

Geometric singularities of Feynman integrals

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We provide a new method to calculate the full microlocal description of the singularities of Feynman integrals. This is done by associating a unique constructible function to the system of partial differential equations (PDEs) annihilating the integral and from this function the singularities can directly be read-off. This function can be constructed explicitly even if the system of PDEs is unknown and describes both the location of the singularities and the number of (generalized) master integrals on them. Our framework is flexible enough to perform the calculation in any of the Lee-Pomeransky, Feynman, or momentum representations.

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I. INTRODUCTION

Feynman integrals are an essential ingredient for calculations of observables in quantum field theory. Today their utility reaches far beyond their origin in high-energy physics and they are used for, e.g., calculations of gravitational waves and critical exponents in statistical field theory. Mathematically they strike the perfect balance of being concrete enough to allow for many explicit calculations to be carried out and general enough to have a rich and varied mathematical structure [1].

The most basic invariant of any object is its singularities. For Feynman integrals this is a classical story [2]; culminating in the S -matrix program of the 60's [3,4]. Today there is vast renewed interest in studying these singularities [5–14], with much focus placed on mathematical rigor and computability. In this paper we provide a new and simple way to describe these singularities as a constructible function. The pairing between a Feynman integral and this constructible function is not only rigorous but also explicitly computable on a computer.

The main results in the present paper are based on our recent article [15] and are given here without proof. Describing the necessary D -module theory for a rigorous proof of this construction distracts from its simplicity and is given in [15]. To emphasize the practicality of our approach we have implemented these methods in version 2.25 of the

WhitneyStratifications [16] Macaulay2 [17] package which may be downloaded at the link [18].

The paper is arranged as follows. We first provide some background on Whitney stratifications and related geometric objects. We then define constructible functions and characteristic cycles and show how they can be calculated for Feynman integrals. Our method is then applied to different representations of the Feynman integral and we conclude the paper with a two-loop example.

II. GEOMETRIC PRELIMINARIES

Before we define the main players in this paper; constructible functions and characteristic cycles, we introduce the objects necessary to both describe and calculate them. In the following, let Y be a subvariety of dimension d_Y in a manifold X . In this paper we take $X = \mathbb{C}^n$.

Whitney stratification. A fundamental tool for the study of singular spaces is a Whitney stratification. For a variety Y this stratification divides it into smooth pieces which join together in a particularly desirable way. This allows for predictable behaviors of limiting tangent planes and secant lines between these pieces; by a *secant line* between two pieces, M , N we will mean a straight line $[p - q]$ joining two designated points $p \in M$, $q \in N$.

More precisely, a *Whitney stratification* of Y is a subdivision into a finite number of smooth disjoint connected manifolds S_α , called *strata*, such that $Y = \sqcup_\alpha S_\alpha$ and such that *Whitney's condition B* holds for all pairs $M = S_\alpha$, $N = S_\beta$, of these manifolds. A pair of strata (M, N) , $M \subseteq \bar{N}$ (where $\bar{N}$ is the closure in Zariski topology), satisfy *Whitney's condition B* [19][Sec. 19] at a point $x \in M$ if for every sequence $\{p_\ell\} \subset M$ and $\{q_\ell\} \subset N$ with $\lim p_\ell = \lim q_\ell = x$, the following holds:

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If we have

$$T_{q_\ell} N \rightarrow \tau \quad \text{and} \quad [p_\ell - q_\ell] \rightarrow l$$

where the arrows imply convergence in the appropriate Grassmannians $\mathbf{Gr}(d_Y, n) \ni \tau$ and $\mathbf{Gr}(1, n) = \mathbb{P}^{n-1} \ni l$, then the limit of secant lines l is contained in the limit of tangent planes τ at x (see Fig. 1). When computing Whitney stratifications, and representing them on a computer, it is best to represent them as a flag Y of varieties $Y_0 \subset \cdots \subset Y_{d_Y} = Y$ where the strata are the connected components of the successive differences $Y_i - Y_{i-1}$.

Conormal variety. Consider an algebraic variety $Y = \mathbf{V}(f_1, \dots, f_r) := \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_r(x)\}$. We will denote its *singular locus* as Y_{Sing} , recall this is the collection of all points in Y for which the Jacobian matrix of the polynomials $f_1, \dots, f_r$ has rank strictly less than $n - d_Y$. We then define the (open) manifold of *regular* or *smooth* points in Y as $Y_{\text{reg}} = Y - Y_{\text{Sing}}$. The *conormal variety* is the subvariety in $Y \times \mathbb{P}^{m-1}$ given by the Zariski closure

$$\mathbf{Con}(Y) := \overline{\{(y, \eta) \mid y \in Y_{\text{reg}} \quad \text{and} \quad T_y Y_{\text{reg}} \subset \eta^\perp\}}.$$

Conceptually speaking the conormal variety catalogs all tangent planes and limiting tangent planes to each point y in Y . A point (y, η) in $\mathbf{Con}(Y)$ indicates that there is a tangent plane or limiting tangent plane to Y at the point y with normal vector η .

