

Fully relativistic evolution of vacuum tensor inhomogeneities during inflation

Ericka Florio^{*} and E. Paul S. Shellard[†]

*Centre for Theoretical Cosmology, Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*



(Received 26 May 2025; accepted 23 December 2025; published 21 January 2026)

We present a complete method for the initialization and extraction of first-order inflationary tensor perturbations for fully relativistic simulations which incorporate gravitational backreaction. We outline a correspondence between the cosmological perturbation theory framework and the numerical relativity variables in the appropriate limit. We describe a generation method for stochastic tensorial initial conditions, inspired by the standard scalar initial condition used from inflation and implemented in lattice cosmology. We discuss the implementation of this procedure in the GRChombo/GRTeclyn code, and demonstrate the detailed quantitative correspondence between the linearized and fully nonlinear solutions in the perturbative limit, through the evolution of the background and the tensor power spectrum. We also validate the methodology by showing that energy and momentum constraints are introduced and preserved to second-order or better. We provide some preliminary indicative results probing tensorial non-Gaussianity using the skewness and kurtosis. The computational pipeline presented here will be used to study the emergence of a primordial tensor bispectra and cross spectra that incorporate the effect of nonlinear gravitational couplings with the metric, which has potential applications for the analysis of next-generation cosmic microwave background observations.

DOI: [10.1103/48w-z27g](https://doi.org/10.1103/48w-z27g)

I. INTRODUCTION

Single-field slow-roll inflation, when added to the standard Λ CDM model of cosmology, explains many properties of the cosmic microwave background (CMB) signal, such as its phase coherence and Gaussianity. However, many standard single-field models have been heavily constrained by the combined BiCEP/Planck experiments (see Refs. [1–3], among others). This result has renewed interest in the investigation of nonstandard models of inflation, particularly into their distinguishing observable signals.

A primordial stochastic gravitational wave background (PGWB) is expected to be produced by most inflationary models, whether minimally via slow-roll inflation (Ref. [4]), or through nonlinear mechanisms such as preheating (Ref. [5]). These primordial tensoral perturbations are model dependent, and therefore carry information about the evolution of inflaton that is complementary to scalar perturbations. A stochastic gravitational wave

background (SGWB) has already been detected by the NANOGrav Collaboration (Ref. [6]). Although the NANOGrav signal is most likely generated by supermassive black hole binaries, signals like this could have been generated by early Universe processes (see for example Refs. [7–9]).

The PGWB, if detected, would provide key hints about the nature of the inflationary epoch. For instance, the amplitude of the primordial tensor power spectrum is directly related to the Hubble factor H during the inflationary period, and would serve as a measure of the energy scale of inflation (Ref. [10]). To date, the most robust constraint on the amplitude of PGWB produced by standard inflation was measured by the Planck 2018 data release and the BiCEP experiment. Combined, these datasets were able to constrain the error in the tensor-to-scalar ratio, δr , to less than 0.03 at a 95% confidence level (Ref. [11]). The BiCEP Collaboration has now released data which give an uncertainty of $\sigma(r) < 0.009$ (Ref. [12]) that will continue to improve, while the Simons Observatory, which saw first light recently, is projected to measure $\sigma(r)$ on the order of 0.002 (Ref. [13]). Additionally, the LiteBIRD experiment, aiming to launch in 2028, will be able to push current measurements of the scalar to tensor ratio to around 0.001 (Ref. [14]).

In addition to the SGWB, the detection of higher-order statistics in scalar and tensor fields could drastically

^{*}Contact author: eaf49@cam.ac.uk

[†]Contact author: eps1@cam.ac.uk

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

constrain the space of inflationary models. Here we focus on the primordial bispectrum, which is expected to be generated for all but the simplest models of inflation. The bispectrum can theoretically be measured in the scalar field alone, using the three-point scalar correlator $\langle\zeta\zeta\zeta\rangle$; in the scalar and tensor field together, via for example $\langle\zeta\zeta\gamma\rangle$; or in the tensor field alone, via $\langle\gamma\gamma\gamma\rangle$. The primordial bispectrum is particularly useful in distinguishing multi-field models [15], which can provide a high-quality power spectrum fit while introducing a unique bispectrum signature.

Both the PGWB and the primordial bispectrum have the potential to constrain the current parameter space of inflationary models, but only if we have developed robust predictions for their spectral shapes in advance. Historically, these spectra have computed using the linearized equations of motion as the sources for the inflaton field and the metric (for example, Ref. [16]), or using lattice simulations which resolve the nonlinear dynamics of the matter sector, but assume the gravitational sector evolves according to linear theory (Refs. [17–21]). However, when perturbations to the metric couple at beyond leading order, as can happen in multifield inflation (see Ref. [22] for example), it is reasonable to expect that beyond-linear-order dynamics in both the matter sector and the metric may become important in the final shape of these spectra.

Many numerical algorithms have been developed recently which can fully capture all aspects of these nonlinear signals. This new discipline of lattice cosmology has produced a variety of codes (Refs. [23–27]), which have been employed most often to study (p)reheating scenarios (see for example Refs. [24,28–30]), phase transitions (for example, Ref. [31]), and inhomogeneous inflationary initial conditions (for example, Ref. [32]). Some lattice codes focus on parametric resonance in the matter sector, which can be generated during single-field inflation for nonstandard models, such as ultra-slow-roll (Refs. [33,34]), and during multi-particle inflation and preheating (Refs. [30,35–37]). In these cases, nonlinear interactions between the various matter fields are all resolved, however the back-reaction onto the metric is assumed to be negligible. In a few cases, a fully-nonlinear evolution of both the metric *and* the matter fields has been employed, for instance to study primordial black hole formation during preheating scenarios (Refs. [38–40]) and their interactions with domain walls (Ref. [41]).

While other authors have used fully-nonlinear methods to constrain models based on their ability to start inflation (Refs. [32,42–44]), and from enhanced particle production and compact object formation at the end of inflation (Refs. [38,40,45–47]), the use of fully-relativistic lattice simulations to examine dynamics *during* inflation has been limited. This is in part due to the great success of linear theory in describing higher-order dynamics in the slow-roll

regime¹; however, multi-field models are known to produce potentially large non-Gaussian signals where there is a strong bend in the field-space trajectory (Refs. [15,16,49]), a scenario which numerical tools are well-suited to study. Additionally, second-order coupling between scalar and tensor perturbations has been shown to produce unique spectral shapes in the PGWB (Ref. [50]).

In this work, we use the Numerical Relativity (NR) code GRChombo to fully evaluate the dynamics of the metric and the inflaton during inflation. GRChombo has previously been used to show that inflating patches can be generically generated from large scalar field fluctuations (Refs. [42,51]), for various potential shapes (Refs. [39,43,52]). We will focus here on a detailed, quantitative study of the inflationary epoch itself, both incorporating and improving similar tools developed by others in the lattice cosmology community (see for example Ref. [53]) to connect our numerical results with perturbative methods arising from the quantum treatment of the inflationary problem (Refs. [54,55]). GRChombo was also recently used to study the evolution of scalar fluctuations during inflation, in both the linearized and nonperturbative regimes, in Ref. [56]. This work showed agreement in the linear regime, and presented a departure from linear evolution in the background and two-point function during ultra-slow-roll inflation and axion monodromy inflation. We perform a similar validation for the case of tensor perturbations; additionally, we present a rigorous test of the initial conditions of the tensor polarization fields.

This work is organized as follows. In Sec. II, we review the relevant theory of gravitational wave propagation on Minkowski and Friedmann–Lemaître–Robertson–Walker (FLRW) backgrounds, as well as the quantization of these perturbations. In Sec. III, we present a gauge-agnostic dictionary which can be used to translate first-order cosmological perturbations to and from the fully relativistic Baumgarte-Shapiro-Shibata-Nakamura (BSSN) variables in the linear regime. In Sec. IV, we describe our particular numerical implementation in GRChombo,² including our choice of gauge and the initial conditions for the background spacetime and perturbations. In Sec. V, we present validation results for the evolution of the background in the superhorizon and horizon-crossing regimes, and of the evolution of the tensor power spectrum. We demonstrate the constraint-satisfying properties of our initial data, and provide an improvement to the standard stochastic initialization used in the lattice cosmology literature. We show convergence test results, and give a first look into how this program could capture the dynamic emergence of higher-order correlators. In Sec. VI, we summarize our results and outline the immediate next steps of our research program.

¹For instance, Ref. [48] derives the second-order equations of motion for perturbations of single-field power-law type inflation.

²We expect our implementation in GRChombo will follow this implementation closely.

Throughout this work we use the mostly plus signature of the metric, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, and natural units³ $c = \hbar = 1$, giving $G = m_{pl}^{-2}$. When translating the background equations of motion into the GRChombo/GRTeclyn program, we will define “code units” which absorb this factor of m_{pl} into our physical quantities, making them unitless.⁴

II. PRIMORDIAL GRAVITATIONAL WAVES

Gravitational waves are typically discussed as transverse-traceless perturbations of the Minkowski metric. However, gravitational waves can also propagate on an FLRW metric, as described by the cosmological perturbation theory (CPT) framework. In this section, we review the description and construction of gravitational waves, in both Minkowskian and cosmological backgrounds. We then solve the first-order equations of motion for tensor perturbations in CPT, which will be used to initialize our fully relativistic simulation. The purpose of this section is to present the necessary details of these gravitational waves in perturbation theory, in order to formulate the CPT-BSSN correspondence central to our computational pipeline and which is described in Sec. III.

A. Gravitational waves in Minkowski spacetime

Gravitational waves represent 2 free, propagating degrees of freedom in the metric. They are traditionally formulated in terms of a perturbation on the Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where the evolution equations of the perturbations $h_{\mu\nu}$ follow from Einstein’s equations. In order to simplify the evolution equations, it is common to consider the trace-reversed metric perturbation (Ref. [58]), given by

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

One must also choose a gauge, which fixes the degrees of freedom associated with the choice of coordinates. The typical gauge choice in Minkowski space involves first taking the partial gauge-fixing known as the de-Donder gauge (Ref. [58]), where

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \quad (1)$$

Decomposing onto plane-wave solutions, one can then show that the remaining degrees of freedom can be

eliminated by prescribing certain components of the plane-wave solution. The typical choice here is the transverse-traceless (TT) condition (Ref. [58]), which sets

$$h_{0\mu} = h_\mu^\mu = 0. \quad (2)$$

In this gauge, the metric perturbation is fully spatial, and the full metric can be simplified to

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j. \quad (3)$$

The spatial metric perturbation h_{ij} is traceless and transverse in spatial derivatives, meaning $h_i^i = \partial^j h_{ij} = 0$. Under these gauge conditions, it can then be shown that in vacuum Einstein’s equations reduce to the massless wave equation for $\tilde{h}_{\mu\nu}$ (Ref. [58]):

$$\square \tilde{h}_{\mu\nu} = 0.$$

In the TT gauge, the metric perturbation contains only 2 degrees of freedom, which we choose to decompose into plus and cross polarizations. Consider a gravitational wave propagating along the $\mathbf{k}$ direction, and consider two vectors $\mathbf{m}$ and $\mathbf{n}$ which form an orthonormal basis together with $\mathbf{k}$. One can construct two polarization basis tensors, given by

$$\begin{aligned} e_{ij}^+ &= m_i m_j - n_i n_j \\ e_{ij}^\times &= m_i n_j + n_i m_j, \end{aligned} \quad (4)$$

which obey the orthonormality condition

$$\sum_{i,j} e_{ij}^s e_{ij}^{s'} = 2\delta^{ss'},$$

where $s, s' \in \{+, \times\}$, and where the sum does *not* assume symmetric indexing on i and j (Ref. [59]). The gravitational wave perturbation can then be constructed from these basis tensors as

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times = \sum_s h_s e_{ij}^s, \quad (5)$$

where h_+ and $h_\times$ represent the 2 true degrees of freedom in the metric (Ref. [10]), which we call the *mode functions* of the metric. These mode functions can be recovered from the spatial metric using the orthonormality condition,

$$h_s = \frac{1}{2} \sum_{i,j} h_{ij} e_{ij}^s.$$

B. Gravitational waves in cosmological spacetimes

1. The inflationary background

Homogeneous and isotropic spacetimes are described by the FLRW metric,

³For more information on how m_{pl} is used to set the scale of spacetime in GRChombo, see Appendix A in Ref. [57].

⁴For instance, in natural units the background inflaton field $\tilde{\phi}$ has units of mass, and so the corresponding “program” field is given by $\tilde{\phi}_{pr} \equiv \tilde{\phi}/m_{pl}$.

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j.$$

Applying this metric to Einstein's equations gives the Friedman equation for the scale factor, $a(t)$, where $H \equiv \dot{a}/a$. Further assuming spatial flatness, we have

$$H^2 = \frac{8\pi}{3m_{pl}^2}\rho. \quad (6)$$

To generate an inflationary spacetime, we assume the matter energy density ρ arises from a scalar inflaton field ϕ , which evolves according to the Klein-Gordon equation, Eq. (22). Consider the standard perturbative description of this field,

$$\phi = \bar{\phi} + \delta\phi.$$

We will first solve only for the background, $\bar{\phi}$. Applying the FLRW metric to Eq. (22) and neglecting spatial gradients, we find

$$\ddot{\bar{\phi}} - 3H\dot{\bar{\phi}} - \frac{dV}{d\bar{\phi}}\bigg|_{\bar{\phi}} = 0. \quad (7)$$

The energy density of the inflaton field is given by the 00 component of the energy-momentum tensor,

$$\rho = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}). \quad (8)$$

Deep in the inflationary regime we define an early epoch of *slow roll*, where the dimensionless slow-roll parameters (SRPs)

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \delta \equiv -\frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} \quad (9)$$

are small compared to 1.⁵ In this regime, $\ddot{\bar{\phi}} \ll H\dot{\bar{\phi}}$, and the Klein-Gordon equation simplifies to

$$3H\dot{\bar{\phi}} + \frac{dV}{d\bar{\phi}}\bigg|_{\bar{\phi}} = 0, \quad (10)$$

which, together with Eqs. (6) and (8), forms a complete set of differential equations which can be solved for the background solution.

2. The gauge-agnostic CPT formalism

Various tools developed for the flat-space formulation of gravitational waves will aid us in developing an

understanding of tensoral perturbations of cosmological spacetimes. Cosmological gravitational waves propagate on a quasi-de Sitter background, whereas astrophysical gravitational waves propagate on nearly Minkowski spacetime. As de Sitter spacetime has an inherent preferred time direction, whereas Minkowski does not, the formulation of the TT gauge for primordial gravitational waves (PGWs) should be carefully considered.⁶ We describe PGWs linearly using the CPT decomposition of the metric (see for example Ref. [10]). The CPT metric is based on a scalar-vector-tensor decomposition of perturbations evolving on an FLRW background, and formulates these perturbations in a gauge-agnostic manner. We follow the notation used in Ref. [10], and begin by perturbing each component of the metric:

$$ds^2 = -\bar{N}^2(1 + 2A)dt^2 + 2a\bar{N}B_idtdx^i + a^2(\delta_{ij} + 2E_{ij})dx^i dx^j, \quad (11)$$

where $\bar{N}$ chooses the time slicing of the background.⁷ This formulation captures all possible first-order metric perturbations.

