrt{x(1-x)}} \left[\int_0^x dy \left(\frac{1}{1+y} - \frac{\sqrt{1-y}}{1+y} \right) \right]. \tag{71}$$

This equals

$$\frac{d}{dx} T_2 = \frac{1}{\sqrt{x(1-x)}} \int_0^x dy_1 \frac{1}{1+y_1} \int_0^{y_1} dy_2 \frac{1}{\sqrt{1-y_2}}, \tag{72}$$

which can be seen differentiating Eqs. (71), (72).

H. Non first order factorizing iterated integrals

In quantum field theoretic calculations also iterated integrals contribute, the alphabets of which contain higher transcendental letters. These functions are also called

iterating noniterative integrals [104,105]. The higher transcendental letters are solutions of nonfactorizing differential equations at least of order two [106–114]. The simplest structures are ${}_2F_1$ -solutions of second order differential equations, which are related to complete elliptic integrals and modular forms [112]. However, higher genus solutions contribute as well. Various of those result from Calabi-Yau differential equations, cf., e.g., Ref. [115]. The μ -extension of these integrals requires different techniques than the ones used in the present paper. General algorithms to address these in Mellin space are not yet available.

IV. CLASSES OF NESTED SUMS

The iterated integrals discussed in Sec. III are related to a series of classes of nested sums by the Mellin transform, Eq. (4). We discuss the following classes of nested sums:

the harmonic sums [73,74], the generalized harmonic sums [68,69], the cyclotomic harmonic sums [70], and the binomial harmonic sums [72].

A. Nested harmonic sums

The single harmonic sums are given by

$$S_{b,\vec{a}}(N) = \sum_{m=1}^N \frac{(\text{sign}(b))^m}{m^{|b|}} S_{\vec{a}}(k), \quad a_i, b \in \mathbb{Z} \setminus \{0\}, \quad S_{\emptyset} = 1, \quad (73)$$

see Refs. [73,74]. The corresponding alphabet $\mathfrak{S}_H$ is

$$\mathfrak{S}_H = \left\{ \frac{(\text{sign}(c))^m}{m^{|c|}} \right\}, \quad c \in \mathbb{Z} \setminus \{0\}, \quad m \in \mathbb{N} \setminus \{0\}. \quad (74)$$

B. Generalized nested harmonic sums

The generalization of nested harmonic sums are implied by the alphabet $\mathfrak{S}_{GH}$

$$\mathfrak{S}_{GH} = \left\{ \frac{c^m}{m^a} \right\}, \quad c \in \mathbb{C} \setminus \{0\}, \quad m, a \in \mathbb{N} \setminus \{0\}. \quad (75)$$

These sums are given by

$$S_{b,\vec{a}}(d, \vec{c}; N) = \sum_{m=1}^N \frac{d^m}{m^b} S_{\vec{a}}(\vec{c}; k), \quad b, a_i \in \mathbb{N} \setminus \{0\}, \quad d, c_i \in \mathbb{C} \setminus \{0\}, \quad S_{\emptyset} = 1, \quad (76)$$

cf. Refs. [68,69]. Furthermore, one has

$$S_0(\{d\}; N) = \sum_{k=1}^N d^k = d \frac{d^N - 1}{d - 1}. \quad (77)$$

C. Nested cyclotomic harmonic sums

The alphabet for this class, $\mathfrak{S}_C$, is given by,

$$\mathfrak{S}_C = \left\{ \frac{s_1^k}{(a_1 k + b_1)^{c_1}} \right\}, \quad a_i, c_i \in \mathbb{N} \setminus \{0\}, \quad b_i \in \mathbb{N}, \quad s_1 = \pm 1, \quad a_i > b_i. \quad (78)$$

cf. Ref. [70]. The cyclotomic sums are given iteratively by

$$\begin{aligned} S_{\{a_1, b_1, c_1\}, \dots, \{a_l, b_l, c_l\}}(s_1, \dots, s_l)(N) \\ = \sum_{k=1}^N \frac{s_1^k}{(a_1 k + b_1)^{c_1}} S_{\{a_2, b_2, c_2\}, \dots, \{a_l, b_l, c_l\}}(s_2, \dots, s_l)(k), \\ S_{\emptyset} = 1. \end{aligned} \quad (79)$$

These are the real representations. In the expansion of the cyclotomic harmonic polylogarithms around $x = 0$ related sums are occurring, which are generalized harmonic sums with complex-valued numerator weights. These representations can be obtained by using the command `CSTORUS`. An example is

$$\begin{aligned} \sum_{k=1}^N \frac{S_1(k)}{(-2+3k)} &= \frac{1}{6(1+3N)^2} \left[-2(2+9N) + (4+9N+i\sqrt{3}-2(e^{4\pi i/3})^{3N})(e^{2\pi i/3})^{3N} + i(i+\sqrt{3})(e^{2\pi i/3})^{6N} \right. \\ &\quad + (4+9N-i\sqrt{3}-i(-i+\sqrt{3})(e^{4\pi i/3})^{3N})(e^{4\pi i/3})^{3N} \left. \right] - \frac{3(1+N)S_1(N)}{2(1+3N)} \\ &\quad + \frac{1}{6(1+3N)} (5+9N+2(e^{2\pi i/3})^{3N}+2(e^{4\pi i/3})^{3N})S_1(3N) + \frac{1}{3}S_{1,1}(3N) \\ &\quad + \left[\frac{1+(e^{2\pi i/3})^{3N}+(e^{4\pi i/3})^{3N}}{3(1+3N)} - \frac{1}{2}\sqrt[3]{-1} \right] S_1(e^{2\pi i/3}, 3N) \\ &\quad + \left[\frac{1+(e^{2\pi i/3})^{3N}+(e^{4\pi i/3})^{3N}}{3(1+3N)} + \frac{1}{2}(-1)^{2/3} \right] S_1(e^{4\pi i/3}, 3N) + \frac{1}{3}S_{1,1}(1, e^{2\pi i/3}, 3N) \\ &\quad + \frac{1}{3}S_{1,1}(1, e^{4\pi i/3}, 3N) - \frac{1}{3}\sqrt[3]{-1}S_{1,1}(e^{2\pi i/3}, 1, 3N) - \frac{1}{3}\sqrt[3]{-1}S_{1,1}(e^{2\pi i/3}, e^{2\pi i/3}, 3N) \\ &\quad - \frac{1}{3}\sqrt[3]{-1}S_{1,1}(e^{2\pi i/3}, e^{4\pi i/3}, 3N) + \frac{1}{3}(-1)^{2/3}S_{1,1}(e^{4\pi i/3}, 1, 3N) \\ &\quad + \frac{1}{3}(-1)^{2/3}S_{1,1}(e^{4\pi i/3}, e^{2\pi i/3}, 3N) + \frac{1}{3}(-1)^{2/3}S_{1,1}(e^{4\pi i/3}, e^{4\pi i/3}, 3N). \end{aligned} \quad (80)$$

D. Nested sums containing central binomials

The alphabet for this class of sums $\mathfrak{A}_B$, cf. Ref. [72], may contain the letters out of the forgoing alphabets, which are multiplied by the terms

$$\frac{1}{4^k} \binom{2k}{k}, \quad \text{and/or} \quad 4^k \binom{2k}{k}. \quad (81)$$

This also applies to subsums.