Polar variety. While the conormal variety above catalogs all the tangents and limiting tangents to a variety, it also requires us to work in a larger ambient space. In many cases it is enough to understand which points in Y have prescribed directions tangent to them, and in particular, to understand which points in Y have tangent directions contained in a generically chosen linear space of some dimension. The object which catalogs these points in Y with prescribed tangent directions is called a polar variety. Consider a flag $L_\bullet$ of length $d_Y + 1$; $L_\bullet = (L_{d_Y+1} \supset L_{d_Y} \supset \cdots \supset L_1)$, where each $L_i \subset \mathbb{C}^n$ is an i -dimensional linear subspace. For each i in $\{0, 1, \dots, d_Y\}$, the dimension- i *polar variety* of Y along the flag $L_\bullet$ is defined as the Zariski closure

$$P_i(Y; L_\bullet) := \overline{\{y \in Y_{\text{reg}} \mid \dim(T_y Y_{\text{reg}} \cap L_{i+1}^\perp) > d_Y - i - 1\}}.$$

Note that $P_{d_Y}(Y; L_\bullet) = Y$.

Euler obstruction. For a Whitney stratification $Y = \sqcup S_\alpha$, we define the *Euler obstruction* of $\bar{S}_\gamma$ along S_α , denoted $\mathbf{Eu}_{\bar{S}_\gamma}(S_\alpha)$, recursively as follows. We define $\mathbf{Eu}_{\bar{S}_\alpha}(S_\alpha) = 1$ and for $S_\alpha \not\subset \bar{S}_\gamma$ we put $\mathbf{Eu}_{\bar{S}_\gamma}(S_\alpha) = 0$. The remaining recursion is given by

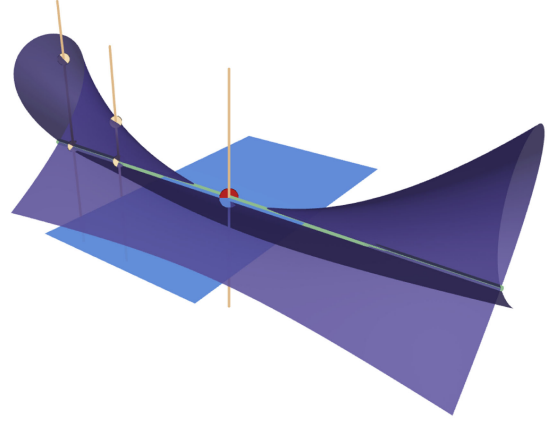


FIG. 1. We see that for the illustrated sequences approaching the origin (illustrated as the central red point) the limiting secant between the teal colored singular line (where the surface self-intersects) and the surface itself is not contained in the limiting tangent (illustrated in light blue); hence condition B fails for this pair at 0 and 0 must be taken as a separate strata.

$$\begin{aligned} \mathbf{Eu}_{\bar{S}_\gamma}(S_\alpha) &= \sum_{S_\alpha \prec S_\beta \preceq S_\gamma} \mathbf{Eu}_{\bar{S}_\gamma}(S_\beta) c(S_\alpha, S_\beta), \\ c(S_\alpha, S_\beta) &= \chi(B(x, \epsilon) \cap S_\beta \cap H_\eta), \end{aligned} \quad (1)$$

where $c(S_\alpha, S_\beta)$ is the topological Euler characteristic of a ball centered at $x \in S_\alpha$ with radius ϵ intersected with S_β intersected with a generic linear affine space of codimension $\dim(S_\alpha) + 1$ at a distance η from x , where $\epsilon \gg \eta > 0$.

All the above constructions are closely related. Both conormal and polar varieties allow for algorithms to compute Whitney stratifications, see [20] resp. [21]. In fact, it has been shown by Teissier [22] that two strata (M, N) , $M \subseteq \bar{N}$ satisfy Whitney's condition B if and only if the sequence of Hilbert-Samuel multiplicities [23]

$$\begin{aligned} \text{mult}_z(\bar{N}, z) \\ = \{\text{mult}_z(P_0(\bar{N}; L_\bullet)), \dots, \text{mult}_z(P_{d-1}(\bar{N}; L_\bullet)), \text{mult}_z(\bar{N})\} \end{aligned}$$

is constant for every $z \in M$. Given a Whitney stratification $\sqcup S_\alpha$ of Y , the above multiplicities are constant on strata and can therefore be indexed by the strata instead of z ; that is for $z \in S_\alpha$ we write $\text{mult}_\alpha(P_k(\bar{S}_\beta))$ for the constant value of $\text{mult}_z(P_k(\bar{S}_\beta))$. We can now use the multiplicities above to calculate the Euler obstruction [24]

$$\mathbf{Eu}_{\bar{S}_\beta}(S_\alpha) = \sum_{k=d_\alpha+1}^{d_\beta} (-1)^{d_\beta-k} \text{mult}_\alpha(P_k(\bar{S}_\beta; L_\bullet)). \quad (2)$$

