

$QQ\bar{Q}\bar{Q}$ quark system and gauge/string dualityOleg Andreev ^{*}*Arnold Sommerfeld Center for Theoretical Physics, LMU-München,
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(Received 7 November 2025; accepted 16 January 2026; published 6 February 2026)

We propose a stringy description of a system composed of two heavy quarks and two heavy antiquarks, mimicking that in pure $SU(3)$ gauge theory. We present both analytical and numerical studies of the string configurations for rectangular geometries. As an application, we analyze the two lowest Born-Oppenheimer potentials. Our results suggest that, depending on the geometry and quark ordering, the ground state of the $QQ\bar{Q}\bar{Q}$ system may be a hadronic molecule, a tetraquark state, or a superposition of the two. For general geometries, we derive the asymptotic expression for the energy of the tetraquark configuration in the infrared limit and extend this result to multi-quark configurations. Here we also demonstrate the universality of the string tension.

DOI: [10.1103/kzxf-gvyn](https://doi.org/10.1103/kzxf-gvyn)**I. INTRODUCTION**

Since the proposal of the quark model by Gell-Mann [1] and Zweig [2], many exotic hadrons have been discovered [3]. However, the long standing question of how quarks are bound inside exotic hadrons still remains open.

The discovery of the J/ψ meson, consisting of the charm quark and the charm antiquark, in 1974 marked the beginning of the modern era of high energy physics [4].¹ Since then, it has taken almost fifty years to observe two exotic hadrons $T_{c\bar{c}c\bar{c}}$ (6600) and $T_{c\bar{c}c\bar{c}}$ (6900), composed of four charm quarks and known as fully heavy tetraquark mesons [5].

One approach to the fully heavy tetraquark system is as follows. Given the large quark masses, it seems reasonable to apply the Born-Oppenheimer (B-O) approximation, originally developed for use in atomic and molecular physics [6].² Within this framework, the corresponding B-O potentials are defined as the energies of stationary configurations of the gluon and light quarks fields in the presence of static heavy quark sources. The hadron spectrum is then determined by solving the Schrödinger equation with these potentials.

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¹This is often referred to as the November revolution.

²For further elaboration on these ideas in the context of QCD, see [7].

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The calculation of the B-O potentials is strongly influenced by nonperturbative effects and therefore cannot be performed within perturbative QCD. Although lattice gauge theory is one of the basic tools for studying nonperturbative phenomena in QCD and has made significant progress in the study of the fully heavy tetraquark systems [8–10], the limited results and the need to understand the physics behind computational complexity motivate the use of effective field and string theories. A special class of string models, called AdS/QCD (holographic) models, has received much attention in the last years. The hope is that the gauge/string duality provides new theoretical tools for studying strongly coupled gauge theories.³ In these models the string configurations for tetraquarks were discussed qualitatively in [12,13]. Nevertheless, the existing literature notably lacks a comprehensive discussion on such systems. Bridging these gaps is the main objective of the present paper.

The paper continues our study of the $Q\bar{Q}$ and $QQ\bar{Q}\bar{Q}$ heavy quark systems [14–16]. It is organized as follows. In Sec. II, we briefly review some preliminary results and establish the framework for the reader's convenience. Then in Secs. III and IV, we construct and analyze string configurations in five dimensions that provide a dual description of the lowest B-O potentials in the heavy quark limit. Here, we restrict ourselves to the symmetric case, when the quark sources are located at the vertices of a rectangular. In Sec. V, we examine the two lowest B-O potentials of the system and also discuss the length scales that characterize transitions between different configurations. These length scales are related to various types of

³For the further development of these ideas in the context of QCD, see the book [11] and references therein.

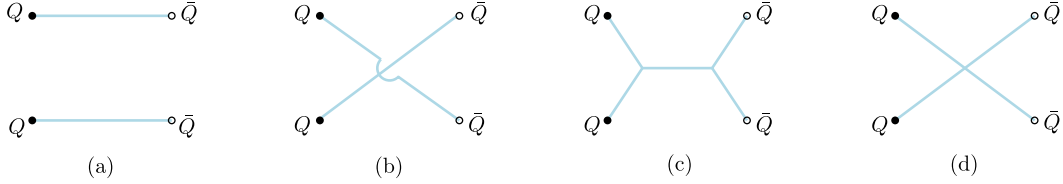


FIG. 1. Four-dimensional string configurations with type-A ordering: (a)–(b) disconnected configurations, (c) a tetraquark configuration, and (d) a pinched tetraquark configuration.

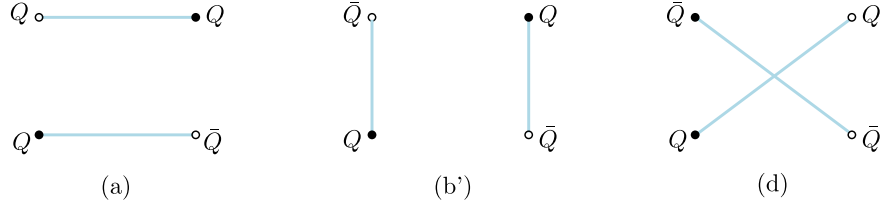


FIG. 2. Four-dimensional string configurations with type-B ordering: (a)–(b') disconnected configurations and (d) a pinched tetraquark configuration. Here, the tetraquark configuration (c') formally coincides with the pinched configuration (d'), but this no longer holds in five dimensions.

string interactions, including string junction annihilation. In Sec. VI, we consider the IR limit for a special class of multi-quark string configurations and explore the asymptotic behavior of their energies. We conclude in Sec. VII with several comments on the implications of our findings and discussing directions for future work. Appendix A contains our notation and definitions. To make the paper self-contained, Appendixes B and C provide the necessary results on Nambu-Goto strings in five-dimensional space and on the $Q\bar{Q}$ system. Finally, in Appendixes D and E, we discuss the five-dimensional analogs of the tetraquark string configuration.

II. PRELIMINARIES

A. General procedure

To study the fully heavy tetraquark system, we use an approach based on a correlation matrix.⁴ In this approach, the diagonal elements of the matrix are determined by the energies of stationary string configurations, while the off-diagonal elements describe transitions between them. The potentials are determined by the eigenvalues of this matrix.

We begin by specializing to the case of pure $SU(3)$ gauge theory and a geometry in which the quark sources are placed at the vertices of a rectangle. There are two possible orderings of these sources; type-A ordering, where the quarks (antiquarks) occupy adjacent vertices, and type-B ordering (bipartite), where each site is connected only to one quark and one antiquark. Let us start with string configurations in four dimensions, where the string picture has long been established [17,18]. The simplest configurations are

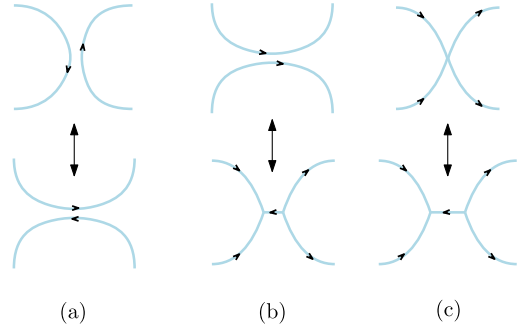


FIG. 3. Some string interactions: (a) reconnection, (b) junction annihilation, and (c) pinching.

disconnected (mesonic) ones, as shown in Figs. 1 and 2. Each consists of two quark-antiquark pairs joined by strings. We assume that other configurations are constructed by adding extra string junctions.⁵ This yields the connected tetraquark configurations, both regular and pinched, shown in the figures. In the latter case the string junctions coincide. In contrast to [17,18], we include the pinched tetraquark configurations, as their five-dimensional analogs turn out to be relevant for describing the first excited B-O potential. In what follows, we refer to hadronic molecules as being described by the disconnected string configurations, and tetraquarks by the connected ones.

Transitions between the configurations arise due to string interactions. In Fig. 3, we sketch three types of interactions that will be discussed in the following sections. These represent only a small part of the broader dynamics of QCD

⁴For more on this in lattice QCD, see [10].

⁵Besides string junctions, other possibilities include excited strings and glueball states, which, however, are not relevant here.

strings. Later, we will introduce the notion of a critical length, which characterizes a transition between two configurations. This helps to deepen our understanding of the physics of QCD strings, the structure of B-O potentials, and, importantly, the nature of multi-quark states.

B. A short account of the five-dimensional model

In our study of the fully heavy tetraquark system, we will use the formalism mainly developed in [15]. This formalism is general and can be adapted to any model of AdS/QCD, although we illustrate it by performing calculations in one of the simplest models, the so called soft wall model. In this model, the background five-dimensional geometry is chosen to be a one-parameter deformation of Euclidean AdS_5 space of radius R ,

$$ds^2 = e^{sr^2} \frac{R^2}{r^2} (dt^2 + (dx^i)^2 + dr^2). \quad (2.1)$$

Here r denotes the fifth (radial) coordinate, and s is a deformation parameter. The boundary lies at $r = 0$, while the soft wall at $r = 1/\sqrt{s}$. The wall effectively prevents strings from penetrating deep into the bulk.

Our analysis relies on two key components. The first is a fundamental string governed by the Nambu-Goto action,

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{\gamma^{(2)}}, \quad (2.2)$$

where γ is an induced metric, α' is a string parameter, and ξ^i are world sheet coordinates. The second component is a high-dimensional analogue of the string junction, commonly referred to as the baryon vertex.⁶ In the context of AdS/CFT, the vertex is a five brane wrapped on an internal space $\mathbf{X}$ [19]. From the five-dimensional viewpoint, this object looks pointlike. As shown in [15], the action for the baryon vertex written in the static gauge,

$$S_{\text{vert}} = \tau_v \int dt \frac{e^{-2sr^2}}{r}, \quad (2.3)$$

yields the results in very good agreement with lattice QCD calculations of the three-quark potential. Actually, S_{vert} represents the world volume of the brane, assuming $\tau_v = \mathcal{T}_5 \text{Rvol}(\mathbf{X})$ with $\mathcal{T}_5$ the brane tension. Unlike AdS/CFT, we treat τ_v as a free parameter to account for α' -corrections as well as the possible effects of other background fields.⁷ At zero baryon chemical potential, it is natural to suggest the same action for the antibaryon vertex, such that $S_{\text{vert}}^{\bar{b}} = S_{\text{vert}}$.

We also need to specify the model parameters. For the purposes of this paper, it is natural to use those from [15], which were obtained by fitting lattice QCD data for the

⁶We use this terminology, to distinguish between the four-dimensional string junction and its five-dimensional counterpart.

⁷In analogy with AdS/CFT, we expect the presence of Ramond-Ramond background fields on $\mathbf{X}$.

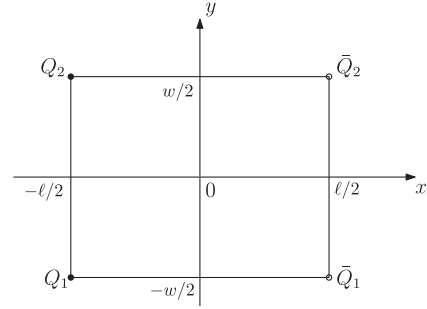


FIG. 4. A rectangle of length ℓ and width w . The quarks (antiquarks) are placed at the vertices according to type-A ordering.

QQQ quark system to the string model under consideration. Accordingly, we set $\mathbf{g} = \frac{R^2}{2\pi\alpha'} = 0.176$, $\mathbf{s} = 0.44 \text{ GeV}^2$, and $\mathbf{k} = \frac{\tau_v}{3\mathbf{g}} = -0.083$. These values will be used in all subsequent estimates unless stated otherwise. Crucially, no additional parameters are introduced to describe the tetraquark system.

III. STRING CONFIGURATIONS IN FIVE DIMENSIONS FOR TYPE-A ORDERING

We focus here on the simplest geometry when the quark sources are at the vertices of a rectangle. Without loss of generality, we can place the rectangle in the xy -plane with its center at the origin and the x - and y -axes serving as symmetry axes, as shown in Fig. 4.

In the following analysis, we keep the ratio of length to width fixed, so that

$$\ell = \eta w, \quad (3.1)$$

with η a positive real number. We refer to Eq. (3.1) as the geometrical constraint. It reduces the number of parameters by one that significantly simplifies the analysis.

A. The disconnected configurations

We begin with the disconnected configurations, each of which can be interpreted as a hadronic molecule composed of two heavy mesons $Q\bar{Q}$. We therefore suppose that the energy equals the sum of the rest energies of the quark-antiquark pairs.⁸ In five dimensions, the configuration consists of two disconnected parts, each having the form shown in Fig. 22 on the right.

1. Configuration (a)

In this case, the quarks and antiquarks are separated by a distance ℓ , and therefore the energy of configuration (a) is simply

⁸This is, of course, a simplification, as it neglects a binding energy. We return to this point in Sec. V.

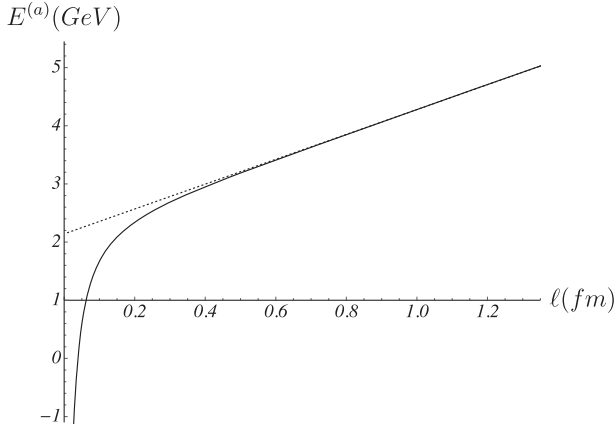


FIG. 5. $E^{(a)}$ vs ℓ . The dotted line represents the large- ℓ asymptotics (3.4).

$$E^{(a)} = 2E_{Q\bar{Q}}(\ell), \quad (3.2)$$

where $E_{Q\bar{Q}}$ and ℓ are given parametrically by Eq. (C1).

Using the asymptotics for $E_{Q\bar{Q}}(\ell)$ from Appendix C, we can easily obtain the behavior of $E^{(a)}$ for small and large ℓ . Explicitly,

$$E^{(a)}(\ell) = -\frac{\alpha^{(a)}}{\ell} + 4c + o(1), \quad \text{with} \quad \alpha^{(a)} = 2\alpha_{Q\bar{Q}}, \quad (3.3)$$

and

$$E^{(a)}(\ell) = 2\sigma\ell + 2C_{Q\bar{Q}} + o(1). \quad (3.4)$$

Here $\alpha_{Q\bar{Q}}$ and $C_{Q\bar{Q}}$ are given by (C2) and (C3), respectively, c is a normalization constant, and $\sigma = \text{ges}$ is the string tension [14].

In Fig. 5 we plot $E^{(a)}$ as a function of ℓ . Notably, this function becomes nearly linear for $\ell \gtrsim 0.6$ fm, as in the case of the $Q\bar{Q}$ system.

2. Configuration (b)

Configuration (b) can be treated similarly, but there is one difference. The quarks and antiquarks are now separated by a distance $\sqrt{1 + \eta^{-2}}\ell$. Therefore, the energy is

$$E^{(b)} = 2E_{Q\bar{Q}}\left(\sqrt{1 + \eta^{-2}}\ell\right). \quad (3.5)$$

This implies that its asymptotic expansions can be obtained from those in (3.3) and (3.4) by rescaling $\ell \rightarrow \sqrt{1 + \eta^{-2}}\ell$. Thus,

$$E^{(b)}(\ell) = -\frac{\alpha^{(b)}}{\ell} + 4c + o(1), \quad \text{with} \quad \alpha^{(b)} = \frac{2}{\sqrt{1 + \eta^{-2}}} \alpha_{Q\bar{Q}}, \quad (3.6)$$

for small ℓ , and

$$E^{(b)}(\ell) = 2\sqrt{1 + \eta^{-2}}\sigma\ell + 2C_{Q\bar{Q}} + o(1) \quad (3.7)$$

for large ℓ . Here, $E^{(b)}$ is proportional to twice the diagonal length.

We conclude with an important remark. At finite η , configuration (b) always has a higher energy than configuration (a). The reason is as follows. Both functions $\mathcal{L}^+(0, \lambda)$ and $\mathcal{E}^+(0, \lambda)$ increase monotonically in the interval $[0, 1]$. For a given ℓ , the value of $\lambda^{(b)}$ must be larger than that of $\lambda^{(a)}$ because $\sqrt{1 + \eta^{-2}} > 1$. Hence, $\mathcal{E}^+(0, \lambda^{(b)}) > \mathcal{E}^+(0, \lambda^{(a)})$. Consequently, configuration (b) is irrelevant for the ground state potential and contributes only to higher B-O potentials. While this is obvious for large ℓ , it is less clear for smaller ℓ .

B. The tetraquark configuration

We now turn to the tetraquark configuration (c) in five dimensions. Its form was already suggested in [12], and here we make it precise. The detailed construction depends nontrivially on η . In practice, it is convenient to define two basic configurations, shown in Fig. 6, and to construct the tetraquark configuration from them.

Before proceeding, let us make a few comments.