The perturbations B_i and E_{ij} contain multiple types of degrees of freedom—for instance, B_i contains an inherent scalar (magnitude) and vector (direction) perturbation. We decompose B_i and E_{ij} into their inherent scalar, vector and tensor degrees of freedom (Ref. [10]):

$$B_i = \partial_i B + \hat{B}_i \\ E_{ij} = C\delta_{ij} + \partial_{(i}\partial_{j)}E + \partial_{(i}\hat{E}_{j)} + \hat{E}_{ij},$$

where $|\hat{B}_i| = |\hat{E}_i| = 1$; where $\partial_{(i}\partial_{j)}$ represents the trace-free derivative of E ,

$$\partial_{(i}\partial_{j)}E = \left(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij}\right)E;$$

and where $\hat{E}_{ij} \equiv 1/2h_{ij}$, as given by Eq. (3). We can see that the CPT metric truly contains four scalar perturbations (A, B, C, E), two vector perturbations ($\hat{B}_i, \hat{E}_i$), and one tensor perturbation, $\hat{E}_{ij}$.

It is well known that in inflationary spacetimes, linear vector perturbations evolve along decaying solutions (Refs. [10,61]). Thus we will neglect vector perturbations of the metric by assuming they are vanishing on our initial slice, and use the scalar-tensor form of the CPT metric given by

⁵Note that we do *not* assume that the slow-roll assumption holds in our definition of these parameters, and instead use their most “general” form which also applies in nonlinear regimes and at late times during inflation.

⁶For a review of how NR gauges have been adapted to study cosmological spacetimes, see Ref. [60].

⁷Commonly one chooses $\bar{N} = 1$ (cosmic time) or $\bar{N} = a$ (conformal time).

$$ds^2 = -\bar{N}^2(1 + 2A)dt^2 + 2a\bar{N}(\partial_i B)dt dx^i + a^2[(1 + 2C)\delta_{ij} + 2\partial_{(i}\partial_{j)}E + 2\hat{E}_{ij}]dx^i dx^j. \quad (12)$$

The purpose of this present work is to construct a complete methodology for initializing and extracting tensor perturbations from cosmological NR simulations. As we mention in Sec. VI, we plan to extend this work to cover both scalar and tensoral perturbations in the near future (see for example Ref. [62]), and will present a complete dictionary between scalar and tensor CPT perturbations and BSSN variables here. However, for all of the numerical results presented in this work, we do not initialize scalar perturbations, and thus the initial metric is given simply by

$$ds^2 = -\bar{N}^2 dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j. \quad (13)$$

Note that this corresponds to the geodesic gauge choice in NR, which is equivalent to synchronous gauge in cosmological perturbation theory (Ref. [60]).

C. Vacuum tensor perturbations

The transverse-traceless part of the spatial metric can be thought of as a massless spin-two field—a graviton. If we assume that the graviton field is in its ground state and that the field was in a Minkowskian vacuum state in the far past, then its mode functions will evolve according to the Mukhanov-Sasaki equation (Ref. [10]). In this section we briefly describe the results of the Mukhanov-Sasaki equation, which we will use to generate a stochastic realization of the graviton field on the lattice.

1. Fourier convention

The Mukhanov-Sasaki formulation is written in Fourier space. We write the transform from Fourier space to configuration space as

$$f(\mathbf{x}) = \int d^3k f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

and the transform from configuration space to Fourier space as

$$f(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3x f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

In GRChombo, we use the discrete definition for each component of $\mathbf{k}$ given by

$$k_j = \frac{2\pi j}{L},$$

where L is the length of the box in code units, and j is the unitless index running from $[0, N - 1]$, where N gives the total number of points on one edge of the lattice.

See Appendix C for details of the implementation of this transform using the FFTW library.

2. The Mukhanov-Sasaki solution

The Einstein-Hilbert action in natural units is given by

$$S = \frac{m_{pl}^2}{16\pi} \int d^4x R.$$

Applying the tensoral CPT line element Eq. (13), and writing the tensoral perturbation as

$$f_{ij} = \sum_s f_s \epsilon_{ij}^s, \quad (14)$$

where $f_s \equiv ah_s$ is the cosmological mode function analogous to the gravitational mode function in Minkowski space, we can derive the first-order tensoral contribution to the action (Refs. [63,64]),

$$S^T = \frac{m_{pl}^2}{16\pi} \sum_{s \in [+,\times]} \int d\tau d^3x \left[(f_s')^2 - (\nabla f_s)^2 + \frac{a''}{a} f_s^2 \right].$$

By varying this action we arrive at the Mukhanov-Sasaki equation, which in Fourier space reads as

$$f_s'' + \left(k^2 - \frac{a''}{a} \right) f_s = 0. \quad (15)$$

We note that we can transform this equation such that it solves for h_s rather than f_s , and from conformal time to cosmic time, which gives

$$\ddot{h}_s + 3H\dot{h}_s + \left(\frac{k}{a} \right)^2 h_s = 0. \quad (16)$$

In the subhorizon limit, where $a''/a \ll k^2$, Eq. (15) becomes a wave equation for f_s . Thus in this regime, we can promote the tensoral degrees of freedom f_s to quantum operators

$$\hat{f}_s(\mathbf{k}) = f_s(k) \hat{a}_{\mathbf{k}} + f_s^*(k) \hat{a}_{-\mathbf{k}}^\dagger,$$

where the quantum mode functions f_s satisfy the classical wave equation at tree level (Ref. [54]). The vacuum state for a quantum operator $\hat{f}_s$ satisfying this equation is the Bunch-Davies vacuum

$$f_s(k, \tau) = \frac{e^{-ik\tau}}{m_{pl}\sqrt{2k}},$$

where we have used $\omega^2 = k^2$, corresponding to a massless Bunch-Davies state, and where we have chosen the positive-frequency solution. This can be used as an initial

condition for Eq. (15). If we assume small slow-roll parameters, we can approximate $a''/a \sim 2/\tau^2$, giving the solution to Eq. (15) in the slow-roll regime as

$$f_s(k, \tau) = \frac{e^{-ik\tau}}{m_{pl}\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right). \quad (17)$$

This solution will be valid as long as our slow-roll parameters, defined in Eq. (9), remain small ($\sim 10^{-3}$) compared to 1. We ensure this condition by measuring the background slow-roll parameters throughout the simulations, and confirming that they begin and remain sufficiently small. See Fig. 2 for a typical example of the evolution of the first SRP.

Taking the conformal time derivative of this mode function gives the appropriate solution for the velocity field,

$$f'_s(k, \tau) = \frac{e^{-ik\tau}}{m_{pl}\sqrt{2k}} \left(\frac{i}{k\tau^2} - ik - \frac{1}{\tau} \right). \quad (18)$$

We will use Eqs. (17) and (18) to build stochastic initial conditions for the metric.

III. GRAVITATIONAL WAVES IN NUMERICAL RELATIVITY

We wish to use the perturbative evolution of tensor perturbations to inform the development of our numerical simulations, where the interaction between various metric degrees of freedom can contribute at any order. First, we formulate the perturbative CPT variables in terms of the nonperturbative BSSN variables, in the regime where the two sets can be identified. This will allow us to phrase the initial conditions of our simulation in terms of a semiclassical realization of the CPT variables, and to create a basic extraction method for these variables.

A. The BSSN formalism

The BSSN formalism (Refs. [65,66]) is a particular decomposition of the Arnowitt–Deser–Misner (Ref. [67]) form of Einstein’s equations, which casts Einstein’s equations into a Cauchy problem for the spatial metric on a set of hypersurfaces Σ_t . The BSSN equations are found by plugging the ADM form of the metric,

$$ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j, \quad (19)$$

into Einstein’s equations. Note that γ_{ij} , the spatial metric, is typically chosen to represent all physical degrees of freedom, and α (the lapse) and β^i (the shift) represent the gauge degrees of freedom.

We also need the definition of the extrinsic curvature, which takes the form

$$K_{ij} = -\frac{1}{2\alpha} (\partial_t \gamma_{ij} - D_{(i} \beta_{j)}) \quad (20)$$

on the hypersurface Σ_t , where D_i is the covariant four derivative projected onto the hypersurface by γ_{ij} . This is the first ADM equation. Applying Eqs. (19) and (20) to Einstein’s equations gives the second ADM equation,

$$\begin{aligned} \partial_t K_{ij} = & \alpha (R_{ij} - 2K_{ik} K_j^k + K K_{ij}) \\ & - D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{k(i} \partial_{j)} \beta^k \\ & - \frac{8\pi\alpha}{m_{pl}^2} \left(S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right), \end{aligned} \quad (21)$$

where R_{ij} are the components of the Ricci tensor on the spatial hypersurface, and S_{ij} , S and ρ are derived from the stress-energy tensor (Refs. [68,69]). In the case of inflation, the stress-energy tensor is often taken to be that of a scalar field with some potential $V(\phi)$, which evolves according to the Klein-Gordon equation (Ref. [57]),

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{dV}{d\phi} = 0. \quad (22)$$

These equations of motion are accompanied by a set of constraint equations, which must be satisfied on every hypersurface:

$$R + K^2 + K_{ij} K^{ij} = 16\pi G \rho \quad (23)$$

$$D_j (K^{ij} - \gamma^{ij} K) = 8\pi G S^i. \quad (24)$$

The BSSN decomposition defines a separate “scalar” and “tensor” component for the γ_{ij} and K_{ij} variables, and reformulates Eqs. (20) and (21) in terms of these new components. This reformulation transforms the ADM equations into a well-posed system of partial differential equations (Ref. [68]). The spatial metric is decomposed into a conformal factor $\chi \equiv (\det[\gamma_{ij}])^{-1/3}$ and a conformally rescaled metric $\tilde{\gamma}_{ij}$ such that

$$\gamma_{ij} = \frac{1}{\chi} \tilde{\gamma}_{ij} \quad (25)$$

and $\det[\tilde{\gamma}_{ij}] = 1$ (Ref. [69]). The extrinsic curvature is decomposed onto a trace $K \equiv K^i_i$ and traceless metric $\tilde{A}_{ij}$ such that

$$K_{ij} = \chi \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right) \quad (26)$$

and $\tilde{A}^i_i = 0$ (Ref. [69]). Applying this decomposition to Eqs. (20) and (21) yields a set of four coupled first-order PDEs, called the BSSN equations:

$$\begin{aligned}
\partial_t \chi &= \beta^k \partial_k \chi + \frac{2}{3} \chi (\alpha K - \partial_k \beta^k) \\
\partial_t \tilde{\gamma}_{ij} &= \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i} \partial_{j)} \beta^k - 2\alpha \tilde{A}_{ij} - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k \\
\partial_t K &= \beta^k \partial_k K + \alpha (R + K^2) - \gamma^{kl} D_k D_l \alpha \\
&\quad + \frac{4\pi\alpha}{m_{pl}^2} (S - 3\rho) \\
\partial_t \tilde{A}_{ij} &= \beta^k \partial_k \tilde{A}_{ij} + \chi \left(-D_i D_j \alpha + \alpha \left(R_{ij} - \frac{8\pi}{m_{pl}^2} S_{ij} \right) \right)^{TF} \\
&\quad + \tilde{A}_{ij} \left(\alpha K - \frac{2}{3} \partial_k \beta^k \right) + 2\tilde{A}_{k(i} \partial_{j)} \beta^k - 2\alpha \tilde{\gamma}^{kl} \tilde{A}_{ik} \tilde{A}_{jl},
\end{aligned}$$

where $(-)^{TF}$ represents the tracefree part of the quantity in brackets, and where $R = \gamma^{ij} R_{ij}$ (Ref. [69]). The variables which must be specified on the initial slice of the simulation are $\{\chi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \alpha, \beta^i\}$. The lapse and the shift will be initialized according to our gauge choice, but the remainder will require physically motivated initial data.

The constraint equations (23) and (24) can be rewritten in terms of these BSSN variables as

$$\mathcal{H} \equiv R + \frac{2}{3} K^2 - \tilde{A}_{ij} \tilde{A}^{ij} - 16\pi G \rho = 0 \quad (27)$$

$$\begin{aligned}
\mathcal{M}_i &\equiv \tilde{\gamma}^{kl} \left(\partial_k \tilde{A}_{il} - 2\tilde{\Gamma}_{l(i}^m \tilde{A}_{k)m} - 3\tilde{A}_{ik} \frac{\partial_l \chi}{2\chi} \right) \\
&\quad - \frac{2}{3} \partial_i K - 8\pi G S_i = 0,
\end{aligned} \quad (28)$$

where we have introduced the BSSN constraint-tracking variables $\mathcal{H}, \mathcal{M}_i$ (Ref. [69]).⁸

B. The CPT-BSSN correspondence

We present here a complete correspondence between the BSSN and CPT variables in the linear regime. This correspondence has been noted previously for specific gauge choices, for example in Ref. [38] for the Newtonian gauge. However, we have developed this correspondence in a *gauge-agnostic* manner, such that any particular gauge choice can be applied directly to the correspondence equations. We anticipate that this dictionary may facilitate comparisons between the results of lattice calculations which have been formulated in separate gauges.

This correspondence was made by comparing the perturbative metric, Eq. (12), to the fully relativistic metric, Eq. (19), and by calculating the extrinsic curvature to first order in perturbative variables. First, we can assign the shift term according to the $dt dx^i$ component of Eq. (12),

$$\beta^i = 2a\bar{N}\partial_i B.$$

Then the $\beta^i \beta_i$ term in Eq. (19) is second order in perturbations, and can be ignored. We assign the lapse to be

$$\alpha = \bar{N} \sqrt{1 + 2A} \approx \bar{N}(1 + A),$$

where the approximation denotes a Taylor expansion truncated at first order. The spatial metric is assigned simply as

$$\gamma_{ij} = a^2[(1 + 2C)\delta_{ij} + 2\partial_{(i}\partial_{j)}E + 2\hat{E}_{ij}]. \quad (29)$$

The extrinsic curvature can now be calculated according to Eq. (20), where we note that at zeroth order the Christoffel coefficients vanish, and as β_i is strictly first order in perturbations, we can replace $D_i \rightarrow \partial_i$. This gives

$$\begin{aligned}
K_{ij} &= a^2 \left[H \left(A\delta_{ij} - \frac{\gamma_{ij}}{a^2} \right) + \frac{1}{\bar{N}} \left(\partial_t \frac{\gamma_{ij}}{a^2} \right) - \frac{1}{a} \partial_i \partial_j B \right] \\
&= a^2 \left[H((A - 2C - 1)\delta_{ij} - 2\partial_{(i}\partial_{j)}E - 2\hat{E}_{ij}) \right. \\
&\quad \left. + \frac{1}{\bar{N}} (\dot{C}\delta_{ij} + \partial_{(i}\partial_{j)}\dot{E} + \hat{E}_{ij}) - \frac{1}{a} B_{,ij} \right].
\end{aligned} \quad (30)$$

We can now calculate the BSSN variables. Using the fact that $\partial_{(i}\partial_{j)}E$ and $\hat{E}_{ij}$ are traceless, the conformal factor becomes

$$\chi = \frac{1}{a^2} (1 + 6C)^{-\frac{1}{3}} \approx \frac{1}{a^2} (1 - 2C). \quad (31)$$

Then the conformally rescaled metric is

$$\tilde{\gamma}_{ij} = \chi \gamma_{ij} \approx \delta_{ij} + 2\partial_{(i}\partial_{j)}E + 2\hat{E}_{ij}. \quad (32)$$

The trace of the extrinsic curvature is calculated by finding γ^{ij} to first order, raising the index of K_{ij} , and then taking the trace. This gives

$$K \approx -3H(1 - A) - 3\frac{\dot{C}}{\bar{N}} - \frac{1}{a} \nabla^2 B, \quad (33)$$

where we have again used the fact that $\partial_{(i}\partial_{j)}E$ and $\hat{E}_{ij}$ are trace free. The trace-free extrinsic curvature follows simply:

$$\begin{aligned}
\tilde{A}_{ij} &= \frac{1}{\chi} K_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} K \\
&\approx \frac{1}{\bar{N}} \left[\partial_{(i}\partial_{j)} \left(\dot{E} + \frac{B}{a} \right) + \hat{E}_{ij} \right].
\end{aligned} \quad (34)$$

Equations (31)–(34) represent the full translation between CPT and BSSN variables, in the case where scalar

⁸Note that in this work, $\mathcal{H}$ does *not* refer to the conformal-time Hubble parameter.

and tensor perturbations are included. We note also that this work is compatible with the work presented in Ref. [62] for the case of stochastic evolution of scalar perturbations during inflation. However, as we will be concerned primarily with tensorial perturbations in this work, we neglect scalar perturbations, giving the tensor-only correspondence:

$$\begin{aligned}\chi &= \frac{1}{a^2} \\ \tilde{\gamma}_{ij} &= \delta_{ij} + 2\hat{E}_{ij} = \delta_{ij} + h_{ij} \\ K &= -3H \\ \tilde{A}_{ij} &= \frac{1}{\tilde{N}}\hat{E}_{ij} = \frac{1}{2\tilde{N}}\dot{h}_{ij}.\end{aligned}\quad (35)$$

Thus any method for prescribing the cosmological variables $\{a, H, h_{ij}, \dot{h}_{ij}\}$ is sufficient to initialize the BSSN variables in the linear regime.