Example. Let $Y = \mathbf{V}(x^2 + z^3 - y^2 z^2) \subset \mathbb{C}^3$ be the Whitney cusp in Fig. 1. The minimal Whitney stratification of this variety is given by the flag

$$\mathbf{V}(x, y, z) \subset \mathbf{V}(x, z) \subset Y.$$

Indexing the strata by their dimension, the only two nontrivial lists of polar multiplicities are

$$\text{mult.}(Y, S_1) = \{0, 0, 2\} \quad \text{and} \quad \text{mult.}(Y, S_0) = \{0, 1, 2\}.$$

Using (2) we find the Euler obstructions (note $\bar{S}_2 = Y$)

	$\mathbf{Eu}_{\bar{S}_0}$	$\mathbf{Eu}_{\bar{S}_1}$	$\mathbf{Eu}_{\bar{S}_2}$
S_0	1	1	1
S_1	0	1	2
S_2	0	0	1

III. CONSTRUCTIBLE FUNCTIONS AND CHARACTERISTIC CYCLES

A *constructible function* α on X is a function of the form $\alpha(x) = \sum_j a_j \mathbb{1}_{Y_j}(x)$ where $Y_j \subset X$ are subvarieties, $\mathbb{1}_{Y_j}$ denotes the characteristic function (which equals 1 for points in Y_j and 0 otherwise) and a_j are integers. The Euler obstructions (1) form another basis for the constructible functions, this is easily seen since $\mathbf{Eu}_{Y_j}(\cdot)$ generically equals one.

Let $Q = \sum_\beta a_\beta(x) \partial^\beta$ denote a differential operator in local coordinates on X of degree m where β is a multi-index. Then we define the *principal symbol* to be $\sigma(Q)(x, \xi) = \sum_{|\beta|=m} a_\beta(x) \xi^\beta$. For a system of PDEs $\mathcal{M}$ on X given by $Q_1 u = \dots = Q_r u = 0$ the *characteristic variety* is given by

$$\text{Char}(\mathcal{M})$$

$$= \{(x, \xi) \in X \times \mathbb{P}^{n-1} \mid \sigma(Q)(x, \xi) = 0 \quad \forall Q \in D_X \langle Q_1, \dots, Q_r \rangle\}$$

where $D_X \langle Q_1, \dots, Q_r \rangle$ denotes the left-ideal of differential operators in the Weyl algebra D_X generated by $Q_1, \dots, Q_r$. If $\text{Char}(\mathcal{M}) = \cup \Lambda_\alpha$ denotes the irreducible decomposition of $\text{Char}(\mathcal{M})$, then every component is of the form $\Lambda_\alpha = \mathbf{Con}(Y_\alpha)$ for some subvariety $Y_\alpha \subset X$. The unique *characteristic cycle* $CC(\mathcal{M})$ is now given by the formal sum $\sum_\alpha m_\alpha \mathbf{Con}(\bar{S}_\alpha)$, $m_\alpha \in \mathbb{Z}$. This is an example of a *conical Lagrangian cycle* [25].

There is a simple isomorphism between the groups (or rather between the functors) of constructible functions and conical Lagrangian cycles

$$\mathbf{Eu}_{Y_\alpha}(\cdot) \simeq (-1)^{d_X - d_\alpha} \mathbf{Con}(Y_\alpha) \quad (3)$$

where $d_X = \dim(X)$ and $d_\alpha = \dim(Y_\alpha)$. For a hypersurface $Y \subset X$ with Whitney stratification $\sqcup S_\alpha$ we get explicitly

$$\begin{aligned} \mathbb{1}_Y(y) &= \sum_\alpha (-1)^{d_Y - d_\alpha} m_\alpha \mathbf{Eu}_{\bar{S}_\alpha}(y) \simeq \\ CC(\mathbb{1}_Y) &= - \sum_\alpha m_\alpha \mathbf{Con}(\bar{S}_\alpha). \end{aligned} \quad (4)$$

It follows from Kashiwara's index theorem [26] that for a system of PDEs there exists a constructible function with the same characteristic cycle.

Example (cont.). Let $f = x^2 + z^3 - y^2 z^2 \in \mathbb{C}[x, y, z]$, then $Y = \mathbf{V}(f)$ is again the Whitney cusp. To find the constructible function $\mathbb{1}_Y$ we treat (4) as a linear system and solve for m_α

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The constant function $\mathbb{1}_Y$ can then be decomposed as.

$$\mathbb{1}_Y(y) = \mathbf{Eu}_{\bar{S}_3}(y) - \mathbf{Eu}_{\bar{S}_2}(y) + \mathbf{Eu}_{\bar{S}_0}(y).$$

Using the isomorphism (3) this defines a Lagrangian cycle. If $\mathcal{M}$ denotes the system of PDEs that annihilate f^λ , with λ being a generic fixed number, the characteristic cycle $CC(\mathcal{M})$ constructed from $\mathbb{1}_Y$ using (6)

$$CC(\mathcal{M}) = \mathbf{Con}(0) - \mathbf{Con}(Y) + \mathbf{Con}(\bar{S}_1) - \mathbf{Con}(\bar{S}_0)$$

where $\mathbf{Con}(0)$ denotes the zero-section of the cotangent bundle $T^*\mathbb{C}^3$.