First, the total action governing the tetraquark configuration is the sum of the Nambu-Goto actions for each of the five strings and the actions for the two vertices,

$$S = \sum_{i=1}^5 S_{\text{NG}}^{(i)} + 2S_{\text{vert}}. \quad (3.8)$$

In Appendix B, it is shown how to extremize the Nambu-Goto actions to find the solutions that describe the strings. We will explain how to do so for the vertices shortly.

Second, the following simple but useful relations are

$$\ell = |Y\bar{Y}| + 2\sin\beta|Q_1Y|, \quad w = 2\cos\beta|Q_1Y|, \quad (3.9)$$

valid for $0 \leq \beta \leq \arctan\eta$.

Third, at $\beta = \frac{\pi}{6}$, the points Y and $\bar{Y}$ coincide with the Steiner points of the rectangle. In this case, the length of the Steiner tree is $|\bar{S}S| + 4|Q_1S| = (1 + \sqrt{3}\eta^{-1})\ell$.⁹ For $\eta \leq \frac{1}{\sqrt{3}}$, the Steiner points collapse to the rectangle center.

Finally, if the vertices coincide, basic configuration (I) doesn't exist, while configuration (II) reduces to the pinched tetraquark configuration (d). Moreover, we treat the tetraquark configurations (c) and (d) separately because configuration (d) also exists for $\eta > \frac{1}{\sqrt{3}}$, where it differs from configuration (c).

⁹Note that the string configurations are not exactly the classical Steiner trees because quarks must connect only to baryon vertices, while antiquarks connect only to antibaryon vertices.

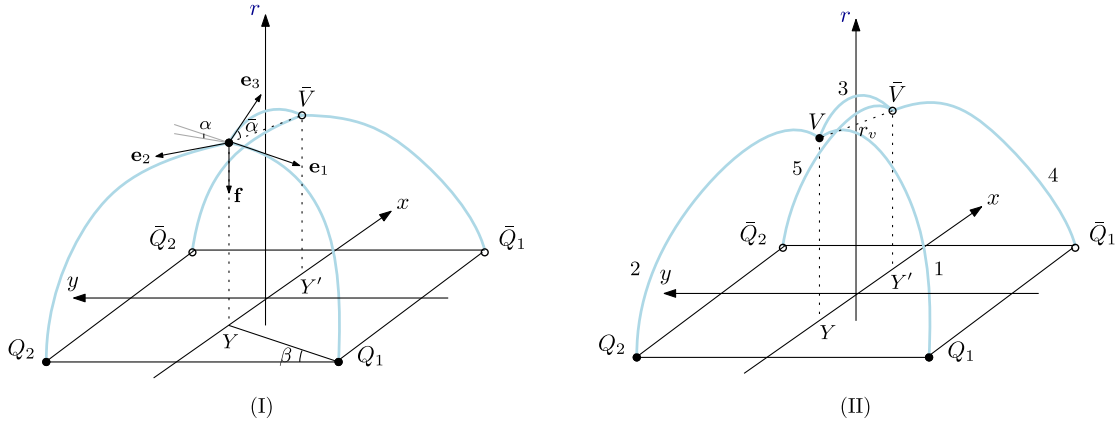


FIG. 6. The basic tetraquark configurations in five dimensions. The baryon vertices are located at $r = r_v$ in the bulk, with Y and Y' being their projections onto the x -axis, and $\beta = \angle YQ_1Q_2$. For the tangent angles (see also Fig. 20), we use the symmetry-based abbreviation: $\alpha_{1,2} = \bar{\alpha}_{1,2} = \alpha$ and $\alpha_3 = \bar{\alpha}$. (I): $\alpha \geq 0$. The arrows indicate the forces acting on V , with the gravitational force $\mathbf{f}$ directed downward. (II): $\alpha \leq 0$. The strings are labeled by 1, ..., 5.

1. Strings joining at a baryon vertex

The string solutions described in Appendix B serve as the building blocks for the basic configurations. These blocks, however, must satisfy certain gluing conditions, which we now describe.

From the physical viewpoint, the gluing conditions express the requirement that the net force vanishes at any vertex where strings meet, as illustrated in Fig. 6 on the left. Translating this into mathematical terms is straightforward; extremizing the total action of the Nambu-Goto strings and baryon vertices with respect to the location of the vertex V yields the force balance equation at that vertex,

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{f} = 0. \quad (3.10)$$

The corresponding string tensions can be read off from the expressions (B7) and (B24),

$$\begin{aligned} \mathbf{e}_1 &= -\mathbf{g}w(r_v)(\sin \beta \cos \alpha, \cos \beta \cos \alpha, \sin \alpha), \\ \mathbf{e}_2 &= -\mathbf{g}w(r_v)(\sin \beta \cos \alpha, -\cos \beta \cos \alpha, \sin \alpha), \\ \mathbf{e}_3 &= \mathbf{g}w(r_v)(\cos \bar{\alpha}, 0, \sin \bar{\alpha}). \end{aligned} \quad (3.11)$$

Note that $\bar{\alpha}$ is always non-negative, as explained in Appendix B.

The gravitational force acting on the vertex is given by $\mathbf{f}_r = -\frac{1}{T} \frac{\delta V_{\text{vert}}}{\delta r_v}$, and therefore has only one nonzero component,

$$\mathbf{f} = \left(0, 0, \tau_v(1 + 4sr_v^2) \frac{e^{-2sr_v^2}}{r_v^2} \right). \quad (3.12)$$

The force balance equation then takes the component form,

$$\begin{aligned} 2 \sin \beta \cos \alpha - \cos \bar{\alpha} &= 0, \\ 2 \sin \alpha - \sin \bar{\alpha} - 3k(1 + 4v)e^{-3v} &= 0. \end{aligned} \quad (3.13)$$

Obviously, the same analysis applies to the vertex $\bar{V}$, with identical tangent angles.

2. Configuration (I)

We begin with configuration (I), characterized by non-negative tangent angles. Using (B8) and (B25), we get $|Q_1Y| = \frac{1}{\sqrt{s}} \mathcal{L}^+(\alpha, v)$ and $|Y\bar{Y}| = \frac{1}{\sqrt{s}} \mathcal{L}(\bar{\lambda}, v)$. Here λ_3 is denoted by $\bar{\lambda}$. Thus, the geometrical constraint (3.1) takes the form,

$$\mathcal{L}(\bar{\lambda}, v) = 2(\eta \cos \beta - \sin \beta) \mathcal{L}^+(\alpha, v), \quad (3.14)$$

where the functions $\mathcal{L}$ and $\mathcal{L}^+$ are defined in Appendix A, and $\bar{\lambda} = -\text{ProductLog}(-ve^{-v}/\cos \bar{\alpha})$, as follows from (B16).

The expression for ℓ follows from Eq. (3.9) with the help of (3.1), while the energy is obtained from Eqs. (2.3), (B11), and (B26). As a result, we find

$$\begin{aligned} \ell^{(I)} &= \frac{2}{\sqrt{s}} \eta \cos \beta \mathcal{L}^+(\alpha, v), \\ E^{(I)} &= \mathbf{g}\sqrt{s} \left(4\mathcal{E}^+(\alpha, v) + \mathcal{E}(\bar{\lambda}, v) + 6k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c. \end{aligned} \quad (3.15)$$

Here c is the normalization constant. When combined with the geometrical constraint and the force balance equations, this gives the function $E(\ell)$ in parametric form; $\ell = \ell^{(I)}(v)$ and $E = E^{(I)}(v)$.

Let us now consider the small- ℓ behavior of $E^{(I)}$. This limit corresponds to $v \rightarrow 0$, as follows from the asymptotic behavior of $\mathcal{E}^+$. In this case, the force balance equation and the geometric constraint become

$$\begin{aligned} 2 \sin \beta \cos \alpha - \cos \bar{\alpha} &= 0, & 2 \sin \alpha - \sin \bar{\alpha} - 3k &= 0, \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \sqrt{\cos \alpha} B\left(\sin^2 \bar{\alpha}, \frac{1}{2}, \frac{3}{4}\right) \\ &= (\eta \cos \beta - \sin \beta) \sqrt{\cos \bar{\alpha}} B\left(\cos^2 \alpha, \frac{3}{4}, \frac{1}{2}\right), \end{aligned} \quad (3.17)$$

where $B(z; a, b)$ denotes the incomplete beta function. For a given η , the three angles can be determined from these equations. Taking the limit $v \rightarrow 0$ in Eq. (3.15) and using Eqs. (A2), (A12), and (A18), we arrive at

$$\begin{aligned} E^{(I)} &= -\frac{\alpha^{(I)}}{\ell} + 4c + o(1), \quad \text{with} \\ \alpha^{(I)} &= -2\eta \cos \beta \mathcal{L}_0^+(\alpha) (4\mathcal{E}_0^+(\alpha) + \mathcal{E}_0(\bar{\alpha}) + 6k)g. \end{aligned} \quad (3.18)$$

Here the coefficients $\mathcal{L}_0^+$, $\mathcal{E}_0^+$, and $\mathcal{E}_0$ are defined in Appendix A. The notation $\mathcal{E}_0(\bar{\alpha})$ is used for $\mathcal{E}_0(\cos \bar{\alpha})$. For our parameter values, the factor $4\mathcal{E}_0^+ + \mathcal{E}_0 + 6k$ is negative, and therefore $\alpha^{(I)}$ is positive.

It is also instructive to discuss the large- ℓ behavior. If η is finite, the first equation in (3.15) implies that $\alpha = 0$ and $v = 1$. In this case, string 3 has $\bar{\alpha} = 0$. However, the second equation in (3.13) has no solution with $\alpha = \bar{\alpha} = 0$ in the interval $0 \leq v \leq 1$. Thus, the large- ℓ limit does not exist for finite η . If η is infinite, the limit exists and coincides with the diquark limit. The point is that in this case string 3 becomes infinitely long, while the others remain finite. In practice, it is more convenient to use the geometrical constraint $w = \text{const}$ to explore the diquark limit. We will return to this issue in [20].

3. Configuration (II)

Since this configuration is governed by the same action as configuration (I), the force balance equations (3.13) still apply. The expressions for the geometrical constraint, the length and the energy can be obtained by replacing $\mathcal{L}^+$ and $\mathcal{E}^+$ with $\mathcal{L}^-$ and $\mathcal{E}^-$ corresponding to negative α . This gives

$$\mathcal{L}(\bar{\lambda}, v) = 2(\eta \cos \beta - \sin \beta) \mathcal{L}^-(\lambda, v), \quad (3.19)$$

and

$$\begin{aligned} \ell^{(II)} &= \frac{2}{\sqrt{s}} \eta \cos \beta \mathcal{L}^-(\lambda, v), \\ E^{(II)} &= g\sqrt{s} \left(4\mathcal{E}^-(\lambda, v) + \mathcal{E}(\bar{\lambda}, v) + 6k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c. \end{aligned} \quad (3.20)$$

Here, due to symmetry, we set $\lambda = \lambda_1 = \lambda_2 = \bar{\lambda}_1 = \bar{\lambda}_2$. For a given η , these equations, together with Eqs. (3.13), provide a parametric representation of E as a function of ℓ ; $\ell = \ell^{(II)}(v)$ and $E = E^{(II)}(v)$.

The small- ℓ behavior of $E^{(II)}(\ell)$ can be analyzed in a straightforward way. For $v \rightarrow 0$, the force balance equations (3.16) apply to this case as well, while the geometrical constraint becomes

$$\begin{aligned} & \sqrt{\cos \alpha} I\left(\sin^2 \bar{\alpha}, \frac{1}{2}, \frac{3}{4}\right) \\ &= (\eta \cos \beta - \sin \beta) \sqrt{\cos \bar{\alpha}} \left(1 + I\left(\sin^2 \alpha, \frac{1}{2}, \frac{3}{4}\right) \right), \end{aligned} \quad (3.21)$$

where I denotes the regularized beta function. Given η , the three angles can be determined from Eqs. (3.16) and (3.21). We also replace $\mathcal{L}_0^+$ and $\mathcal{E}_0^+$ with $\mathcal{L}_0^-$ and $\mathcal{E}_0^-$ in (3.18), getting

$$\begin{aligned} E^{(II)} &= -\frac{\alpha^{(II)}}{\ell} + 4c + o(1), \quad \text{with} \\ \alpha^{(II)} &= -2\eta \cos \beta \mathcal{L}_0^-(\alpha) (4\mathcal{E}_0^-(\alpha) + \mathcal{E}_0(\bar{\alpha}) + 6k)g. \end{aligned} \quad (3.22)$$

Here the coefficients $\mathcal{L}_0^-$ and $\mathcal{E}_0^-$ are defined in Appendix A. For brevity, the argument is written as α instead of $\cos \alpha$.

Now we turn to the case of large ℓ . There are two subcases to consider; $\lambda, \bar{\lambda} \rightarrow 1$, where all the strings become infinitely long, and $\lambda \rightarrow 1$ with fixed $\bar{\lambda} < 1$, where string 3 remains finite in length.

We begin with the former case, which leads to two distinct Steiner points and, as a result, to the term $\sigma L_{\min}$ in $E^{(II)}$. From (B15) it follows that in this limit $\cos \alpha = \cos \bar{\alpha} = v e^{1-v}$ and therefore $\alpha = -\bar{\alpha}$. Using this relation, the second equation in (3.13) can be readily solved, yielding $\sin \alpha = k(1 + 4v)e^{-3v}$. Combining both expressions for α , we find,¹⁰

$$1 - k^2(1 + 4v)^2 e^{-6v} - v^2 e^{2(1-v)} = 0, \quad (3.23)$$

whose solution provides the upper bound on v . We denote it by v_1 . Importantly, v_1 is independent of η and belongs to the interval $[0, 1]$. Numerically, $v_1 = 0.978$. It is noteworthy that in this large- ℓ limit the first equation in (3.13) yields $\sin \beta = \frac{1}{2}$ and thus $\beta = \frac{\pi}{6}$. Hence the points Y and Y' coincide with the Steiner points of the rectangle.

As $\lambda, \bar{\lambda} \rightarrow 1$, the strings become infinitely long. The singularities arise from the functions $\mathcal{L}^-$ and $\mathcal{L}$, as seen from Eqs. (A6) and (A9). With this in mind, we keep only the singular terms in ℓ and $E^{(II)}$. The resulting equations are then

$$\begin{aligned} \ell &= -\sqrt{\frac{3}{s}} \ln(1 - \lambda) + O(1), \\ E^{(II)} &= -g e \sqrt{s} (4 \ln(1 - \lambda) + \ln(1 - \bar{\lambda})) + O(1). \end{aligned} \quad (3.24)$$

¹⁰An important point is that this is a general equation. It holds at any vertex where three infinitely long strings meet. For example, in the QQQ system it was obtained in [15,16]. We return to this issue in Sec. VI.

The relation between λ and $\bar{\lambda}$ follows from the geometrical constraint. Expressing $\bar{\lambda}$ in terms of λ , we obtain

$$1 - \bar{\lambda} = (1 - \lambda)^{\sqrt{3}\eta^{-1}}. \quad (3.25)$$

Importantly, this relation makes sense only if $\eta > \frac{1}{\sqrt{3}}$. This implies that the Steiner points do not coincide, as expected when an infinite long string terminates on them. Eliminating the parameter from Eq. (3.24) then gives

$$E^{(\text{II})} = (1 + \sqrt{3}\eta^{-1})\sigma\ell + O(1). \quad (3.26)$$

This is the desired behavior with the minimal length of the string network (Steiner tree) $L_{\min} = (1 + \sqrt{3}\eta^{-1})\ell$.

To find the next-to-leading term, consider the difference between $E^{(\text{II})}$ and $\sigma L_{\min}$

$$\begin{aligned} E^{(\text{II})} - \sigma L_{\min} &= \mathbf{g}\sqrt{\mathbf{s}} \left(4(\mathcal{E}^-(\lambda, v) - \mathbf{e}\mathcal{L}^-(\lambda, v)) + \mathcal{E}(\bar{\lambda}, v) \right. \\ &\quad \left. - \mathbf{e}\mathcal{L}(\bar{\lambda}, v) + 6\mathbf{k}\frac{\mathbf{e}^{-2v}}{\sqrt{v}} \right) \\ &\quad + 4c + 2\mathbf{g}\mathbf{e}\sqrt{\mathbf{s}}(2 - \sin\beta - \sqrt{3}\cos\beta)\mathcal{L}^-(\lambda, v). \end{aligned} \quad (3.27)$$

Taking the limit $\lambda, \bar{\lambda} \rightarrow 1$ ($v \rightarrow v_1$) and using the formulas (A24) and (A26), we get

$$E^{(\text{II})} - \sigma L_{\min} = \mathbf{g}\sqrt{\mathbf{s}} \left(-4\mathcal{I}(v_1) - \mathcal{J}(v_1) + 6\mathbf{k}\frac{\mathbf{e}^{-2v_1}}{\sqrt{v_1}} \right) + 4c. \quad (3.28)$$

Here the functions $\mathcal{I}$ and $\mathcal{J}$ are defined in Appendix A. The last term in (3.27) vanishes because the factor $2 - \sin\beta - \sqrt{3}\cos\beta$ behaves like a power law in $1 - \lambda$, whereas $\mathcal{L}^-$ like a logarithm. Thus,

$$\begin{aligned} E^{(\text{II})} &= (1 + \sqrt{3}\eta^{-1})\sigma\ell + C^{(\text{II})} + o(1), \quad \text{with} \\ C^{(\text{II})} &= 4c - \mathbf{g}\sqrt{\mathbf{s}} \left(4\mathcal{I}(v_1) + \mathcal{J}(v_1) - 6\mathbf{k}\frac{\mathbf{e}^{-2v_1}}{\sqrt{v_1}} \right). \end{aligned} \quad (3.29)$$

Note that, unlike the linear term, the constant term does not depend on η and is, therefore, universal from this viewpoint.