If we apply the correspondence, Eq. (35), to the Hamiltonian constraint, Eq. (27), we find

$$\mathcal{H} = R(h) + 6H^2 - \frac{1}{4}\dot{h}_{kl}\dot{h}^{kl} - 16\pi G\rho = \mathcal{O}(h^2),$$

since R and $\tilde{A}^{ij}\tilde{A}_{ij}$ only receive contributions at second order and above. Applying (35) to (28) gives

$$\mathcal{M}_i = -\frac{1}{2}\delta^{kl}(\partial_k\dot{h}_{li} - 2\tilde{\Gamma}_{l(i}^m\dot{h}_{k)m}) - 8\pi GS_i,$$

and given that $\tilde{\Gamma}_{jk}^i$ is first order in perturbations, that $S_i = 0$ and that $\dot{h}_{ij}$ is transverse, this gives $\mathcal{M}_i = \mathcal{O}(h^2)$. Thus as long as constraint violations remain at or below second order in h , we can be confident in our first-order results.

IV. NUMERICAL METHODS AND IMPLEMENTATION

In this section, we provide details of how our initial data were constructed, and how we extracted tensorial perturbations from GRChombo.

A. Gauge choice

In NR, a gauge is chosen by specifying initial data for the lapse and shift, and then evolving these data according to some first-order evolution equation. The evolution equations for α and β_i in GRChombo are represented by the general Bona-Masso family of slicing conditions, first formulated in Ref. [70]:

$$\begin{aligned}\partial_t\alpha &= a_1\beta^k\partial_k\alpha - a_2\alpha^{a_3}K \\ \partial_t\beta^i &= b_1\beta^k\partial_k\beta^i + b_2B^i \\ \partial_tB^i &= b_1(\beta^k\partial_kB^i - \beta^k\partial_k\tilde{\Gamma}^i) + \partial_t\tilde{\Gamma}^i - \eta B^i,\end{aligned}$$

where $\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}_{jk}^i$ are the contracted Christoffel coefficients on the rescaled spatial metric, and where η is a freely specifiable parameter controlling the damping of B^i (Ref. [69]).

A particular gauge within the Bona-Masso family is selected by choosing $\{a_i, a_2, a_3, b_1, b_2, \eta\}$. In this work, we must use a geodesic gauge in either cosmic or conformal time, as this is required by our initial metric, Eq. (13). We chose to use cosmic time, which sets $\alpha = 1$ and $\beta^i = 0$ on the initial slice, and chose all Bona-Masso coefficients to be zero so that the lapse and the shift should not evolve from these values.⁹

We may wish to move to conformal time, where $\tilde{N} = a(\tau)$ rather than 1. This can be accomplished in the Bona-Masso family by setting $\alpha = a^2(0)$ on the initial slice, and choosing $a_1 = 0$, $a_2 = 2/3$ and $a_3 = 1$. This ensures

$$\partial_t\alpha = -\frac{2}{3}\alpha K \sim 2a^2H,$$

as expected, where $\alpha(\tau) = a^2(\tau)$.

B. Initial data construction

The background initial conditions are determined by the choice of inflaton model, and describe which period of inflation we are evolving. The perturbation initial conditions will be generated similarly to the standard initialization scheme used in cosmological lattice simulations, which seeks to approximate the quantum nature of these perturbations at a very large occupation number with a semiclassical stochastic realization (see for example Refs. [23,38,53]).

1. Inflationary model choice

The initial conditions of the background arise from the Friedman and Klein-Gordon equations. We choose to start in a regime of slow roll, where we can use the slow-roll Klein-Gordon equation, as outlined in Sec. IV B. One can combine Eqs. (6) and (10) and, together with Eq. (8) and a choice for the form of $V(\phi)$, evaluate these on the initial slice. Taking the large-field toy model $V = \frac{1}{2}m^2\phi^2$, we find

$$\begin{aligned}H_0^2 &= \frac{8\pi}{3m_{pl}^2} \left(\frac{1}{2}\dot{\phi}_0^2 + \frac{1}{2}m^2\phi_0^2 \right), \\ 3H_0\dot{\phi}_0 &= -m^2\phi_0.\end{aligned}$$

In order to ensure we are deep in the inflationary epoch, we choose $\phi_0 = 4m_{pl}$ with mass parameter $m = 10^{-4}m_{pl}$.

⁹As the Christoffel coefficients vanish to leading order for TT tensorial perturbations, ∂_tB^i should remain zero as well. Any deviation from this path could indicate a departure from a perfect TT gauge.

This choice is similar to the choice made in Ref. [42]. The initial Hubble parameter and scalar field velocity are found by solving the above equations, and are given by $H_0 \approx 8.1934 \times 10^{-4} m_{pl}$ and $\dot{\phi}_0 \approx -1.6273 \times 10^{-5} m_{pl}^2$. This set of parameters gives an initial SRP of $\epsilon_0 \approx 2 \times 10^{-3}$. Note that we feed these initial conditions into GRChombo with 15 decimal places of accuracy, in order to reduce any error present in the incorrect assignment of these values to the level of floating point accuracy.

2. Gaussian random field generation

Inflation is driven by a quantum field, the inflaton. However, parts of the inflationary epoch may be made amenable to the techniques of classical field theory, where a “quantum-to-classical” transition is justifiable. This transition is usually associated with modes crossing outside the horizon (see, e.g., Refs. [10] or [71]). Perturbations much smaller than the horizon size must be treated as quantum to ensure continuity with the Bunch-Davies vacuum state in the far past. On the other hand, after horizon crossing these perturbations are frozen into the background spacetime¹⁰ and then follow entirely classical equations of motion.

However, as shown explicitly in Ref. [72], there is an important distinction between the quantum initial conditions and the ensuing evolution of this vacuum state, which has both quantum and classical contributions. Thus the classical approximation may have applications beyond superhorizon evolution. For instance, predicted observables like the power spectrum from single field inflation are reproduced using classical evolution on the initial Bunch-Davies vacuum state (e.g. Ref. [63]), while higher-order correlators like the bispectrum may also be dominated by evolutionary contributions that are classical rather than quantum, though this is a model-dependent statement [72].

Another previously studied quantum-to-classical transition is the end of inflation, where $\tau \rightarrow 0$ as reheating takes hold. As argued in Ref. [54], in this limit the mode function of the inflaton field will become a stochastic realization of the underlying quantum system, and will obey fully classical equations of motion. References [5,23,38,73] use this fact to construct scalar initial data for lattice simulations of the preheating period. We will use a similar method to construct stochastic initial data for tensorial perturbations in the near-horizon regime, though we note, in Sec. VB, a key difference in the treatment of the phase between our initialization method and previous methods.

Consider the quantum state whose mode functions obey the Mukhanov-Sasaki solution, Eq. (15). The two-point correlator of this state is given by

$$\langle \hat{f}_s(\mathbf{k}) \hat{f}_s(\mathbf{k}') \rangle \equiv \langle 0 | \hat{f}_s(\mathbf{k}) \hat{f}_s(\mathbf{k}') | 0 \rangle.$$

Inserting the decomposition of the mode function onto raising and lowering operators gives the relationship to the power spectrum,

$$\langle \hat{f}_s(\mathbf{k}) \hat{f}_s(\mathbf{k}') \rangle = (2\pi)^2 P_h(k) \delta(\mathbf{k} - \mathbf{k}').$$

In the case of reheating, oscillatory modes are most important to the dynamics of interest, as the interaction between these modes produces the resonance that characterizes the thermalization of the postinflationary Universe (Refs. [5,24,38,46,74,75]). Thus in these cases, all modes are subhorizon on the initial slice, in which case the power spectrum is well approximated by the Bunch-Davies spectrum, $P_{BD} = 1/2\omega_k$. Here, we are interested in modes that are very near the horizon, and that cross the horizon within the dynamic range of our simulation. Therefore we will base our power spectrum on the solution to the Mukhanov-Sasaki equation presented in Eq. (17), which accounts for alterations to the mode function arising from the presence of a Hubble scale:

$$\begin{aligned} P_h(k) &= |h_k(\tau)|^2 = \frac{1}{a^2 m_{pl}^2} |f_k(\tau)|^2 \\ &= \frac{1}{a^2 m_{pl}^2} \left(\frac{1}{2k} + \frac{(aH)^2}{2k^3} \right). \end{aligned} \quad (36)$$

Note here that we have used the de Sitter approximation $\tau = aH$, as we know H_0 on the initial slice and therefore can construct this solution analytically. This power spectrum transitions smoothly between the expected scaling for subhorizon modes, $P_h \propto 1/k$, and the expected scaling for superhorizon modes, $P_h \propto 1/k^3$.

We will represent the tensor initial conditions as a Gaussian random field (GRF) generated using this power spectrum. GRFs can be constructed by decomposing the field at each point in Fourier space into a magnitude and phase,

$$h_s(k) = M_h e^{i\theta_h}, \quad (37)$$

where M_h is drawn from a Rayleigh distribution¹¹ with size parameter $\sigma_h = \sqrt{P_h(k)}$, and where θ_h represents the phase of the quantum state. In order for the resulting field to be Gaussian distributed, θ_h must contain a random variable drawn independently from the uniform distribution $\mathcal{U}(0, 2\pi)$ at each point in Fourier space. However, we have found that the inherent phase present in the mode function Eq. (17) is essential to accurately recover the correct spectral evolution

¹¹Note that, as C++ has no in-built function to draw from a Rayleigh distribution, we use an equivalent Rayleigh draw given by

$$R(P) = \sqrt{-2 \ln(U) P(k)},$$

where $U \sim \mathcal{U}(0, 1)$ is a random variable drawn from the standard uniform distribution.

¹⁰At least for slow-roll single-field inflation.

predicted by the Mukhanov-Sasaki equation (15). See Sec. VB for more details.

The configuration-space polarization fields are found via the inverse Fourier transform:

$$h_s(\mathbf{x}) = \int d^3k h_s(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

where the reality of the initial conditions is enforced by imposing Hermitian symmetry on $h_s(\mathbf{k})$. The Fourier-space tensor object can be constructed from the mode functions using the polarization basis tensors,¹²

$$h_{ij}(\mathbf{k}) = \sum_s h_s(k) \epsilon_{ij}^s(\mathbf{k}).$$

The appropriate initial conditions for the rescaled spatial metric are then

$$h_{ij}(\mathbf{x}) = \int d^3k h_{ij}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (38)$$

Note that $h_{ij}(\mathbf{x})$ is not found by combining $h_s(\mathbf{x})$ and a set of ϵ_{ij} constructed in configuration space. This is because the Fourier transform does not distribute over the multiplication of the mode functions and the basis tensors. In the same vein, the configuration-space mode functions are not to be found by using the orthonormality condition on $h_{ij}(\mathbf{x})$. The Fourier-space tensor object $h_{ij}(\mathbf{k})$ is calculated first, in both cases.

We have developed a new initial condition class of GRChombo which generates this random tensor perturbation on the lattice.¹³ This program also verifies that the tensor field is trace free to a specified level of precision. The polarization fields h_s are extracted from GRChombo's output files, using a similar program to the initial condition class but which is currently external to GRChombo (but which shortly will be combined). An initial polarization field generated by this class is shown in panel (a) of Fig. 7.

To fully initialize GRChombo we need an initial profile for $\dot{h}_{ij}$. In order to derive the power spectrum of $\dot{h}$, we must translate our expression for the comoving tensor perturbation f_s into an expression for h_s , and we must change our time coordinate from conformal to cosmic time. Performing these operations, we find

$$\begin{aligned} \dot{h}(t) &= \frac{1}{a^2} \left(f'(\tau) - \frac{a'}{a} f(\tau) \right) \\ &= -\frac{i}{a^2} \sqrt{\frac{k}{2}} e^{-ik\tau}, \end{aligned}$$

¹²See Appendix B for further details on how these polarization basis tensors are constructed on the grid in Fourier space.

¹³We expect that this class will be made public in an upcoming release of GRChombo/GRTEclyn; however if you would like access to this functionality before such time, please contact the authors.

meaning that the power spectrum for the velocity field has the same scaling with k as in Minkowski space:

$$\begin{aligned} P_{\dot{h}}(k) &= |f'_k(\tau)|^2 \\ &= \frac{1}{a^4 m_{pl}^2} \frac{k}{2}. \end{aligned} \quad (39)$$

Note that this power spectrum also recovers the super-horizon limit, as where k is very small, the power spectrum approaches zero. The random velocity field is generated by applying this power spectrum to the exact same random amplitude and phase draw as are used to generate $h_s(\mathbf{k})$.

3. Window function

Random initial data can have large spatial derivatives on small length scales. However, our finite grid resolution limits the precision with which we can resolve the derivative between two nearby points. Initial data which are too “noisy” on small scales can introduce numerical instabilities early, and the derivative stencil will artificially inflate power in high modes as the simulation progresses. One common way to avoid this issue is to apply a window function to the initial data (See Refs. [38,46]). This suppresses the power leakage into high modes by damping their power on the first slice.

We chose to use a tanh-like window function,

$$W_{k_*, \Delta_k}(k) = \frac{1}{2} [1 - \tanh(\Delta_k(k - k_*))],$$

where k_* is the cutoff mode and Δ_k is the window width. We measure k_* in terms of the unit length of the isotropic power spectrum, $2\pi/L$, and measure Δ_k in terms of L .

C. Extraction of transverse-traceless perturbations

As long as we remain in the perturbative regime, we should be able to extract the tensor perturbations at successive slices using the simple relation

$$h_{ij}(t) = \tilde{\gamma}_{ij}(t) - \delta_{ij}. \quad (40)$$

We employ this scheme for the results shown here, and ensure that the perturbations remain at least trace free throughout the simulation with the same tests that were run on the initial conditions. We note that this method avoids the complication laid out in Ref. [76], which details how the transverse nature of a gravitational wave signal evolved on the lattice depends on the spatial derivative stencil used in the evolution scheme.¹⁴

¹⁴We are, however, able to ensure our initial conditions are transverse on the first slice, before any spatial derivative stencil has altered them.