IV. GEOMETRIC SINGULARITIES OF INTEGRALS

We take the Lee-Pomeransky [27] form as our representation of Feynman integrals when describing our construction

$$\mathcal{I} = \int_{\mathbb{R}_+^n} \left(\prod_{i=1}^n \frac{x_i^{\nu_i} dx_i}{x_i \Gamma(\nu_i)} \right) \frac{1}{(\mathcal{U}(x) + \mathcal{F}(x, z))^{D/2}}. \quad (5)$$

In this representation there is one integration variable x_i for each edge of the corresponding Feynman graph and the variables z_j are kinematic invariants like Mandelstam variables and masses. The parameters ν_i are propagator powers and D denotes the space-time dimension. We always assume that these are generic but fixed numbers over $\mathbb{C}$ or $\mathbb{R}$. We will see in the next section this framework is flexible enough to deal with other representations besides the Lee-Pomeransky one.

To get all singularities we lift the integration variables x to projective space using the Cheng-Wu theorem [1]. Let $Y \subset \mathbb{P}_x^n \times \mathbb{C}_z^m$ be the hypersurface defined by the vanishing of the homogenized integrand $x_0 \cdots x_n (\mathcal{U}x_0 + \mathcal{F}) = 0$. As every factor in the integrand is raised to a parametric power, all of Y is the singular locus for the system of PDEs annihilating the integrand. The rank of this system is one since the function defining Y spans the solution space. So the constructible function that reproduces this system's characteristic variety is $\mathbb{1}_{\mathbb{P}^n \times \mathbb{C}^m} - \mathbb{1}_Y$. Expanding this we find

$$(\mathbb{1}_{\mathbb{P}^n \times \mathbb{C}^m} - \mathbb{1}_Y)(y) = 1 - \sum_{\alpha} (-1)^{d_Y - d_{\alpha}} m_{\alpha} \mathbf{Eu}_{\bar{S}_{\alpha}}(y) \quad (6)$$

where $\sqcup S_{\alpha}$ is a Whitney stratification of Y . We note that we have all information necessary to calculate the integers m_{α} . This is done by the following steps:

- (i) Calculate a Whitney stratification of Y .
 - (ii) Calculate the Euler obstructions using polar varieties (2).
 - (iii) Treat (6) as a linear system and solve for the m_{α} .
- If $\mathcal{M}$ denotes the system of PDEs annihilating the integrand, then $\int \mathcal{M}$ denotes the system annihilating the integral. For the characteristic variety we now have

$$\begin{aligned} \text{Char}(\mathcal{M}) &= \mathbf{Con}(0) \cup (\cup_i \mathbf{Con}(\bar{S}_{\alpha_i})) \\ &\subset \mathbb{P}_x^n \times \mathbb{C}_z^m \times \mathbb{P}_{\xi}^n \times \mathbb{P}_{\zeta}^{m-1} \end{aligned}$$

where α_i are the indices with nonzero m_{α_i} in (6). Let $I_{\text{Char}(\mathcal{M})}$ be an ideal in $\mathbb{C}[x, z, \xi, \zeta]$ defining $\text{Char}(\mathcal{M})$, then we define the elimination ideal

$$I_{\text{Char}(\int \mathcal{M})} = (I_{\text{Char}(\mathcal{M})} + \langle \xi_0, \dots, \xi_n \rangle) \cap \mathbb{C}[z, \zeta] \quad (7)$$

whose variety is precisely the characteristic variety $\text{Char}(\int \mathcal{M})$, see [15] and references therein.

We think of integration as being represented by the canonical projection map $\varphi: \mathbb{P}_x^n \times \mathbb{C}_z^m \rightarrow \mathbb{C}_z^m$. The functorial properties of characteristic cycles and constructible functions means that we can obtain the constructible function associated to $\int \mathcal{M}$ from $\mathcal{M}$ by the proper push-forward

$$\varphi_*(\mathbb{1}_V)(z) = \chi(\varphi^{-1}(z) \cap V) \quad (8)$$

for any variety $V \subset \mathbb{P}^n \times \mathbb{C}^m$. Note that this assignment is unique [28]. Specifically we obtain

$$\begin{aligned} \varphi_*(\mathbb{1}_{\mathbb{P}^n \times \mathbb{C}^m} - \mathbb{1}_Y)(z) &= \chi((\mathbb{P}^n \times \mathbb{C}^m) \cap \varphi^{-1}(z)) \\ &\quad - \chi(Y \cap \varphi^{-1}(z)) \\ &= n + 1 - \chi(Y \cap \varphi^{-1}(z)) \end{aligned} \quad (9)$$

and we denote $(-1)^n$ times the function in (9) as $\chi_{\int \mathcal{M}}(z)$.