V. THE μ -EXTENSION: BASIC RELATIONS

Let us now turn to the μ -extension of the special functions discussed in the previous Sections. We first summarize basic relations occurring in the μ -extension, which form the basic building blocks to compute the μ -extensions of the iterated integrals and nested sums.

The μ -extension of the number n is given by [43,116]

$$\{n\}_\mu = \frac{n}{1 + \mu n}. \quad (82)$$

This is also sometimes called the μ bracket. The original functions are obtained in the limit $\mu \rightarrow 0$ for the corresponding μ -extensions. Polynomial terms and summands in the nested sums contain terms of the type

$$\frac{1}{\{n\}_\mu^l} = \sum_{k=0}^{l-1} \binom{l}{k} \mu^k \frac{1}{n^{l-k}} + \mu^l. \quad (83)$$

The term μ^l causes singular contributions in the limit $x \rightarrow 1$ in several cases discussed below.

Since sums over the terms Eq. (83) will be performed, the term μ^l will introduce *integer sums* instead of *harmonic sums*, the former ones are divergent in the limit $N \rightarrow \infty$. The μ -extension Eq. (82) destroys the structural property of harmonic summation by the term $\propto \mu^l$. This divergence transforms into the one of the x -space functions like the μ -extended polylogarithms, Sec. VII A, by the contribution of $\text{Li}_0(x)$, and similarly to the other functions.

The μ -extension of the factorial n is given by

$$\{n\}_\mu! = \prod_{k=1}^n \frac{k}{1 + \mu k}. \quad (84)$$

The μ -extension of the central binomial $\binom{2n}{n}$ reads

$$\binom{2n}{n} \Big|_\mu = \frac{\{2n\}_\mu!}{(\{n\}_\mu!)^2}. \quad (85)$$

The μ derivative is given by

$$\begin{aligned} \mathcal{D}_x^{(\mu)} &= \frac{d}{dx} \left[1 + \mu x \frac{d}{dx} \right]^{-1} \\ &= \frac{d}{dx} \left[1 - \mu \left(x \frac{d}{dx} \right) + \mu^2 \left(x \frac{d}{dx} \right)^2 - \dots \right] \end{aligned} \quad (86)$$

to be understood as expanded series in μ .⁸ One obtains

$$\mathcal{D}_x^{(\mu)} x^n = \{n\}_\mu x^{n-1}. \quad (87)$$

The inverse operation is

$$\int_0^x t^{n-1} \mathcal{D}_t^{(\mu)} = \frac{x^n}{\{n\}_\mu}. \quad (88)$$

VI. THE μ -EXTENSION OF NESTED SUMS

We first consider the μ -extension of several types of nested sums, which will later occur in the μ -extension of the different iterated integrals. These are the harmonic sums, the generalized harmonic sums with complex numerator weights, and the finite (inverse) binomial sums.

A. μ -extension of harmonic sums

The μ -extension of the single harmonic sums is given by

$$S_m(N; \mu) = \sum_{n=1}^N \frac{1}{(\{n\}_\mu)^m} = \sum_{k=0}^m \binom{m}{k} \mu^k S_{m-k}(N), \quad m > 0, \quad m \in \mathbb{N}, \quad (89)$$

$$S_{-|m|}(N; \mu) = \sum_{n=1}^N \frac{(-1)^n}{(\{n\}_\mu)^{|m|}} = \sum_{k=0}^{|m|} \binom{|m|}{k} \mu^k S_{-|m|+k}(N), \quad m < 0, \quad m \in \mathbb{Z}. \quad (90)$$

⁸We note that this μ -deformed derivative was involved in the respective μ -deformed generalization of the Lane-Emden equation enabling the successful modeling [42] of the rotation curves of a number of dwarf galaxies. From two distinct (though equivalent) μ -deformed versions of the Lane-Emden equation, a novel deformation of Heisenberg algebra has been derived leading to unusual uncertainty relations [45].

Here also $S_0(N)$ ⁹

$$S_0(N) = N \quad (91)$$

contributes and

$$S_1(N) \propto \ln(N) + \gamma_E, \quad \text{for } N \rightarrow \infty, \quad (92)$$

where γ_E is the Euler-Mascheroni constant. Note that the inverse Mellin transforms of integer powers of N are differential operators and no longer, like in the case of nested sums, regular functions or $+$ -distributions, cf., e.g., [74].

Since both the harmonic sums $S_1(N)$ and $S_0(N)$ contribute to the μ -extensions of all single harmonic sums, a definition of $\lim_{N \rightarrow \infty} S_m(N; \mu) = \zeta_m(\mu)$, $m \geq 2$ is not possible.

For the nested harmonic sums one obtains

$$S_{b,\vec{a}}(N; \mu) = \sum_{n=1}^N \frac{(\text{sign}(b))^n}{(\{n\}_\mu)^{|b|}} S_{\vec{a}}(N; \mu), \quad b, a_i \in \mathbb{Z} \setminus \{0\}. \quad (93)$$

By binomial decomposition also $S_{b,\vec{a}}(N; \mu)$ can be represented as a polynomial in μ over nested harmonic sums.

For $\forall a_i > 0$ one obtains

$$S_{\vec{a}}(N; \mu) = \sum_{\alpha_1=0}^{a_1} \binom{a_1}{\alpha_1} \dots \sum_{\alpha_m=0}^{a_m} \binom{a_m}{\alpha_m} \mu^{\alpha_1 + \dots + \alpha_m} \times S_{a_1 - \alpha_1, \dots, a_m - \alpha_m}(N). \quad (94)$$

The case of values of $a_i < 0$ will also be handled as for the generalized harmonic sums. Furthermore, one has

$$S_{\underbrace{0, \dots, 0}_k}(N) = \frac{(N+k-1)!}{(N-1)!k!}, \quad (95)$$

which diverge like $\propto N^k$ for $N \rightarrow \infty$. This behavior translates into a corresponding singularity for $x \rightarrow 1$ for the associated iterative integrals.