However, if the metric were to develop a significant trace or transverse part, as will be the case when we include initial scalar perturbations, we would need a more robust extraction mechanism for tensor modes, which can adequately separate each component of the metric and which takes into account the derivative stencil used. We can construct an improved extraction method by using the decomposition, outlined clearly in Ref. [77], which explicitly removes the trace and the transverse component of a generic metric. Let g_{ij} be a general perturbation of the full metric $\tilde{\gamma}_{ij}$, which may or may not be transverse traceless.¹⁵ We can write g_{ij} as

$$g_{ij} = \frac{1}{3}\delta_{ij}g + 2D^{(i}V^{j)} + h_{ij}, \quad (41)$$

where $g \equiv g_i^i$ and

$$V^i \equiv \gamma^{il}\partial^m \left(g_{ml} - \frac{1}{3}\delta_{ml}g \right),$$

representing the transverse violation of the trace-free perturbation (Ref. [77]). Then the true transverse-traceless signal can be extracted most generally from the rescaled spatial metric $\tilde{\gamma}_{ij}$ by calculating

$$h_{ij} = \tilde{\gamma}_{ij} - \delta_{ij} - \frac{1}{3}\delta_{ij}g - 2D^{(i}V^{j)}, \quad (42)$$

where D_i is approximated using the same stencil which was used in the evolution scheme. This reduces to the simple extraction method in the case where $g = V^i = 0$. We expect to show further results using this improved method of extraction in an upcoming publication.

Additionally we expect that, beyond linear order, both of the extraction schemes above may not be gauge invariant. We are currently developing an extraction scheme based on the Weyl tensor, described for instance in Ref. [78], which we expect to be gauge invariant to at least second order. The full details of this scheme will be presented in Ref. [79].

V. COSMOLOGICAL EVOLUTION AND VALIDATION

Here, we provide validation of our GRChombo example in the superhorizon and horizon-crossing regimes. We define these two cases according to the size of the mode corresponding to the comoving Hubble diameter, $k_{aH} \equiv \pi aH$, with respect to the minimum and maximum resolvable modes in the box. Given a box length L , the smallest resolvable mode is given by the “DC” mode¹⁶

¹⁵In fact, on an FLRW background g_{ij} would correspond to the tensor perturbation E_{ij} presented in Eq. (11).

¹⁶Note that we include the volume correction $\sqrt{3}$ here, as this corresponds to the largest mode propagating *diagonally* through the box.

$k_{DC} = 2\pi\sqrt{3}/L$, and the largest resolvable mode is given by the Nyquist mode, $k_{Nyq} = \pi N/L$.

For the superhorizon case, we validate the background dynamics. For the horizon-crossing case, we additionally demonstrate agreement with the Mukhanov-Sasaki equation (17) in the evolution of the tensor perturbations. This comparison allowed us to validate our results against the linear solution, as well as to uncover a more faithful representation of semiclassical initial conditions. We also present convergence tests on the constraints and the evolution of the tensor mode function at high k , and we show preliminary calculations of the skewness and kurtosis of the polarization fields.

A. Superhorizon validation

The superhorizon run initially satisfies $k_{Nyq} \sim k_{aH}$, such that all modes are initialized above the horizon scale. Given the model choice described in Sec. IV B, $k_{aH} \approx 2.57 \times 10^{-3} m_{pl}$. We choose to use $N = 128$ grid points as the standard resolution, and choose $L = 16 \times 10^4 m_{pl}^{-1}$, as this places the Hubble scale close to the grid resolution on the initial slice. This gives a mode range of

$$\begin{aligned} k_{DC} &\approx 6.80 \times 10^{-5} m_{pl} \\ k_{Nyq} &\approx 2.51 \times 10^{-3} m_{pl}. \end{aligned}$$

Figure 1 shows the phases of evolution that the superhorizon run passes through, characterized by the location of the comoving k_{aH} with respect to k_{Nyq} . We use the following parameters for the window function:

$$\begin{aligned} k_* &= 3/4 k_{Nyq} \approx 1.88 \times 10^{-3} m_{pl} \\ \Delta_k &= L/30 \approx 5333 m_{pl}^{-1}. \end{aligned}$$

We choose this value of k_* so that most of our modes are resolved, but modes near k_{Nyq} are excluded. Our choice of

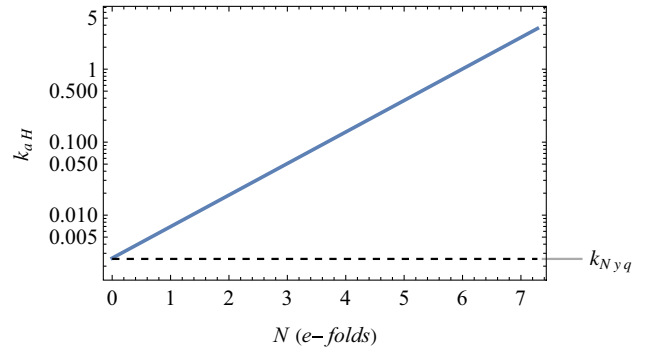


FIG. 1. The evolution of the Hubble mode k_{aH} (blue) as a function of e-folds, in the superhorizon case. The black dashed line shows the smallest resolvable mode, k_{Nyq} , for a grid size $N = 128$.

Δ_k was made such that our modes were suppressed smoothly, in order to avoid any instabilities which may arise in k space from having too discontinuous of a cutoff. The shape of this cutoff function can be seen in Fig. 6. Additionally, we expect that the window will not be entirely necessary in preserving the stability of the evolution as the dynamics are extremely suppressed on superhorizon scales. We include it here in order to make contact with the horizon-crossing results presented later.

We choose a background field which is described by the initial conditions laid out in Sec. II B. We extract background quantities as box averages of the corresponding BSSN quantities,

$$H \approx -\frac{1}{3}\bar{K}, \quad a \approx \frac{1}{\sqrt{\bar{\chi}}}; \quad (43)$$

however we do note some possible issues with this procedure have recently been brought forward in the literature, for instance in Ref. [60]. We suspect that this procedure is appropriate in our case, as we do not expect perturbations to become large in amplitude, and thus the spacetime should remain roughly homogeneous on the scale of the box.

We write the first slow-roll parameter under the slow-roll approximation as

$$\epsilon \approx \frac{3/2\dot{\phi}^2}{\bar{\rho}}, \quad (44)$$

since we can obtain $\dot{\phi}$ and $\bar{\rho}$ easily from GRChombo. Figure 2 shows a comparison in the evolution of the first slow-roll parameter between GRChombo and the Friedman equations. We note that these two evolution schemes agree to a high precision, as expected from our model choice. We also observe that the first slow-roll parameter remains small ($\sim 10^{-3}$) over a dynamic range which captures full horizon-crossing behavior, $\gtrsim 7$ e-folds.

B. Single mode validation

We now turn to the evolution of tensoral perturbations in this cosmological spacetime. In order to ensure a thorough match between our results in GRChombo and those produced by the Mukhanov-Sasaki equation, we performed a series of “single mode” tests, where we initialized only one plane wave in the box, and matched the power of this mode though the course of the simulation with the prediction from linear theory.¹⁷ We performed this test using multiple methods of initialization, some of which have been

¹⁷Note that this is not the power *spectrum*, as the power spectrum in this case is not continuous, but rather a delta function in Fourier space.

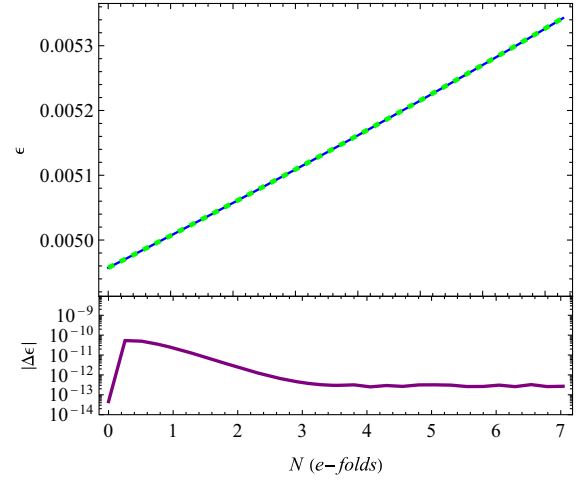


FIG. 2. The first slow-roll parameter evolution for the super-horizon case, extracted directly from GRChombo (solid blue) and from the Friedman equations (dashed green) using the same background initialization. The difference between these curves is shown in purple.

employed in the literature for the study of semiclassical perturbations in the early Universe.

We present the most insightful of these comparisons in Fig. 3. In this test we compared the power evolution for two initialization methods, which we will call the decoherent method (a), and the mode function method (b), for short. The decoherent method initializes the field at each point in Fourier space in modulus/argument form as

$$h_s(k) = \sqrt{P_{h_s}(k)} e^{i\theta_s}, \quad (45)$$

where $P_{h_s}(k)$ is the power spectrum given by Eq. (36), and θ_s is a stochastic phase drawn from $\mathcal{U}(0, 2\pi)$. For this example we use the standard velocity corresponding to a free wave:

$$\dot{h}_s(k) = -ikh_s(k).$$

The mode function method initializes the real and imaginary parts of the field exactly as given by Eqs. (17) and (18). Transforming these functions into modulus/argument form, we find this method follows the structure

$$h_s(k) = \sqrt{P_{h_s}(k)} e^{i(\theta_{MS}(k) + \theta_s)}, \quad (46)$$

where we note the addition of a k -dependent phase coming directly from the Mukhanov-Sasaki solution.¹⁸ This θ_{MS}

¹⁸The functional form of θ_{MS} is given in Appendix A.

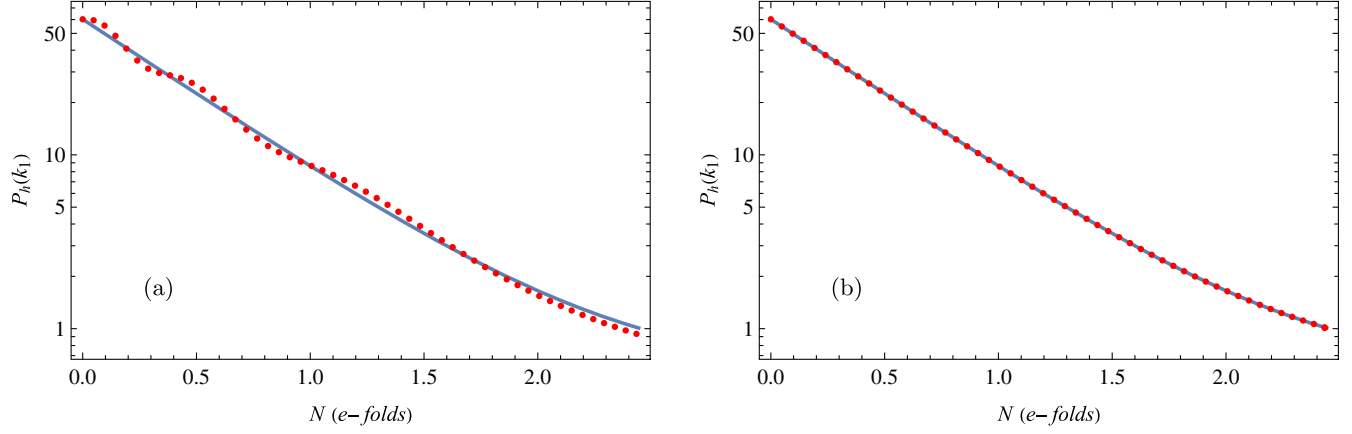


FIG. 3. The evolution of the power $h_+^* h_+$ at a single mode $k_1 = 4\pi/L$ where $L = 3000 m_{pl}^{-1}$. This test was performed in GRChombo using the “decoherent” (a) and mode function (b) initialization methods. Both GRChombo datasets (red, dotted) are compared against the Mukhanov-Sasaki solution (blue, solid) solved using the mode function initialization method at k_1 . This mode remains subhorizon until approximately e-fold 2.

variable introduces a phase offset between the real and imaginary components *at a particular point* in Fourier space. However, its inclusion does not require that separate points in Fourier space be phase coherent, and the standard procedure of assigning a random phase to each point in Fourier space can still be used in addition to this variable. The mode function for $\dot{h}_s$ is similarly decomposed where, as in the decoherent method, we use the same random draw to set the stochastic elements of the velocity field and the position field at each point.

We note that the decoherent method is similar to the method commonly used in lattice studies of inflation, for example in Refs. [40,46,53]. In these works, the magnitude of h_s is set using a Rayleigh distribution centered on the power spectrum, as described in Sec. IV B. We neglect this aspect of the stochastic realization, as it only serves to introduce a difference in the overall scale of the GRChombo and Mukhanov-Sasaki solutions. We note, however, that the inclusion of the Rayleigh draw does not change our conclusion.

As shown in Fig. 3, we recover the smooth decay of the power expected in the subhorizon regime *only* for the mode function method; or, in other words, when we take into account the k -dependent phase difference between the real and imaginary parts of both the initial field and velocity profile. This is because the real and imaginary components are both oscillatory in time, and so only with the correct relative phase will they add in quadrature to produce a smooth power spectrum evolution. The oscillations in the decoherent method power evolution appear when this relative phase is not correctly assigned on the initial slice.

C. Horizon-crossing validation

The horizon-crossing case begins where $k \geq k_{aH}$ for all resolvable k , meaning that all modes start off dynamical. The Hubble diameter then moves into the box, freezing out

modes from the largest to the smallest. Horizon crossing ends where $k < k_{aH}$ for all resolvable k .

We choose $N = 128$ and $L = 3 \times 10^3 m_{pl}^{-1}$ as the standard case. This gives a mode range of

$$k_{DC} \approx 3.63 \times 10^{-3} m_{pl}$$

$$k_{Nyq} \approx 0.134 m_{pl}.$$

Figure 4 demonstrates these phases, in the same style as Fig. 1. Horizon crossing begins at about e-fold 0.5, and lasts until about e-fold 4. Thus we should expect to see oscillatory behavior in the polarization fields initially, and then a gradual transition to a frozen state, completed by e-fold 4.

We use the same parameters for the window function as were used in the superhorizon case, taking into account the change in L :

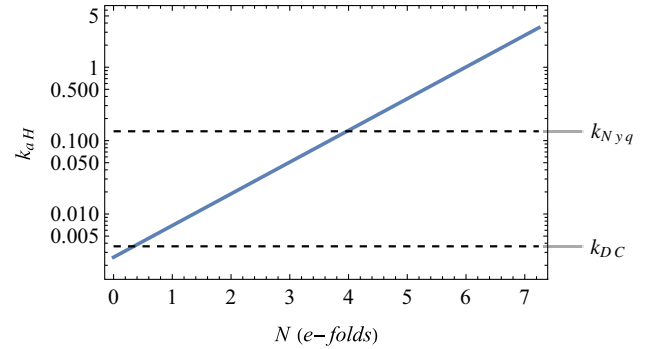


FIG. 4. The Hubble mode k_{aH} as a function of e-folds, for the horizon crossing run. The dotted black lines correspond to the size of the lowest resolvable mode, k_{DC} , and the highest resolvable mode, k_{Nyq} .