This is a constructible function on $\mathbb{C}_z^m$ meaning that it has a decomposition into Euler obstructions. The singular locus Z of $\int \mathcal{M}$, i.e., the *Landau singularities* of the integral, is given by $Z = \overline{\pi_z(\text{Char}^*(\int \mathcal{M}))}$ where $\text{Char}^*(\cdot)$ is the characteristic variety with the zero-section $\mathbf{Con}(0)$ removed and π_z the canonical projection map $\mathbb{C}_z^m \times \mathbb{P}_{\xi}^{m-1} \rightarrow \mathbb{C}_z^m$. Given a Whitney stratification $\sqcup W_{\alpha}$ of Z , the constructible function is written as

$$\begin{aligned} \chi_{\int \mathcal{M}}(z) &= (-1)^n (n + 1 - \chi(Y \cap \varphi^{-1}(z))) \\ &= \mu_0 - \sum_{\alpha} (-1)^{d_Z - d_{\alpha}} \mu_{\alpha} \mathbf{Eu}_{\bar{W}_{\alpha}}(z) \end{aligned} \quad (10)$$

where $d_Z = \dim(Z)$, $d_{\alpha} = \dim(W_{\alpha})$, and $\mu_i \in \mathbb{Z}$. By our sign convention we have $\mu_i \geq 0$, then (10) proves that $\chi_{\int \mathcal{M}}(z)$ has to drop on all irreducible components of the singular locus Z . In particular, given an overcomplete list of components in Z , the Euler characteristic may be used to remove superfluous components. The Euler characteristic $\chi(Y \cap \varphi^{-1}(z))$ is constant for $z \in W_{\alpha}$, hence this number is readily calculable using e.g., [29]. We therefore have all information needed to calculate the multiplicities μ_{α} explicitly following the steps:

- (iv) Calculate the singular locus Z as the variety of the ideal $(I_{\text{Char}(\int \mathcal{M})} : \langle \zeta_1, \dots, \zeta_m \rangle^{\infty}) \cap \mathbb{C}[z]$. This variety Z is also the Landau singularities.
- (v) Whitney stratify Z and calculate the Euler obstructions using polar varieties (2).
- (vi) Calculate the Euler characteristic $\chi(Y \cap \varphi^{-1}(z))$ for a z in each strata W_{α} and for a $z \in \mathbb{C}_z^m - Z$, the latter equals μ_0 .
- (vii) Treat (10) as a linear system and solve for the μ_{α} .

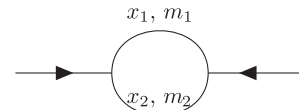
Let α_i be the indices such that $\mu_{\alpha_i} \neq 0$, then we have the characteristic cycle

$$CC\left(\int \mathcal{M}\right) = \mu_0 \mathbf{Con}(0) + \sum_i \mu_{\alpha_i} \mathbf{Con}(\bar{W}_{\alpha_i}).$$

As we can determine the characteristic cycle directly from (10) and can write down this function for any Feynman integral, this function constitutes our main result. This function encodes significantly more than just the Landau singularities. First, the description of the singularities is in the cotangent bundle of $\mathbb{C}_z^m$ so we get an upper bound on the set where solutions to the system $\int \mathcal{M}$ fail to be micro-analytic. Second, μ_0 is the number of generalized master integrals, i.e., the dimension of the solution space for generic D and $\nu_1, \dots, \nu_n$. In fact, $\chi_{\int \mathcal{M}}(z)$ is the dimension of the solution space at every z so we also obtain the number of generalized master integrals on singularities.

V. ONE DIAGRAM, THREE INTEGRALS, SAME SINGULARITIES

To illustrate the method just described we calculate the singularities of the one-loop bubble using three different representations of the Feynman integral. Instead of using z we denote the kinematic invariants as $\{p^2, m_1^2, m_2^2\} \in \mathbb{C}^3$.



Lee-Pomeransky. In the Lee-Pomeransky representation we first have to homogenize the integrand. We have $Y = \mathbf{V}(x_0 x_1 x_2 (\mathcal{U} x_0 + \mathcal{F})) \subset \mathbb{P}_x^2 \times \mathbb{C}^3$ with the Symanzik polynomials $\mathcal{U} = x_1 + x_2$ and $\mathcal{F} = (m_1^2 + m_2^2 - p^2)x_1 x_2 + m_1^2 x_1^2 + m_2^2 x_2^2$. There are 24 strata in the minimal Whitney stratification of this surface and the seven strata of dimension zero and one appear with multiplicity zero in the characteristic cycle. Using (7) and eliminating the conormal variables we find the Landau singularities $Z = \mathbf{V}(p^2 m_1^2 m_2^2 ((m_1^2 + m_2^2 - p^2)^2 - 4m_1^2 m_2^2))$. The minimal Whitney stratification of this variety contains eleven strata where only the four codimension-one strata appear with nonzero multiplicity after solving (10). Let $\lambda = \lambda(a, b, c) = (a + b - c)^2 - 4ab$ be the Källén function. The constructible function and characteristic cycle of the bubble integral is

