

# Theorem on vacuum stability

André Milagre<sup>\*</sup> and Luís Lavoura<sup>†</sup>

*Universidade de Lisboa, Instituto Superior Técnico, CFTP, 1049-001 Lisboa, Portugal*



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We consider an extension of the Standard Model with one or more scalar multiplets beyond the Higgs doublet  $\Phi$ . The additional scalar multiplets are supposed to carry arbitrary hypercharges (the precise conditions on the scalar potential are less stringent). We prove that, in such a model, if the field configuration where only  $\Phi$  has a nonzero vacuum expectation value (VEV) is a local minimum of the potential, then it has a lower value of the potential than any extremum where both  $\Phi$  and other scalar multiplets have nonzero VEVs.

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## I. INTRODUCTION

In 2019 Ferreira and Gonçalves (FG) [1] considered the extension of the Standard Model where the scalar sector consists of one  $SU(2)$  doublet  $\Phi$  and one  $SU(2)$  triplet  $\Delta$ :

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Delta = \begin{pmatrix} -\delta_2/\sqrt{2} & \delta_1 \\ -\delta_3 & \delta_2/\sqrt{2} \end{pmatrix}, \quad (1)$$

where  $\phi_1, \phi_2, \delta_1, \delta_2$ , and  $\delta_3$  are complex scalar fields. They assumed that the scalar potential is

$$V = m^2 \Phi^\dagger \Phi + M^2 \text{tr}(\Delta^\dagger \Delta) + \lambda_1 (\Phi^\dagger \Phi)^2 + \lambda_2 [\text{tr}(\Delta^\dagger \Delta)]^2 \quad (2a)$$

$$+ \lambda_3 \text{tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta) + \lambda_4 (\Phi^\dagger \Phi) \text{tr}(\Delta^\dagger \Delta) + \lambda_5 \Phi^\dagger \Delta \Delta^\dagger \Phi, \quad (2b)$$

which is invariant under  $\Phi \rightarrow U\Phi$ ,  $\Delta \rightarrow U\Delta U^\dagger$ , where  $U$  is an  $SU(2)$  matrix. A crucial feature of the potential in Eq. (2) is that it is also invariant under *independent* rephasings of  $\Phi$  and  $\Delta$ , viz., both under  $\Phi \rightarrow \Phi \exp(i\chi)$  and under  $\Delta \rightarrow \Delta \exp(i\varpi)$ . FG also assumed that there is a stationary point of  $V$  where either one or both fields in  $\Phi$  have nonzero vacuum expectation values (VEVs), while all the fields in  $\Delta$  have zero VEVs—we call this a “type-I extremum”—and that this stationary point is a *local minimum*, i.e., that it leads to all the physical particles—i.e., all but the Goldstone

bosons—having *positive* masses-squared. Under these assumptions, FG proved that type-I extremum has a lower value of  $V$  than *any* stationary point of  $V$  where *both*  $\Phi$  and  $\Delta$  have nonzero VEVs—we call this a “type-III extremum.” FG emphasized, though, that an extremum where only fields in  $\Delta$  have nonzero VEVs while both  $\phi_1$  and  $\phi_2$  have zero VEVs—we call this a “type-II extremum”—may have a lower value of  $V$  than the type-I extremum.

One may exploit this observation and require the parameters of a scalar potential to be such that the global minimum corresponds to a vacuum configuration in which only the neutral component of the Higgs doublet acquires VEV. Such theoretical constraints have been used to build consistent minimal extensions of the Standard Model. For instance, the authors of Refs. [2,3] have analyzed the vacua of the most general two-Higgs-doublet model (2HDM) and proved that if a charge-preserving minimum exists, then any extremum that spontaneously breaks either charge-conservation,  $C$ , or  $CP$  symmetries is necessarily a saddle point lying above that minimum. Stability conditions of neutral vacua have also been derived for the real [4] and complex [5] scalar singlet models, the left-right symmetric model [6], the Georgi-Machacek model [7,8], the next-to-minimal 2HDM [9], some symmetry-constrained realizations of the three-Higgs-doublet model [10–13], and the  $N$ -Higgs-doublet model (NHDM) [14–16]. Notably, Ref. [14] has asserted that, in the NHDM, the depth of the potential at a stationary point that breaks a given symmetry, relative to a neutral minimum, is related to the mass-squared matrix of the scalar particles directly linked with that symmetry. We recover that result in the present paper.

In particular, in dark matter models, the dark matter scalar multiplets must be VEV-less in order not to break their stabilizing symmetries. This requires vacuum stability conditions, viz., one should make sure that the desired vacuum structure corresponds to the true global minimum

<sup>\*</sup>Contact author: [andre.milagre@tecnico.ulisboa.pt](mailto:andre.milagre@tecnico.ulisboa.pt)

<sup>†</sup>Contact author: [balio@cftp.tecnico.ulisboa.pt](mailto:balio@cftp.tecnico.ulisboa.pt)

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of the potential. The present paper gives one step that may be useful for this purpose.