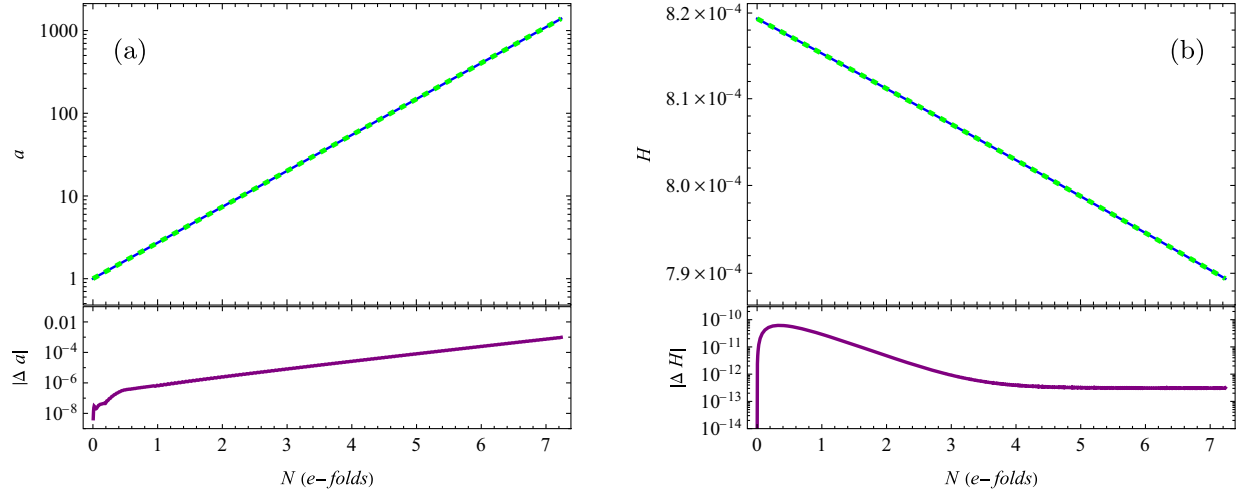


FIG. 5. The scale factor (a) and Hubble parameter (b) evolution in the horizon-crossing case, as calculated by GRChombo (blue, solid) and by the Friedman equations (green, dashed) for the same initial conditions. The differences between these solutions are shown in purple.

$$k_* = 3/4 k_{Nyq} \approx 0.101 m_{pl}$$

$$\Delta_k = L/30 = 100 m_{pl}^{-1}.$$

This produces the initial spectral shape shown in the blue line of Fig. 6. We set A_{ij} according to the spectrum found in Sec. IV B, and apply the same window function to this spectrum.

Figure 5 shows a comparison between a and H as calculated by GRChombo, and as calculated by the Friedman equations with the same initial conditions. We note that for all of these quantities, the error remains small over the course of the simulation, reflecting the fact that we are able to successfully recover the background solution, even in the dynamical regime.

1. Recovery of spectral turnover

We wish to match our solution for $h_s(k)$ to the solution of the Mukhanov-Sasaki equation, Eq. (16), as we have chosen our model such that the slow-roll parameters are small. In order to accomplish this, we evolved the Mukhanov-Sasaki equation using the initial spectrum given in Eq. (17). We do *not* apply randomization to this spectrum in the linearized case, but we do apply the same window function as is used in GRChombo. We note the random draw for the modulus in the GRF formalism can cause disagreement with the results of Mukhanov-Sasaki, particularly where only a few modes are examined. However, we choose to compare the isotropic power spectrum from GRChombo with the results of Mukhanov-Sasaki along the same range of modes, in which case the power from a large number of modes with the same $|\mathbf{k}|$ is integrated. We find that in all but the first bin (where the effects of cosmic variance are most relevant), this average is sufficiently well

sampled that we can recover the magnitude given by Mukhanov-Sasaki prediction well.

Figure 6 compares the evolution of the spectrum between the Mukhanov-Sasaki solution and the numerical solution. The solid lines represent the Mukhanov-Sasaki power spectrum at different time slices, from blue on the initial slice to yellow on the final slice. Eleven slices are shown in total, sampled evenly between e-fold 0 and 7, meaning that one spectrum is shown for every ~ 0.7 e-folds. The dark blue dots represent the isotropic binned power spectrum extracted from GRChombo at the corresponding time slice. We include all bins between $k_2 = 2 \cdot \frac{2\pi}{L}$ and $k_{30} = 30 \cdot \frac{2\pi}{L}$, after which the window function dominates the signal. The initial and final spectra shown in particular demonstrate the recovery of the correct subhorizon and superhorizon scaling with k , as described by Eq. (36). The power spectrum

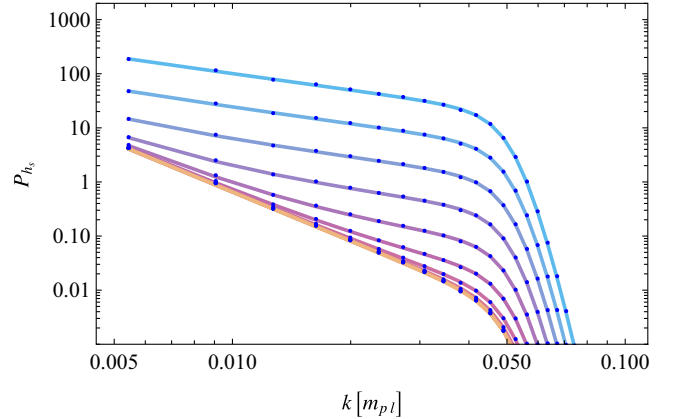


FIG. 6. Evolution of the power spectrum of $h_+(k)$, from e-fold 0 (light blue) to e-fold 7 (yellow). The binned power spectrum extracted from GRChombo is shown in dark blue dots, and the linearized spectrum evolution is shown by the solid lines.

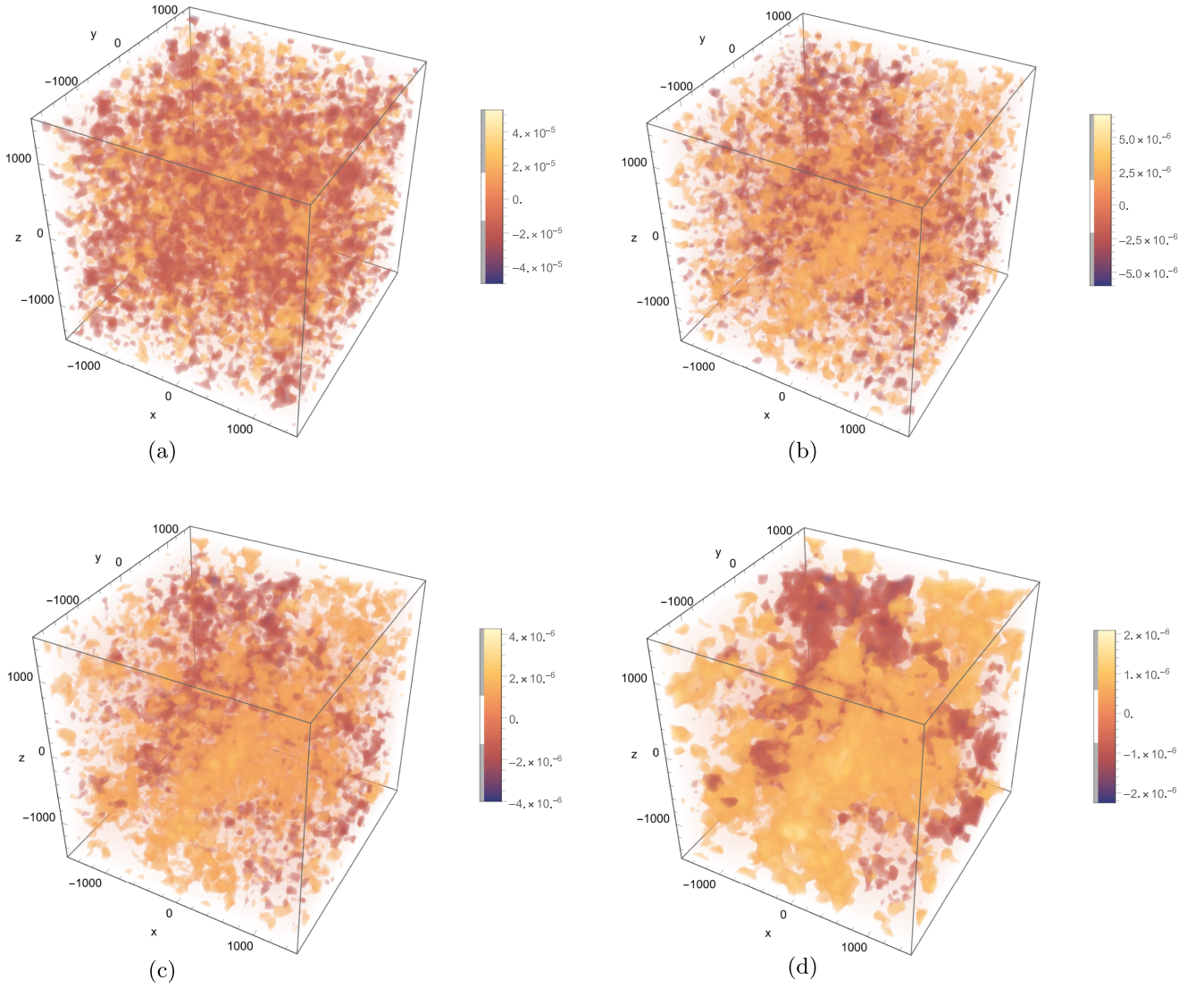


FIG. 7. Density plots of the amplitude of $h_+(\mathbf{x})$, extracted from GRChombo at four points in time: the initial slice (a), the final slice (d), and two slices during horizon crossing: where $N = 2.1$ (b), and where $N = 2.9$ (c). We have allowed the color scale to vary in each plot, in order to highlight the change of structure which occurs in these polarization fields separately from the overall scale change. Note that in our units, length is measured in units of m_{pl}^{-1} .

derived from GRChombo matches the Mukhanov-Sasaki power spectrum quite well for all modes shown, which are the modes best protected from the numerical issues.

Figure 7 shows the configuration-space representation of this spectral phenomenon. Here we present four density plots showing the amplitude of $h_+(\mathbf{x})$ through the course of the simulation. We note the emergence of structure at low modes, corresponding to the transition to a $1/k^3$ scaling. We find that this transition occurs quite obviously between approximately e-folds 2 and 3.5, a period where many of the intermediate modes are frozen out. As the skewness and kurtosis parameters are calculated directly from the field depicted here, the transition that this figure demonstrates will be key to understanding the evolution of the skewness and kurtosis, presented in the final section.

D. Constraints, stability and convergence

For a perfect solution to Einstein's equations, $\mathcal{H} = \mathcal{M}_i = 0$. However, numerical solutions to Einstein's equations can never perfectly replicate the continuum solution, and so some violation of these constraint equations is expected. Stable NR simulations are characterized by constraint variables which are initially small, compared to some relevant physical scale, and which do not grow with time at any point during the simulation. This ensures that the numerical solution stays close to the continuum solution surface.

In order for a realistic numerical solution to be found, the initial data must satisfy the constraints to an acceptable degree. Where the initial curvature of the metric may be large, the initial data can be found by solving the constraints

iteratively for some initial ansatz metric and matter configuration.¹⁹ However, as we plan to stay within perturbation theory for the sake of the following validation, we do not expect large initial curvature to be present. As illustrated at the end of Sec. III B, as long as h_{ij} and $\dot{h}_{ij}$ remain transverse, traceless, and small relative to the background, we expect that the constraints will be satisfied at first order, meaning that they should have a magnitude on the order of σ_h^2 where σ_h measures the size of the tensor perturbations. We choose the sample standard deviation of the polarization fields as a measure of their amplitude on the lattice:

$$\sigma_h = \sqrt{\frac{1}{N^3} \sum_{i,j,k} h_s^2(i, j, k)},$$

where we have eliminated the mean of the h_s from the standard definition as the mean of these fields lies at machine precision.

NR simulations also typically utilize some form of dynamical constraint damping. However, as detailed in Appendix D, dynamical constraint damping can have a strong effect on the spectrum, which could introduce nonphysical signals in our analysis. Additionally, de Sitter space is naturally damping, in that it suppresses higher-order terms in the constraints²⁰ by a factor of a^2 . Therefore we have disabled all forms of dynamical dissipation for the purpose of validation. This means that our simulation could be more prone to leaving the constrained surface of classical solutions. We demonstrate here that we can robustly reproduce the linear results without the use of constraint damping, and that our constraint violation does not grow even in the quasilinear regime.

1. Constraint characterization

Since the BSSN constraint variables $\mathcal{H}, \mathcal{M}_i$ become zero on the solution surface, rarely is there an obvious method to compare their magnitude to some relevant physical quantity. In fact, the appropriate comparison for these constraints will often differ depending on the system one is attempting to solve. In arriving at this comparison scheme, it is important to consider what values these constraints can possibly take on. The Hamiltonian constraint could take on positive or negative values, depending on the arrangement of the perturbation field. The momentum constraint, however, is characterized in GRChombo by its vector magnitude,

$$\mathcal{M} \equiv |\mathcal{M}_i| = \sqrt{\sum_i \mathcal{M}_i^2},$$

and so is always positive.

GRChombo defines two alternative constraint measures which can be used to normalize $\mathcal{H}$ and $\mathcal{M}$. These are the “absolute” constraint variables, defined to be the sum of the absolute values of all terms in the constraint equations, and are given by

$$\begin{aligned} \mathcal{H}_{abs} &\equiv \left| R + \frac{2}{3} K^2 + \tilde{A}_{ij} \tilde{A}^{ij} + 16\pi G\rho \right| \\ \mathcal{M}_{abs} &\equiv \left| \tilde{\gamma}^{kl} \left(\partial_k \tilde{A}_{il} + 2\tilde{\Gamma}_{l(i}^m \tilde{A}_{k)m} + 3\tilde{A}_{ik} \frac{\partial_l \chi}{2\chi} \right) \right. \\ &\quad \left. + \frac{2}{3} \partial_i K + 8\pi G S_i \right|. \end{aligned}$$

Note that $\mathcal{H}_{abs} \neq |\mathcal{H}|$, as $|\mathcal{H}|$ allows for the cancellation between different terms at each point, whereas $\mathcal{H}_{abs}$ does not.

Under the CPT-BSSN correspondence, we note that the mean of $\mathcal{H}_{abs}$ should measure twice the energy density. This energy density will be made up of a background component, captured by the Friedman equation, and a perturbative part arising from the presence of tensor perturbations. Where this gravitational wave energy density is fully accounted for in the initial matter sector (see discussion in Ref. [77]), the constraints may be satisfied beyond second order. However, in our case, we only wish to show that these tensors contribute to the Hamiltonian constraint violation at second order. Thus we define a diagnostic for the Hamiltonian constraint violation due to the presence of tensor perturbations as

$$\mathcal{H}_h \equiv \mathcal{H}_{abs} - 12H^2. \quad (47)$$

This definition eliminates the homogeneous contribution to $\mathcal{H}_{abs}$, leaving only the tensoral component, which will go like σ_h^2 if the constraints are satisfied to second order.