These representations can be used to express products of μ -extended harmonic sums $S_{\vec{a}}(N; \mu)$ as polynomials in μ of related nested harmonic sums $S_{\vec{b}}(N)$. The latter ones form a quasishuffle algebra [68,74,81,82] and due to the shuffle product a Hopf algebra [117], see Refs. [118–121].

B. μ -extension of the Stirling numbers of the first kind

The μ -deformed Stirling numbers of the first kind emerge in the general representation of the Nielsen integrals, Eq. (34). As we consider them only in this context, the counter p is not μ extended, but remains an integer. The respective determinant

$$\left[\begin{matrix} n \\ p \end{matrix} \right]_\mu = \frac{1}{(p-1)!} \begin{vmatrix} S_1(n-1; \mu) & 1 & 0 & 0 & 0 & \dots & 0 \\ S_2(n-1; \mu) & S_1(n-1; \mu) & 2 & 0 & 0 & \dots & 0 \\ S_3(n-1; \mu) & S_2(n-1; \mu) & S_1(n-1; \mu) & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{p-1}(n-1; \mu) & \dots & \dots & \dots & \dots & S_1(n-1; \mu) \end{vmatrix} \quad (96)$$

remains the one of a $(p-1) \times (p-1)$ dimensional matrix. The μ -deformed single harmonic sums needed are given in Eq. (90).

C. μ -extension of generalized harmonic sums

The generalized harmonic sums [69] have the following μ -extension

$$\begin{aligned} S_{b,\vec{a}}(d, \vec{c}, N; \mu) &= \sum_{n=1}^N \frac{d^n}{(\{n\}_\mu)^b} S_{\vec{a}}(\vec{c}, N; \mu), \quad b, a_i \in \mathbb{N} \setminus \{0\}, d, c_i \in \mathbb{C} \setminus \{0\} \\ &= \sum_{k=1}^b \binom{b}{k} \mu^k \sum_{n=1}^N \frac{d^n}{n^{b-k}} S_{\vec{a}}(\vec{c}, N; \mu). \end{aligned} \quad (97)$$

⁹This index is usually not considered for harmonic sums, but needs to be included here.

One obtains

$$S_{\vec{a}}(\vec{c}, N; \mu) = \sum_{\alpha_1=0}^{a_1} \binom{a_1}{\alpha_1} \dots \sum_{\alpha_m=0}^{a_m} \binom{a_m}{\alpha_m} \mu^{\alpha_1 + \dots + \alpha_m} \times S_{a_1 - \alpha_1, \dots, a_m - \alpha_m}(\vec{c}, N). \quad (98)$$

Again the μ -extended generalized harmonic sums are spanned by the generalized harmonic sums and their quasishuffle algebra implies that they form a Hopf algebra.

D. μ -extension of finite binomial sums

The generalized μ -extended harmonic sums are extended by factors of

$$\left(\binom{2n}{n} \right) \Big|_{\mu} = \frac{\prod_{k=n+1}^{2n} (k + \frac{1}{\mu})}{\prod_{k=1}^n (k + \frac{1}{\mu})} = \frac{(n + 1 + \frac{1}{\mu})_n}{(1 + \frac{1}{\mu})_n}, \quad (99)$$

which is a (growing) rational function in μ of numerator and denominator degree n .

We consider the most simple sum for which we obtain

$$\sum_{n=1}^N \frac{1}{4^n} \left(\binom{2n}{n} \right) \Big|_{\mu} = \frac{(1 + 2\mu)}{4(1 + \mu)} {}_3F_2 \left[\begin{matrix} 1 & \frac{3}{2} + \frac{1}{2\mu} & 2 + \frac{1}{2\mu} \\ 2 + \frac{1}{\mu} & 2 + \frac{1}{\mu} \end{matrix}; 1 \right] - \frac{4^{-N} \Gamma(1 + \frac{1}{\mu}) \Gamma(1 + \frac{1}{\mu} + 2N)}{\Gamma(1 + \frac{1}{\mu} + N)^2} \times \left[-1 + {}_3F_2 \left[\begin{matrix} 1 & \frac{1}{2} + \frac{1}{2\mu} + N & 1 + \frac{1}{2\mu} + N \\ 1 + N + \frac{1}{\mu} & 1 + N + \frac{1}{\mu} \end{matrix}; 1 \right] \right]. \quad (100)$$

It is evident that the μ -extension results in higher transcendental functions, like the generalized hypergeometric functions here [5,6]. The recurrences of these functions are not first order factorizable, similar to the functions discussed in Sec. III H. This structure will persist for more involved sums. In this case the μ -extension moves out of the algebra for the μ -free expressions.

In the present paper, we are focusing on sums, the recurrences of which are first order factorizing. For small values of μ one may consider a Taylor expansion for the sums containing central binomials. This is discussed in Appendix B.

VII. THE μ -EXTENSION OF ITERATED INTEGRALS

The μ -extensions of iterated integrals are obtained from their analytic expansion around $x = 0$, Eq. (5). Some of the representations can be given in closed form in the general case. For others one can compute the individual cases algorithmically.

A. The μ deformation of the polylogarithm

One obtains

$$\text{Li}_k(x; \mu) = \sum_{n=1}^{\infty} \frac{x^n}{(\{n\}_{\mu})^k} = \sum_{l=0}^k \binom{k}{l} \mu^l \text{Li}_{k-l}(x), \quad x \in [-1, 1]. \quad (101)$$

Examples are

$$\begin{aligned} \text{Li}_2(x; \mu) &= \sum_{k=1}^{\infty} \left(\frac{1 + \mu k}{k} \right)^2 x^k \\ &= \text{Li}_2(x) + 2\mu \text{Li}_1(x) + \mu^2 \text{Li}_0(x), \end{aligned} \quad (102)$$

$$\text{Li}_1(x; \mu) = \sum_{k=1}^{\infty} \left(\frac{1 + \mu k}{k} \right) x^k = \text{Li}_1(x) + \mu \text{Li}_0(x), \quad (103)$$

$$\text{Li}_0(x; \mu) = \sum_{k=1}^{\infty} x^k = \text{Li}_0(x). \quad (104)$$

Because of this, the μ -deformed polylogarithm¹⁰ is a linear function of all classical polylogarithms for $k \geq 0$, with a polynomial in μ . In the limit $x \rightarrow 1$, $\text{Li}_k(x; \mu)$ diverges like

$$\text{Li}_k(x; \mu) \propto \mu^k \frac{x}{1-x} - k\mu^{k-1} \ln(1-x). \quad (105)$$

Therefore, one cannot derive the μ -extended ζ values from this representation in the limit $x \rightarrow 1$. Yet another form of a μ -deformed polylogarithm has been introduced in Ref. [34,41].