In Sec. II of this paper we state a theorem that generalizes FG's result to the case where  $\Delta$  is an  $SU(2)$  multiplet—or, indeed, a set of  $SU(2)$  multiplets—with *any* isospin. In Sec. III we prove the theorem. We summarize our findings in Sec. IV. We use the Appendix to apply our theorem to the  $U(1)$ -symmetric 2HDM [17].

## II. THE THEOREM

We consider an extension of the Standard Model with one  $SU(2)$  scalar doublet  $\Phi$  and  $N$   $SU(2)$  scalar multiplets  $\Psi_k$  ( $k = 1, \dots, N$ ). We write the field components of  $\Phi$  and of its charge-conjugate  $SU(2)$  doublet  $\tilde{\Phi}$  as

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}. \quad (3)$$

Each multiplet  $\Psi_k$  has isospin  $J_k$  and dimension  $n_k = 2J_k + 1$ . Note that the isospins and hypercharges of the multiplets  $\Psi_k$  are constrained by partial wave unitarity bounds, as found in Refs. [18,19]. We label the field components of  $\Psi_k$  as  $\psi_{k,I}$ , where  $I$  is the third component of isospin, ranging from  $-J_k$  to  $J_k$  in discrete steps  $\Delta I = 1$ . Explicitly,

$$\Psi_k = \begin{pmatrix} \psi_{k,J_k} \\ \psi_{k,J_k-1} \\ \psi_{k,J_k-2} \\ \vdots \\ \psi_{k,1-J_k} \\ \psi_{k,-J_k} \end{pmatrix}, \quad \tilde{\Psi}_k = \begin{pmatrix} \psi_{k,-J_k}^* \\ -\psi_{k,1-J_k}^* \\ \psi_{k,2-J_k}^* \\ \vdots \\ (-1)^{n_k} \psi_{k,J_k-1}^* \\ -(-1)^{n_k} \psi_{k,J_k}^* \end{pmatrix}. \quad (4)$$

The scalar fields  $\phi_1$ ,  $\phi_2$ , and  $\psi_{k,I}$  are complex, and their third components of isospin are

$$I(\phi_1) = \frac{1}{2}, \quad I(\phi_2) = -\frac{1}{2}, \quad I(\psi_{k,I}) = I, \quad (5)$$

respectively.

Let  $V$  be the scalar potential of  $\Phi$  and the  $\Psi_k$ . We make the following assumptions:

- (1)  $V$  is invariant under the gauge symmetry  $SU(2) \times U(1)$ .
- (2)  $V$  is renormalizable, i.e., it does not contain any terms with more than four fields.
- (3)  $V$  does not contain any linear terms. This is a trivial consequence of  $SU(2) \times U(1)$ -invariance in the case of a  $\Psi_k$  that is not  $SU(2) \times U(1)$ -invariant. Else, it may be achieved, for instance, through an additional symmetry  $\Psi_k \rightarrow -\Psi_k$  for each  $\Psi_k$  which is  $SU(2) \times U(1)$ -invariant; or else by collecting all such  $\Psi_k$  in a nontrivial representation of some non-Abelian horizontal-symmetry group.
- (4)  $V$  does not contain any trilinear terms. This may be a consequence either of the  $SU(2) \times U(1)$  symmetry or of some additional (nongauge) symmetry. For instance, if a  $\Psi_k$  has half-integer  $J_k$ , then  $SU(2)$ -invariance forbids any trilinear term only with  $\Phi$  and that  $\Psi_k$ . Another instance: if a  $\Psi_k$  is a triplet of  $SU(2)$ , then  $SU(2) \times U(1)$ -invariance forbids any trilinear term of  $\Phi$  and that  $\Psi_k$  if  $\Psi_k$  has hypercharge different from either 0 or  $\pm 1$ ; or else, there may be an additional symmetry  $\Psi_k \rightarrow -\Psi_k$  that forbids the trilinear coupling  $\Phi \Phi \Psi_k$ .
- (5)  $V$  does not contain any bilinear terms with two different multiplets. These may be terms of the form either  $\Phi \otimes \Psi_k$  or  $\Psi_k \otimes \Psi_{k'}$  with  $k \neq k'$ ; they may be eliminated either by the gauge symmetry (for instance, if  $\Phi$  and all the  $\Psi_k$  have different isospins), or by some additional symmetry, like a separate rephasing symmetry  $\Psi_k \rightarrow \Psi_k \exp(i\vartheta_k)$  for each  $k$ .
- (6)  $V$  also does not contain any bilinear terms of the form  $\Psi_k \otimes \Psi_k$ , but for

$$F_k \equiv \sum_{I=-J_k}^{J_k} |\psi_{k,I}|^2. \quad (6)$$

For instance, in the case of the triplet  $\Delta$  of the previous section, there might be a term  $\text{tr}(\Delta^2) = (\delta_2)^2 - 2\delta_1\delta_3$  in the scalar potential; we assume that some symmetry eliminates that term. Once again, a  $U(1)$  symmetry  $\Psi_k \rightarrow \Psi_k \exp(i\vartheta_k)$  for each  $k$  is the handiest way to eliminate such terms.