We note that the momentum constraint does not have a contribution from the background. Thus the raw momentum constraint should go like σ_h^2 . We note also that a factor of $1/a^2$ multiplies the second-order tensoral contributions, which arises from the background term in the inverse spatial metric. Thus we define the *comoving* momentum constraints as

$$\mathcal{M}_{cm} \equiv a^2 \mathcal{M}.$$

These measure the constraint violation on comoving scales.

Figure 8 shows the evolution of the sample mean of $\mathcal{H}_h$ (a) and $\mathcal{M}_{cm}$ (b) as a function of e-folds, in the horizon-crossing case. These runs were performed with $L = 3000 m_{pl}^{-1}$ and $H_0 \approx 8.2 \times 10^{-4} m_{pl}$, which produces

¹⁹The recently developed program GRTresna can accomplish this task for GRChombo simulations. See Refs. [80,81] for further details.

²⁰In other words, geodesics naturally separate in de Sitter, and thus their crossing becomes much less likely. For a more detailed argument, see Ref. [60].

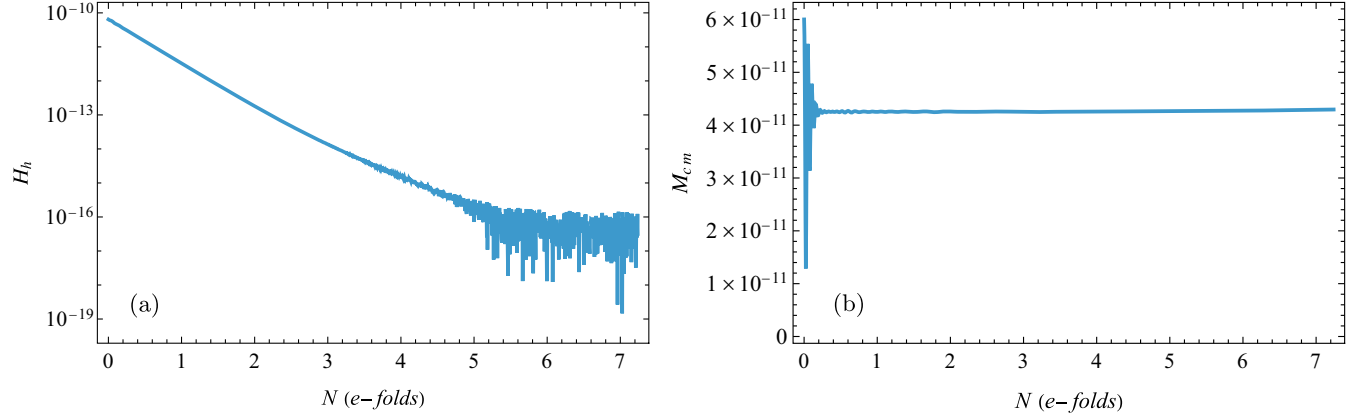


FIG. 8. Sample averages of the constraints in the horizon-crossing case for $N = 128$. Panel (a) shows the residual Hamiltonian $\mathcal{H}_h$, and panel (b) shows the comoving momentum constraint, $\mathcal{M}_{cm}$.

an initial tensor polarization field with $\sigma_h \sim 10^{-5}$. We note that the initial Hamiltonian and momentum constraint violations lie within an order of magnitude of σ_h^2 . The constraints continue to either damp away with a^2 , in the case of the raw constraint measure, or to remain constant, in the case of the comoving constraint measure, and at no point do the constraints show consistent growth. Thus our simulations satisfy this check of numerical stability.

We point out that since the raw constraints scale with a^{-2} , they may decrease sufficiently quickly such that they will reach floating-point precision within the dynamical range of the simulation. This is demonstrated in the evolution of $\mathcal{H}_h$, where we note that in modern C++ the floating-point precision for double-type variables is 10^{-15} . This effect may produce an apparent amplification in the constraints, particularly in the comoving constraints, that is *not* due to an instability in the evolution scheme itself. Parallel analysis of the raw and comoving constraints is therefore recommended, in order to avoid this pitfall.

2. Convergence tests

We have performed convergence tests on the evolution of the power spectrum, at three resolutions: $N_c = 64$, $N_m = 128$ and $N_f = 256$. The initial conditions for these runs were coarse grained off of the finest resolution, so that each run represented the same draw from the same statistical distribution.²¹ In each case we used a window function characterized by

$$k_* = \pi N_c / L \times 3/4,$$

²¹This means, for instance, that the initial conditions for N_c were found by generating the initial conditions for N_f , and placing every fourth point on the GRChombo grid.

such that increasing resolution did not add any new modes with significant power, but only increased the accuracy with which our initial range of modes was resolved. Thus the results of these tests should converge purely on the classical dynamics; they should not contain spurious noise due to the inherent stochasticity of the initial conditions; and any numerical error due to the evolution of modes close to the Nyquist frequency should be increasingly suppressed.

Figure 9 shows convergence on the evolution of one bin in the power spectrum. We choose a high bin for this test, $k_{28} = 28\sqrt{3} \times 2\pi/L$, as high modes are less numerically stable, and so convergent behavior here can be clearer. Note that for each run, k_{28} represents the evolution of a mode at approximately 88% (N_c), 44% (N_m) and 22% (N_f) of the Nyquist frequency. As shown, we recover strong visual

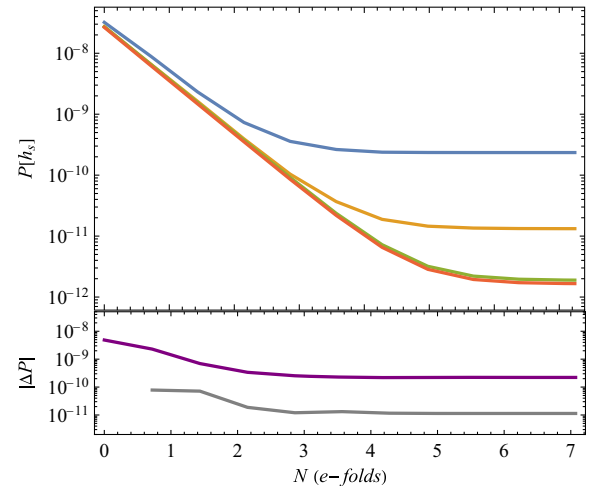


FIG. 9. The evolution of the power in $h_+(k)$ at the isotropic bin k_{28} , for N_c (blue), N_m (orange) and N_f (green). This is compared with the prediction from Mukhanov-Sasaki, solved at k_{28} (red). The differences $|N_m - N_c|$ (purple) and $|N_f - N_m|$ (gray) are shown in the inset.

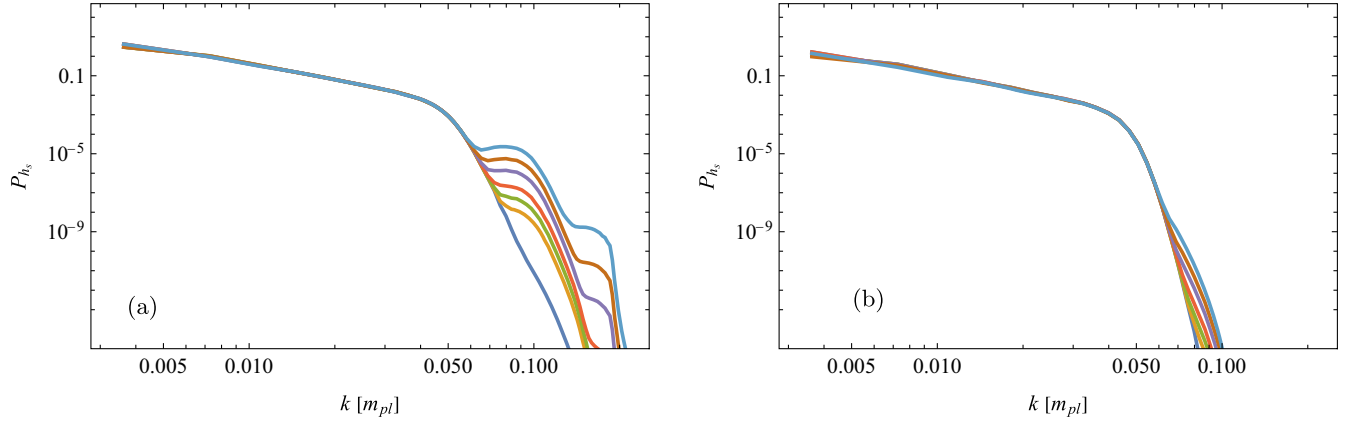


FIG. 10. The plus polarization field spectrum on the final slice, for initial tensor amplitudes $A = 1$ (dark blue), 25 (orange), 50 (green), 100 (red), 250 (purple), 500 (brown) and 1000 (light blue). In panel (a), the window function was applied only to the initial data, whereas in panel (b), the window was also applied during the extraction of the spectrum. Both plots show only the power at or above double precision (10^{-15}).

convergence toward the Mukhanov-Sasaki solution, even at this high mode. We expect second-order convergence in time, as in GRChombo the time-evolution algorithm is second-order accurate (Ref. [69]). We show that our solution converges at second order with the residuals between each curve.

E. First look at non-Gaussianity

A key goal of our research program is to model tensoral bispectra and cross spectra arising from higher-order couplings in Einstein's equations, which can distinguish between the predictions of a wide variety of inflationary scenarios. We seek to combine our results from GRChombo with the efficient MODAL bispectrum estimation pipeline (Refs. [82,83]), where the separable methodology has been applied to constrain many inflation models using Planck satellite data [84]; in this case, the 3D implementation will be applied first to extract primordial bispectra [85,86]. However, as the purpose of the present paper is to implement and validate the tensor evolution equations with gravitational backreaction, together with appropriate initial conditions, we will only give some indicative results on the emergence of higher-point correlators.

We sought an emergent nonlinear signal by introducing an artificial amplitude A for the tensoral perturbations,

$$\tilde{\gamma}_{ij} = \delta_{ij} + A h_{ij}.$$

The case of $A = 1$ represents the fully perturbative example, and is shown in all results presented in previous sections. We increased the initial tensoral amplitude A logarithmically up to the moderately nonlinear value $A = 1000$ (giving $\sigma_h \sim 10^{-2}$ initially), while also varying the initial random seed, in order to draw from the distribution of possible trajectories for the skewness and

kurtosis parameters (see Fig. 11). Note that we remove A in the Fourier normalization during extraction. This implies that initial data generated at different values of A have been rescaled so that polyspectra in Fourier space appear to have the same amplitude, with any differences due to nonlinear effects.

Figure 6 shows that for $A = 1$, our window function applied only to the initial slice kills off numerical noise at high modes sufficiently well to reproduce the predicted linear power spectrum evolution at low modes. However, we found that as we increased A , the resulting increase in power at high modes introduced numerical noise on the grid resolution that grew and leaked into lower modes, creating spurious features above $k = 0.05 m_{pl}$ on the final slice. This is illustrated in panel (a) of Fig. 10. These spurious features remained subdominant and did not impact the predicted linear power spectrum evolution for $k < 0.05$ but, in principle, they could contribute to higher-order correlators. We addressed this issue by applying our window function to the polarization fields as they were extracted.²² The resulting spectra, shown in panel (b), display a much lower amplification of power at high k . This dataset was used to extract the higher-order statistics presented here.

We identify the presence of the bispectrum and trispectrum by simply calculating the skewness and kurtosis parameters of each polarization field in configuration space. The skew parameter is defined as

$$S_3 \equiv \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{E[(h_s(\mathbf{x}) - \bar{h}_s)^3]}{E[(h_s(\mathbf{x}) - \bar{h}_s)^2]^{3/2}},$$

²²Note that we applied our window *only* to the data we extracted. It was not applied to the data that were used within the evolution scheme.

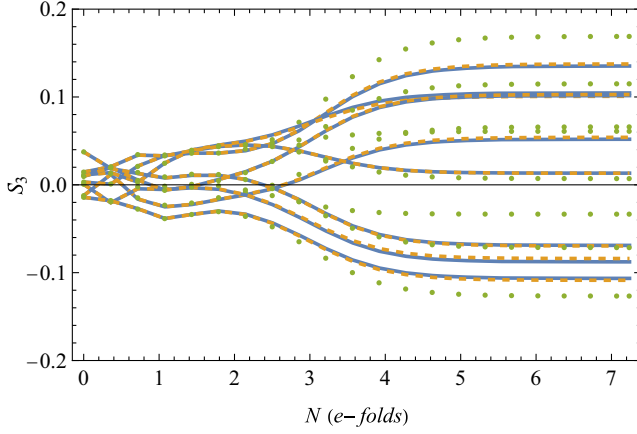


FIG. 11. The skewness “trajectories” of $h_+(\mathbf{x})$ and $h_\times(\mathbf{x})$, as a function of e-folds, for four runs with different random seeds. These runs were performed for tensor amplitude values $A = 1$ (blue, solid), $A = 100$ (orange, dashed) and $A = 1000$ (green, dotted). We note that the paths for $A = 1$ and $A = 100$ appear similar in each run, and the paths for $A = 1000$ differ more clearly.

where κ_n represents the n^{th} cumulant of the field h_s . The kurtosis is defined as

$$S_4 \equiv \frac{\kappa_4}{\kappa_2^2} = \frac{E[(h_s(\mathbf{x}) - \bar{h}_s)^4]}{E[(h_s(\mathbf{x}) - \bar{h}_s)^2]^2}.$$

We first noticed a common evolution pattern in the skewness trajectories, where the skewness oscillated with a small variance in the subhorizon regime, then froze out with a larger variance in the superhorizon regime. Most of this growth can be understood by considering the evolution of the power spectrum $P(k)$ from the Bunch-Davies scaling of k^{-1} to the superhorizon scaling k^{-3} . The skewness corresponds to a summation over the bispectrum from large to small physical scales, normalized by $P(k)$, where the

sum is equivalent to an integration over the measure $k^2 dk$. When the power spectrum is k^{-1} , there is more weight given to the highly populated, large k modes, which also contain less “shot noise.” However, in the k^{-3} scaling regime there is greater weight at small k , which enhances the stochastic noise contribution; hence, in linear theory, the transition shown in Fig. 11 from small to larger skewness is a result of the power spectrum being transformed at horizon crossing.

We note also that for a time after horizon crossing, the tensor amplitude remains “frozen,” indicating that it is a nonlinearly conserved quantity, analogous to the scalar curvature perturbation ζ which is nonlinearly conserved for single field inflation. We note in particular that this apparent growth and the final freeze-out value occurs where the superhorizon and horizon-crossing runs overlap and at the same value for small amplitude A , suggesting that this process is largely caused by linear-order dynamics; that is, the linear evolution of the skewness by the time t_f gives the same result as the initial conditions set at the time t_f for the same random seed.