In studying the latter case, we proceed as follows. The angle α is still given by $\cos\alpha = v\mathbf{e}^{1-v}$, but this no longer holds for the angle $\bar{\alpha}$, which is now determined by the geometrical constraint. First, since the left-hand side of (3.19) is finite, the factor $(\eta\cos\beta - \sin\beta)$ must vanish in this limit that implies $\tan\beta = \eta$. Then, the first equation in (3.13) gives $\cos\bar{\alpha} = \frac{2}{\sqrt{1+\eta^{-2}}}v\mathbf{e}^{1-v}$. Substituting α and $\bar{\alpha}$ into

the second equation in (3.13), one finds that

$$\begin{aligned} &2\sqrt{1 - v^2\mathbf{e}^{2(1-v)}} + \sqrt{1 - \frac{4}{1+\eta^{-2}}v^2\mathbf{e}^{2(1-v)} + 3\mathbf{k}(1+4v)\mathbf{e}^{-3v}} \\ &= 0, \end{aligned} \quad (3.30)$$

which manifestly depends on η . Accordingly, its solution is η -dependent and establishes the upper bound on v . We denote this solution in the interval $[0, 1]$ as v_η . The allowed range of η is governed by the two conditions $\bar{\lambda} = 1$ and $\bar{\alpha} = 0$, as discussed in Appendix D.

Using Eqs. (A6) and (A16), we obtain the singular terms in ℓ and $E^{(\text{II})}$,

$$\begin{aligned} \ell &= -\frac{2}{\sqrt{\mathbf{s}}} \frac{\ln(1 - \lambda)}{\sqrt{1 + \eta^{-2}}} + O(1), \\ E^{(\text{II})} &= -4\mathbf{g}\mathbf{e}\sqrt{\mathbf{s}}\ln(1 - \lambda) + O(1). \end{aligned} \quad (3.31)$$

It follows immediately that

$$E^{(\text{II})} = 2\sqrt{1 + \eta^{-2}}\sigma\ell + O(1), \quad (3.32)$$

with the same string tension as before. In this case the length of the string network is equal to twice the length of the diagonal $d = \sqrt{1 + \eta^{-2}}\ell$.

To compute the next term in the expansion consider the difference,

$$\begin{aligned} E^{(\text{II})} - 2\sigma d &= \mathbf{g}\sqrt{\mathbf{s}} \left(4(\mathcal{E}^-(\lambda, v) - \mathbf{e}\mathcal{L}^-(\lambda, v)) + \mathcal{E}(\bar{\lambda}, v) \right. \\ &\quad \left. + 2\mathbf{e}\frac{1 - \sqrt{1 + \eta^{-2}}\cos\beta}{\eta\cos\beta - \sin\beta}\mathcal{L}(\bar{\lambda}, v) + 6\mathbf{k}\frac{\mathbf{e}^{-2v}}{\sqrt{v}} \right) \\ &\quad + 4c. \end{aligned} \quad (3.33)$$

Now we take the limit $\lambda \rightarrow 1$ ($v \rightarrow v_\eta$), getting

$$\begin{aligned} E^{(\text{II})} - 2\sigma d &= \mathbf{g}\sqrt{\mathbf{s}} \left(-4\mathcal{I}^-(v_\eta) + \mathcal{E}(\lambda_\eta, v_\eta) \right. \\ &\quad \left. - \frac{2\mathbf{e}}{\sqrt{1 + \eta^{-2}}}\mathcal{L}(\lambda_\eta, v_\eta) + 6\mathbf{k}\frac{\mathbf{e}^{-2v_\eta}}{\sqrt{v_\eta}} \right) \\ &\quad + 4c, \end{aligned} \quad (3.34)$$

where $\lambda_\eta = \lim_{v \rightarrow v_\eta} \bar{\lambda} = -\text{ProductLog}\left(-\frac{\sqrt{1+\eta^{-2}}}{2\mathbf{e}}\right)$. Here we used that $\lim_{\beta \rightarrow \arctan\eta} \frac{1 - \sqrt{1+\eta^{-2}}\cos\beta}{\eta\cos\beta - \sin\beta} = -\frac{1}{\sqrt{1+\eta^{-2}}}$. Finally, the expansion of $E^{(\text{II})}$ reads,

$$\begin{aligned}
E^{(\text{II})} &= 2\sqrt{1 + \eta^{-2}\sigma\ell} + C_\eta^{(\text{II})} + o(1), \quad \text{with} \\
C_\eta^{(\text{II})} &= 4c - g\sqrt{s} \left(4\mathcal{I}(v_\eta) - \mathcal{E}(\lambda_\eta, v_\eta) \right. \\
&\quad \left. + \frac{2e}{\sqrt{1 + \eta^{-2}}} \mathcal{L}(\lambda_\eta, v_\eta) - 6k \frac{e^{-2v_\eta}}{\sqrt{v_\eta}} \right). \quad (3.35)
\end{aligned}$$

The notable feature here is that the constant term depends explicitly on the ratio η .

We have discussed the basic configurations, from which any tetraquark configuration (c) can be constructed. This construction, however, is cumbersome due to its η -dependence, so we describe it explicitly in Appendix D.

C. The pinched tetraquark configuration

In contrast to the tetraquark configuration, the pinched tetraquark configuration exists for all values of η and is generally distinct from it.¹¹ Visually, one may think of the pinched configuration as the tetraquark configuration with string 3 pinched into a point, see Fig. 7. It is governed by the action

$$S = \sum_{i=1}^4 S_{\text{NG}}^{(i)} + 2S_{\text{vert}}. \quad (3.36)$$

In this case, the force balance equation at $r = r_v$ takes the form

$$\mathbf{e}_1 + \mathbf{e}_2 + \bar{\mathbf{e}}_1 + \bar{\mathbf{e}}_2 + 2\mathbf{f} = 0. \quad (3.37)$$

Its only nontrivial component is the r -component, which allows us to explicitly express the tangent angle α in terms of the parameter v ,

$$\sin \alpha = \frac{3}{2} k(1 + 4v)e^{-3v}. \quad (3.38)$$

Note that α is negative for negative k .

A quick way to get some formulas is to set $\bar{\alpha} = 0$ and $\tan \beta = \eta$ in the corresponding formulas of Sec. III B. So, the length and the energy are given by

$$\begin{aligned}
\ell &= \frac{2}{\sqrt{s}} \frac{\mathcal{L}^-(\lambda, v)}{\sqrt{1 + \eta^{-2}}}, \\
E^{(d)} &= 2g\sqrt{s} \left(2\mathcal{E}^-(\lambda, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c, \quad (3.39)
\end{aligned}$$

where v varies from 0 to v_p , defined shortly. For a fixed η , these parametric equations determine the function $E^{(d)}(\ell)$.

The small- ℓ behavior of $E^{(d)}$ can be read off from Eq. (3.22). Thus,

¹¹For explicit examples, see Sec. V.

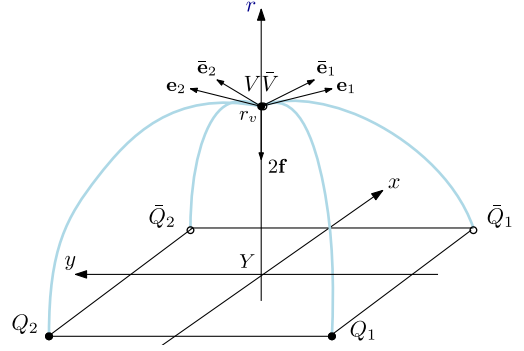


FIG. 7. A pinched tetraquark configuration in five dimensions. Both baryon vertices are located at $r = r_v$ on the r -axis, and Y is at the center of the rectangle. The arrows indicate the forces acting on the vertices. By symmetry, all the tangent angles of the strings are equal.

$$\begin{aligned}
E^{(d)} &= -\frac{\alpha^{(d)}}{\ell} + 4c + o(1), \quad \text{with} \\
\alpha^{(d)} &= -\frac{4}{\sqrt{1 + \eta^{-2}}} \mathcal{L}_0^-(\alpha) (2\mathcal{E}_0^-(\alpha) + 3k)g. \quad (3.40)
\end{aligned}$$

The factor $2\mathcal{E}_0^-(\alpha) + 3k$ is negative for our parameter values, so the coefficient $\alpha^{(d)}$ is positive.

Next, let us consider the large- ℓ behavior. This corresponds to the limit $\lambda \rightarrow 1$, where the tangent angle is given by $\cos \alpha = ve^{1-v}$. Combining this with (3.38) we get the equation,

$$1 - v^2 e^{2(1-v)} - \frac{9}{4} k^2 (1 + 4v)^2 e^{-6v} = 0, \quad (3.41)$$

whose solution v_p in the interval $[0, 1]$ gives the upper bound on v . Numerically, $v_p = 0.967$.

From the foregoing, it is clear that at leading order in ℓ , $E^{(d)}$ is proportional to twice the diagonal length,

$$E^{(d)} = 2\sqrt{1 + \eta^{-2}\sigma\ell} + O(1), \quad (3.42)$$

as $E^{(\text{II})}$ in Eq. (3.32). To compute the constant term, consider the difference between $E^{(d)}$ and the linear term,

$$\begin{aligned}
E^{(d)} - 2\sigma d &= 2g\sqrt{s} \left(2\mathcal{E}^-(\lambda, v) - 2e\mathcal{L}^-(\lambda, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) \\
&\quad + 4c. \quad (3.43)
\end{aligned}$$

After taking the limit $\lambda \rightarrow 1$, the right-hand side becomes $2g\sqrt{s}(-2\mathcal{I}(v_p) + 3k \frac{e^{-2v_p}}{\sqrt{v_p}}) + 4c$, where we have used (A24). Thus, we arrive at

$$\begin{aligned}
E^{(d)} &= 2\sqrt{1 + \eta^{-2}\sigma\ell} + C^{(d)} + o(1), \quad \text{with} \\
C^{(d)} &= 4c - 2g\sqrt{s} \left(2\mathcal{I}(v_p) - 3k \frac{e^{-2v_p}}{\sqrt{v_p}} \right). \quad (3.44)
\end{aligned}$$

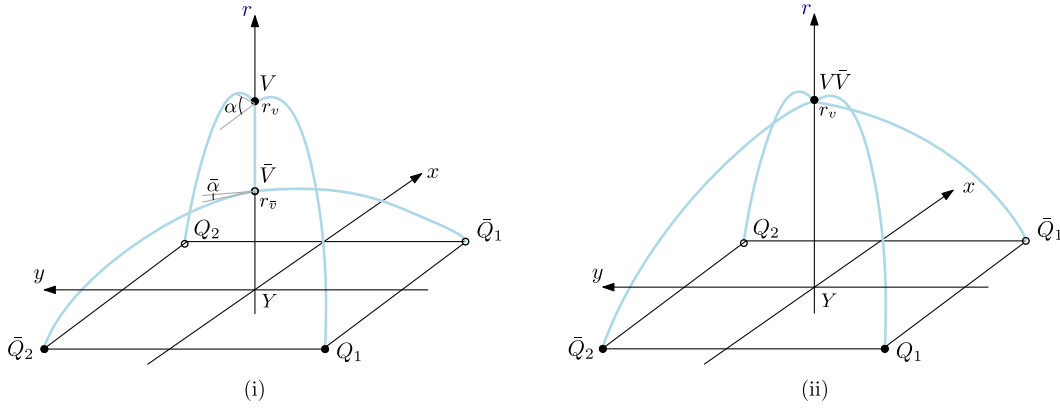


FIG. 8. Basic configurations for type-B ordering in five dimensions. The baryon vertices lie on the r -axis, with Y located at the origin. Because the configuration is symmetric under a permutation of the quarks (antiquarks), we use the shorthand $\alpha_{1,2} = \alpha$ and $\bar{\alpha}_{1,2} = \bar{\alpha}$ for the tangent angles. (i): $r_v > r_{\bar{v}}$. (ii): $r_v = r_{\bar{v}}$. Shown here is a basic configuration (ii), where $\bar{\alpha} \geq 0$.

An important feature of this expression is that the constant term $C^{(d)}$ is independent of η .

IV. STRING CONFIGURATIONS IN FIVE DIMENSIONS FOR TYPE-B ORDERING

The above analysis can be straightforwardly extended to the string configurations with type-B ordering. As before, the quarks (antiquarks) are placed at the vertices of the rectangle shown in Fig. 4, but with Q_2 and $\bar{Q}_2$ interchanged. Obviously, nothing happens with configurations (a) and (d). The main subtlety arises for the analog of configuration (c). While it is puzzling in four dimensions, in five dimensions a possible resolution is to align string 3 along the radial direction.

A. The disconnected configuration (b')

We start with configuration (b'). Here, the quarks and antiquarks are separated by a distance w , and the energy is given by

$$E^{(b')} = 2E_{Q\bar{Q}}(\eta^{-1}\ell). \quad (4.1)$$

The function $E_{Q\bar{Q}}$ is defined parametrically by $\ell = \frac{2\eta}{\sqrt{s}}\mathcal{L}^+(0, \lambda)$ and $E_{Q\bar{Q}} = 2g\sqrt{s}\mathcal{E}^+(0, \lambda) + 2c$.

The asymptotic expansions follow directly from the expressions (3.3) and (3.4) after rescaling $\ell \rightarrow \frac{\ell}{\eta}$. Thus, for small ℓ one finds

$$E^{(b')}(\ell) = -\frac{\alpha^{(b')}}{\ell} + 4c + o(1) \quad \text{with} \quad \alpha^{(b')} = 2\eta\alpha_{Q\bar{Q}}, \quad (4.2)$$

and for large ℓ ,

$$E^{(b')}(\ell) = \frac{2}{\eta}\sigma\ell + 2C_{Q\bar{Q}} + o(1). \quad (4.3)$$

B. The tetraquark configuration (c')

The problem of constructing the tetraquark configuration for type-B ordering is slightly tricky. One way to do so is to imagine “cutting” the strings attached to the quarks in configuration (c), while keeping those attached to the antiquarks intact, and then reconnecting them to the quarks reordered according to type-B ordering. The two resulting configurations are shown in Fig. 8.¹² An unusual feature is that string 3 is stretched along the r -axis. When projected onto the xy -plane, it becomes indistinguishable from configuration (d) in Fig. 2. This makes the tetraquark configuration difficult to visualize in four dimensions.

This tetraquark configuration may be analyzed along the lines of Sec. III by introducing basic configurations and then constructing the tetraquark configuration from them. The total action is again given by the sums of the Nambu-Goto actions and the vertex contributions, as in (3.8) and (3.36).

1. Configuration (i)

The first basic configuration is shown on the left of Fig. 8. We must extremize the corresponding action with respect to $x(r)$, which describes the string profiles, and with respect to r_v and $r_{\bar{v}}$, which specify the position of the vertices. In Appendix B, we outline how this is done for the strings. Varying r_v and $r_{\bar{v}}$ results in the force balance equations of the form (3.10) at V and $\bar{V}$. Due to symmetry, only the r -component is nonzero. Using (B7) and (B24) together with (3.12), we get

$$\begin{aligned} 2 \sin \alpha + 1 - 3k(1 + 4v)e^{-3v} &= 0, \\ 2 \sin \bar{\alpha} - 1 - 3k(1 + 4\bar{v})e^{-3\bar{v}} &= 0, \end{aligned} \quad (4.4)$$

¹²A third configuration is obtained from configuration (ii) by changing the signs of the tangent angles of the strings attached to the antiquarks.

which allow us to express the tangent angles in terms of v and $\bar{v}$, where $\bar{v} = \mathbf{s}r_{\bar{v}}^2$. For $k = -0.083$, α is negative in the interval $[0, 1]$ while $\bar{\alpha}$ is positive.

The form of the expression for ℓ is the same as that in the case of configuration (d). For the energy, however, one must take into account string 3 and the different tangent angle of the strings attached to the antiquarks. Putting all together with the help of (B13), we find

$$\begin{aligned} \ell^{(i)} &= \frac{2}{\sqrt{\mathbf{s}}} \frac{\mathcal{L}^-(\lambda, v)}{\sqrt{1 + \eta^{-2}}}, \\ E^{(i)} &= g\sqrt{\mathbf{s}} \left(2\mathcal{E}^-(\lambda, v) + 2\mathcal{E}^+(\bar{\alpha}, \bar{v}) + \mathcal{Q}(v) \right. \\ &\quad \left. - \mathcal{Q}(\bar{v}) + 3k \frac{e^{-2v}}{\sqrt{v}} + 3k \frac{e^{-2\bar{v}}}{\sqrt{\bar{v}}} \right) + 4c. \end{aligned} \quad (4.5)$$

This is not the whole story. A geometrical constraint arises from the relation $|\bar{Q}_1 Y| = |\bar{Q}_1 Y|$, which yields

$$\mathcal{L}^-(\lambda, v) = \mathcal{L}^+(\bar{\alpha}, \bar{v}). \quad (4.6)$$

Using this, one can express $\bar{v}$ in terms of v and write the energy in parametric form as $E = E^{(i)}(v)$ and $\ell = \ell^{(i)}(v)$.