However, one may also define the ζ function using the Dirichlet η function,

¹⁰Note that a version of the μ polylogarithm was used in conjunction with the μ -Bose gas model [34]. In Reference [41] the representation $\text{Li}_s(x; \mu) = \sum_{k=1}^{\infty} x^k / k^s (1 + \mu k)$ has been used. The associated ζ values are described in the Appendix C. Also in Ref. [9] deviating definitions in the case of the q -dilogarithm are reported, where not all powers of n are changed to $(1 - q^n)/(1 - q)$.

$$\zeta_s = \frac{1}{1-2^{1-s}} \eta_s, \quad s \in \mathbb{Z} \setminus \{1\}, \quad (106)$$

where η_s is defined by

$$\eta_s = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = \text{Li}_s(-1), \quad (107)$$

see Refs. [74,122–124]. The μ -extended η function may be obtained as follows. First one calculates the μ -extension of $\text{Li}_s(-x)$ for $x > -1$,

$$\text{Li}_s(-x; \mu) = - \sum_{i=0}^s \binom{s}{i} \mu^i \text{Li}_{s-i}(-x), \quad (108)$$

with $\text{Li}_0(-x) = x/(1+x)$ and

$$\text{Li}_s(-1) = \left(1 - \frac{1}{2^{s-1}}\right) \zeta_s, \quad s \geq 2, \quad \text{Li}_1(-1) = \ln(2). \quad (109)$$

$\text{Li}_0(-1)$ is now obtained as analytic continuation $\text{Li}_0(-1) = 1/2$, by which

$$\eta_s(\mu) = \text{Li}_s(-1; \mu). \quad (110)$$

We finally apply the derivative (86) to the μ -extended polylogarithm using Eq. (87)

$$x \mathcal{D}_x^{(\mu)} \text{Li}_m(x; \mu) = \sum_{n=1}^{\infty} \frac{x^n}{(\{n\}_{\mu})^{m-1}} = \text{Li}_{m-1}(x; \mu). \quad (111)$$

Equation (111) defines the set of μ -extended polylogarithms hierarchically.

B. The μ deformation of the Nielsen integrals

By using Eq. (35) and replacing the single harmonic sums and the denominator function by their μ -deformed values one obtains the μ -deformed Nielsen integrals $S_{n,p}(x; \mu)$,

$$S_{n,p}(x; \mu) = \sum_{k=p}^{\infty} \left[\begin{matrix} k \\ p \end{matrix} \right]_{\mu} \frac{x^k}{\{k\}_{\mu}^{n+1}}. \quad (112)$$

An example is

$$\begin{aligned} S_{n,2}(x; \mu) &= \sum_{k=2}^{\infty} S_1(k-1; \mu) \frac{x^k}{\{k\}_{\mu}^{n+1}} \\ &= \sum_{k=2}^{\infty} [S_1(k-1) + \mu(k-1)] \frac{x^k}{\{k\}_{\mu}^{n+1}} \\ &= \sum_{l=0}^{n+1} \binom{n+1}{l} \mu^l \sum_{k=2}^{\infty} [S_1(k-1) + \mu(k-1)] \frac{x^k}{k^{n+1-l}} \\ &= \sum_{l=0}^{n+1} \binom{n+1}{l} \mu^l [S_{n-l,2}(x) + \mu(\text{Li}_{n-l}(x) - \text{Li}_{n+1-l}(x))]. \end{aligned} \quad (113)$$

By resumming the individual monomials one obtains harmonic polylogarithms over the alphabet $\mathfrak{A}_p$ in the order requested by the Nielsen integrals and logarithmic derivatives thereof. We consider the example $S_{2,3}(x; \mu)$, which reads

$$\begin{aligned} S_{2,3}(x; \mu) &= - \left\{ -\text{H}_{0,0,1,1,1}(x) + \mu(-\text{H}_{0,1,1}(x) + 2\text{H}_{0,0,1,1}(x) - 3\text{H}_{0,1,1,1}(x)) + \mu^2 \left(-\frac{1}{2} \text{H}_1(x) \right. \right. \\ &\quad \left. \left. - \frac{3}{2} \text{H}_1^2(x) - \frac{1}{2} \text{H}_1^3(x) + \frac{3}{2} \text{H}_{0,1}(x) - \text{H}_{0,0,1}(x) + 6\text{H}_{0,1,1}(x) \right) + \mu^3 \left(-\frac{3x}{2(1-x)} \right. \right. \\ &\quad \left. \left. - \frac{3(-3+5x)\text{H}_1(x)}{2(1-x)} - \frac{(-6+7x)\text{H}_1(x)^2}{2(1-x)} - 3\text{H}_{0,1}(x) \right) + \mu^4 \left(-\frac{x(-6+11x)}{2(1-x)^2} \right. \right. \\ &\quad \left. \left. + \frac{(-3+7x-5x^2)\text{H}_1(x)}{(1-x)^2} \right) - \mu^5 \frac{x^3}{(1-x)^3} \right\}. \end{aligned} \quad (114)$$

In calculating expressions of this type some of the sums have to be computed using the command `ComputeGeneratingFunction`, which establishes and solves a differential equation for the problem at hand. Also in the case of the Nielsen integrals one observes a divergent behavior in the limit $x \rightarrow 1$. For $S_{2,3}(x; \mu)$ the contributions from $O(\mu^3)$ on are concerned.

The hierarchy equations for the μ -extended Nielsen integrals are

$$x\mathcal{D}_x^{(\mu)} S_{n,p}(x;\mu) = S_{n-1,p}(x;\mu), \quad (115)$$

$$\int_0^x S_{n-1,p}(t;\mu) \frac{\mathcal{D}_t^{(\mu)}}{t} = S_{n,p}(x;\mu). \quad (116)$$

C. The μ deformation of other iterated integrals

It has been shown in Secs. III A–III G that the functions $f[n]$ are polynomials of

$$\frac{1}{\{n\}^l}, \quad S_{\vec{a}}(\vec{b}, n), l, \quad a_i \in \mathbb{N} \setminus \{0\}, \quad b_i \in \mathbb{C} \setminus \{0\}. \quad (117)$$

Therefore the μ -extension of $f[n]$, $f[n;\mu]$, can polynomially be represented by linear combinations of the terms

$$\frac{1}{\{n\}_\mu^l} = \left(\frac{1}{n} + \mu\right)^l, \quad S_{\vec{a}}(\vec{b}, n; \mu), \quad l, a_i \in \mathbb{N} \setminus \{0\}, \quad b_i \in \mathbb{C} \setminus \{0\}. \quad (118)$$

The functions $f[n;\mu]$ are therefore polynomials in μ , $1/n^m$, and generalized μ -extended harmonic sums. Each polynomial coefficient in μ of $f[n;\mu]x^n$, $f[n;\mu, k]x^n$, can then be summed over $n \in [1, \infty)$. The result is a polynomial in μ and generalized harmonic polylogarithms for the classes of harmonic polylogarithms, generalized harmonic polylogarithms, cyclotomic harmonic polylogarithms, and iterative integrals implied by quadratic forms.