Although a particular skewness path follows a stochastic trajectory, increasing A may cause an alteration in all trajectories. To test this idea, we calculated the absolute difference between the skewness and kurtosis trajectories as a function of increasing A , i.e. $\Delta S_3(A) \equiv |S_3(A) - S_3(1)|$ and $\Delta S_4(A) \equiv |S_4(A) - S_4(1)|$. Figure 12 shows these difference measures, where each color corresponds to a different value of A . We have performed runs for four different initial configurations, and we can average over polarization type as the polarization fields evolve independently in the linear regime, giving a sample set of eight independent skew and kurtosis trajectories for each value of A . We found a discernible difference in the skewness ΔS_3 (and kurtosis ΔS_4) above $A > 100$ which grows further with increasing amplitude A . We note that the standard deviation σ_h of both measures on the initial slice also

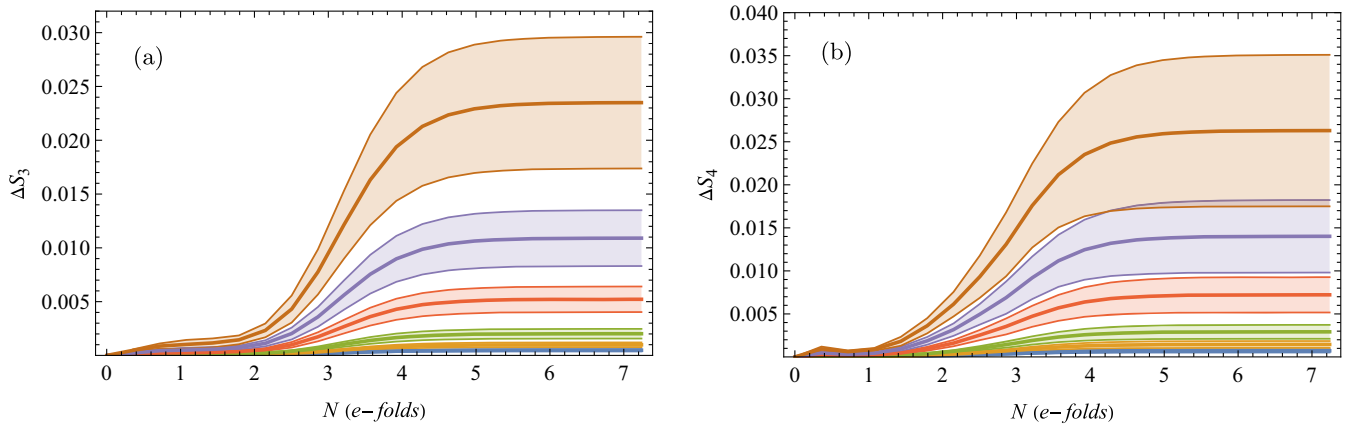


FIG. 12. The difference between the skew (a) and kurtosis (b) paths as a function of e-folds, for $A = 25$ (blue), 50 (orange), 100 (green), 250 (red), 500 (purple) and 1000 (brown), where in each case the difference is taken with respect to the $A = 1$ solution. For each dataset, the mean (solid) and first standard deviation (filled bands) over random field configurations are shown.

increases linearly with A . We take this behavior as a hint that, at sufficiently high tensor amplitudes, the resulting bispectrum can be affected by including nonlinear back-reaction from the metric. Note that we will undertake a quantitative study of non-Gaussianity by estimating the tensoral bispectrum directly for a variety of inflationary scenarios in a future publication [87].

VI. SUMMARY AND FUTURE DIRECTIONS

Primordial non-Gaussian signals are expected to provide unique insights in the effort to uncover the true nature of the inflationary epoch. In order to fully understand how the coupling between different perturbations alters the resulting spectral profile, we should include terms at all orders in the equations of motion. Numerical relativity allows us to solve Einstein's equations entirely nonperturbatively, enabling the inclusion of classical gravitational contributions at all orders. Nevertheless, inflation is expected to be a largely perturbative process, and while nonlinear signals may allow us to distinguish inflationary scenarios, these signals are expected to be small relative to the background. This has necessitated the development of a new set of tools for the use of NR in cosmology, which are adapted to problems where nonlinear physics may play a relatively modest role and then a significant role at different points in the evolution.

In this work, we have presented a gauge-agnostic dictionary which translates between the perturbative description of inflation and the fully relativistic variables of NR, while the NR simulation remains near the linear regime. We have used this dictionary with a straightforward gauge choice to develop a new version of the GRChombo code which evolves tensoral perturbations on an inflating background. We have validated the dynamics of the inflationary background, in both the superhorizon and horizon-crossing case. For tensor perturbations with given inflationary initial conditions, we have accurately matched the evolution of the two-point correlator or power spectrum to the results of the Mukhanov-Sasaki solution. Thus, we have demonstrated that our NR simulation is able to match the evolution of gravitational waves during inflation predicted by linear theory.

Furthermore, this investigation has presented a series of tests which can be used to validate the accuracy of our results, as well as other NR codes which evolve inflationary spacetimes. Through these tests, we have uncovered the presence of a k -dependent phase in the Mukhanov-Sasaki solution which yields initial conditions on subhorizon scales that more closely reproduce the power spectrum evolution from linear theory.

We have analyzed the behavior of the constraints for a perturbative gravitational wave background. We have found that these constraints are initially second order in magnitude or smaller, and remain stable or diminish, depending on the particular constraint diagnostic used.

We have also demonstrated the expected second-order convergence of the tensor power evolution. We then gave a first look into how our pipeline can be used to capture the emergence of non-Gaussian signals from higher-order gravitational couplings to the metric.

We calculated the amplitude of the bispectrum, encapsulated by the skewness parameter, for a series of runs with different random configurations, and measured the change in the amplitude of the bispectrum as we increased the amplitude of the gravitational-wave perturbations. We will use this pipeline to investigate the emergence of non-Gaussianity in models of inflation that exhibit significant nonlinear dynamics, such as axion monodromy or ultra-slow roll (Ref. [62]). We also plan to combine our work with that of Ref. [56], in order to create an NR program which can evolve scalar and tensor perturbations together, incorporating their interaction fully into the equations of motion. This will allow us to measure mixed bispectrum signals, such as $\langle \zeta \zeta \gamma \rangle$, which may be dominant over the tensor-only bispectrum in future observations. Once this is complete, we will be able to combine our program with the MODAL pipeline to produce full bispectrum estimates which can be used to analyze next-generation CMB data.

ACKNOWLEDGMENTS

The authors thank Dr. Juliana Kwan, who has provided expert programming guidance and developmental help with GRChombo/GRTecln. We would also like to thank Yoann Launay for many insightful comments on stochastic inflation and Ulrich Sperhake for his expertise in numerical relativity, along with David Baker and Bowei Zhang for helpful discussions. We would like to acknowledge the members of the GRTL community for maintenance and improvement of the GRChombo/GRTecln codebase, especially Katy Clough, Eugene Lim, Josu Aurrekoetxea, Miren Radia, and Amelia Drew. E. F. acknowledges financial support from the GAPSTI Foundation and an Isaac Newton Scholarship, both held within the STFC CDT in Data Intensive Science. E. P. S. acknowledges partial support from STFC Consolidated Grants No. ST/X000664/1 and No. ST/P000673/1 and from the CTC Intel oneAPI Centre. Part of this work was undertaken on the Cambridge CSD3 part of the STFC DiRAC HPC Facility ([88]) funded by BEIS capital funding via STFC Capital Grants No. ST/P002307/1 and No. ST/R002452/1 and STFC Operations Grant No. ST/R00689X/1.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: MODULUS/ARGUMENT DECOMPOSITION OF THE MUKHANOV-SASAKI SOLUTIONS

In order to properly use the GRF generation method described in Sec. IV B, we must first break Eqs. (17) and (18) into a modulus-argument form:

$$\begin{aligned} h_s(k) &= M_h e^{i\theta_h}, \\ \dot{h}_s(k) &= M_{\dot{h}} e^{i\theta_{\dot{h}}}. \end{aligned}$$

The moduli M_h and $M_{\dot{h}}$ can be used as the basis for the Rayleigh draw. These moduli are given simply by the square root of the corresponding power spectrum. The argument can be found by splitting the complex exponential into sines and cosines using Euler's formula. Let $k' = k/H_0$. Then

$$\begin{aligned} \theta_h &= \arctan\left(\frac{\cos(k') + k' \sin(k')}{k' \cos(k') - \sin(k')}\right) \\ \theta_{\dot{h}} &= -\arctan\left(\frac{\cos(k')}{\sin(k')}\right). \end{aligned}$$

Note that one must use the arctan which takes into account the sign of the numerator and denominator.

APPENDIX B: CONSTRUCTION OF PLUS/CROSS BASIS IN FOURIER SPACE

Since we will be generating the perturbation field h_{ij} on the lattice, we will need a method of determining ϵ_{ij}^s at each point in the box, as a function of $\mathbf{k}$, the three-dimensional wave number. For each point in the box, let $\mathbf{k} = (i, j, k)$ represent the propagation of a particular gravitational wave, where i, j , and k are integers in the interval $(0, N-1)$.²³ We now choose $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ such that they are orthogonal to $\hat{\mathbf{k}}$ and to each other. Note that there exists a degeneracy in the choice of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, and therefore we must choose a particular construction of these vectors.

Let $\hat{\mathbf{m}} = \hat{\mathbf{k}} \times (0, 0, k)$. Then $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ can be expressed in terms of the integer coordinates as

$$\begin{aligned} \hat{\mathbf{m}} &= \frac{(j, -i, 0)}{\sqrt{i^2 + j^2}} \\ \hat{\mathbf{n}} &= \frac{(ik, jk, i^2 + j^2)}{\sqrt{k^2(i^2 + j^2) + (i^2 + j^2)^2}}, \end{aligned} \quad (\text{B1})$$

using the fact that $\hat{\mathbf{n}} = \hat{\mathbf{k}} \times \hat{\mathbf{m}}$. Note that this choice is only valid where $i, j, k > 0$, which I will call region 1. There are

three other regions in which we will need to choose a different set of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$: (2) $k > 0$ but $i = j = 0$; (3) $k = 0$ but $j > 0$; and (4) $k = j = 0$ but $i > 0$. In two of these regions, the choice can be made trivially: for region 2, $\hat{\mathbf{k}} = (0, 0, 1)$ and thus,

$$\hat{\mathbf{m}} = (1, 0, 0), \quad \hat{\mathbf{n}} = (0, 1, 0). \quad (\text{B2})$$

In region 4, $\hat{\mathbf{k}} = (1, 0, 0)$ and thus

$$\hat{\mathbf{m}} = (0, 1, 0), \quad \hat{\mathbf{n}} = (0, 0, 1). \quad (\text{B3})$$

Now we simply choose a construction for region 3. In this case, let $\hat{\mathbf{m}} = \hat{\mathbf{k}} \times (i + j, 0, 0)$. Then

$$\hat{\mathbf{m}} = (0, 0, -1), \quad \hat{\mathbf{n}} = \frac{(-j, i, 0)}{\sqrt{i^2 + j^2}}. \quad (\text{B4})$$

Note that we can generate a new choice of decomposition by rotating $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$. Let $\alpha \in (0, 2\pi)$. Then the new basis,

$$\hat{\mathbf{m}}' = \cos(\alpha)\hat{\mathbf{m}} + \sin(\alpha)\hat{\mathbf{n}}, \quad \hat{\mathbf{n}}' = -\sin(\alpha)\hat{\mathbf{m}} + \cos(\alpha)\hat{\mathbf{n}},$$

can also be used to construct a valid set of basis tensors, ϵ_{ij}^+ and $\epsilon_{ij}^\times$.

APPENDIX C: FOURIER CONVENTIONS IN FFTW

The Fourier conventions given in Sec. II C represent the standard Quantum Field Theory (QFT) normalization of the Fourier transform (FT). We implemented our Fourier transform with the FFTW package (Ref. [89]), which is designed to transform “unitless” grids. This appendix lays out how we can transform to and from the FFTW conventions and the QFT conventions.

As we require that the perturbation field $h_{ij}(\mathbf{x})$ be real, we use the “r2c” and “c2r” Fourier transforms available in the FFTW package. These functions neatly take care of the Hermitian symmetry necessary to form a real field. As an example, the one-dimensional “c2r” Fourier transform calculates

$$\mathbb{Y}_l = \sum_{j=0}^{N-1} \Delta j \mathbb{X}_j e^{2\pi(jl)\sqrt{-1}/N}, \quad (\text{C1})$$

where $j \in [0, N-1]$ and $l \in [0, N-1]$ are the unitless Fourier and configuration space axes, respectively. We want to find a way to transform this into the standard discrete QFT transform,

$$\mathbb{Y}_{x_l} = \sum_{j=0}^{N-1} \Delta k_j \mathbb{X}_{k_j} e^{ik_j x_l}, \quad (\text{C2})$$

²³In order to avoid confusion, I will denote the unit distance along the third axis in $\mathbf{k}$ space as k , and write the magnitude of $\mathbf{k}$ as $|\mathbf{k}|$.

where $k_j = 2\pi j/L$ and $x_l = l \cdot D$ for $D = L/N$. This will require two normalizations, one for the change of variables $j \rightarrow k$ in the inverse FT, and one for the change of variables $x \rightarrow l$ in the FT.

We start with the inverse FT, Eq. (C1). We will change variables in the integral into the QFT unitful variables, and then use the resulting norm as a multiplicative constant on the output of the FFTW algorithm.

First, note that we can transform $j \rightarrow k$ using

$$k = \frac{2\pi j}{L} = \frac{2\pi j}{N \cdot D} = \mathbb{K}_j \frac{1}{D},$$

where we have defined the intermediate unitless mode magnitude to be $\mathbb{K}_j = 2\pi j/N$. Thus

$$\mathbb{Y}(l) = \sum_{j=0}^{N-1} \Delta j \mathbb{X}_j e^{2\pi(jl)\sqrt{-1}/N} = \frac{N}{2\pi} \sum_{j=0}^{N-1} \Delta \mathbb{K}_j \mathbb{X}_{\mathbb{K}_j} e^{i\mathbb{K}_j l},$$

where the prefactor comes from the discrete measure. Now we use

$$\Delta k_j = \frac{1}{D} \Delta \mathbb{K}_j \quad \text{and} \quad l = \frac{x_l}{D}$$

to convert to a unitful mode integrand:

$$\begin{aligned} \frac{N}{2\pi} \sum_{j=0}^{N-1} \Delta \mathbb{K}_j \mathbb{X}_{\mathbb{K}_j} e^{i\mathbb{K}_j l} &= D \frac{N}{2\pi} \sum_{j=0}^{N-1} \Delta k_j \mathbb{X}_{k_j} e^{ik_j x_l/D} \\ &= \frac{L}{2\pi} \sum_{j=0}^{N-1} \Delta k_j \mathbb{X}_{k_j} e^{ik_j x_l}. \end{aligned}$$

This gives

$$\mathbb{Y}_{x_l} \equiv \sum_{j=0}^{N-1} \Delta k_j \mathbb{X}_{k_j} e^{ik_j x_l} = \frac{2\pi}{L} \mathbb{Y}_l,$$

where we have just moved the conversion factor to the lhs of the above set of equations.

We can repeat this process for the “r2c” FT,

$$\mathbb{X}_j = \sum_{l=0}^{N-1} \Delta l \mathbb{Y}_l e^{-i(2\pi l j)/N}.$$

Using $\Delta l = \Delta x_l/D$ gives

$$\mathbb{X}_j = \frac{1}{D} \sum_{l=0}^{N-1} \Delta x_l \mathbb{Y}_{x_l} e^{-ix_l(2\pi j/ND)} = \frac{1}{D} \sum_{l=0}^{N-1} \Delta x_l \mathbb{Y}_{x_l} e^{-ix_l k_j},$$

which means

$$\mathbb{X}_{k_j} = \mathbb{X}_j \cdot D.$$

We can generalize this easily to three dimensions by raising the transformation factor to the power of the number of dimensions:

$$\mathbb{Y}_{x_l} = \left(\frac{2\pi}{L}\right)^3 \mathbb{Y}_l \quad (\text{C3})$$

$$\mathbb{X}_{\mathbf{k}_j} = \left(\frac{L}{N}\right)^3 \cdot \mathbb{X}_j. \quad (\text{C4})$$

APPENDIX D: THE EFFECT OF KO DISSIPATION ON SPECTRA

GRChombo uses two constraint damping mechanisms, Kreiss-Oliger (KO) dissipation (see Ref. [90]) and the damping terms inherent in the CCZ4 formalism, an extension to the BSSN formalism, (see Refs. [69,91,92] for reviews), in order to stay close to the desired solution surface. However, dissipation can leave nonphysical markers on systems where precision in the solution is essential. To demonstrate this effect, we ran the horizon-crossing case for various values of the unitless KO dissipation parameter σ_{KO} . We performed this run with a light window, given by

$$k_* = 50 \times \frac{2\pi}{L} \approx 0.1 m_{pl}, \quad \epsilon = 100,$$

where $N = 64$, in order to best illustrate the effects of damping on high modes.