A numerical analysis shows that, for any given v (except $v = 0$), there is no solution to Eqs. (4.4) and (4.6) satisfying $\bar{v} \leq v$. Thus, the basic configuration (i) does not exist for our parameter values.

2. Configuration (ii)

The second basic configuration is shown on the right of the figure. It may be viewed as configuration (i) with string 3 pinched into a point, so that $\bar{v} = v$ but $\alpha \neq \bar{\alpha}$. The latter distinguishes it from configuration (d).

In this case, the geometrical constraint (4.6) simplifies to

$$\mathcal{L}^-(\lambda, v) = \mathcal{L}^+(\bar{\alpha}, v). \quad (4.7)$$

The force balance equation takes the form of (3.37), with only the r -component nonzero. A simple way to obtain it is to add the equations in (4.4) and then to set $\bar{v} = v$. This gives

$$\sin \alpha + \sin \bar{\alpha} - 3k(1 + 4v)e^{-3v} = 0. \quad (4.8)$$

The expression for the length in terms of v is the same as in (4.5), but that for the energy becomes

$$E^{(ii)} = 2g\sqrt{\mathbf{s}} \left(\mathcal{E}^-(\lambda, v) + \mathcal{E}^+(\bar{\alpha}, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c. \quad (4.9)$$

For a given v , one may determine the tangent angles from the geometrical constraint and the force balance equation.

Numerics, however, shows that there are no solutions with $\alpha \neq \bar{\alpha}$. Thus, the basic configuration (ii) also does not exist.

3. Configuration (iii)

This basic configuration has the same structure as (ii), but with all the tangent angles negative.¹³ Hence, all the formulas can be obtained by replacing $\mathcal{L}^+$ and $\mathcal{E}^+$ with $\mathcal{L}^-$ and $\mathcal{E}^-$, corresponding to negative $\bar{\alpha}$. In this way, from (4.7) we get

$$\mathcal{L}^-(\lambda, v) = \mathcal{L}^-(\bar{\lambda}, v). \quad (4.10)$$

Note that $\bar{\lambda} \neq \lambda$ because $\bar{\alpha} \neq \alpha$. The above constraint becomes trivial at $\bar{\alpha} = \alpha$, as in configuration (d). Similarly, for the energy, we have

$$E^{(iii)} = 2g\sqrt{\mathbf{s}} \left(\mathcal{E}^-(\lambda, v) + \mathcal{E}^-(\bar{\lambda}, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c. \quad (4.11)$$

Meanwhile, the force balance equation (4.8) remains unchanged.

In principle, the geometrical constraint and force balance equation allow one to express the tangent angles in terms of v . A simple numerical analysis, however, show that for a given v the only solution has $\alpha = \bar{\alpha}$. In other words, the result reduces to configuration (d). Thus, the basic configuration (iii) also does not exist. We conclude that for our parameter set configuration (c') does not exist either. One might suspect that this conclusion holds for other geometries as well, but this is not the case; configuration (c') does exist when the diagonals are unequal. A concrete example is presented in Appendix E. This marks an important difference between four- and five-dimensional string models.

We can summarize all this by saying that for our parameter values, only three types of configurations exist in the case of type-B ordering.

V. THE POTENTIALS OF THE $QQ\bar{Q}\bar{Q}$ SYSTEM

We can gain insight into the lowest Born-Oppenheimer potentials by following the approach used in lattice QCD. Consider a model Hamiltonian, which in the general case is an $n \times n$ matrix,

$$\mathcal{H}(\ell) = \begin{pmatrix} E_1 & & \Theta_{ij} \\ & \ddots & \\ \Theta_{ij} & & E_n \end{pmatrix}, \quad (5.1)$$

¹³It may be thought of as an asymmetric version of configuration (d) with $\alpha \neq \bar{\alpha}$.

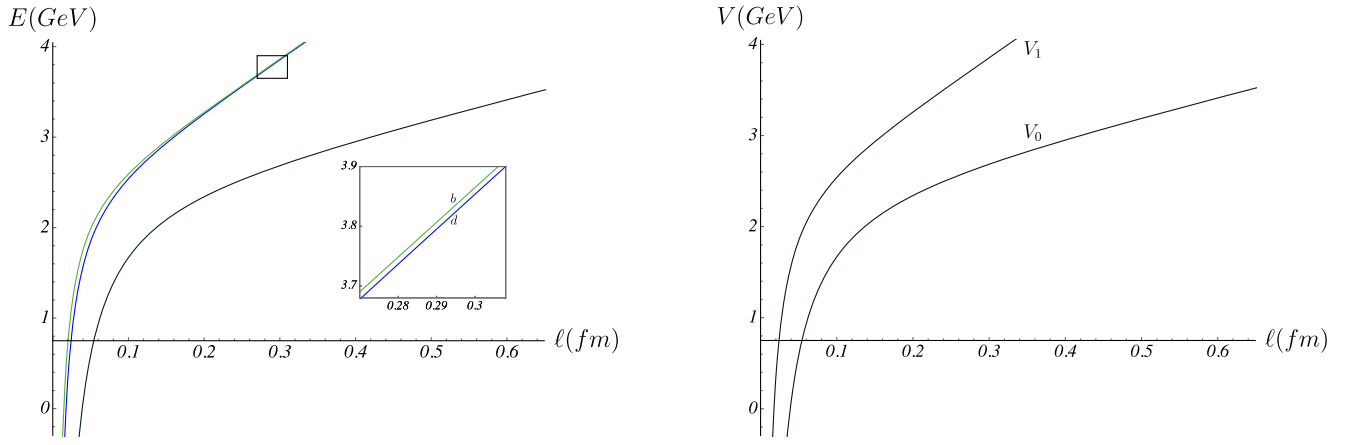


FIG. 9. Various plots at $\eta = 0.4$. Left: The E 's vs ℓ . The black, green, and blue curves correspond respectively to configurations (a), (b), and (d), here and below. Right: The two lowest potentials. The energies of irrelevant configurations are omitted, here and below.

where the diagonal elements represent the energies of string configurations, and the off-diagonal elements describe the strength of mixing between them.¹⁴ The lowest B-O potentials are given by the smallest eigenvalues of $\mathcal{H}$. Note that if Θ_{ij} vanishes for the states i and j , then there is no mixing between them, and the plots of E_i and E_j do not intersect. Although in principle the Hamiltonian can be determined from a correlation matrix in lattice QCD, it remains challenging to compute the off-diagonal elements within the effective string model. Because of this, it is not possible to visualize the exact shape of the potentials. Nevertheless, valuable insight can be gained by treating Θ_{ij} as free parameters, for example in terms of the approximate magnitudes of the Θ values near the transition points. In this case, we may assume that the Θ 's are approximately constant, with values of about 40–60 MeV, as in [10,21].

As a special case of (5.1), consider $n = 2$. The corresponding potentials are given by

$$V_{0,1} = \frac{1}{2}(E_1 + E_2) \mp \sqrt{\frac{1}{4}(E_1 - E_2)^2 + \Theta_{12}^2}. \quad (5.2)$$

Here $V_0 = \min\{E_1, E_2\}$ and $V_1 = \max\{E_1, E_2\}$. This formula is useful for plotting the potentials in the vicinity of intersection points.

A. The case of type-A ordering

Once the string configurations are constructed in Sec. III, we will use them to analyze the two lowest B-O potentials, denoted as V_0 and V_1 . However, the complexity of configuration (c) makes the following analysis somewhat cumbersome.

¹⁴Importantly, the binding energies of disconnected configurations are partially encoded in the Θ 's.

1. $0 < \eta < \frac{1}{\sqrt{3}}$

In this range, the tetraquark configuration (c) does not exist that considerably simplifies the analysis. The ground-state potential is provided by configuration (a) and the first excited one by configuration (d), namely $V_0 = E^{(a)}$ and $V_1 = E^{(d)}$. Thus, if the Θ 's are sufficiently small, we expect the ground state to correspond to a hadronic molecule, as the potential V_0 is described by the disconnected configuration, whereas the first excited potential gives rise to tetraquark states.

As an illustration, Fig. 9 shows the results of numerical computations for $\eta = 0.4$. The plots of $E^{(b)}$ and $E^{(d)}$ do not intersect. Although the linear asymptotics (3.7) and (3.44) have the same slopes, their intercepts differ by about 8 MeV.¹⁵

2. $\eta_p < \eta \leq 0.5840$

As in Appendix D, we skip the very narrow range $\frac{1}{\sqrt{3}} \leq \eta \leq \eta_p$, where the construction of configuration (c) is puzzling, and proceed to the next interval. Here the lower bound η_p is defined by (D1). The ground state potential is still provided by configuration (a), while the first excited potential by two configurations (c) and (d). Thus, $V_0 = E^{(a)}$ and $V_1 = \min\{E^{(c)}, E^{(d)}\}$, where configuration (d) dominates at small ℓ and configuration (c) at large. This does not change the physical interpretation compared to the previous range; the ground state may still correspond to a hadronic molecule, depending on the strength of mixing.

As the next example, consider $\eta = 0.58$. The results are presented in Fig. 10. The plots of $E^{(c)}$ and $E^{(d)}$ are indistinguishable because the difference between them is extremely small. At small ℓ , $E^{(c)}$ is larger than $E^{(d)}$, with the Coulomb coefficients satisfying $\alpha^{(c)}/\alpha^{(d)} = 1 - 1.5 \times 10^{-3}$,

¹⁵This is true for any η .

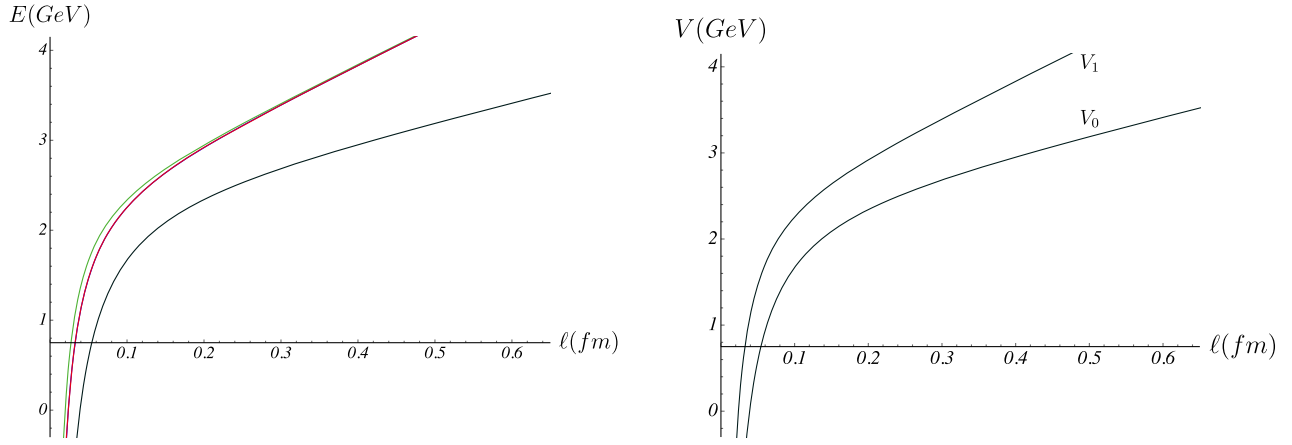


FIG. 10. Various plots at $\eta = 0.58$. Left: The E 's vs ℓ . The red curve corresponds to configuration (c), here and below. Right: Shown here are the two lowest potentials. The approximate formula (5.2) with $\Theta_{12} = 50$ MeV is used to plot the potentials near the transition points, here and below.

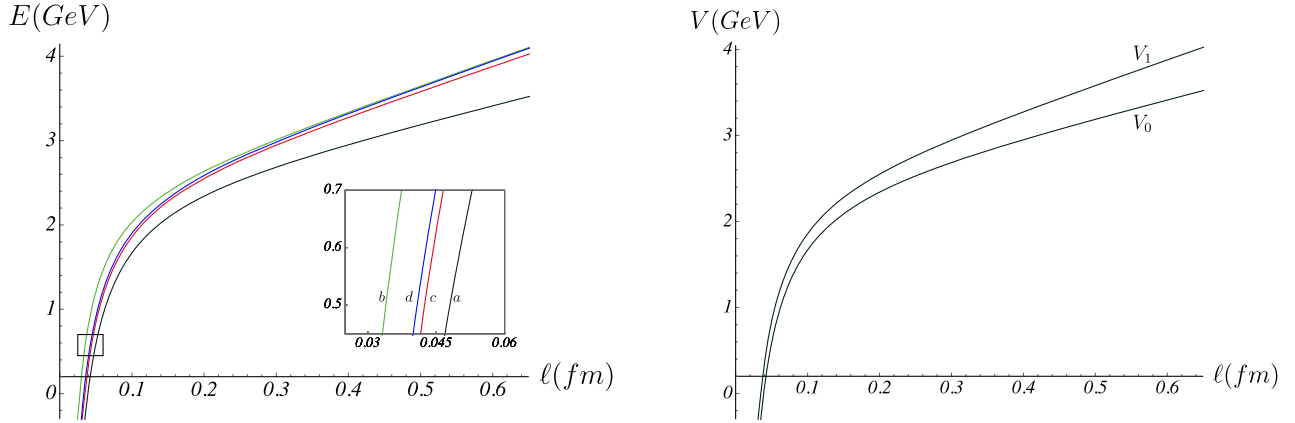


FIG. 11. Various plot at $\eta = 1$. Left: The E 's vs ℓ . Right: The two lowest potentials.

whereas at large ℓ the situation is reversed, with the slope ratio $0.5(\sqrt{3} + \eta)/\sqrt{1 + \eta^2} = 1 - 2 \times 10^{-6}$. The transition between these configurations can be interpreted as pinching (see Fig. 3). To make this more quantitative, we define a critical length by equating the energies

$$E^{(c)}(\ell_c) = E^{(d)}(\ell_c), \quad (5.3)$$

and similarly for other transitions. At sufficiently large ℓ , one may estimate the critical length using the linear asymptotics (3.29) and (3.44). For $\eta = 0.58$ this yields $\ell_c = 13.6$ fm. Before proceeding further, we note that the upper bound of this range corresponds to $\ell_c = 0$, where the Coulomb coefficients become equal.

3. $0.5840 < \eta < 1.17937$

This range is easier to analyze. The ground state potential is given by configuration (a) and the first excited potential

by configuration (c). Thus, $V_0 = E^{(a)}$ and $V_1 = E^{(c)}$. As in the previous ranges, we expect the ground state to be a hadronic molecule.

To illustrate these statements, let us choose $\eta = 1$. Figure 11 shows the resulting plots.

4. $1.17937 \leq \eta \leq 1.434792$

At the lower bound of this range, the Coulomb coefficients of $E^{(a)}$ and $E^{(c)}$ coincide, implying an intersection point between them at $\ell = 0$. The coordinate of the intersection point increases with η . This implies that a transition occurs between these configurations, which can be interpreted as the process of string junction annihilation. As before, we define the critical length by equating the energies, $E^{(a)} = E^{(c)}$. At the upper bound, the Coulomb coefficients of $E^{(a)}$ and $E^{(d)}$ coincide, indicating another intersection at $\ell = 0$. The potentials are therefore described by two configurations (a) and (c) as $V_0 = \min\{E^{(a)}, E^{(c)}\}$

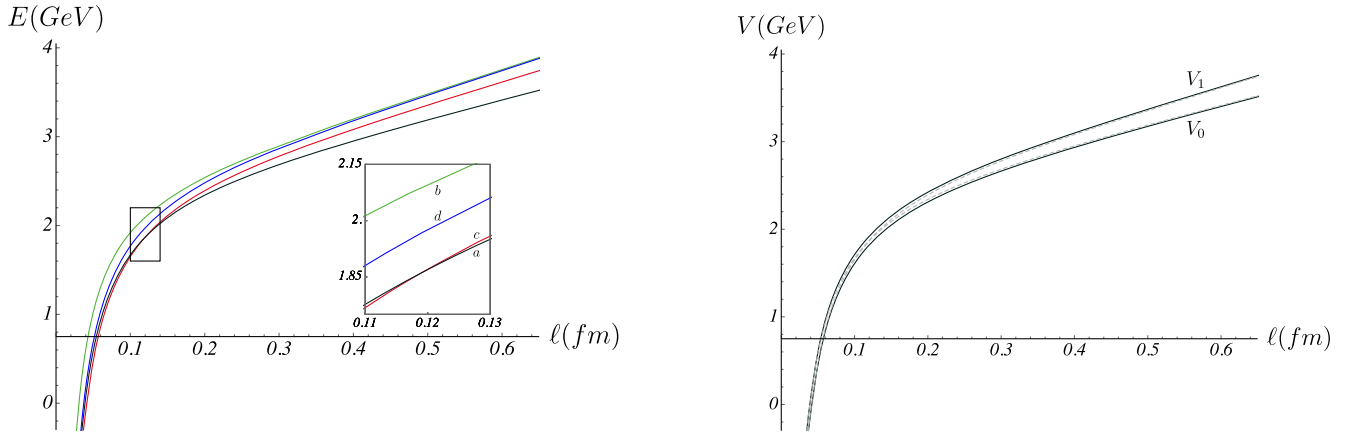


FIG. 12. Various plot at $\eta = 1.3$. Left: The E 's vs ℓ . Right: The two lowest potentials. The dashed curves indicate the plots of the relevant E 's, here and below.