Let us sum, as an example,

$$T_3 = \sum_{n=1}^{\infty} \frac{S_{-2,1}(n)}{n^2} x^n \quad (119)$$

as one of the potential terms. The command

```
(HarmonicSumsSum[S[-2,1,n]/
n^2*x^n, {n,1,Infinity}]] // Trans-
formToSSums// SinfToH // Variable-
ToArgument) /. H[a_, -x] ->
TransformH[H[a, -x], x] // Reduce-
ToHBasis
```

yields

$$\begin{aligned} T_3 = & -H_{0,1}(x) \cdot H_{0,0,-1}(x) + H_{0,1}(x) \cdot H_{0,-1,-1}(x) - H_{0,0,0,-1}(x) + 3H_{0,0,0,1,-1}(x) \\ & + 3H_{0,0,0,-1,1}(x) + 2H_{0,0,1,0,-1}(x) - 2H_{0,0,1,-1,-1}(x) - 2H_{0,0,-1,1,-1}(x) - 2H_{0,0,-1,-1,1}(x) \\ & - H_{0,-1,0,1,-1}(x) - H_{0,-1,0,-1,1}(x) - H_{0,-1,-1,0,1}(x) + H_{0,0,0,-1,-1}(x) + H_{0,0,-1,0,1}(x). \end{aligned} \quad (120)$$

In the following we consider a number of examples for the different classes of iterated integrals.

D. The μ deformation of harmonic polylogarithms

We consider the example $H_{-1,0,1}(x;\mu)$. The expansion of $H_{-1,0,1}(x)$ around $x = 0$ is given by

$$H_{-1,0,1}(x) = \sum_{n=1}^{\infty} \left[\frac{1}{n^3} - \frac{(-1)^n}{n} S_{-2}(n) \right] x^n. \quad (121)$$

The μ -extension is obtained by

$$\begin{aligned} H_{-1,0,1}(x;\mu) = & H_{-1,0,1}(x) + \mu \left(\frac{xH_{0,1}(x)}{1+x} + 2H_{-1,1}(x) \right) \\ & + \frac{1}{2}\mu^2 \left(\frac{(1+5x)H_1(x)}{1+x} - H_{-1}(x) \right) \\ & + \frac{\mu^3 x^2}{1-x^2}. \end{aligned} \quad (122)$$

To see the hierarchy for μ -extended harmonic polylogarithms we also look at the derivative by using the series representation

$$\begin{aligned} \mathcal{D}_x^{(\mu)} H_{-1,0,1}(x;\mu) &= \sum_{n=1}^{\infty} \frac{n}{1+\mu n} f[n;\mu] x^{n-1} \\ &= \frac{1}{1+x} \left[H_{0,1}(x) + 2\mu H_1(x) + \mu^2 \frac{x}{1-x} \right]. \end{aligned} \quad (123)$$

This compares to the μ -extension of $H_{0,1}(x)/(1+x)$, which is given by

$$\sum_{n=1}^{\infty} (-1)^n S_{-2}(n;\mu) x^n = \frac{1}{1+x} \left[H_{0,1}(x) + 2\mu H_1(x) + \mu^2 \frac{x}{1-x} \right]. \quad (124)$$

E. The μ deformation of generalized harmonic polylogarithms

We consider the μ -extension of the generalized harmonic polylogarithm $H[-1/2, 2, -1, x]$ as a typical example and solve this algorithmically. The associated function $f[n]$ reads

$$f[n] = \frac{(-1)^n}{n^3} - \frac{2^{-n}}{n^2} S_1(-2, n) - \frac{(-2)^n}{n} S_2\left(\frac{1}{2}, n\right) + \frac{(-2)^n}{n} S_{1,1}\left(-\frac{1}{4}, -2, n\right). \quad (125)$$

It is obtained by applying the command `GetMoment`. Note that Eq. (125) contains a global factor of $1/n$.

Given the structure of the alphabet, we obtain the μ -extension by using `ComputeGeneratingFunction`.

$$\begin{aligned} H_{-\frac{1}{2}, 2, -1}(x)|_{\mu} &= H_{-\frac{1}{2}, 2, -1}(x) + \frac{\mu}{15(1+2x)} \left[4(1+2x) \left(\frac{5}{2} H_{-\frac{1}{2}, 2}(x) - 2H_{-\frac{1}{2}, -1}(x) \right) \right. \\ &\quad \left. + 6(2+9x)H_{2, -1}(x) \right] + \frac{\mu^2}{15(-2+x)(1+2x)} \left[3(-2+x)(1+2x)H_{-\frac{1}{2}}(x) \right. \\ &\quad \left. + 4(-2+x)(2+9x)H_2(x) - 10(-2-5x+6x^2)H_{-1}(x) \right] \\ &\quad - \frac{2\mu^3 x^3}{(-2+x)(1+x)(1+2x)}. \end{aligned} \quad (126)$$

The expansion coefficient of the derivative has $2H[2, -1, x]/(1+2x)$ and is

$$f[n] = (-2)^{1+n} \left[S_{1,1}\left(-\frac{1}{4}, -2, n\right) - S_2\left(\frac{1}{2}, n\right) \right], \quad (127)$$

resulting in

$$\frac{2H_{2, -1}(x)}{(1+2x)}|_{\mu} = \frac{2}{1+2x} \left[H_{2, -1}(x) + \mu \frac{2}{3} \left(H_2(x) - \frac{(1-2x)H_{-1}(x)}{2-x} \right) + \frac{\mu^2 x^2}{(2-x)(1+x)} \right]. \quad (128)$$

The μ derivative of (126) agrees with (128).