- (7) The only  $SU(2) \times U(1)$ -invariant terms in  $V$  with both  $\Phi$  and  $\Psi_k$  are constructed out of

$$(\Phi \otimes \tilde{\Phi} \otimes \Psi_k \otimes \tilde{\Psi}_k). \quad (7)$$

From the product of  $\Phi$  and  $\tilde{\Phi}$  we get both a singlet and a triplet of  $SU(2)$ :

$$F_0 \equiv (\Phi \otimes \tilde{\Phi})_1 = |\phi_1|^2 + |\phi_2|^2, \quad (8a)$$

$$(\Phi \otimes \tilde{\Phi})_3 = \begin{pmatrix} \phi_1 \phi_2^* \\ \frac{|\phi_2|^2 - |\phi_1|^2}{\sqrt{2}} \\ -\phi_1^* \phi_2 \end{pmatrix}. \quad (8b)$$

As a consequence, only two  $SU(2)$  invariants may be built from the expression (7):

$$(\Phi \otimes \tilde{\Phi} \otimes \Psi_k \otimes \tilde{\Psi}_k)_1 = (\Phi \otimes \tilde{\Phi})_1 (\Psi_k \otimes \tilde{\Psi}_k)_1 \\ \oplus [(\Phi \otimes \tilde{\Phi})_3 \otimes (\Psi_k \otimes \tilde{\Psi}_k)_3]_1, \quad (9)$$

where  $(\Psi_k \otimes \tilde{\Psi}_k)_1$  is the quantity  $F_k$  given in Eq. (6). Using Clebsch-Gordan coefficients in the usual manner, we derive

$$F_{0k} \equiv \left[ (\Phi \otimes \tilde{\Phi})_3 \otimes (\Psi_k \otimes \tilde{\Psi}_k)_3 \right]_1 = \frac{|\phi_1|^2 - |\phi_2|^2}{2} \sum_{I=-J_k}^{J_k} I |\psi_{k,I}|^2 + \frac{z + z^*}{2}, \quad (10)$$

where

$$z = \phi_1 \phi_2^* \sum_{I=1-J_k}^{J_k} \psi_{k,I}^* \psi_{k,I-1} \sqrt{J_k^2 - I^2 + J_k + I}. \quad (11)$$

Putting all these conditions together, one concludes that

$$V = \mu_0^2 F_0 + \frac{\lambda_{00}}{2} F_0^2 + \sum_{k=1}^N (\lambda_{0k} F_0 F_k + \lambda'_{0k} F_{0k}) \quad (12a)$$

$$+ \sum_{k=1}^N \mu_k^2 F_k + \frac{1}{2} \sum_{k=1}^N \sum_{k'=1}^N \lambda_{kk'} F_i F_{k'} + \text{other terms quartic in the } \Psi_k, \quad (12b)$$

where  $F_0$ ,  $F_k$ , and  $F_{0k}$  have been defined in Eqs. (6), (8a), and (10), respectively;  $\mu_0^2$  and  $\mu_k^2$  are real coefficients with squared-mass dimension; and  $\lambda_{00}$ ,  $\lambda_{0k}$ ,  $\lambda'_{0k}$ , and  $\lambda_{kk'}$  are real dimensionless coefficients (note that  $\lambda_{kk'} = \lambda_{k'k}$  for all pairs  $k \neq k'$ ).<sup>1</sup>

The scalar potential may have various extrema. We classify those extrema in the following way:

(i) The type-0 extremum has

$$\langle \Phi \rangle = 0, \quad \forall k \langle \Psi_k \rangle = 0. \quad (13)$$

(ii) A type-I extremum has

$$\langle \Phi \rangle \neq 0, \quad \forall k \langle \Psi_k \rangle = 0. \quad (14)$$

(iii) A type-II extremum has

$$\langle \Phi \rangle = 0, \quad \exists \langle \Psi_k \rangle \neq 0. \quad (15)$$

(iv) A type-III extremum has

$$\langle \Phi \rangle \neq 0, \quad \exists \langle \Psi_k \rangle \neq 0. \quad (16)$$

<sup>1</sup>In the case where  $\Psi_1$  is an  $SU(2)$  triplet and there are no  $\Psi_k$  with  $k > 1$ , one may compare the potential (12) with FG's one in Eq. (2). One sees that  $\mu_0^2 = m^2$ ,  $\mu_1^2 = M^2$ ,  $\lambda_{00} = 2\lambda_1$ ,  $\lambda_{11} = 2\lambda_2$ ,  $\lambda_{01} = \lambda_4 + \lambda_5/2$ , and  $\lambda'_{01} = \lambda_5$ . Moreover, the "other terms quartic in the  $\Psi_k$ " of Eq. (12) consist just of the term with coefficient  $\lambda_3$  in Eq. (2).