$$\begin{aligned} \chi_{\int \mathcal{M}}(z) &= 3 - \mathbf{Eu}_{\mathbf{V}(p^2)}(z) - \mathbf{Eu}_{\mathbf{V}(m_1^2)}(z) \\ &\quad - \mathbf{Eu}_{\mathbf{V}(m_2^2)}(z) - (z) \mathbf{Eu}_{\mathbf{V}(\lambda)}(z) \\ CC\left(\int \mathcal{M}\right) &= 3\mathbf{Con}(0) + \mathbf{Con}(\mathbf{V}(p^2)) + \mathbf{Con}(\mathbf{V}(m_1^2)) \\ &\quad + \mathbf{Con}(\mathbf{V}(m_2^2)) + \mathbf{Con}(\mathbf{V}(\lambda)). \end{aligned} \quad (11)$$

This shows that the bubble generically has three master integrals and that the number drops by one on each singularity.

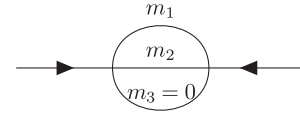
Feynman. In the Feynman representation the integrand is already homogenized and given by the product of the Symanzik polynomials. For the bubble we have $Y = \mathbf{V}(x_1 x_2 \mathcal{U} \mathcal{F}) \subset \mathbb{P}_x^1 \times \mathbb{C}_z^3$. The Whitney stratification of Y consists of 19 strata. All strata of dimension zero and one (in total seven strata) have multiplicity zero. The remaining strata have multiplicity one except $\mathbf{V}(x_1, x_2)$ which has multiplicity four. Using (7) and eliminating the conormal variables we find the singular locus $Z = \mathbf{V}(p^2 m_1^2 m_2^2 ((m_1^2 + m_2^2 - p^2)^2 - 4m_1^2 m_2^2))$, the same as in the Lee-Pomeransky representation.

Momentum. The integrand in momentum representation is a product of propagators $q_i^2 - m_i^2$ where q_i is a linear combination of internal and external momenta. This is not

homogeneous in the loop momenta, i.e., the integration variables, which will be denoted k_i (with the homogenization variable denoted k_h) below. In momentum representation the number of integration variables and kinematic parameters depend on the intended number of space-time dimensions. For computational simplicity we assume that $m_1 = m_2 = m$. In two dimensions we have $Y_{2D} = \mathbf{V}(k_h((k_0^2 - k_1^2) - k_h^2 m^2)((k_0 + k_h p_0)^2 - (k_1 + k_h p_1)^2) - k_h^2 m^2) \subset \mathbb{P}_k^2 \times \mathbb{C}^3$ while in three dimensions we have $Y_{3D} \subset \mathbb{P}_k^3 \times \mathbb{C}^4$ and similarly for higher dimensions. There are 18 strata in the minimal Whitney stratification of Y_{2D} and the singular locus is $Z = \mathbf{V}(m^2(p_0^2 - p_1^2)(p_0^2 - p_1^2 - 4m^2))$ which is the same as in the other representations with $m_1 = m_2$.

VI. TWO-LOOP EXAMPLE

For the one loop bubble we observed that the characteristic variety is just the union of the conormal varieties of each irreducible component of the singular locus. This is expected as the one-loop bubble satisfies a Gel'fand-Kapranov-Zelevinsky (GKZ) system of PDEs [30]. Outside the GKZ setting, the characteristic variety and cycle can contain components coming from any codimension strata of the singular locus. To illustrate this we will look at a simple two-loop example



Following steps (i)–(iv) above we arrive at the singular locus

$$Z = \mathbf{V}(z_1 z_2 z_3 ((z_1 + z_2 - z_3)^2 - 4z_1 z_2))$$

with the identification $\{z_1, z_2, z_3\} \leftrightarrow \{p^2, m_1^2, m_2^2\}$. This is precisely the same singular locus as the one-loop bubble. However, it has a very different characteristic cycle. The Whitney stratification of Z contains eleven components, calculating the Euler characteristics according to (vi) and solving the linear system (10) [i.e., step (vii)] yields the constructible function and characteristic cycle

$$\begin{aligned} \chi_{\int \mathcal{M}}(z) &= 4 - \mathbf{Eu}_{\mathbf{V}(\lambda)}(z) - 2 \sum_{i=1}^3 \mathbf{Eu}_{\mathbf{V}(z_i)}(z) + \sum_{1 \leq i < j \leq 3} \mathbf{Eu}_{\mathbf{V}(z_i, z_j)}(z) - \mathbf{Eu}_{\mathbf{V}(z_1, z_2, z_3)}(z) \\ CC\left(\int \mathcal{M}\right) &= 4\mathbf{Con}(0) + \mathbf{Con}(\mathbf{V}(\lambda)) + 2 \sum_{i=1}^3 \mathbf{Con}(\mathbf{V}(z_i)) + \sum_{1 \leq i < j \leq 3} \mathbf{Con}(\mathbf{V}(z_i, z_j)) + \mathbf{Con}(\mathbf{V}(z_1, z_2, z_3)). \end{aligned}$$

Here the characteristic cycle does not just contain the four irreducible components of Z but also components of codimension two and three. Note that still not all eleven strata contribute.