Figure 13 shows the spectrum on the final slice, for parameter choices $\sigma_{KO} = 0.3, 0.1$ and 0.05 . We note that $\sigma_{KO} = 0.3$ is a typical value for black hole simulations. We found that at high k , even a relatively low value of σ_{KO} can strongly suppress some modes below the Nyquist bump.

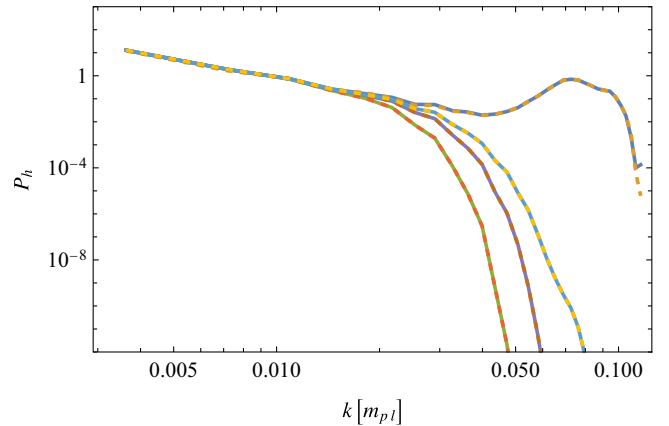


FIG. 13. The plus (solid) and cross (dashed) power spectra on the final slice, for $\sigma_{KO} = 0.3$ (green/red), $\sigma_{KO} = 0.1$ (purple/brown), and $\sigma_{KO} = 0.05$ (light blue/yellow), compared to a run where KO dissipation was not used (dark blue/yellow).

Lower values of σ_{KO} may be useful in damping power near the Nyquist frequency for cosmological simulations. However, KO dissipation is dynamical, and thus its impact on the power spectrum would be difficult to disentangle

from the physical dynamical processes one has set out to study. This issue is avoided when “static” damping techniques, such as applying a window function on the initial data, are used.

-
- [1] S. Tsujikawa, J. Ohashi, S. Kuroyanagi, and A. De Felice, *Phys. Rev. D* **88**, 023529 (2013).
 - [2] J. Martin, C. Ringeval, R. Trotta, and V. Vennin, *J. Cosmol. Astropart. Phys.* **03** (2014) 039.
 - [3] BICEP/Keck Collaboration, *Phys. Rev. Lett.* **127**, 151301 (2021).
 - [4] A. A. Starobinskiĭ, *JETP Lett.* **30**, 682 (1979), <https://ui.adsabs.harvard.edu/abs/1979JETPL...30..682S/abstract>.
 - [5] R. Easther, J. T. Giblin, Jr., and E. A. Lim, *Phys. Rev. Lett.* **99**, 221301 (2007).
 - [6] NANOGrav Collaboration, *Astrophys. J. Lett.* **951**, L8 (2023).
 - [7] A. Drew, T. Kinowski, and E. P. S. Shellard, *Phys. Rev. D* **110**, 043513 (2024).
 - [8] Y. Tada and M. Yamada, *Phys. Lett. B* **855**, 138854 (2024).
 - [9] S. Choudhury, K. Dey, A. Karde, S. Panda, and M. Sami, *Phys. Lett. B* **856**, 138925 (2024).
 - [10] D. Baumann, *Cosmology* (Cambridge University Press, Cambridge, 2022).
 - [11] D. Paoletti, F. Finelli, J. Valiviita, and M. Hazumi, *Phys. Rev. D* **106**, 083528 (2022).
 - [12] BICEP/Keck Collaboration, The latest constraints on inflationary B-modes from the BICEP/Keck telescopes, [arXiv:2203.16556](https://arxiv.org/abs/2203.16556).
 - [13] N. Galitzki *et al.* (Simons Observatory Collaboration), *Astrophys. J. Suppl. Ser.* **274**, 33 (2024).
 - [14] LiteBIRD Collaboration, *Prog. Theor. Exp. Phys.* **2023**, 042F01 (2023).
 - [15] F. Bernardeau and J.-P. Uzan, *Phys. Rev. D* **66**, 103506 (2002).
 - [16] X. Chen and Y. Wang, *Phys. Rev. D* **81**, 063511 (2010).
 - [17] J.-F. Dufaux, A. Bergman, G. Felder, L. Kofman, and J.-P. Uzan, *Phys. Rev. D* **76**, 123517 (2007).
 - [18] J.-F. Dufaux, G. Felder, L. Kofman, and O. Navros, *J. Cosmol. Astropart. Phys.* **03** (2009) 001.
 - [19] L. R. Price and X. Siemens, *Phys. Rev. D* **78**, 063541 (2008).
 - [20] J. García-Bellido and D. G. Figueroa, *Phys. Rev. Lett.* **98**, 061302 (2007).
 - [21] S. Khlebnikov and I. Tkachev, *Phys. Rev. D* **56**, 653 (1997).
 - [22] L. Iacconi and D. J. Mulryne, *J. Cosmol. Astropart. Phys.* **09** (2023) 033.
 - [23] G. Felder and I. Tkachev, *Comput. Phys. Commun.* **178**, 929 (2008).
 - [24] J. T. Giblin, Jr., L. R. Price, and X. Siemens, *J. Cosmol. Astropart. Phys.* **08** (2010) 012.
 - [25] Z. Huang, *Phys. Rev. D* **83**, 123509 (2011).
 - [26] K. Clough, P. Figueras, H. Finkel, M. Kunesch, E. A. Lim, and S. Tunyasuvunakool, *Classical Quantum Gravity* **32**, 245011 (2015).
 - [27] D. G. Figueroa, A. Florio, F. Torrenti, and W. Valkenburg, *Comput. Phys. Commun.* **283**, 108586 (2023).
 - [28] J. van de Vis, R. Nguyen, E. I. Sfakianakis, J. T. Giblin, and D. I. Kaiser, *Phys. Rev. D* **102**, 043528 (2020).
 - [29] P. Adshead, J. T. Giblin, and A. Tishue, *Phys. Rev. D* **110**, 043536 (2024).
 - [30] D. G. Figueroa and N. Loayza, *J. Cosmol. Astropart. Phys.* **03** (2025) 073.
 - [31] J. Dahl, M. Hindmarsh, K. Rummukainen, and D. J. Weir, *Phys. Rev. D* **106**, 063511 (2022).
 - [32] M. Corman and W. E. East, *J. Cosmol. Astropart. Phys.* **10** (2023) 046.
 - [33] A. Caravano, G. Franciolini, and S. Renaux-Petel, *Phys. Rev. D* **111**, 063518 (2025).
 - [34] A. Caravano, K. Inomata, and S. Renaux-Petel, *Phys. Rev. Lett.* **133**, 151001 (2024).
 - [35] D. G. Figueroa, J. Lizarraga, N. Loayza, A. Urrio, and J. Urrestilla, *Phys. Rev. D* **111**, 063545 (2025).
 - [36] R. Nguyen, J. van de Vis, E. I. Sfakianakis, J. T. Giblin, and D. I. Kaiser, *Phys. Rev. Lett.* **123**, 171301 (2019).
 - [37] M. Bastero-Gil, M. Tristram, J. Macías-Pérez, and D. Santos, *Phys. Rev. D* **77**, 023520 (2008).
 - [38] J. T. Giblin, Jr. and A. J. Tishue, *Phys. Rev. D* **100**, 063543 (2019).
 - [39] C. Joana, [arXiv:2211.03534](https://arxiv.org/abs/2211.03534).
 - [40] P. Adshead, J. T. Giblin, R. Grutkoski, and Z. J. Weiner, *J. Cosmol. Astropart. Phys.* **03** (2024) 017.
 - [41] Z.-Y. Yuwen, C. Joana, S.-J. Wang, and R.-G. Cai, *Phys. Rev. Res.* **7**, 023180 (2025).
 - [42] K. Clough, R. Flauger, and E. A. Lim, *J. Cosmol. Astropart. Phys.* **05** (2018) 065.
 - [43] J. C. Aurrekoetxea, K. Clough, R. Flauger, and E. A. Lim, *J. Cosmol. Astropart. Phys.* **05** (2020) 030.
 - [44] A. Ijjas, W. G. Cook, F. Pretorius, P. J. Steinhardt, and E. Y. Davies, *J. Cosmol. Astropart. Phys.* **08** (2020) 030.
 - [45] X.-X. Kou, J. B. Mertens, C. Tian, and S.-Y. Zhou, *Phys. Rev. D* **105**, 123505 (2022).
 - [46] J. C. Aurrekoetxea, K. Clough, and F. Muia, *Phys. Rev. D* **108**, 023501 (2023).
 - [47] R. Grutkoski, H. J. Macpherson, J. T. Giblin, Jr., and J. Frieman, *Phys. Rev. D* **112**, 023541 (2025).
 - [48] B. Wang and Y. Zhang, *Gen. Relativ. Gravit.* **56**, 29 (2024).
 - [49] R. N. Raveendran and L. Sriramkumar, *J. Cosmol. Astropart. Phys.* **07** (2017) 035.

- [50] P. Bari, N. Bartolo, G. Domènech, and S. Matarrese, *Phys. Rev. D* **109**, 023509 (2024).
- [51] K. Clough and J. C. Niemeyer, *Classical Quantum Gravity* **35**, 187001 (2018).
- [52] C. Joana, *Phys. Rev. D* **106**, 023504 (2022).
- [53] D. G. Figueroa, A. Florio, F. Torrenti, and W. Valkenburg, *J. Cosmol. Astropart. Phys.* **04** (2021) 035.
- [54] S. Yu. Khlebnikov and I. I. Tkachev, *Phys. Rev. Lett.* **77**, 219 (1996).
- [55] J. Maldacena, *J. High Energy Phys.* **05** (2003) 013.
- [56] Y. L. Launay, G. I. Rigopoulos, and E. P. S. Shellard, *Phys. Rev. D* **112**, 043518 (2025).
- [57] K. Clough, *Scalar Fields in Numerical General Relativity*, Springer Theses (Springer International Publishing, Cham, 2018).
- [58] S. M. Carroll, [arXiv:gr-qc/9712019](https://arxiv.org/abs/gr-qc/9712019).
- [59] M. Isi, *Classical Quantum Gravity* **40**, 203001 (2023).
- [60] J. C. Aurrekoetxea, K. Clough, and E. A. Lim, *Living Rev. Relativity* **28**, 5 (2025).
- [61] A. Golovnev and V. Vanchurin, *Phys. Rev. D* **79**, 103524 (2009).
- [62] Y. L. Launay, G. I. Rigopoulos, and E. P. S. Shellard, *Phys. Rev. D* **109**, 123523 (2024).
- [63] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
- [64] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1 (1984).
- [65] T. W. Baumgarte and S. L. Shapiro, [arXiv:gr-qc/9810065](https://arxiv.org/abs/gr-qc/9810065).
- [66] M. Shibata, *Phys. Rev. D* **52**, 5428 (1995).
- [67] R. Arnowitt, S. Deser, and C. W. Misner, [arXiv:gr-qc/0405109](https://arxiv.org/abs/gr-qc/0405109).
- [68] T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer* (Cambridge University Press, Cambridge, England, 2010).
- [69] M. Radia, U. Sperhake, A. Drew, K. Clough, P. Figueras, E. A. Lim, J. L. Ripley, J. C. Aurrekoetxea, T. França, and T. Helfer, *Classical Quantum Gravity* **39**, 135006 (2022).
- [70] C. Bona, J. Massó, E. Seidel, and J. Stela, *Phys. Rev. Lett.* **75**, 600 (1995).
- [71] D. Polarski and A. A. Starobinsky, *Classical Quantum Gravity* **13**, 377 (1996).
- [72] Y. L. Launay, G. I. Rigopoulos, and E. P. S. Shellard, *J. Cosmol. Astropart. Phys.* **05** (2025) 071.
- [73] R. Easther and E. A. Lim, *J. Cosmol. Astropart. Phys.* **04** (2006) 010.
- [74] J. T. Deskins, J. T. Giblin, Jr., and R. R. Caldwell, *Phys. Rev. D* **88**, 063530 (2013).
- [75] D. Battefeld, T. Battefeld, and J. T. Giblin, *Phys. Rev. D* **79**, 123510 (2009).
- [76] D. G. Figueroa, J. García-Bellido, and A. Rajantie, *J. Cosmol. Astropart. Phys.* **11** (2011) 015.
- [77] A. Ota, H. J. Macpherson, and W. R. Coulton, *Phys. Rev. D* **106**, 063521 (2022).
- [78] A. Ijjas, *J. Cosmol. Astropart. Phys.* **06** (2023) 061.
- [79] E. Florio, C. Loew, and E. P. S. Shellard (to be published).
- [80] J. C. Aurrekoetxea, K. Clough, and E. A. Lim, *Classical Quantum Gravity* **40**, 075003 (2023).
- [81] J. C. Aurrekoetxea, S. E. Brady, L. Aresté-Saló, J. Bamber, L. Chung-Jukko, K. Clough, E. de Jong, M. Elley, P. Figueras, T. Helfer, E. A. Lim, M. Radia, A. Waeming, and Z. Wang, [arXiv:2501.13046](https://arxiv.org/abs/2501.13046).
- [82] J. R. Fergusson, M. Liguori, and E. P. S. Shellard, *J. Cosmol. Astropart. Phys.* **12** (2012) 032.
- [83] J. R. Fergusson and E. P. S. Shellard, *Phys. Rev. D* **76**, 083523 (2007).
- [84] Y. Akrami *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A10 (2020).
- [85] J. R. Fergusson, D. M. Regan, and E. P. S. Shellard, *Phys. Rev. D* **86**, 063511 (2012).
- [86] J. Hung, J. R. Fergusson, and E. P. S. Shellard, [arXiv:1902.01830](https://arxiv.org/abs/1902.01830).
- [87] E. Florio and E. P. S. Shellard (to be published).
- [88] www.dirac.ac.uk.
- [89] M. Frigo and S. Johnson, *Proc. IEEE* **93**, 216 (2005).
- [90] H. Kreiss and J. Oliger, *Methods for the Approximate Solution of Time Dependent Problems* (International Council of Scientific Unions, World Meteorological Organization, Geneva, 1973).
- [91] C. Bona, T. Ledvinka, C. Palenzuela, and M. Žáček, *Phys. Rev. D* **67**, 104005 (2003).
- [92] D. Alic, C. Bona-Casas, C. Bona, L. Rezzolla, and C. Palenzuela, *Phys. Rev. D* **85**, 064040 (2012).