F. The μ deformation of cyclotomic harmonic polylogarithms

Let us consider the cyclotomic harmonic polylogarithm

$$G\left[\left\{\frac{1}{1-\tau+\tau^2}, \frac{1}{1+\tau}\right\}, x\right] \equiv \int_0^x d\tau_1 \frac{1}{1-\tau_1+\tau_1^2} \int_0^{\tau_1} d\tau_2 \frac{1}{1+\tau_2}. \quad (129)$$

The associated function $f[n]$ is obtained by

$$f[n] = -\frac{(-1)^{2/3}(-(-1)^{2/3})^n}{(1+\sqrt[3]{-1})n} S_1(-\sqrt[3]{-1}, n) + \frac{(-1)^{2/3+n/3}}{(1+\sqrt[3]{-1})n} S_1((-1)^{2/3}, n). \quad (130)$$

The μ -extension $f[n; \mu]$ reads

$$\begin{aligned}
f[n; \mu] = & \frac{\mu^2}{3} (((-1 + (-1)^{n/3})(3 + i\sqrt{3})2^{2n} - 2i\sqrt{3}(i + \sqrt{3})^{2n})i^n \\
& + ((3 - i\sqrt{3})(-1 + (-1)^{2/3})^n)2^n + 2i\sqrt{3}(1 - i\sqrt{3})^n)(-1)^n(i + \sqrt{3})^n) \\
& \times 2^{-1-3n}(i + \sqrt{3})^n \\
& + \frac{\mu}{3n} (((-3 + (3 + i\sqrt{3})(-1)^{n/3} - i\sqrt{3} + 2in\sqrt{3}S_1(\sqrt{3} - 1, n))2^{2n} \\
& - 4i\sqrt{3}(i + \sqrt{3})^{2n})i^n + ((-3 + i\sqrt{3} + (3 - i\sqrt{3})(-1)^{2/3})^n \\
& - 2in\sqrt{3}S_1(-(-1)^{2/3}, n))2^n + 4i\sqrt{3}(1 - i\sqrt{3})^n)(-1)^n(i + \sqrt{3})^n)2^{-1-3n} \\
& \times (i + \sqrt{3})^n + i\frac{1}{n^2} ((-i + \sqrt{3})^{2n} + n2^{2n}S_1(\sqrt{3} - 1, n))i^n + ((1 - i\sqrt{3})^n \\
& - n2^nS_1(-(-1)^{2/3}, n))(-1)^n(i + \sqrt{3})^n) \frac{2^{-3n}(i + \sqrt{3})^n}{\sqrt{3}}. \tag{131}
\end{aligned}$$

By the methods described in the previous Sections, the μ -extension in x space is obtained as

$$\begin{aligned}
G\left[\left\{\frac{1}{1-\tau+\tau^2}, \frac{1}{1+\tau}\right\}, x\right]\Big|_{\mu} = & \frac{(-1)^{2/3}}{1+\sqrt{3}-1} \left[G\left(\frac{1}{(-1)^{1/3}+\tau}, \frac{1}{1+\tau}; x\right) - G\left(\frac{1}{-(-1)^{2/3}+\tau}, \frac{1}{1+\tau}; x\right) \right] \\
& + \mu \left[-\frac{(1+x^2)H_{-1}(x)}{1+x+x^2} + \frac{\sqrt{6}-1H_{-1}(-(-1)^{2/3}x)}{\sqrt{3}} - \frac{1}{3}(-1+(-1)^{2/3})H_{-1}((-1)^{1/3}x) \right] \\
& + \mu^2 \frac{x^2}{(1+x)(1+x+x^2)}. \tag{132}
\end{aligned}$$

The μ -extension of the derivative $H_{-1}(x)/(1-x+x^2)$ is given by

$$\left. \frac{H_{-1}(x)}{1-x+x^2} \right|_{\mu} = \frac{H_{-1}(x)}{1-x+x^2} + \mu \frac{x}{(1+x)(1-x+x^2)}. \tag{133}$$

It agrees with $\mathcal{D}_x^{(\mu)} G[\{\frac{1}{1-\tau+\tau^2}, \frac{1}{1+\tau}\}, x]$.

G. The μ deformation of iterated integrals implied by quadratic forms

We perform the μ -extension of the iterative integral (64). It reads

$$\begin{aligned}
H_{((1,1,2),0),((0,1,-1),0),((0,1,1),0)}(x)\Big|_{\mu} = & \frac{1}{3}H_1(x)H_{-1,1}(x) - \frac{2}{3}H_{-1,1,1}(x) + \frac{1}{3}H_{-2,-1,1}(x) + \mu \left[-\frac{1}{6}H_{-1}(x)H_1(x) + \frac{1}{6}H_1(x)^2 \right. \\
& + \frac{(2-4x-x^2)H_{-1,1}(x)}{3(-1+x)(2+x)} + \frac{5}{6}H_{-2,1}(x) - \frac{1}{6}H_{-2,-1}(x) \Big] \\
& + \mu^2 \left[-\frac{x}{6(-1+x)} + \frac{(14-11x-68x^2-7x^3)H_1(x)}{36(-1+x)(1+x)(2+x)} + \frac{(-2+3x+x^2)H_{-1}(x)}{4(-1+x)(2+x)} \right. \\
& \left. - \frac{4}{9}H_{-2}(x) \right] + \mu^3 \frac{x^3}{(-1+x)^2(1+x)(2+x)}. \tag{134}
\end{aligned}$$

Here the term $O(\mu^0)$ is written in an equivalent representation to the one in Eq. (64).

The μ -extension of the derivative of (64) reads

$$-\left. \frac{H_{-1,1}(x)}{x^2+x-2} \right|_{\mu} = -\frac{H_{-1,1}(x)}{x^2+x-2} - \mu \frac{(1-3x)H_1(x) - (1+x)H_{-1}(x)}{2(x^2+x-2)(1+x)} - \mu^2 \frac{x^2}{(1-x^2)(x^2+x-2)}. \tag{135}$$

It agrees with $\mathcal{D}_x^{(\mu)} H_{((1,1,2),0),((0,1,-1),0),((0,1,1),0)}(x)\Big|_{\mu}$.

H. μ deformation of iterated integrals with square-root valued letters

The functions $f[n]$ for iterated integrals with square-root valued letters contains central binomials. Their μ -extension is not polynomial in μ , like in the previous cases. The μ -extended central binomials are (growing) rational functions in μ . Variable transformations will imply real-valued sum quantifiers. Already in very simple cases one obtains higher transcendental functions, as the example

$$\sum_{n=1}^{\infty} \frac{4^{-n} x^n (1 + \frac{1}{\mu} + n)_n}{(1 + \frac{1}{\mu})_n} = \frac{(1 + 2\mu)x}{4(1 + \mu)} {}_3F_2 \left[\begin{matrix} 1 & \frac{3}{2} + \frac{1}{2\mu} & 2 + \frac{1}{2\mu} \\ 2 + \frac{1}{\mu} & 2 + \frac{1}{\mu} \end{matrix}; x \right] \quad (136)$$

shows. In any case one may write an infinite sum, after the μ -extension of all terms occurring in $f[n]$ in the central binomial case. Algorithms, which resum this representation are not yet known in general.