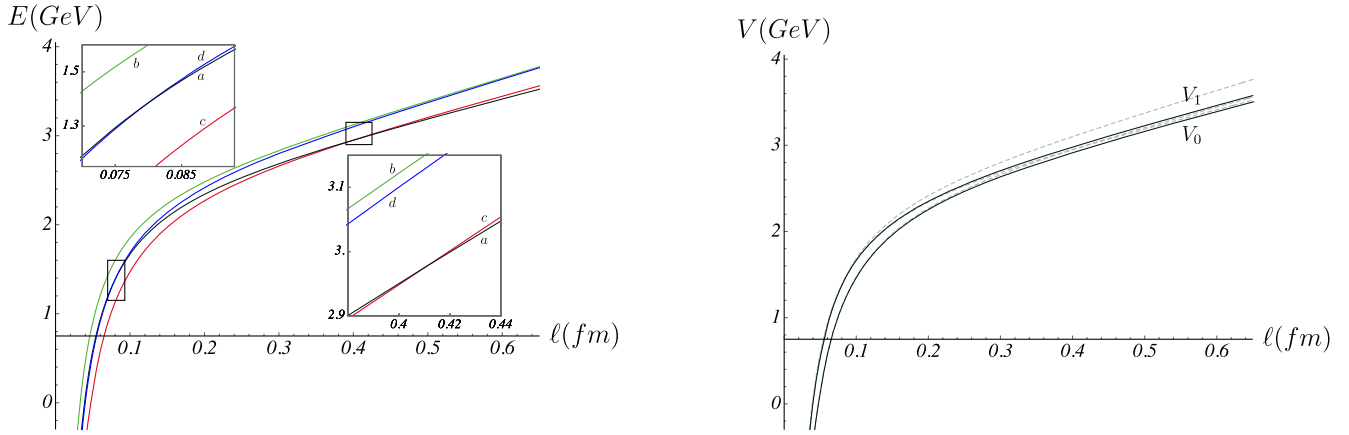


FIG. 13. Various plots at $\eta = 1.6$. Left: The E 's vs ℓ . Right: Sketched here are the two lowest potentials.

and $V_1 = \max\{E^{(a)}, E^{(c)}\}$. This suggests that the ground state is no longer a pure state but a mixed state of a hadronic molecule and a tetraquark state.

As an illustration, Fig. 12 shows the results of numerical computations for $\eta = 1.3$. Here the critical length is $\ell_c = 0.118$ fm. For larger ℓ , configuration (c) has a higher energy, while for smaller ℓ the energy of configuration (a) is higher.

5. $1.434792 \leq \eta \leq \sqrt{3}$

As η increases, the intersection points between $E^{(a)}$ and $E^{(d)}$ as well as between $E^{(a)}$ and $E^{(c)}$ shift to larger values of ℓ . At the upper bound, the slopes of $E^{(a)}$ and $E^{(c)}$ become equal and, as a consequence, their plots no longer intersect at finite ℓ . This leads to a rather entangled pattern of the energies, as illustrated in Fig. 13 on the left. The ground state potential is described by configurations (a) and (c), while the first excited potential involves configurations (a), (c), and (d). Explicitly, $V_0 = \min\{E^{(a)}, E^{(c)}\}$

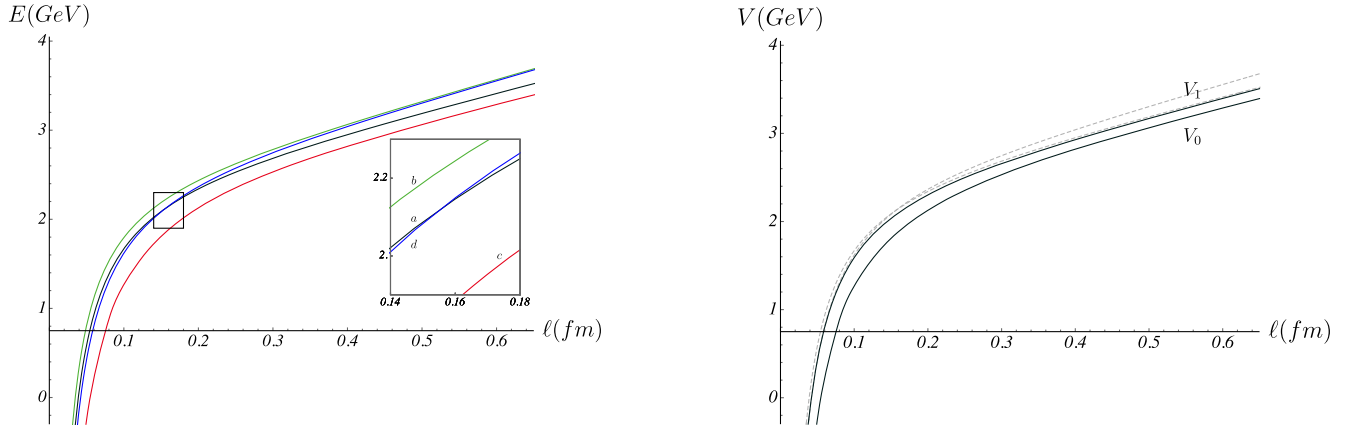
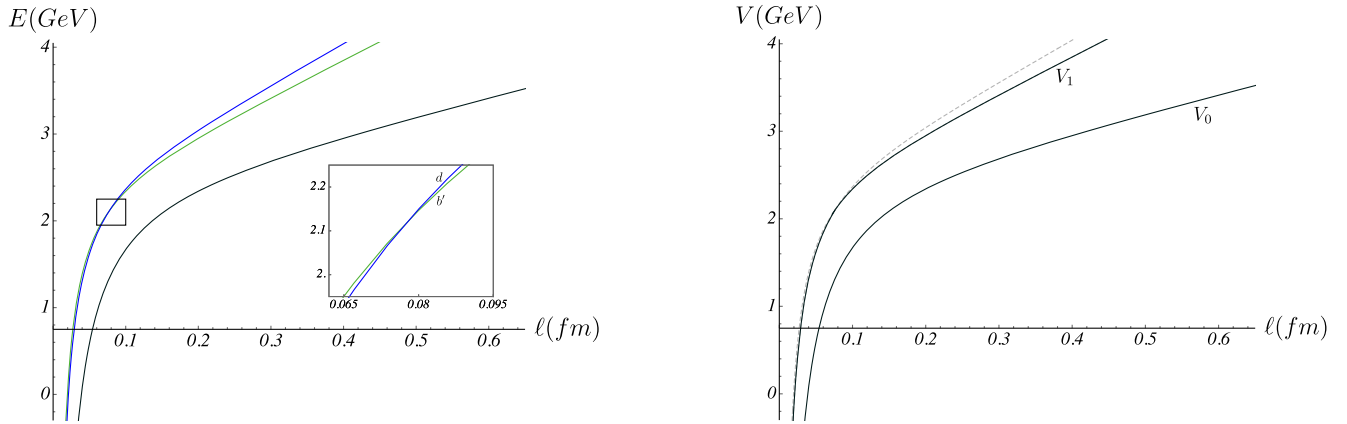
and $V_1 = \min\{E^{(a)}, E^{(c)}, E^{(d)}\}$.¹⁶ Hence, the ground state is expected to be a mixed state, involving both hadronic and tetraquark components.

As an example, consider $\eta = 1.6$. The resulting plots are presented in Fig. 13. The plot of $E^{(a)}$ intersects with $E^{(c)}$ and $E^{(d)}$. We define the corresponding critical length by equating the energies. So, we have $\ell_c = 0.411$ fm and $\ell_c = 0.080$ fm, respectively. Both transitions can be interpreted as string junction annihilation.

6. $\sqrt{3} < \eta$

In this range, $E^{(c)}$ does not intersect with the others, while $E^{(a)}$ and $E^{(d)}$ intersect with each other. Thus, the ground state potential is given simply by $V_0 = E^{(c)}$ and the first excited one by $V_1 = \min\{E^{(a)}, E^{(d)}\}$. Because of this, we expect the ground state to correspond to a tetraquark state.

¹⁶The minimum in V_0 must be taken first.

FIG. 14. Various plots at $\eta = 2$. Left: The E 's vs ℓ . Right: The two lowest potentials.FIG. 15. Various plots at $\eta = 0.5$. Left: The E 's vs ℓ . Right: The two lowest potentials.

We illustrate this with the example $\eta = 2$. Figure 14 shows the corresponding numerical results. The transition between configurations (a) and (d) occurs at $\ell_c = 0.156$ fm and can be interpreted as string junction annihilation. Note that ℓ_c increases with η and tends to infinity in the diquark limit ($\eta \rightarrow \infty$).

B. Type-B ordering

As we saw in Sec. IV, the tetraquark configuration does not exist for this ordering of the quark sources.¹⁷ This, together with an additional symmetry, constitutes a special feature that significantly simplifies the following analysis. On the other hand, the analog of configuration (b), namely configuration (b'), now becomes relevant for the potential V_1 .

1. $0 < \eta \leq 0.6969651$

A numerical analysis shows that in this range, the ground state potential is described by a single configuration, (a),

¹⁷Note that this does not exclude the existence of tetraquark states, since the pinched tetraquark configuration still exists for type-B ordering.

whereas the first excited potential involves two configurations, (b') and (d). In the latter case, the tetraquark (pinched) configuration dominates at small ℓ , while configuration (a) dominates at large ℓ . The transition between them occurs due to string junction annihilation. So, we have $V_0 = E_a$ and $V_1 = \min\{E_{b'}, E_d\}$, and we expect the ground state to be a hadronic molecule.

To illustrate this, let us choose $\eta = 0.5$. The corresponding plots are shown in Fig. 15. The critical length characterizing the transition between configuration (b') and (d) is given by $\ell_c = 0.078$ fm.

2. $0.6969651 < \eta < 1$

As η increases, the critical length ℓ_c decreases and eventually vanishes at $\eta = 0.6969651$, where the Coulomb coefficients $\alpha^{(b')}$ and $\alpha^{(d)}$ become equal. This defines the lower bound. For larger values of η , $E^{(b')}$ and $E^{(d)}$ no longer intersect. Hence, $V_0 = E_a$ and $V_1 = E_{b'}$. In this case, the ground state is also expected to correspond to a hadronic molecule.

As a typical example, consider $\eta = 0.85$. Figure 16 shows the resulting plots.

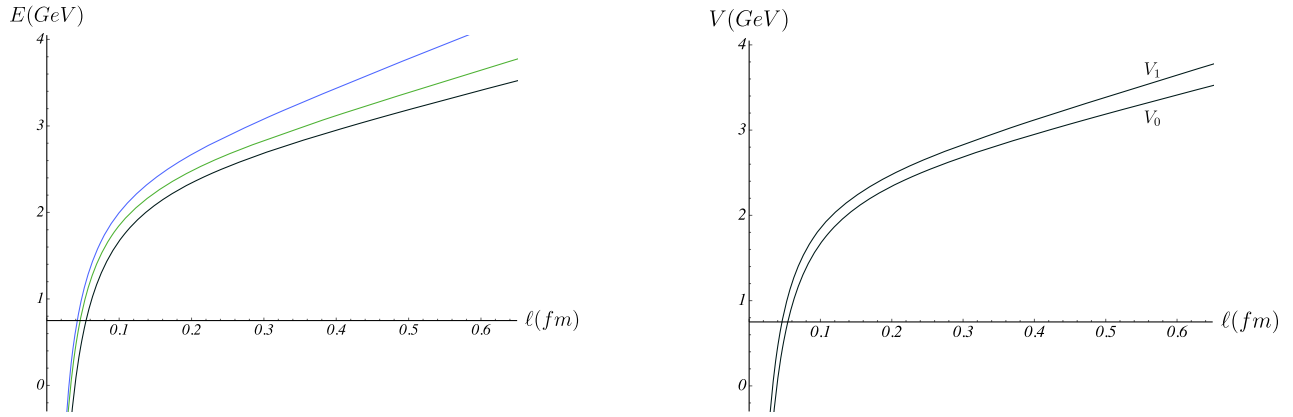


FIG. 16. Various plots at $\eta = 0.85$. Left: The E 's vs ℓ . Right: The two lowest potentials.

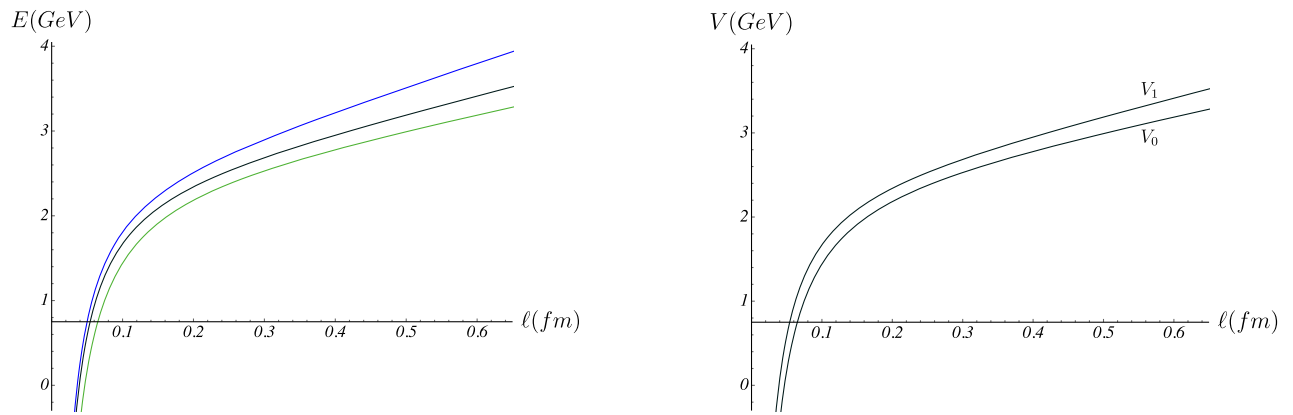


FIG. 17. Various plots at $\eta = 1.2$. Left: The E 's vs ℓ . Right: The two lowest potentials.

3. $1 \leq \eta$

At $\eta = 1$, the rectangle becomes a square, and hence configurations (a) and (b') are indistinguishable. The potentials are then simply given by $V_0 = E_a$ and $V_1 = E_d$. Thus, the ground state remains a hadronic molecule.

The behavior of the potentials for larger values of η can be understood purely on symmetry grounds. The key point is that for type-B ordering there is an additional symmetry which exchanges the length and the width. It acts on η and ℓ as $\eta \rightarrow \eta^{-1}$ and $\ell \rightarrow \eta^{-1}\ell$, with a fixed point at $\eta = 1$. In terms of configurations, the symmetry exchanges the meson configurations $E^{(a)}$ and $E^{(b')}$.¹⁸

In this description, the range $0.6969651 < \eta < 1$ is mapped to $1 < \eta < 1.43479$. The corresponding potentials are simply $V_0 = E_{b'}$ and $V_1 = E_a$. Therefore, as before,

¹⁸In the literature, such an exchange is referred to as a flip-flop of strings [22]. It is, in fact, a special case of string reconnection (see Fig. 3). Although the flip-flop is hidden when the constraint $\ell/w = \text{const}$ is imposed, it becomes manifest under the constraint $w = \text{const}$ and has been observed in lattice simulations (see, e.g., [23] and reference therein). We will return to this issue in [20].

the ground state is expected to be a hadronic molecule. For illustration, the resulting plots for $\eta = 1.2$ are shown in Fig. 17.

Similarly, the range $\eta < 0.6969651$ is mapped to the range $1.43479 < \eta$. The potentials are then written as $V_0 = E^{(b')}$ and $V_1 = \min\{E^{(a)}, E^{(d)}\}$. For V_1 , the tetraquark configuration dominates at small ℓ , as in the range $\eta < 0.6969651$. The ground state is again expected to correspond to a hadronic molecule. As an example, we choose $\eta = 2$. The corresponding plots are shown in Fig. 18. The transition between configurations (a) and (d) occurs at $\ell_c = 0.156$ fm.