The four types of extrema are qualitatively different, and one cannot continuously transition from one to another. In Eqs. (13)–(16),  $\langle \rangle$  denotes a field's VEV.

We may now state

*Theorem 1.* A local type-I minimum of  $V$  in Eq. (12) has a lower expectation value of the potential than any type-0 or type-III extremum.

*Corollary 1.1.* Either the global minimum of the potential is that type-I local minimum, or it is a type-II local minimum (if there is any).

### III. PROOF

#### A. The scalar potential at an extremum

Given the structure of  $V$ , one may rewrite Eq. (12) as

$$V = L^T X + \frac{1}{2} X^T M X + \sum_{k=1}^N \lambda'_{0k} F_{0k} + \text{other terms quartic in the } \Psi_k, \quad (17)$$

where

$$X = (F_0, F_1, \dots, F_N)^T, \quad (18a)$$

$$L = (\mu_0^2, \mu_1^2, \dots, \mu_N^2)^T, \quad (18b)$$

$$M = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \cdots & \lambda_{0N} \\ \lambda_{01} & \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{0N} & \lambda_{1N} & \cdots & \lambda_{NN} \end{pmatrix}. \quad (18c)$$

Note that the matrix  $M$  is symmetric.

In Eq. (17) one sees that

$$\frac{\partial V}{\partial X} = \left( \frac{\partial V}{\partial F_0}, \frac{\partial V}{\partial F_1}, \dots, \frac{\partial V}{\partial F_N} \right)^T \quad (19a)$$

$$= L + X^T M, \quad (19b)$$

$$\frac{\partial V}{\partial F_{0k}} = \lambda'_{0k}. \quad (19c)$$

The potential is the sum of two homogeneous functions<sup>2</sup> of the scalar fields: a homogeneous function of degree 2 (which we call  $V_{(2)}$ ) and a homogeneous function of degree 4 (which we call  $V_{(4)}$ ). Indeed,

<sup>2</sup>A function  $f: \mathbb{C}^n \rightarrow \mathbb{R}$  is said to be *homogeneous of degree*  $d \in \mathbb{Z}$  if

$$f(\alpha x_1, \dots, \alpha x_n) = \alpha^d f(x_1, \dots, x_n), \quad (20)$$

for all  $\alpha \in \mathbb{R}$  and all  $(x_1, \dots, x_n) \in \mathbb{R}^n$  [20].

$$V = V_{(2)} + V_{(4)}, \quad (21a)$$

$$V_{(2)} = L^T X, \quad (21b)$$

$$V_{(4)} = \frac{1}{2} X^T M X + \sum_{k=1}^N \lambda'_{0k} F_{0k} + \text{other terms quartic in the } \Psi_k. \quad (21c)$$

We define a vector  $\varphi$  containing all the fields in the theory:

$$\varphi = (\phi_1, \phi_2, \psi_{1,J_1}, \dots, \psi_{1,-J_1}, \psi_{2,J_2}, \dots, \psi_{N,-J_N})^T. \quad (22)$$

It follows from Euler's Theorem for homogeneous functions that [20]

$$\sum_{\alpha} \left( \varphi_{\alpha} \frac{\partial V_{(2)}}{\partial \varphi_{\alpha}} + \varphi_{\alpha}^* \frac{\partial V_{(2)}}{\partial \varphi_{\alpha}^*} \right) = 2V_{(2)}, \quad (23a)$$

$$\sum_{\alpha} \left( \varphi_{\alpha} \frac{\partial V_{(4)}}{\partial \varphi_{\alpha}} + \varphi_{\alpha}^* \frac{\partial V_{(4)}}{\partial \varphi_{\alpha}^*} \right) = 4V_{(4)}, \quad (23b)$$

hence,

$$\sum_{\alpha} \left( \varphi_{\alpha} \frac{\partial V}{\partial \varphi_{\alpha}} + \varphi_{\alpha}^* \frac{\partial V}{\partial \varphi_{\alpha}^*} \right) = 2V_{(2)} + 4V_{(4)}. \quad (24)$$

At any extremum of  $V$ , the fields have a certain vacuum configuration  $\varphi = \bar{\varphi}$ . Let  $\bar{V}$ ,  $\bar{V}_{(2)}$ , and  $\bar{V}_{(4)}$  denote the values of  $V$ ,  $V_{(2)}$ , and  $V_{(4)}$ , respectively, when  $\varphi = \bar{\varphi}$ . The stationarity conditions imply that, for every field  $\varphi_{\alpha}$ ,

$$\left. \frac{\partial V}{\partial \varphi_{\alpha}} \right|_{\varphi=\bar{\varphi}} = \left. \frac{\partial V}{\partial \varphi_{\alpha}^*} \right|_{\varphi=\bar{\varphi}} = 0. \quad (25)$$