As an example we consider

$$\begin{aligned} T_4 &= \sum_{i=1}^{\infty} \frac{(4x)^i}{i^2 \binom{2i}{i}} S_1(i) = \frac{1}{3} x (6 - 45x + 88x^2 - 42x^3) + 64G\left(\frac{1}{\tau}, \sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}; x\right) \\ &+ 64G\left(\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau}, \sqrt{1-\tau}\sqrt{\tau}; x\right) + 24(1-2x)\sqrt{(1-x)x}G(\sqrt{1-\tau}\sqrt{\tau}; x) \\ &+ 16(1-2x)\sqrt{(1-x)x}G\left(\frac{1}{1-\tau}, \sqrt{1-\tau}\sqrt{\tau}; x\right) + 32G(\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}; x). \end{aligned} \quad (137)$$

Its μ -extension is given by

$$T_4|_{\mu} = \sum_{i=1}^{\infty} \frac{(4x)^i (1 + \mu i)^2}{i^2 \binom{2i}{i}} [S_1(i) + \mu i]. \quad (138)$$

The derivative of T_4 is

$$\begin{aligned} \frac{d}{dx} T_4 &= \frac{1}{x} \left[2x(1 - 3x + 8x^2 - 4x^3) + 4(3 - 12x + 8x^2) \frac{\sqrt{x}}{\sqrt{1-x}} G(\sqrt{1-\tau}\sqrt{\tau}; x) \right. \\ &\left. + 8 \frac{\sqrt{x}}{\sqrt{1-x}} G\left(\frac{1}{1-\tau}, \sqrt{1-\tau}\sqrt{\tau}; x\right) + 64G(\sqrt{1-\tau}\sqrt{\tau}, \sqrt{1-\tau}\sqrt{\tau}; x) \right]. \end{aligned} \quad (139)$$

By applying (87) one obtains

$$\mathcal{D}_x^{(\mu)} T_4|_{\mu} = \sum_{i=1}^{\infty} 4 \frac{(4x)^{i-1} (1 + \mu i)}{i \binom{2i}{i}} [S_1(i) + \mu i], \quad (140)$$

which agrees with the μ -extension of (139). For small values of μ , one may perform a Taylor expansion, as outlined in the Appendix B.

VIII. CONCLUSIONS

In quantum field theory several classes of iterative integrals and nested sums play a central role in analytic higher order calculations. They form shuffle or quasishuffle algebras or subsystems of those. By virtue of the shuffle product, these algebras are also Hopf algebras [117]. These

functions obey first-order factorizing differential or difference equations.

In the present paper we have derived the μ -extension for these functions, inspired by Ref. [28]. Except for the case of nested sums containing central binomial sums and iterated integrals containing square-root valued letters, the μ -extension maps into the same Hopf algebra as the nondeformed functions. In these cases the μ -extension adds lower weight functions, up to the weight $w = 0$ contributions. The latter terms imply the singular behavior at $x = 1$ for the iterative integrals and for $N \rightarrow \infty$ for the nested sums, beyond logarithmic singularities. The point $x = 1$ in the non- μ -extended case is already a branch point of the functions. In the case of iterated integrals with square-root valued letters the central function $f[n]$ contains central binomials. Due to this, their μ -extension implies growing rational functions in μ and n . The x representation is then an infinite

sum over these terms, which forms a higher transcendental function. Thus, the μ -extension leads out of the space of the nondeformed functions. For small values of μ , one may consider the series expansion in μ , the terms of which belong to the nondeformed algebras, while the general solutions do not.

We have shown that the μ -extensions obey analogous differentiation hierarchies like the μ -free cases by replacing the differential operator $\frac{d}{dx}$ with the μ derivative $\mathcal{D}_x^{(\mu)}$.

It should be emphasized that the set of the μ -deformed special functions constructed and studied in this paper is completely new. We believe that these μ -extensions should play the same important role as their nondeformed counterparts, in the corresponding μ -deformed analogs of customary quantum field theory models that have yet to be elaborated. After being completed, the corresponding results will be presented elsewhere in the future. This will supplement the existing amount of already developed diverse applications of μ -deformed calculus along with its physical realization in the realm of thermo-statistical problems corresponding to very distinct scales, from hadronic to galactic ones.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DIFFERENT q EXTENSIONS

Related to the μ -extension, there is also a number of different q extensions in use. In what follows we summarize some of their basic features for completeness.

1. Asymmetric q extension

The q extension of a number a is given by

$$\{a\}_q \equiv \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} \setminus \{1\}. \quad (\text{A1})$$

This approach has been used for the q -basic functions in Refs. [3–12]. In the case $q \in \mathbb{N} \setminus \{0\}$ one has

$$\{n\}_q \equiv \frac{1 - q^n}{1 - q} = \sum_{l=0}^{n-1} q^l. \quad (\text{A2})$$

Equation (A2) allows one to define the q -factorial

$$\{n\}_q! \equiv \prod_{k=1}^n \{k\}_q, \quad \{0\}_q! := 1. \quad (\text{A3})$$

In various sum representations integer powers of complex numbers c

$$c^n, \quad c \in \mathbb{C} \quad (\text{A4})$$

contribute. This n dependence is not q extended, similar to that for the monomial x^n , see Eq. (5).

The Jackson q derivative is defined by [125–128]

$$\mathcal{D}_q[f](x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad (\text{A5})$$

with

$$\mathcal{D}_q[x^\alpha] = \frac{(1 - q^\alpha)}{(1 - q)} x^{\alpha-1}, \quad (\text{A6})$$

for the power function.

2. Symmetric q extension

Here the extension of a number a reads, see, e.g., Refs. [15,129,130],

$$\{a\}_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (\text{A7})$$

The q derivative is given by

$$\mathcal{D}_q[f](x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \quad (\text{A8})$$

and

$$\mathcal{D}_q[x^\alpha] = \frac{q^\alpha - q^{-\alpha}}{(q - q^{-1})} x^{\alpha-1}. \quad (\text{A9})$$

3. q extension using two variables

This extension has been considered in Refs. [131–133]. The extension of a number a is given by

$$\{a\}_{q,r} = \frac{q^a - r^a}{q - r}. \quad (\text{A10})$$

The q, r derivative is given by

$$\mathcal{D}_{q,r}[f(x)] = \frac{f(qx) - f(rx)}{(q - r)x}, \quad (\text{A11})$$

resulting in

$$\mathcal{D}_{q,r}[x^n] = \{n\}_{q,r} x^{n-1}. \quad (\text{A12})$$

Three further generalizations have been discussed in Ref. [134], see also Refs. [135–137]. Note that all the different deformed brackets shown herein may involve, besides a number a , also an operator, e.g., the excitation number operator N —in this case the corresponding “bracket” is meant as formal power series.