VI. THE IR LIMIT OF SOME STRING CONFIGURATIONS AND STEINER TREES

Consider now the general case in which N quark sources are placed at arbitrary points on the boundary of the five-dimensional space. We impose several restrictions on a string configuration connecting these sources. First, we assume that the configuration is connected. Second, it is tree, meaning that it contains no string loops. We also assume that in the IR limit, when the quark sources are infinitely separated, its projection onto the boundary forms

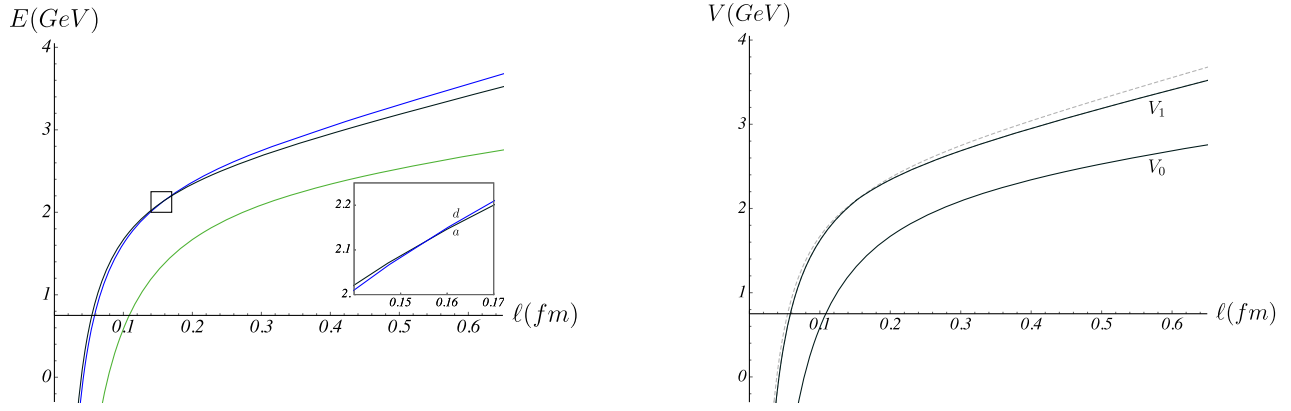


FIG. 18. Various plots at $\eta = 2$. Left: The E 's vs ℓ . Right: The two lowest potentials.

a regular Steiner tree whose Steiner points do not coincide either with each other or with the sources. The physical meaning of the last assumption is straightforward; in such a limit, all strings become infinitely long, so that baryon vertices are infinitely separated from one another and from the quark sources.

Within the present model, the asymptotic expansion of the energy can be easily obtained. The leading term comes from the strings. Summing over them gives

$$E_{\text{NQ}} = \sigma L_{\text{min}} + O(1), \quad (6.1)$$

where L_{min} is the length of the Steiner tree. Importantly, the string tension is universal; it is the same for all the strings, as follows from the asymptotic formulas (B23) and (B27). This is the well-known asymptotic behavior. Our goal is to determine the leading correction to it.¹⁹

Before getting to arbitrary N , recall that for $N = 3$ the desired correction was computed in [15,16]. Explicitly,

$$C_{3Q} = 3c - g\sqrt{s} \left(3\mathcal{I}(v_1) - 3k \frac{e^{-2v_1}}{\sqrt{v_1}} \right). \quad (6.2)$$

Here v_1 is a solution to Eq. (3.23), and c is the normalization constant, as before. Two important remarks are in order. First, (6.2) is meaningful only if none of the triangle's angles formed by the quarks exceeds $\frac{2}{3}\pi$. In other words, the Steiner tree is regular.²⁰ Second, the correction is universal in the sense that it does not depend on the triangle's angles.

The case $N = 4$ may be analyzed in a similar way. The corresponding string configuration resembles those of Fig. 6, but with the quark sources placed at arbitrary points on the boundary. Consider the force balance equation (3.10) at V . It is convenient to decompose the vectors $\mathbf{e}$ as $\mathbf{e} = (\vec{e}, e_r)$,

where e_r is the radial component, and the remaining x, y , and z components are grouped into $\vec{e}$. Then $|\vec{e}| = g w(r_v) \cos \alpha$ and $e_r = -g w(r_v) \sin \alpha$, as follows from (B7) and (B24). The force balance equation can then be written in component form as²¹

$$\begin{aligned} \vec{e}_1 + \vec{e}_2 + \vec{e}_3 &= 0, \\ \sin \alpha_1 + \sin \alpha_2 - \sin \alpha_3 &= 3k(1 + 4v)e^{-3v}. \end{aligned} \quad (6.3)$$

As explained in Appendixes A and B, string 1 becomes infinitely long as its λ parameter approaches 1. In this IR limit, its tangent angle at the baryon vertex is negative and given by $\cos \alpha_1 = v e^{1-v}$ [see Eq. (B15)]. The same argument applies to the two remaining strings, yielding $\alpha_1 = \alpha_2 = -\alpha_3$. Consequently, all the vectors $\vec{e}$ have the same length. This implies that the angles between them must be $\frac{2}{3}\pi$ to satisfy the first equation in (6.3), and therefore the point Y coincides with the Steiner point S . Moreover, from the second equation it follows that $\sin \alpha_1 = k(1 + 4v)e^{-3v}$. Combining this with the above expression for $\cos \alpha_1$ leads directly to Eq. (3.23), whose solution is v_1 . Clearly, the same argument can be applied to the vertex $\bar{V}$ to show that its projection $\bar{Y}$ coincides with the second Steiner point $\bar{S}$. The important consistency condition is that both vertices are equidistant from the boundary, namely at $r = \sqrt{v_1/s}$. The asymptotic expression for the energy of the four external strings (ending on the quark sources) is given by (B22), while that for the internal string (ending only on the vertices) is given by (B27). So after summing over the strings and adding the vertex contributions, we arrive at

$$\begin{aligned} E_{4Q} &= \sigma L_{\text{min}} + C_{4Q} + o(1), \quad \text{with} \\ C_{4Q} &= 4c - g\sqrt{s} \left(4\mathcal{I}(v_1) + \mathcal{J}(v_1) - 6k \frac{e^{-2v_1}}{\sqrt{v_1}} \right), \end{aligned} \quad (6.4)$$

¹⁹We will not discuss the leading term in detail here. There exists a vast literature on the Steiner tree problem. See, e.g., the book [24] and references therein.

²⁰In this case, the Fermat point of the triangle does not coincide with any of its vertices.

²¹String 3 is internal, as it is stretched between the baryon vertices. Its tangent angle is always positive (see Appendix B).

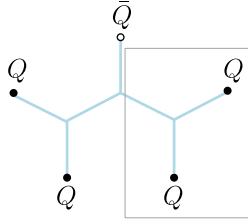


FIG. 19. A schematic illustration of a regular Steiner tree for five points corresponding to the quark positions. It can be obtained from the $N = 4$ tree by cutting one external string together with the quark source at its endpoint and then attaching a giant diquark, the object shown in the box.

which includes the universal constant term. The expression (3.29) obtained in Sec. III for the rectangular geometry is a special case of (6.4).

Technically, the case $N = 5$ differs from the previous one only by the numbers of strings and vertices. Using essentially the same arguments, one can show that in the IR limit the projections of the vertices on the boundary correspond to the Steiner points, and all the vertices lie at the same distance from the boundary. A typical Steiner tree is sketched in Fig. 19. In this case, summing over the strings and the vertices gives

$$E_{5Q} = \sigma L_{\min} + C_{5Q} + o(1), \quad \text{with} \\ C_{5Q} = 5c - g\sqrt{s} \left(5\mathcal{I}(v_1) + 2\mathcal{J}(v_1) - 9k \frac{e^{-2v_1}}{\sqrt{v_1}} \right), \quad (6.5)$$

again with a universal constant term. Note that C_{5Q} is obtained from C_{4Q} by adding the contributions of one external string, one internal string, and one vertex, according to the procedure illustrated in the figure.

We are now in a position to write down the formula for arbitrary N ,

$$E_{NQ} = \sigma L_{\min} + C_{NQ} + o(1), \quad \text{with} \\ C_{NQ} = Nc - g\sqrt{s} \left(N\mathcal{I}(v_1) + (N-3)\mathcal{J}(v_1) - 3(N-2)k \frac{e^{-2v_1}}{\sqrt{v_1}} \right), \quad N \geq 2. \quad (6.6)$$

This is the main result of this section. Note that the above formula applies to the $QQ\bar{Q}$ system, where C_{2Q} reduces to $C_{Q\bar{Q}}$ defined in (C3). However, it should be stressed that the result is valid for a specific class of string configurations rather than for the ground-state potentials of the multi-quark systems.

We conclude our discussion with a few remarks:

- (1) In [15] it was noted that the constant terms appearing in the expansions of the energy of the QQQ system at small and large quark separations are different. More precisely, the constant term in the IR limit is

smaller than that in the UV. The same feature holds for the string configurations considered above, with Nc being the constant term in the UV. It is noteworthy that the difference between these constants is well-defined (scheme independent), and can be straightforwardly estimated using (6.6). For a few small values of N , we get $3c - C_{3Q} = 267$ MeV, $4c - C_{4Q} = 359$ MeV, and $5c - C_{5Q} = 450$ MeV;

- (2) If the normalization constant c is chosen to be sufficiently large so that the first term dominates in C_{NQ} , then

$$\frac{C_{NQ}}{N} \simeq \text{const}, \quad N \geq 2. \quad (6.7)$$

For $N \leq 5$, such a relation has been observed on the lattice [23]. Since the parameter values used here were obtained by fitting the lattice data for the three-quark potential [15], it is worth making some estimates. We obtain $\frac{C_{Q\bar{Q}}}{2} = 0.535$ GeV, $\frac{C_{3Q}}{3} = \frac{C_{4Q}}{4} = \frac{C_{5Q}}{5} = 0.533$ GeV, and $\frac{C_{NQ}}{N} = 0.531$ GeV as $N \rightarrow \infty$; and

- (3) Another special choice is to set $c = g\sqrt{s}I_0$. In this case, the following relations hold²²:

$$C_{Q\bar{Q}} = 0, \quad C_{3Q} = 3g\sqrt{s} \left(k \frac{e^{-2v_1}}{\sqrt{v_1}} - \frac{1}{2}\mathcal{J}(v_1) \right), \\ C_{NQ} = (N-2)C_{3Q}. \quad (6.8)$$

With our parameter values, we find $c = 88$ MeV and $C_{3Q} = -4$ MeV;

- (4) There are other IR limits in which some of the strings remain of finite length or even shrink to points. In such limits, the corresponding Steiner points coincide with each other or with given points (quark positions), as discussed in Sec. III and in [16]. These limits are more complex and generally do not yield universal constant terms.

VII. CONCLUSIONS

We conclude our discussion of the fully tetraquark system with several remarks:

- (1) The results of Sec. V suggest that, depending on the geometry and quark ordering, the ground state of the $QQ\bar{Q}\bar{Q}$ system may be a hadronic molecule, a tetraquark state, or a superposition of the two;
- (2) In general, the tetraquark and pinched tetraquark configurations are distinct and may coexist. In terms of string interactions in ten dimensions, a transition between them can be interpreted as the creation or decay of a brane-antibrane bound state. For the type-B ordering, configurations (c') and (d')

²²The relation $C_{4Q} = 2C_{3Q}$ was suggested by Z. Komargodski.

are indistinguishable in four dimensions but become distinguishable in five dimensions, as explained in Appendix E. This is puzzling and warrants further study. Hopefully, this will shed light on the five-dimensional origin of QCD strings [25];

- (3) For the class of the string configurations discussed in Sec. VI, we can deduce from (6.6) that

$$\frac{C_{NQ}}{N} - \frac{C_{Q\bar{Q}}}{2} = 3 \frac{N-2}{N} \left(\frac{C_{3Q}}{3} - \frac{C_{Q\bar{Q}}}{2} \right), \quad N \geq 3. \quad (7.1)$$

This is a general relation which is well-defined (i.e., scheme independent). It is obviously satisfied if $\frac{C_{NQ}}{N} = \text{const}$, and (6.8) is its special case. It will be interesting to test this prediction through high precision computer simulations;

- (4) In general, for the rectangular geometry the potentials are complicated functions of both length and width. In this work we imposed the constraint $\ell = \eta w$ to simplify the analysis. Another option, used in lattice QCD, is to fix $w = \text{const}$. This is convenient for studying the diquark limit ($\eta \rightarrow \infty$) but is less suitable for exploring the UV and IR limits. We plan to return to this issue in future work [20]; and
- (5) In the light of the recent discovery of the $T_{c\bar{c}c\bar{c}}(6600)$ and $T_{c\bar{c}c\bar{c}}(6900)$ mesons at the LHC, understanding the properties of the tetraquark systems has become a matter of primary importance. The $QQ\bar{Q}\bar{Q}$ system exhibits a rich and complex landscape of physics, with many open questions yet to be answered. Advancing our theoretical understanding and connecting it to hadron spectroscopy will require a concerted effort from the high-energy

physics community. We hope this study provides useful insights and motivation for future research.

ACKNOWLEDGMENTS

We would like to thank Zohar Komargodski for discussions concerning this topic. This work was conducted as part of Program FFWR-2024-0011 at the Landau Institute.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: NOTATION AND USEFUL FORMULAS

Throughout this paper, we denote heavy quarks (antiquarks) by $Q(\bar{Q})$ and baryon (antibaryon) vertices by $V(\bar{V})$. Heavy quark sources are always located on the boundary of the five-dimensional space at $r = 0$, while vertices are located in its interior at $r = r_v(r_{\bar{v}})$. For convenience, we introduce the dimensionless variables $v = \mathbf{sr}_v^2$ and $\bar{v} = \mathbf{sr}_{\bar{v}}^2$, which take values in the interval $[0, 1]$ and quantify the proximity of these objects to the soft wall, located at 1 in such units.

To present the resulting formulas in a compact form, we make use of the basic functions introduced in [26], together with several newly defined ones:

- (i) The function $\mathcal{L}^+$ is defined as

$$\mathcal{L}^+(\alpha, x) = \cos \alpha \sqrt{x} \int_0^1 du u^2 e^{x(1-u^2)} [1 - \cos^2 \alpha u^4 e^{2x(1-u^2)}]^{-\frac{1}{2}}, \quad 0 \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq x \leq 1. \quad (A1)$$

This non-negative function vanishes at $\alpha = \frac{\pi}{2}$ or $x = 0$. For small x , if α tends to α_0 , it behaves as

$$\mathcal{L}^+(\alpha, x) = \sqrt{x} (\mathcal{L}_0^+(\alpha_0) + O(x)), \quad \text{with} \quad \mathcal{L}_0^+(\alpha_0) = \frac{1}{4} \cos^{-\frac{1}{2}} \alpha_0 B\left(\cos^2 \alpha_0; \frac{3}{4}, \frac{1}{2}\right). \quad (A2)$$

Here $B(z; a, b)$ denotes the incomplete beta function. At $(0, 1)$, $\mathcal{L}^+$ develops a logarithmic singularity,

$$\mathcal{L}^+(\alpha, x) = -\frac{1}{2} \ln(1-x) + O(1), \quad \text{as } \alpha \rightarrow 0; \quad (A3)$$

- (ii) The $\mathcal{L}^-$ function is given by

$$\mathcal{L}^-(y, x) = \sqrt{y} \left(\int_0^1 du u^2 e^{y(1-u^2)} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}} + \int_{\sqrt{\frac{x}{y}}}^1 du u^2 e^{y(1-u^2)} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}} \right), \quad 0 \leq x \leq y \leq 1. \quad (A4)$$

This function is non-negative and vanishes at the origin. For $y = x/\rho$, with non-zero ρ as $x \rightarrow 0$, its small- ℓ behavior is

$$\mathcal{L}^-(y, x) = \sqrt{x}(\mathcal{L}_0^-(\rho) + O(x)), \quad \text{with} \quad \mathcal{L}_0^-(\rho) = \frac{1}{4}\rho^{-\frac{1}{2}}\mathcal{B}\left(1 - \rho^2; \frac{1}{2}, \frac{3}{4}\right). \quad (\text{A5})$$

Here $\mathcal{B}(z; a, b) = B(a, b) + B(z; a, b)$. At $y = 1$ it exhibits a logarithmic singularity,

$$\mathcal{L}^-(y, x) = -\ln(1 - y) + O(1), \quad \text{at fixed } x. \quad (\text{A6})$$

The $\mathcal{L}^\pm$ functions are related by $\mathcal{L}^+(0, x) = \mathcal{L}^-(x, x)$;

(iii) The $\mathcal{L}$ function is defined as

$$\mathcal{L}(y, x) = 2\sqrt{y} \int_{\sqrt{y}}^1 du u^2 e^{y(1-u^2)} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}}, \quad 0 < x \leq y. \quad (\text{A7})$$

It is non-negative and vanishes at $y = 0$ and $x = y$. For $y = x/\rho$, with nonzero ρ as $x \rightarrow 0$, it behaves as

$$\mathcal{L}(y, x) = \sqrt{x}(\mathcal{L}_0(\rho) + O(1)), \quad \text{with} \quad \mathcal{L}_0(\rho) = \frac{1}{2}\rho^{-\frac{1}{2}}\mathcal{B}\left(1 - \rho^2; \frac{1}{2}, \frac{3}{4}\right). \quad (\text{A8})$$

It also develops a logarithmic singularity at $y = 1$

$$\mathcal{L}(y, x) = -\ln(1 - y) + O(1), \quad \text{at fixed } x. \quad (\text{A9})$$