Therefore, at any extremum, Eq. (24) reads

$$2\bar{V}_{(2)} + 4\bar{V}_{(4)} = 0, \quad (26)$$

which implies  $\bar{V}_{(4)} = -\bar{V}_{(2)}/2$ , and therefore,

$$\bar{V} = \bar{V}_{(2)} + \bar{V}_{(4)} = \frac{\bar{V}_{(2)}}{2} = \frac{L^T \bar{X}}{2}, \quad (27)$$

where

$$\bar{X} \equiv X|_{\varphi=\bar{\varphi}}. \quad (28)$$

## B. The type-0 extremum

At the type-0 extremum no field has a VEV, hence  $\bar{X} \equiv \bar{X}_0$  is a null vector. If  $\bar{V}_0$  is the value of the potential at the type-0 extremum, it follows from Eq. (27) that

$$\bar{V}_0 = \frac{L^T \bar{X}_0}{2} = 0. \quad (29)$$

## C. Type-I extrema

Starting from a type-I extremum, we use an  $SU(2)$  rotation to place the whole VEV of  $\Phi$  in  $\phi_2$ , while  $\phi_1$  has null VEV. Then, at a type-I extremum,

$$\bar{\varphi}_I = (0, v_{\phi}, 0, \dots, 0)^T, \quad (30a)$$

$$\bar{X}_I = (v_{\phi}^2, 0, \dots, 0)^T. \quad (30b)$$

Without loss of generality we have assumed  $v_{\phi}$  to be real.

At this extremum, the stationarity conditions of Eq. (25) are trivially satisfied for all fields except for  $\phi_2$ . For that field one has

$$\left. \frac{\partial V}{\partial \phi_2} \right|_{\varphi=\bar{\varphi}_I} = 0 \Leftrightarrow (\mu_0^2 + \lambda_{00} v_{\phi}^2) v_{\phi} = 0. \quad (31)$$

Since we assume  $v_{\phi}$  to be nonzero, we must have

$$v_{\phi}^2 = -\frac{\mu_0^2}{\lambda_{00}}. \quad (32)$$

The conditions for  $V$  to be bounded from below require  $\lambda_{00}$  to be positive. Hence, Eq. (32) tells us that  $\mu_0^2$  must be negative for a type-I extremum to exist. From Eq. (27),

$$\bar{V}_I = \frac{L^T \bar{X}_I}{2} = \frac{\mu_0^2 v_{\phi}^2}{2} = \frac{-(\mu_0^2)^2}{2\lambda_{00}}, \quad (33)$$

which is negative because  $\lambda_{00} > 0$ . Therefore, a type-I extremum has a lower value of the potential than the type-0 extremum:

$$\bar{V}_0 > \bar{V}_I. \quad (34)$$

We next search in the potential for the terms quadratic in the fields of  $\Psi_k$  when  $\varphi = \bar{\varphi}_I$ . We find

$$V \supset \sum_{k=1}^N \sum_{I=-J_k}^{J_k} \left( \mu_k^2 + \lambda_{0k} v_{\phi}^2 - \frac{\lambda'_{0k}}{2} I v_{\phi}^2 \right) |\psi_{k,I}|^2. \quad (35)$$

Therefore, at this extremum, the masses-squared of the complex scalar fields  $\psi_{k,I}$  are given by

$$m_{\psi_{k,I}}^2 = \mu_k^2 + \left( \lambda_{0k} - \frac{\lambda'_{0k}}{2} I \right) v_{\phi}^2. \quad (36)$$

We assume that the type-I extremum is a local *minimum* of the potential, which means that all the  $m_{\psi_{k,I}}^2$  are *positive*.

#### D. Type-III extrema

Starting from the most general type-III extremum, we use an  $SU(2)$  rotation to place the whole VEV of  $\Phi$  in  $\phi_2$ , while  $\phi_1$  has null VEV. Then, at a type-III extremum,

$$\bar{\varphi}_{\text{III}} = (0, v_b, v_{\psi_{1,J_1}}, \dots, v_{\psi_{1,-J_1}}, v_{\psi_{2,J_2}}, \dots, v_{\psi_{N,-J_N}})^T, \quad (37a)$$

$$\bar{X}_{\text{III}} = \left( v_b^2, \sum_{I=-J_1}^{J_1} |v_{\psi_{1,I}}|^2, \dots, \sum_{I=-J_N}^{J_N} |v_{\psi_{N,I}}|^2 \right)^T. \quad (37b)$$

Without loss of generality, we have assumed  $v_b$  to be real, while the  $v_{\psi_{k,I}}$  are, in general, complex numbers.