APPENDIX B: THE μ -EXPANSION FOR SQUARE-ROOT VALUED LETTERS

The complete representation in the case of contributions due to central binomials in n space can be considered for small values of μ applying a Taylor expansion in the first few powers,

$$\left(\frac{2n}{n}\right)\Big|_{\mu} = \left(\frac{2n}{n}\right) \{1 + 2\mu n^2[S_1(n) - S_1(2n)] + \mu^2[n^4(2S_1(n)^2 - 4S_1(n)S_1(2n) + 2S_1(2n)^2 - 2S_2(2n) + S_2(n) + \zeta_2) + 2n^3(-S_1(n) + S_1(2n))]\} + O(\mu^3). \quad (\text{B1})$$

There are two new aspects:

- (i) also harmonic sums at argument $2n$ contribute.
- (ii) positive powers of n appear.

For i) one has to synchronize arguments, which will lead to contributions due to cyclotomic harmonic sums [70]. For the sums $S_1(2n)$ and $S_1(n)S_1(2n)$ one obtains

$$S_1(2n) = 1 - \frac{1}{1+2n} + \frac{1}{2}S_1(n) + S_{\{2,1,1\}}(n), \quad (\text{B2})$$

$$S_1(n) \cdot S_1(2n) = -S_1(n) + \frac{2nS_1(n)}{1+2n} - \frac{1}{2}S_2(n) + 2S_{\{2,1,1\}}(n) + S_{1,1}(n) + S_{\{1,0,1\},\{2,1,1\}}(n) + S_{\{2,1,1\},\{1,0,1\}}(n), \quad (\text{B3})$$

$$\begin{aligned} S_1^2(2n) &= 1 + \frac{1}{(1+2n)^2} - \frac{2}{1+2n} - \frac{S_1(n)}{1+2n} + S_{\{2,1,1\}}(n)S_1(n) - \frac{1}{4}S_2(n) + \frac{1}{2}S_{1,1}(n) \\ &\quad + 2S_{\{2,1,1\}}(n) - \frac{2S_{\{2,1,1\}}(n)}{1+2n} - S_{\{2,1,2\}}(n) + 2S_{\{2,1,1\},\{2,1,1\}}(n) + S_1(n), \end{aligned} \quad (\text{B4})$$

by applying the commands `Synchronize` and `LinearExpand`. Here cyclotomic sums of the kind

$$S_{\{1,0,1\},\{2,1,1\}}(n) = \sum_{k=1}^n \frac{1}{k} \sum_{l=1}^k \frac{1}{2l+1} \quad (\text{B5})$$

do appear.

ii) Some of the positive powers n^k in (B1) may contribute under the Mellin transform and can be absorbed by

$$\sum_{n=0}^{\infty} x^n \int_0^1 dy y^n f(y) = \int_0^1 dy \frac{xy}{(1-xy)^2} f(y). \quad (\text{B6})$$

In the general case for factors n^l one obtains

$$\sum_{n=0}^{\infty} n^l z^n = \left(z \frac{d}{dz}\right)^l \frac{1}{1-z}. \quad (\text{B7})$$

By the command `GeneralInvMellin` with the option `ComputeMellinConvolution -> true` one can obtain the n -space expression as a Mellin transform. Techniques of this kind were also used in Refs. [74,138–140]. The final integral can be performed by the command `GLIntegrate`.

A typical example is given by the term

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{2n}{n}\right) S_1(2n)S_1(n)x^n &= \frac{1}{\sqrt{1-4x}} \left\{ 4 + 4x + \left[-4 - H_0(x) + G\left(\frac{\sqrt{1-4\tau}}{\tau}; x\right) \right] \sqrt{1-4x} - 3H_{\frac{1}{4}}(x) - 2H_0(x) \right. \\ &\quad + \frac{1}{2}H_0(x)^2 + 2H_{\frac{1}{4},1}(x) + \frac{3}{2}H_{\frac{1}{4},0}(x) + \frac{3}{2}H_{0,\frac{1}{4}}(x) + 2G\left(\frac{\sqrt{1-4\tau}}{\tau}; x\right) - G\left(\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}; x\right) \\ &\quad \left. - 6G\left(\frac{1}{1-4\tau}, \frac{\sqrt{1-4\tau}}{\tau}; x\right) + \frac{1}{2}G\left(\frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}; x\right) - \frac{1}{2}G\left(\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}; x\right) \right\}. \end{aligned} \quad (\text{B8})$$

Similar sums have been considered in Ref. [141]. The number of terms quickly grows for higher powers in μ , but the above example shows that closed-form expressions using G functions at main argument x can be found. In general, the μ expansion does not substitute the complete result, however.

APPENDIX C: μ -DEFORMED ζ VALUES FROM A MODIFIED μ POLYLOGARITHM

Modified μ polylogarithms allow for a direct calculation of the associated μ -deformed ζ values for $x \rightarrow 1$. We consider

$$\tilde{\text{Li}}_s(x; \mu) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} \frac{1}{1 + k\mu} \quad (\text{C1})$$

of Ref. [41]. Here the μ -dependent part introduces a Hurwitz-type structure, cf. Ref. [142]. Resumming (C1)

for $x = 1$ one obtains

$$\tilde{\zeta}_1(\mu) = \psi\left(1 + \frac{1}{\mu}\right) + \gamma_E, \quad (\text{C2})$$

$$\tilde{\zeta}_2(\mu) = \zeta_2 - \mu \left[\psi\left(1 + \frac{1}{\mu}\right) + \gamma_E \right], \quad (\text{C3})$$

$$\tilde{\zeta}_3(\mu) = \zeta_3 - \mu \zeta_2 + \mu^2 \left[\psi\left(1 + \frac{1}{\mu}\right) + \gamma_E \right], \quad (\text{C4})$$

$$\tilde{\zeta}_4(\mu) = \zeta_4 - \mu \zeta_3 + \mu^2 \zeta_2 - \mu^3 \left[\psi\left(1 + \frac{1}{\mu}\right) + \gamma_E \right], \quad (\text{C5})$$

etc. Here γ_E denotes the Euler-Mascheroni constant and $\psi(z)$ the digamma function. In this modified definition also $\tilde{\zeta}_1(\mu)$ exists, but it diverges logarithmically for $\mu \rightarrow 0$, while for $s > 1$ the usual ζ values are obtained.

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