The $\mathcal{L}$ functions satisfy

$$\mathcal{L}(y, x) = \mathcal{L}^-(y, x) - \mathcal{L}^+(\alpha, x), \quad (\text{A10})$$

valid if $\frac{e^y}{y} = \cos \alpha \frac{e^x}{x}$;

(iv) The function $\mathcal{E}^+$ is given by

$$\mathcal{E}^+(\alpha, x) = \frac{1}{\sqrt{x}} \int_0^1 \frac{du}{u^2} (e^{xu^2} [1 - \cos^2 \alpha u^4 e^{2x(1-u^2)}]^{-\frac{1}{2}} - 1 - u^2), \quad 0 \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq x \leq 1. \quad (\text{A11})$$

It is singular at $x = 0$ and at $(0,1)$. If $\alpha \rightarrow \alpha_0$ as $x \rightarrow 0$, then

$$\mathcal{E}^+(\alpha, x) = \frac{1}{\sqrt{x}} (\mathcal{E}_0^+(\alpha_0) + O(x)), \quad \text{with} \quad \mathcal{E}_0^+(\alpha_0) = \frac{1}{4} \cos^{\frac{1}{2}} \alpha_0 \mathcal{B}\left(\cos^2 \alpha_0; -\frac{1}{4}, \frac{1}{2}\right). \quad (\text{A12})$$

Near $x = 1$, $\alpha \rightarrow 0$, it behaves as

$$\mathcal{E}^+(\alpha, x) = -\frac{1}{2} e \ln(1 - x) + O(1); \quad (\text{A13})$$

(v) The $\mathcal{E}^-$ function is defined as

$$\mathcal{E}^-(y, x) = \frac{1}{\sqrt{y}} \left(\int_0^1 \frac{du}{u^2} (e^{yu^2} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}} - 1 - u^2) + \int_{\sqrt{y}}^1 \frac{du}{u^2} e^{yu^2} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}} \right), \quad 0 \leq x \leq y \leq 1. \quad (\text{A14})$$

This function is singular at $(0,0)$ and at $y = 1$. If $y = x/\rho$ with nonzero ρ as $x \rightarrow 0$, then

$$\mathcal{E}^-(y, x) = \frac{1}{\sqrt{x}} (\mathcal{E}_0^-(\rho) + O(x)), \quad \text{with} \quad \mathcal{E}_0^-(\rho) = \frac{1}{4} \rho^{\frac{1}{2}} \mathcal{B}\left(1 - \rho^2; \frac{1}{2}, -\frac{1}{4}\right). \quad (\text{A15})$$

Near $y = 1$ with fixed x , it is singular,

$$\mathcal{E}^-(y, x) = -e \ln(1 - y) + O(1). \quad (\text{A16})$$

Note that the relation $\mathcal{E}^+(0, x) = \mathcal{E}^-(x, x)$ holds;

(vi) The $\mathcal{E}$ function is defined by

$$\mathcal{E}(y, x) = \frac{2}{\sqrt{y}} \int_{\sqrt{x}}^1 \frac{du}{u^2} e^{yu^2} [1 - u^4 e^{2y(1-u^2)}]^{-\frac{1}{2}},$$

$$0 < x \leq y. \quad (\text{A17})$$

It is singular at the origin. For $y = x/\rho$ with nonzero ρ as $x \rightarrow 0$, it behaves as

$$\mathcal{E}(y, x) = \frac{1}{\sqrt{x}} (\mathcal{E}_0(\rho) + o(x)), \quad \text{with}$$

$$\mathcal{E}_0(\rho) = \frac{1}{2} \rho^{\frac{1}{2}} B\left(1 - \rho^2; \frac{1}{2}, -\frac{1}{4}\right). \quad (\text{A18})$$

It also has a logarithmic singularity at $y = 1$,

$$\mathcal{E}(y, x) = -e \ln(1 - y) + O(1), \quad \text{with fixed } x. \quad (\text{A19})$$

For the $\mathcal{E}$ functions, the analog of (A10) is

$$\mathcal{E}(y, x) = \mathcal{E}^-(y, x) - \mathcal{E}^+(\alpha, x), \quad (\text{A20})$$

which is also valid if $\frac{e^y}{y} = \cos \alpha \frac{e^x}{x}$;

(vii) The $\mathcal{Q}$ function is given by

$$\mathcal{Q}(x) = \sqrt{\pi} \operatorname{erfi}(\sqrt{x}) - \frac{e^x}{\sqrt{x}}, \quad 0 < x \leq 1, \quad (\text{A21})$$

where $\operatorname{erfi}(x)$ stands for the imaginary error function. This is a special case of $\mathcal{E}^+$ with $\alpha = \frac{\pi}{2}$. For small x ,

$$\mathcal{Q}(x) = -\frac{1}{\sqrt{x}} + \sqrt{x} + O(x^{\frac{3}{2}}); \quad (\text{A22})$$

(viii) The $\mathcal{I}$ function is given by

$$\mathcal{I}(x) = I_0 - \int_{\sqrt{x}}^1 \frac{du}{u^2} e^{u^2} [1 - u^4 e^{2(1-u^2)}]^{-\frac{1}{2}}, \quad \text{with } I_0 = \int_0^1 \frac{du}{u^2} (1 + u^2 - e^{u^2} [1 - u^4 e^{2(1-u^2)}]^{-\frac{1}{2}}), \quad 0 < x \leq 1. \quad (\text{A23})$$

In particular, $\mathcal{I}(1) = I_0$, and numerically $I_0 = 0.751$. This function is related to the $\mathcal{L}^-$ and $\mathcal{E}^-$ functions as

$$\mathcal{I}(x) = e\mathcal{L}^-(y, x) - \mathcal{E}^-(y, x) \quad \text{as } y \rightarrow 1 \quad \text{at fixed } x; \quad (\text{A24})$$

(ix) The $\mathcal{J}$ function is defined by

$$\mathcal{J}(x) = 2(\mathcal{I}(x) - I_0) = -2 \int_{\sqrt{x}}^1 \frac{du}{u^2} e^{u^2} [1 - u^4 e^{2(1-u^2)}]^{-\frac{1}{2}}, \quad 0 < x \leq 1. \quad (\text{A25})$$

Analogous to $\mathcal{I}$, it satisfies the relation

$$\mathcal{J}(x) = e\mathcal{L}(y, x) - \mathcal{E}(y, x) \quad \text{as } y \rightarrow 1 \quad \text{at fixed } x. \quad (\text{A26})$$

APPENDIX B: STATIC NAMBU-GOTO STRINGS WITH FIXED ENDPOINTS

The aim here is to describe some facts about static Nambu-Goto strings in the curved geometry (2.1), which are essential for constructing the string configurations discussed in Secs. III and IV. Some of this material is not new and can be found in [15], whose notation we largely adopt.

1. A static string with one endpoint on the boundary

Consider a static string stretched between two fixed points in the xr -plane, as shown in Fig. 20.

In static gauge, where $\xi^1 = t$ and $\xi^2 = x$, the boundary conditions for $r(x)$ describing the string profile are

$$r(0) = 0, \quad r(\ell) = r_v. \quad (\text{B1})$$

The Nambu-Goto action (2.2) takes the form

$$S = Tg \int_0^\ell dx w(r) \sqrt{1 + (\partial_x r)^2}, \quad w(r) = \frac{e^{sr^2}}{r^2}. \quad (\text{B2})$$

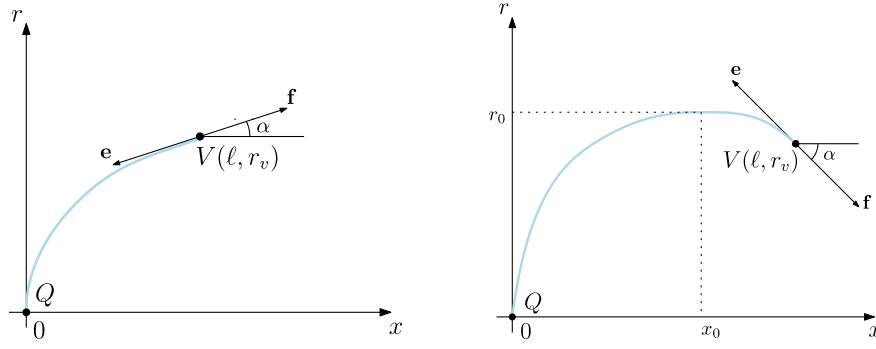


FIG. 20. A string stretched between two points, $Q(0, 0)$ and $V(\ell, r_v)$, where $r_v < 1/\sqrt{s}$. The tangent angle at V is denoted by α , and the arrows indicate the forces acting on this point. The left and right panels show the cases for $\alpha \geq 0$ and $\alpha \leq 0$, respectively.

For convenience, we use the shorthand notation $\mathbf{g} = \frac{R^2}{2\pi\alpha'}$, $T = \int dt$, and $\partial_x r = \frac{dr}{dx}$. Since the integrand does not depend explicitly on x , the equation of motion admits the first integral,

$$I = \frac{w(r)}{\sqrt{1 + (\partial_x r)^2}}. \quad (\text{B3})$$

At point V , it can be written as

$$I = w(r_v) \cos \alpha, \quad (\text{B4})$$

where $\tan \alpha = \partial_x r|_{x=\ell}$ and $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

In this paper, point Q is associated with an infinitely heavy quark, while point V with a baryon vertex. It is therefore natural to consider the forces acting on V in order to maintain equilibrium. The force balance equation is simply,

$$\mathbf{e} + \mathbf{f} = 0, \quad (\text{B5})$$

where $\mathbf{e}$ is the string tension and $\mathbf{f}$ is an external force (see the figure). It is straightforward to compute the r -component of $\mathbf{e}$, which arises from the boundary term in the variation of the action. Indeed, with $\delta r|_{x=\ell} = \delta r_v$, we find that

$$\delta S = Tg \frac{w(r)\partial_x r}{\sqrt{1 + (\partial_x r)^2}} \delta r_v, \quad (\text{B6})$$

which implies $\mathbf{e}_r = -T^{-1}\delta S/\delta r_v = -g w(r_v) \sin \alpha$. Similarly, in static gauge $\xi^1 = t$ and $\xi^2 = r$, the boundary term provides the x -component: $\mathbf{e}_x = -g w(r_v) \cos \alpha$.²³ Putting both components together, we have

$$\mathbf{e} = -g w(r_v) (\cos \alpha, \sin \alpha). \quad (\text{B7})$$

A simple but important observation is that the magnitude of the tension is determined by a radial coordinate, specifically $|\mathbf{e}| = g w(r)$.

In general, the tangent angle α may be positive or negative. For $\alpha \geq 0$, the function $r(x)$ describing a string profile monotonically increases on the interval $[0, \ell]$. Conversely, for $\alpha < 0$, the situation is more intricate; $r(x)$ increases on $[0, x_0]$ and decreases on $[x_0, \ell]$, reaching a local maximum at $x = x_0$. Both cases are depicted in Fig. 20.

Let us examine these cases more systematically, starting with $\alpha \geq 0$. First, we express I in terms of α and r_v . This yields the differential equation $w(r_v) \cos \alpha = w(r)/\sqrt{1 + (\partial_x r)^2}$, which can be integrated over x and r . Using the boundary conditions (B1), we obtain

$$\begin{aligned} \ell &= \cos \alpha \sqrt{\frac{v}{s}} \int_0^1 du u^2 e^{v(1-u^2)} (1 - \cos^2 \alpha u^4 e^{2v(1-u^2)})^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{s}} \mathcal{L}^+(\alpha, v), \end{aligned} \quad (\text{B8})$$

where $v = s r_v^2$, and the function $\mathcal{L}^+$ is defined in Appendix A.

To compute the string energy, we reduce the integral over x in S to an integral over r using the differential equation. Since the resulting expression diverges at $r = 0$, we regularize it by imposing a short-distance cutoff ϵ . This gives

$$E_R = \frac{S_R}{T} = g \sqrt{\frac{s}{v}} \int_{\sqrt{\frac{s}{v}} \epsilon}^1 \frac{du}{u^2} e^{v u^2} [1 - \cos^2 \alpha u^4 e^{2v(1-u^2)}]^{-\frac{1}{2}}. \quad (\text{B9})$$

E_R behaves for $\epsilon \rightarrow 0$ as

$$E_R = \frac{g}{\epsilon} + E + O(\epsilon). \quad (\text{B10})$$

Subtracting the $\frac{1}{\epsilon}$ term and taking $\epsilon = 0$ we obtain a finite expression for the energy,

²³Alternatively, one can choose a gauge in which the boundary terms provide both components [15].

$$E = g\sqrt{\frac{s}{v}} \int_0^1 \frac{du}{u^2} (e^{vu^2} (1 - \cos^2 \alpha u^4 e^{2v(1-u^2)})^{-\frac{1}{2}} - 1 - u^2) + c$$

$$= g\sqrt{s} \mathcal{E}^+(v, \alpha) + c. \quad (\text{B11})$$

Here $\mathcal{E}^+$ is defined in (A11), and c is a normalization constant.

It is worth noting that for $\alpha = \frac{\pi}{2}$, the above expressions simplify to

$$\ell = 0, \quad E = g\sqrt{s} \mathcal{Q}(v) + c, \quad (\text{B12})$$

where $\mathcal{Q}$ is defined in Appendix A. In this case, the string is stretched entirely along the radial direction. If both end-points lie in the bulk, the second expression becomes

$$E = g\sqrt{s} (\mathcal{Q}(v) - \mathcal{Q}(\bar{v})). \quad (\text{B13})$$

Here $\bar{v} = sr_0^2$ such that $r_v > r_{\bar{v}}$.

The above analysis extends straightforwardly to the case $\alpha \leq 0$. A key point, relevant for all formulas below, is that the string configuration involves two segments: one over the interval $[0, x_0]$, where $r(x)$ increases, and another over the interval $[x_0, \ell]$, where $r(x)$ decreases (see Fig. 20). First, we define the first integral at $r = r_0$ so that $I = w(r_0)$, and then integrate the differential equation over both intervals. This yields

$$\ell = \sqrt{\frac{\lambda}{s}} \left[\int_0^1 du u^2 e^{\lambda(1-u^2)} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} + \int_{\sqrt{\frac{\ell}{\lambda}}}^1 du u^2 e^{\lambda(1-u^2)} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} \right]$$

$$= \frac{1}{\sqrt{s}} \mathcal{L}^-(\lambda, v). \quad (\text{B14})$$

Here $\lambda = sr_0^2$, and $\mathcal{L}^-$ is defined in (A4). Importantly, λ , v , and α are not independent. From the first integral it follows that:

$$\frac{e^\lambda}{\lambda} = \frac{e^v}{v} \cos \alpha, \quad (\text{B15})$$

which allows us to express λ in terms of v and α as

$$\lambda = -\text{ProductLog}(-ve^{-v}/\cos \alpha). \quad (\text{B16})$$

Here $\text{ProductLog}(z)$ is the principal solution for w in the equation $z = we^w$ [27].

As before, the string energy is computed by first rewriting the integral over x in S as an integral over r and imposing the short-distance cutoff on r . A direct calculation gives

$$E_R = g\sqrt{\frac{s}{\lambda}} \left[\int_{\sqrt{\frac{\ell}{\lambda}}}^1 \frac{du}{u^2} e^{\lambda u^2} [1 - u^4 e^{2\lambda(1-u^2)}]^{-\frac{1}{2}} + \int_{\sqrt{\frac{\ell}{\lambda}}}^1 \frac{du}{u^2} e^{\lambda u^2} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} \right]. \quad (\text{B17})$$

To obtain a finite result, we subtract the $\frac{1}{\epsilon}$ term and take $\epsilon \rightarrow 0$, yielding

$$E = g\sqrt{\frac{s}{\lambda}} \left[\int_0^1 \frac{du}{u^2} (e^{\lambda u^2} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} - 1 - u^2) + \int_{\sqrt{\frac{\ell}{\lambda}}}^1 \frac{du}{u^2} e^{\lambda u^2} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} \right] + c$$

$$= g\sqrt{s} \mathcal{E}^-(\lambda, v) + c, \quad (\text{B18})$$

where c is the same normalization constant introduced earlier. The function $\mathcal{E}^-$ is defined in Appendix A.

Finally, let us discuss the large- ℓ behavior. In doing so, it is convenient to start with $\alpha \leq 0$. In this case the large- ℓ limit corresponds to $\lambda \rightarrow 1$. From this, it follows that v and α are not independent but subject to the constraint $ve^{1-v} = \cos \alpha$. Explicitly, for a given α , the corresponding v is $v_\alpha = -\text{ProductLog}(-\cos \alpha/e)$. The leading contributions to ℓ and E arise from the logarithmic terms. From Eqs. (A6) and (A16), we immediately find

$$\ell = -\frac{1}{\sqrt{s}} \ln(1 - \lambda) + O(1),$$

$$E = -g\sqrt{s} \ln(1 - \lambda) + O(1). \quad (\text{B19})$$

This leads to the standard result,

$$E = \sigma \ell + O(1) \quad \text{with} \quad \sigma = g\sqrt{s}. \quad (\text{B20})$$

Here σ is the string tension. To extract the subleading term, consider the difference,

$$E - \sigma \ell = g\sqrt{s} (\mathcal{E}^-(\lambda, v) - e\mathcal{L}^-(\lambda, v)) + c. \quad (\text{B21})$$

Taking $\lambda \rightarrow 1$ ($v \rightarrow v_\alpha$) and using (A24), we obtain

$$E = \sigma \ell + c - g\sqrt{s} \mathcal{I}(v_\alpha) + o(1). \quad (\text{B22})$$

For $\alpha \geq 0$, the only way to reach infinite string length is to take $v \rightarrow 1$ at $\alpha = 0$. This implies $v_\alpha = 1$, so (B22) reduces to

$$E = \sigma \ell + c - g\sqrt{s} I_0 + o(1). \quad (\text{B23})$$

Here we used the fact that $\mathcal{I}(1) = I_0$.