At a type-III extremum,

$$\left. \frac{\partial V}{\partial \phi_2} \right|_{\varphi=\bar{\varphi}_{\text{III}}} = 0 \Leftrightarrow \left. \frac{\partial V}{\partial F_0} \frac{\partial F_0}{\partial \phi_2} \right|_{\varphi=\bar{\varphi}_{\text{III}}} + \sum_{k=1}^N \left. \frac{\partial V}{\partial F_{0k}} \frac{\partial F_{0k}}{\partial \phi_2} \right|_{\varphi=\bar{\varphi}_{\text{III}}} = 0 \quad (38a)$$

$$\Leftrightarrow v_b \left. \frac{\partial V}{\partial F_0} \right|_{\varphi=\bar{\varphi}_{\text{III}}} + \sum_{k=1}^N \lambda'_{0k} \left. \frac{\partial F_{0k}}{\partial \phi_2} \right|_{\varphi=\bar{\varphi}_{\text{III}}} = 0 \quad (38b)$$

$$\Leftrightarrow v_b \left( \left. \frac{\partial V}{\partial F_0} \right|_{\varphi=\bar{\varphi}_{\text{III}}} - \sum_{k=1}^N \sum_{I=-J_k}^{J_k} \frac{\lambda'_{0k}}{2} I |v_{\psi_{k,I}}|^2 \right) = 0, \quad (38c)$$

where we have used Eqs. (10) and (19c). Since we *assume*  $v_b$  to be nonzero, we must have

$$\left. \frac{\partial V}{\partial F_0} \right|_{\varphi=\bar{\varphi}_{\text{III}}} = \sum_{k=1}^N \sum_{I=-J_k}^{J_k} \frac{\lambda'_{0k}}{2} I |v_{\psi_{k,I}}|^2. \quad (39)$$

#### E. Comparing the type-I and type-III extrema

Let us now suppose that the potential has two coexisting extrema  $\bar{\varphi}_{\text{I}}$  and  $\bar{\varphi}_{\text{III}}$  of types I and III, respectively. Let us call the values of the scalar potential at those extrema  $\bar{V}_{\text{I}}$  and  $\bar{V}_{\text{III}}$ , respectively. Using Eqs. (19b) and (27) one finds

$$\left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{I}}} \cdot X|_{\varphi=\bar{\varphi}_{\text{I}}} = (L^T + \bar{X}_{\text{I}}^T M) \bar{X}_{\text{III}} \quad (40a)$$

$$= 2\bar{V}_{\text{III}} + \bar{X}_{\text{I}}^T M \bar{X}_{\text{III}}. \quad (40b)$$

$$\left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{III}}} \cdot X|_{\varphi=\bar{\varphi}_{\text{I}}} = (L^T + \bar{X}_{\text{III}}^T M) \bar{X}_{\text{I}} \quad (40c)$$

$$= 2\bar{V}_{\text{I}} + \bar{X}_{\text{III}}^T M \bar{X}_{\text{I}}. \quad (40d)$$

Since  $M$  is symmetric,  $\bar{X}_{\text{I}}^T M \bar{X}_{\text{III}} = \bar{X}_{\text{III}}^T M \bar{X}_{\text{I}}$ . Subtracting Eq. (40d) from Eq. (40b) we find that the relative depth of the two extrema is [1]

$$\bar{V}_{\text{III}} - \bar{V}_{\text{I}} = \frac{1}{2} \left( \left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{I}}} \cdot \bar{X}_{\text{III}} - \left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{III}}} \cdot \bar{X}_{\text{I}} \right). \quad (41)$$

Firstly, from Eqs. (18b), (18c), (19b), and (30b) one sees that

$$\left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{I}}} = L + \bar{X}_{\text{I}}^T M = \begin{pmatrix} \mu_0^2 + \lambda_{00} v_{\phi}^2 \\ \mu_1^2 + \lambda_{01} v_{\phi}^2 \\ \vdots \\ \mu_N^2 + \lambda_{0N} v_{\phi}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_1^2 + \lambda_{01} v_{\phi}^2 \\ \vdots \\ \mu_N^2 + \lambda_{0N} v_{\phi}^2 \end{pmatrix}. \quad (42)$$

Utilizing Eq. (37b), this means that

$$\left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{I}}} \cdot \bar{X}_{\text{III}} = \sum_{k=1}^N \sum_{I=-J_k}^{J_k} (\mu_k^2 + \lambda_{0k} v_{\phi}^2) |v_{\psi_{k,I}}|^2. \quad (43)$$

Secondly, it follows from Eqs. (30b) and (39) that

$$\left. \frac{\partial V}{\partial X} \right|_{\varphi=\bar{\varphi}_{\text{III}}} \cdot \bar{X}_{\text{I}} = v_{\phi}^2 \left. \frac{\partial V}{\partial F_0} \right|_{\varphi=\bar{\varphi}_{\text{III}}} = v_{\phi}^2 \sum_{k=1}^N \sum_{I=-J_k}^{J_k} \frac{\lambda'_{0k}}{2} I |v_{\psi_{k,I}}|^2. \quad (44)$$

Thirdly, by using Eqs. (36) and (41) one finds

$$\bar{V}_{\text{III}} - \bar{V}_{\text{I}} = \frac{1}{2} \sum_{k=1}^N \sum_{I=-J_k}^{J_k} \left( \mu_k^2 + \lambda_{0k} v_{\phi}^2 - \frac{\lambda'_{0k}}{2} I v_{\phi}^2 \right) |v_{\psi_{k,I}}|^2 \quad (45a)$$

$$= \frac{1}{2} \sum_{k=1}^N \sum_{I=-J_k}^{J_k} m_{\psi_{k,I}}^2 |v_{\psi_{k,I}}|^2, \quad (45b)$$

which is positive because we have assumed that the mass-squared  $m_{\psi_{k,I}}^2$  are positive Q.E.D.