We remark that the degree in x of Y equals $(n+1)(L+1)$, where L is the number of loops, and also that the dimension of the kinematic space grows rapidly with the size of the Feynman graph. As such, examples

with higher loop orders and more edges remain challenging; however it is worth emphasizing that our current computer implementation makes use of many general purpose methods which are by no means optimized to take advantage of the special structure of Feynman integrals and hence much larger examples than those presented here should be accessible with additional work on the software implementation.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [18].

APPENDIX: AN EXAMPLE ILLUSTRATING THE USAGE OF THE WHITNEY STRATIFICATION PACKAGE

In this appendix we use the one loop bubble example from Sec. V to demonstrate how the `WhitneyStratifications` Macaulay2 [17] package can be used to carry out the computations described in steps (i)–(vii) of Sec. IV. Note that all code given in the text below is also provided in the Supplemental Material for ease of use [31].

We begin by starting a Macaulay2 session and loading the package:

```
needsPackage "WhitneyStratifications"
```

Recall that for the one loop bubble we have $Y = \mathbf{V}(x_0x_1x_2(\mathcal{U}x_0 + \mathcal{F})) \subset \mathbb{P}_x^2 \times \mathbb{C}^3$ with the Symanzik polynomials $\mathcal{U} = x_1 + x_2$ and $\mathcal{F} = (z_1 + z_2 - z_3)x_1x_2 + z_1x_1^2 + z_2x_2^2$. As above let $\mathcal{M}$ denote the system of PDEs annihilating the integrand $x_0x_1x_2(\mathcal{U}x_0 + \mathcal{F})$.

We first compute the Whitney stratification of Y . Here we opt to use the `whitneyStratifyPol` command with the optional `Algorithm=>"msolve"` argument as this tends to be that fastest option on most Feynman integral examples; we note that to use the `Algorithm=>"msolve"` option one needs to have Macaulay2 version 1.25.06 or higher installed. We also note other options are available (and are described in the package documentation linked to in the introduction). The main drawback with this choice of options is that we always compute unnecessary strata, but

the unnecessary strata are random, hence we run the command twice and keep the strata which are common to both runs. In practice this is still much faster than using other methods which we would only run once.

```
n=2;m=3;
R=QQ[x_0..x_n,z_1..z_m];
U = (x_1+x_2);
F = -z_3*x_1*x_2+U*(z_1*x_1+z_2*x_2);
Y=ideal (x_0*x_1*x_2*(U*x_0+F));
V1=whitneyStratifyPol(Y,Algorithm=>"msolve");
V2=whitneyStratifyPol(Y,Algorithm=>"msolve");
V=new MutableHashTable from
for k in keys V1 list k=>toList(
(set(V1)#k)*(set(V2)#k));
```

The output above is displayed with the command

```
peek V
```

We do not print the output as there are 24 strata. This completes step (i) of the computation. Next we carry out steps (ii) and (iii). These steps are both carried out by a single command which takes as input the output from step (i).

```
CCmults=multCC(V, Algorithm=>"msolve")
```

This command computes the Euler obstruction matrix via a polar multiplicity calculation and then solves the resulting linear system to obtain the coefficients m_α ; the argument `Algorithm=>"msolve"` is again optional but is often faster. The output of this command is a list of length two. The first entry is the Euler obstruction matrix; note that in Macaulay2 lists are indexed from 0.

```
Eu=(CCmults)_0
```

The second entry is a list of lists, with each list representing the strata and their multiplicities in the characteristic cycle $CC(\mathcal{M})$.

```
CCList=(CCmults)_1
```

To illustrate the structure we display an entry of `CCList`, entering

```
CCList_4
outputs
```

```
{0, {1, ideal (z_2, z_1 - z_3, x_2, x_1, x_0)}}
```

This means that entry four in the list is a strata of dimension one (in the Whitney stratification of Y) with closure given by $\overline{S_{\alpha_4}} = \mathbf{V}(z_2, z_1 - z_3, x_2, x_1, x_0)$ and has multiplicity 0 in $CC(\mathcal{M})$, i.e., $\mathbf{Con}(S_{\alpha_4})$ does not appear in $CC(\mathcal{M})$. We can extract the ordered list of strata closures

```
Sbars=for s in CCList list (last s)_1
```

their respective dimensions

```
dimStrats=for s in Cclist list (last s)_0
```

and their corresponding ordered list of multiplicities

```
mult=for s in Cclist list first s
```