We conclude that a *local minimum of type-I has a lower value of the potential than any type-0 or type-III extremum*, cf. Eqs. (34) and (45), which means that either the global minimum of the potential is that type-I local

minimum, or it is a type-II local minimum (if such a local minimum exists). We refer to the Appendix for a pedagogical application of this theorem to a model with two  $SU(2)$  doublets.

#### IV. SUMMARY

In this paper, we have generalized a result demonstrated by FG [1]. They considered an  $SU(2) \times U(1)$  gauge model with an  $SU(2)$  doublet  $\Phi$ , an  $SU(2)$  triplet  $\Delta$ , and an additional symmetry  $\Phi \rightarrow \Phi \exp(i\chi)$ ,  $\Delta \rightarrow \Delta \exp(i\varpi)$ ; we have allowed  $\Delta$  to be *any*  $SU(2)$  multiplet, not necessarily a triplet. As a matter of fact, our theorem is valid for larger scalar sectors with several  $SU(2)$  multiplets, provided the symmetries of the model are such that the scalar potential (SP) is of the form in Eq. (12). We wanted to check the conditions under which a *local* minimum of the SP, where only  $\Phi$  has a nonzero VEV, is the *global* minimum of the potential. We have proved that this local minimum of the SP lies *below* any extremum of the SP where *both*  $\Phi$  and  $\Delta$  have nonzero VEVs. As FG rightly remarked, this still leaves open the possibility that the global minimum of the relevant SP is an extremum where *only*  $\Delta$  has nonzero VEV. Our result reduces the need for a full minimization of the SP of many Standard Model extensions, as some vacuum configurations are immediately disfavored. This is especially convenient in dark matter models such as the  $U(1)$ -invariant 2HDM (see the Appendix). This theorem may also prove useful in the grand unified theories based on the Lie groups  $SU(5)$  and  $SO(10)$ , where one or more scalar quadruplets (and in some cases even quintuplets) of  $SU(2)$ , with various hypercharges, may be present—depending on the scalar multiplets used to effect the spontaneous symmetry breaking.

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#### DATA AVAILABILITY

No data were created or analyzed in this study.

#### APPENDIX: THE $U(1)$ -SYMMETRIC 2HDM

Consider the simple case of two  $SU(2)$  doublets:  $\Phi$  and  $\Psi$ . We write the field components of  $\Phi$ ,  $\Psi$ , and their charge-conjugates  $\tilde{\Phi}$  and  $\tilde{\Psi}$ , respectively, as

$$\Phi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \Psi = \begin{pmatrix} c \\ d \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}, \quad \tilde{\Psi} = \begin{pmatrix} d^* \\ -c^* \end{pmatrix}. \quad (\text{A1})$$

We assume the scalar potential to be the one given in Eq. (12), namely,

$$V = \mu_0^2 F_0 + \mu_1^2 F_1 + \frac{\lambda_{00}}{2} F_0^2 + \frac{\lambda_{11}}{2} F_1^2 + \lambda_{01} F_0 F_1 + \lambda'_{01} F_{01}, \quad (\text{A2})$$

where

$$F_0 \equiv \Phi^\dagger \Phi, \quad (\text{A3a})$$

$$F_1 \equiv \Psi^\dagger \Psi, \quad (\text{A3b})$$

$$F_{01} \equiv \frac{1}{4}(|a|^2 - |b|^2)(|c|^2 - |d|^2) + \frac{1}{2}(ab^*c^*d + a^*bcd^*). \quad (\text{A3c})$$

At a type-I extremum,<sup>3</sup> we use an  $SU(2)$  rotation to place the whole VEV of  $\Phi$  in  $b$ , leaving  $a$  with null VEV. Explicitly, we write

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v_b \end{pmatrix}, \quad \langle \Psi \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (\text{A4})$$

where  $v_b$  is real. Hence, at a type-I extremum, the squared masses of the fields  $c$  and  $d$  read

$$m_c^2 = \mu_1^2 + v_b^2 \left( \lambda_{01} - \frac{1}{4} \lambda'_{01} \right), \quad (\text{A5a})$$

$$m_d^2 = \mu_1^2 + v_b^2 \left( \lambda_{01} + \frac{1}{4} \lambda'_{01} \right). \quad (\text{A5b})$$

At a type-III extremum, the VEV of  $\Phi$  can still be written as in Eq. (A4), but now  $\Psi$  may acquire one of three vacuum configurations:

$$\langle \Psi \rangle_A = \begin{pmatrix} v_c \\ 0 \end{pmatrix}, \quad \langle \Psi \rangle_B = \begin{pmatrix} 0 \\ v_d \end{pmatrix}, \quad \langle \Psi \rangle_C = \begin{pmatrix} v_c \\ v_d \end{pmatrix}, \quad (\text{A6})$$