With this we can compute the characteristic variety $\text{Char}(\mathcal{M})$ as well as the components of the characteristic cycle of $\mathcal{M}$. We can then employ (7) to obtain $\text{Char}(\int \mathcal{M})$ and the singular locus $Z = \pi_z(\text{Char}^*(\int \mathcal{M}))$ where $\text{Char}^*(\cdot)$ is the characteristic variety with the zero-section $\text{Con}(0)$ removed and $\pi_z: \mathbb{C}_z^m \times \mathbb{P}_{\zeta}^{m-1} \rightarrow \mathbb{C}_z^m$ is the canonical projection map onto the first factor. The `WhitneyStratifications` package contains a function called `charVarInt` which computes $\text{Char}(\int \mathcal{M})$ in one step and also returns Z , it is used as follows. First we specify the kinematic variables.

```
params={z_1, z_2, z_3}
```

Then we input the ambient polynomial ring, the kinematic variables, and the list of multiplicity and strata pairs computed above.

```
chars=charVarInt(R,params, Cclist)
chMlist=chars_0
chIntMlist=chars_1
Zlist=last chars
```

The last three lines will return the prime components of the ideals defining $\text{Char}(\mathcal{M})$, $\text{Char}(\int \mathcal{M})$, and Z , respectively, as a list. We can alternately obtain the single ideal defining these varieties via

```
chM=intersect toList chMlist
chIntM=intersect chIntMlist
Z=intersect Zlist
```

We have now completed step (iv) from Sec. IV and obtain that Z is the ideal defined by the polynomial

$$z_1^3 z_2 z_3 - 2z_1^2 z_2^2 z_3 + z_1 z_2^3 z_3 - 2z_1^2 z_2 z_3^2 - 2z_1 z_2^2 z_3^2 + z_1 z_2 z_3^3.$$

To find $CC(\int \mathcal{M})$ we begin by computing the Whitney stratification of Z . Here we opt to simply use the standard `whitneyStratify` method.

```
WZ=whitneyStratify Z
```

As above let $\varphi: \mathbb{P}_x^n \times \mathbb{C}_z^m \rightarrow \mathbb{C}_z^m$ be the canonical projection map onto the second factor. Next we calculate the Euler obstruction matrix associated to Z , $\mathbf{Eu}(Z)$, along with the other quantities which appear in the linear system specified by (10). These quantities are $\mu_0 = (-1)^n((n+1) - \chi(Y \cap \varphi^{-1}(z_0)))$ for a general choice of $z_0 \in \mathbb{C}^m - Z$ and the column vector appearing in the linear system (10), which

has entries given by

$$\mu_0 - (-1)^n((n+1) - \chi(Y \cap \varphi^{-1}(z)))$$

for z taken from each Whitney strata of Z .

The following function computes all these quantities as well as the resulting list of multiplicities μ_{α_i} which are obtained by solving the linear system (10).

```
muList=multMu (Yh,m,WZ)
```

The output is a sequence of length four; the first entry is μ_0 , followed by a vector containing the multiplicities μ_{α_i} . The third entry is the ordered list of Whitney strata of Z used in the function, and hence its ordering corresponds to the ordering of the multiplicities. The fourth entry is a list of results of intermediate calculations: the first and second elements of the list are the matrix (which is a signed version of $\mathbf{Eu}(Z)$) and vector appearing in the linear system (10), while the final entry is the output of the `eulerObsMatrix` command applied to `WZ`.

The above function `multMu` carries out the necessary Euler obstruction and Euler characteristic calculations following the procedure described in [15][Remark 5.7]. We remark that several steps carried out in this function are probabilistic, hence to be confident of the output it is best to run the function twice.

We can now collect the values returned by the `muList` function which we need to obtain $CC(\int \mathcal{M})$. The first output is μ_0 , executing

```
mu0=first muList
```

we see that $\mu_0 = 3$. We can now extract the vector of multiplicities μ_{α_i} appearing in (10).

```
mu=muList_1
```

Outputting `mu` gives

```
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 1 |
| 1 |
| 1 |
| 1 |
```

From this output we see that only the zero section (with weight $\mu_0 = 3$) and the last four strata, each with weight one, contribute to $CC(\int \mathcal{M})$. We obtain the ordered list of strata by executing

```
ZStrataList=muList_2
```

The output from this command is the list of ideals

```
{ideal (z , z , z ), ideal (z , z ),
      3   2   1       3   2

ideal (z , z ), ideal (z , z ),
      3   1       2   1

ideal (z , z - z ), ideal (z , z - z ),
      3   1   2       2   1   3

ideal (z - z , z ), ideal z ,
      2   3   1       3

ideal z , ideal z ,
      2           1

ideal(z - 2z z + z - 2z z - 2z z + z )}
```

Hence we have

$$CC\left(\int \mathcal{M}\right) = 3\mathbf{Con}(0) + \mathbf{Con}(z_3) + \mathbf{Con}(z_2) + \mathbf{Con}(z_1) + \mathbf{Con}(z_1^2 - 2z_1z_2 + z_2^2 - 2z_1z_3 - 2z_2z_3 + z_3^2).$$

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