<sup>3</sup>This vacuum configuration corresponds to the one used in the  $U(1)$ -symmetric inert doublet model [21,22].

with  $v_c$  and  $v_d$  being, in general, complex numbers. According to Eq. (45), the relative depth of the scalar potential between the type-I extremum and any of the three type-III extrema reads

$$\langle \Psi \rangle_A: \bar{V}_{\text{III}}^A - \bar{V}_I = \frac{1}{2} m_c^2 |v_c|^2, \quad (\text{A7a})$$

$$\langle \Psi \rangle_B: \bar{V}_{\text{III}}^B - \bar{V}_I = \frac{1}{2} m_d^2 |v_d|^2, \quad (\text{A7b})$$

$$\langle \Psi \rangle_C: \bar{V}_{\text{III}}^C - \bar{V}_I = \frac{1}{2} m_c^2 |v_c|^2 + \frac{1}{2} m_d^2 |v_d|^2. \quad (\text{A7c})$$

To compare these results with the literature, it is convenient to write the scalar potential of Eq. (A2) in the language of the 2HDM, where the most general scalar potential is usually written as [17]

$$V = -\frac{1}{2} m_{11}^2 \Phi^\dagger \Phi - \frac{1}{2} m_{22}^2 \Psi^\dagger \Psi - (m_{12}^2 \Phi^\dagger \Psi + \text{H.c.}) \quad (\text{A8a})$$

$$+ \frac{1}{2} \lambda_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2} \lambda_2 (\Psi^\dagger \Psi)^2 + \lambda_3 (\Phi^\dagger \Phi) (\Psi^\dagger \Psi) + \lambda_4 (\Phi^\dagger \Psi) (\Psi^\dagger \Phi) \quad (\text{A8b})$$

$$+ \left[ \frac{1}{2} \lambda_5 (\Phi^\dagger \Psi)^2 + \lambda_6 (\Phi^\dagger \Phi) (\Phi^\dagger \Psi) + \lambda_7 (\Psi^\dagger \Psi) (\Phi^\dagger \Psi) + \text{H.c.} \right]. \quad (\text{A8c})$$

Equation (A2) can be cast into this notation by first setting  $m_{12}^2 = 0$ ,  $\lambda_{5,6,7} = 0$ —this corresponds to the  $U(1)$ -symmetric 2HDM [17]—and then by performing the following substitutions:

$$\mu_0^2 \rightarrow -\frac{1}{2} m_{11}^2, \quad \mu_1^2 \rightarrow -\frac{1}{2} m_{22}^2, \quad (\text{A9a})$$

$$\lambda_{00} \rightarrow \lambda_1, \quad \lambda_{11} \rightarrow \lambda_2, \quad \lambda_{01} \rightarrow \lambda_3 + \frac{1}{2} \lambda_4, \quad \lambda'_{01} \rightarrow 2\lambda_4. \quad (\text{A9b})$$

In Ref. [21] the stability of the (inert) type-I vacuum against the type-III vacuum  $\langle \Psi \rangle_B$  has been studied. In Eq. (19), the authors state a condition that guarantees that  $\bar{V}_{\text{III}}^B - \bar{V}_I > 0$ , namely,

$$\bar{V}_{\text{III}}^B - \bar{V}_I = \frac{(m_{22}^2 \lambda_1 - m_{11}^2 (\lambda_3 + \lambda_4))^2}{8\lambda_1 ((\lambda_3 + \lambda_4)^2 - \lambda_1 \lambda_2)}. \quad (\text{A10})$$

However, realizing that

$$v_b = -\frac{\mu_0^2}{\lambda_{00}} = \frac{m_{11}^2}{2\lambda_1}, \quad (\text{A11a})$$

$$v_d = \frac{\mu_0^2 (\lambda_{01} + \frac{1}{4} \lambda'_{01}) - \mu_1^2 \lambda_{00}}{\lambda_{00} \lambda_{11} - (\lambda_{01} + \frac{1}{4} \lambda'_{01})^2} = \frac{m_{22}^2 \lambda_1 - m_{11}^2 (\lambda_3 + \lambda_4)}{2(\lambda_1 \lambda_2 - (\lambda_3 + \lambda_4)^2)}, \quad (\text{A11b})$$

the quantity in Eq. (A10) simply becomes

$$\frac{(m_{22}^2 \lambda_1 - m_{11}^2 (\lambda_3 + \lambda_4))^2}{8\lambda_1 ((\lambda_3 + \lambda_4)^2 - \lambda_1 \lambda_2)} = \frac{1}{2} m_d^2 |v_d|^2, \quad (\text{A12})$$

which is necessarily positive because  $m_d^2$  must be positive for the type-I extremum to be a local minimum of the scalar potential.

This not only confirms the consistency of our results in the context of the  $U(1)$ -symmetric 2HDM but also highlights the usefulness of this theorem in reducing redundancy when minimizing a scalar potential, as some vacuum configurations are immediately disfavored.

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