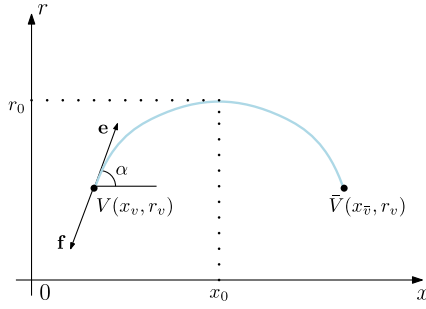


FIG. 21. A string stretched between two symmetric points in the bulk. Here $r_v < \frac{1}{\sqrt{s}}$, α denotes the tangent angle, and the arrows indicate forces acting at V .

2. A static string with endpoints in the bulk

Now consider a static string stretched between two points in the bulk at the same radial coordinate, as shown in Fig. 21. In this case the string profile is symmetric about $x = \ell/2$. Clearly, external forces are required to keep the string in equilibrium. On symmetry grounds, we indicate only the forces acting at V . The force balance equation takes the same form as Eq. (B5), with the string tension given by

$$\mathbf{e} = g w(r_v)(\cos \alpha, \sin \alpha). \quad (\text{B24})$$

This expression for $\mathbf{e}$ follows from Eq. (B7), with $\mathbf{e}$ replaced by $-\mathbf{e}$.

The length of the string can be obtained in the standard way, by integrating the differential equation with respect to x and r , subject to the boundary conditions $r(x_v) = r(x_{\bar{v}}) = r_v$. Alternatively, it can be derived by considering two strings with upper endpoints at V and $\bar{V}$, and lower endpoints on the boundary. Subtracting the length of the first string from that of the second yields,

$$\begin{aligned} \ell &= x_{\bar{v}} - x_v = \frac{1}{\sqrt{s}}(\mathcal{L}^-(\lambda, v) - \mathcal{L}^+(\alpha, v)) \\ &= 2\sqrt{\frac{\lambda}{s}} \int_{\sqrt{\frac{v}{\lambda}}}^1 du u^2 e^{\lambda(1-u^2)} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{s}} \mathcal{L}(\lambda, v). \end{aligned} \quad (\text{B25})$$

The energy can be computed in a similar way, giving

$$\begin{aligned} E &= g\sqrt{s}(\mathcal{E}^-(\lambda, v) - \mathcal{E}^+(\alpha, v)) \\ &= 2g\sqrt{\frac{s}{\lambda}} \int_{\sqrt{\frac{v}{\lambda}}}^1 \frac{du}{u^2} e^{\lambda u^2} (1 - u^4 e^{2\lambda(1-u^2)})^{-\frac{1}{2}} \\ &= g\sqrt{s} \mathcal{E}(\lambda, v), \end{aligned} \quad (\text{B26})$$

where the incomplete $\mathcal{L}$ and $\mathcal{E}$ functions are defined in Appendix A. In our derivation, we used the relation $\cos \alpha = \frac{v}{\lambda} e^{\lambda-v}$. Note that no regularization is needed in (B26), as the integral is finite.

Finally, let us discuss the large- ℓ limit, which corresponds to $\lambda \rightarrow 1$. As before, we consider two strings but now with upper endpoints at $\bar{V}$ and at the turning point (x_0, r_0) . From (B22) and (B23), it follows immediately that

$$E = \sigma \ell - g\sqrt{s} \mathcal{J}(v_\alpha) + o(1), \quad (\text{B27})$$

where in the last step we used (A25). Note that σ is the same as in (B23). This is an important feature of the soft wall model. Its physical meaning is clear: the physical string tension is determined by the expression (B7) or (B24) evaluated precisely at the soft wall's position, $r = 1/\sqrt{s}$. The wall effectively prevents strings from penetrating deeper into the bulk.

APPENDIX C: SOME DETAILS ON THE $Q\bar{Q}$ SYSTEM

This appendix provides a brief summary of key results concerning the heavy quark-antiquark potential, which represents the ground state energy of a static quark-antiquark pair. These results are relevant to the discussions in Secs. III and IV. For standard explanations, see [14], whose conventions we follow unless otherwise stated.

In four dimensions, a static string configuration takes the form shown in Fig. 22 on the left, which is the standard configuration for mesons [17]. In five dimensions, this configuration transforms into the one shown on the right. It involves a string attached to the heavy quark sources on the boundary of five-dimensional space, but in this case a gravitational force pulls the string into the interior [28].

For a Nambu-Goto string in the background geometry (2.1), the relation between the string energy and the quark separation distance is given parametrically as

$$\ell = \frac{2}{\sqrt{s}} \mathcal{L}^+(0, \lambda), \quad E_{Q\bar{Q}} = 2g\sqrt{s} \mathcal{E}^+(0, \lambda) + 2c. \quad (\text{C1})$$

Here c is the normalization constant, as before, and $\lambda = s r_0^2$ is a parameter ranging from 0 to 1. The functions $\mathcal{L}^+$ and $\mathcal{E}^+$ are defined in Appendix A.

The small- ℓ behavior of $E_{Q\bar{Q}}$ is

$$\begin{aligned} E_{Q\bar{Q}}(\ell) &= -\frac{\alpha_{Q\bar{Q}}}{\ell} + 2c + \sigma_{Q\bar{Q}} \ell + o(\ell), \quad \text{with} \\ \alpha_{Q\bar{Q}} &= (2\pi)^3 \Gamma^{-4} \left(\frac{1}{4} \right) g, \\ \sigma_{Q\bar{Q}} &= \frac{1}{2} (2\pi)^{-2} \Gamma^4 \left(\frac{1}{4} \right) g s, \end{aligned} \quad (\text{C2})$$

while the large- ℓ behavior is

$$\begin{aligned} E_{Q\bar{Q}}(\ell) &= \sigma \ell + C_{Q\bar{Q}} + o(1), \quad \text{with} \\ C_{Q\bar{Q}} &= 2c - 2g\sqrt{s} I_0. \end{aligned} \quad (\text{C3})$$

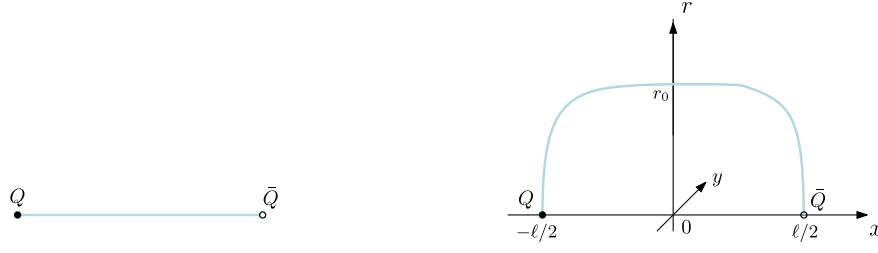


FIG. 22. A connected string configuration in four dimensions (left) and in five dimensions (right). ℓ denotes the distance between the heavy quark sources, and r_0 is the radial coordinate of the turning point at $x = 0$.

Here σ is the string tension, and I_0 is defined in Appendix A. It is worth noting that the coefficients $\sigma_{Q\bar{Q}}$ and σ are different, with their ratio given numerically by $\sigma_{Q\bar{Q}}/\sigma = 0.805$. Furthermore, the difference between the constant terms in the small- and large- ℓ expansions is $2c - C_{Q\bar{Q}} = 175$ MeV, showing that the constant term in the small- ℓ expansion is larger.

APPENDIX D: A DETAILED DESCRIPTION OF CONFIGURATION (c)

For a given η , configuration (c) can be constructed from the basic configurations of Sec. III. However, the construction becomes intricate when it involves multiple transitions between these basic configurations. In this appendix, we describe configuration (c) and some of its properties, under the assumption that η is not very close to $\frac{1}{\sqrt{3}}$.

1. The limiting values of v

As explained in Sec. III, the limiting values of the parameter v , denoted as v_1 , v_η , and v_p , are the solutions to Eqs. (3.23), (3.30), and (3.41). Here we examine how these solutions behave as functions of η .

A quick way to proceed is by numerical analysis. The results are shown in Fig. 23. The solutions v_1 and v_p exist for all η and take values in the interval $[0, 1]$. A simple calculation yields $v_1 = 0.978$ and $v_p = 0.967$. In contrast, the solution v_η exists only in the very narrow interval $\frac{1}{\sqrt{3}} \leq \eta \leq \eta_p$.²⁴ These bounds can be better understood by considering configuration (c). The lower bound corresponds to $\bar{\lambda} = 1$, where string 3 becomes infinitely long and the second large ℓ limit reduces to the first one. The upper bound corresponds to $\bar{\alpha} = 0$, where string 3 collapses to a point, transforming configuration (c) into configuration (d). It can be expressed in terms of v_p as

$$\eta_p = \frac{1}{\sqrt{4v_p^2 e^{2(1-v_p)} - 1}}. \quad (\text{D1})$$

²⁴Numerically, $\eta_p - \frac{1}{\sqrt{3}} = 0.0004$.

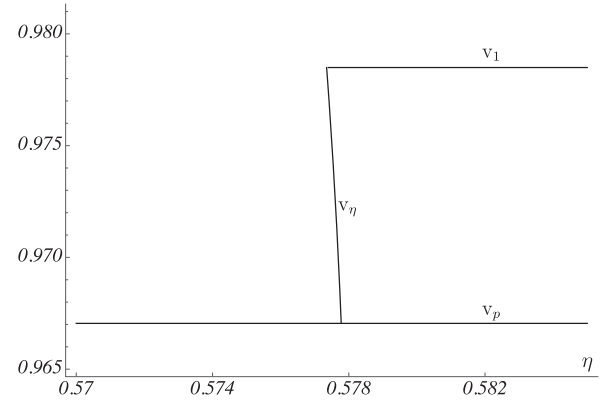


FIG. 23. The solutions v_1 , v_η , and v_p as functions of η in the vicinity of $\eta = \frac{1}{\sqrt{3}}$. Although it is true that the solution v_1 is independent of η , we plot it only for $\eta > \frac{1}{\sqrt{3}}$, where configuration (c) exists.

2. Some preliminary comments

As we saw in Sec. III, the large- ℓ limit of configuration (c) does not exist if $\eta \leq 1/\sqrt{3}$. The same is also true for the small- ℓ limit. A numerical analysis shows that the geometrical constraints (3.17) and (3.21), together with the force balance equations (3.16), have no solutions unless $\bar{\alpha} = 0$, which corresponds to configuration (d). Because of these issues, no physically meaningful tetraquark configuration exists for $\eta \leq \frac{1}{\sqrt{3}}$.

A useful tool for studying transitions between the basic configurations of Sec. III is the function,

$$f_\eta(v) = 1 - 2(\eta \cos \beta - \sin \beta) \frac{\mathcal{L}^+(0, v)}{\mathcal{L}(\bar{\lambda}, v)}, \quad (\text{D2})$$

whose zeros are the solutions of the equation $\alpha(v) = 0$ if $\bar{\lambda}(v)$ and $\beta(v)$ are defined by

$$\sin \bar{\alpha} = -3k(1 + 4v)e^{-3v}, \quad \sin \beta = \frac{1}{2} \cos \bar{\alpha}. \quad (\text{D3})$$

Clearly, at zero values, Eq. (D2) reduces to the geometric constraint (3.14). These zeros play an important role in

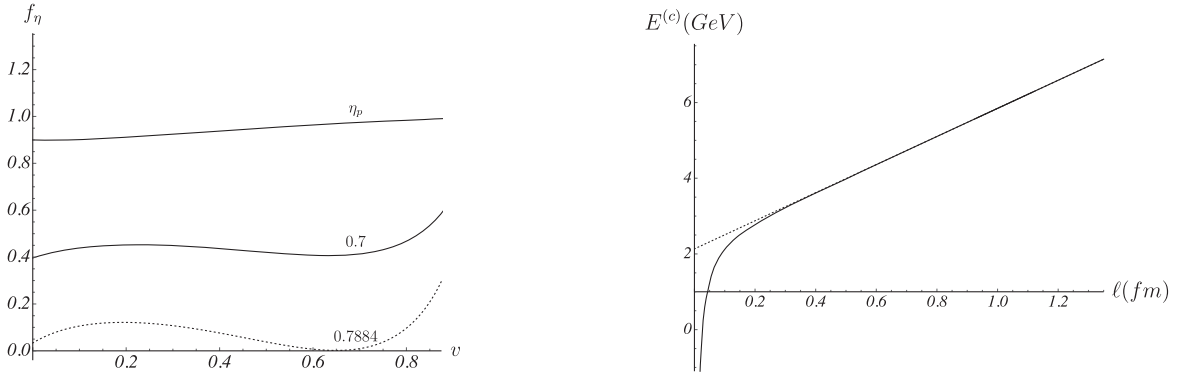


FIG. 24. Left: The function f_η for several values of η . Right: $E^{(c)}$ vs ℓ at $\eta = 0.7$. The dotted line corresponds to the large- ℓ asymptotics (3.29), here and below.

analyzing the tetraquark configuration, as they correspond to transitions between configurations (I) and (II).

The interval $\frac{1}{\sqrt{3}} \leq \eta \leq \eta_p$ is somewhat puzzling. It appears that within this range, there exists two string configurations whose large- ℓ expansions correspond to the different limits of configuration (II). In particular, the configuration with the linear asymptotics (3.29) dominates at very large ℓ (greater than 2000 fm), whereas the configuration with the linear asymptotics (3.35) dominates at smaller ℓ . This suggests that configuration (c) is effectively a superposition of the two. We will not pursue this issue further, as the interval is very narrow and irrelevant for determining the ground state potential of the $QQ\bar{Q}\bar{Q}$ system. It matters only for the excited states.

3. The range $\eta_p < \eta \leq 0.7884$

We are interested in the zeros of $f_\eta(v)$ in the interval $0 < v < v_1$. Numerical analysis shows that this function has no zeros, except for a single one at $\eta = 0.7884$. For illustration, we plot f_η as a function of v in Fig. 24 on the left. However, this zero has no effect on the form of the tetraquark configuration, which is described entirely by configuration (II). Its energy is given parametrically by

$$\ell = \ell^{(II)}(v), \quad E^{(c)} = E^{(II)}(v), \quad 0 \leq v \leq v_1. \quad (\text{D4})$$

We conclude with a simple example: the $\eta = 0.7$ case. Figure 24 presents both $f_\eta(v)$ and $E^{(c)}(\ell)$. A notable feature of $E^{(c)}$ is its near linear behavior at relatively small quark separations, which the figure shows occurs for $\ell \gtrsim 0.3$ fm. This is significantly smaller than $\ell \gtrsim 0.6$ fm observed for configuration (a), where $E^{(a)}$ becomes nearly linear (see Fig. 5).

4. The range $0.7884 < \eta \leq \eta_0$

In this range, the function $f_\eta(v)$ has two zeros, as shown by a straightforward numerical analysis. At the upper bound, however, it has an additional zero at $v = 0$. The value of η_0 can be computed analytically, with the result

$$\eta_0 = \sqrt{\frac{1 - 9k^2}{3 + 9k^2}} \left[1 + 2 \frac{I(9k^2; \frac{1}{2}, \frac{3}{4})}{(1 - 9k^2)^{\frac{3}{4}}} \right]. \quad (\text{D5})$$

A simple calculation gives $\eta_0 = 0.7961$. For illustration, we plot the function f_η in Fig. 25 on the left. The energy of the tetraquark configuration is now a piecewise function, described parametrically in terms of the energies of configurations (I) and (II) as

$$\ell = \begin{cases} \ell^{(II)}(v), & 0 \leq v \leq v_{01}, \\ \ell^{(I)}(v), & v_{01} \leq v \leq v_{02}, \\ \ell^{(II)}(v), & v_{02} \leq v \leq v_1, \end{cases} \quad E^{(c)} = \begin{cases} E^{(II)}(v), & 0 \leq v \leq v_{01}, \\ E^{(I)}(v), & v_{01} \leq v \leq v_{02}, \\ E^{(II)}(v), & v_{02} \leq v \leq v_1. \end{cases} \quad (\text{D6})$$

Here v_{0i} are the solutions to the equation $\alpha(v) = 0$.²⁵

To illustrate (D6), we take $\eta = 0.792$. In this case, a simple calculation gives $v_{01} = 0.5579$ and $v_{02} = 0.7213$. The functions f_η and $E^{(c)}$ are presented in Fig. 25. Again, $E^{(c)}$ is approximately linear for $\ell \gtrsim 0.3$ fm.

²⁵For a single solution we simply use the notation v_0 .

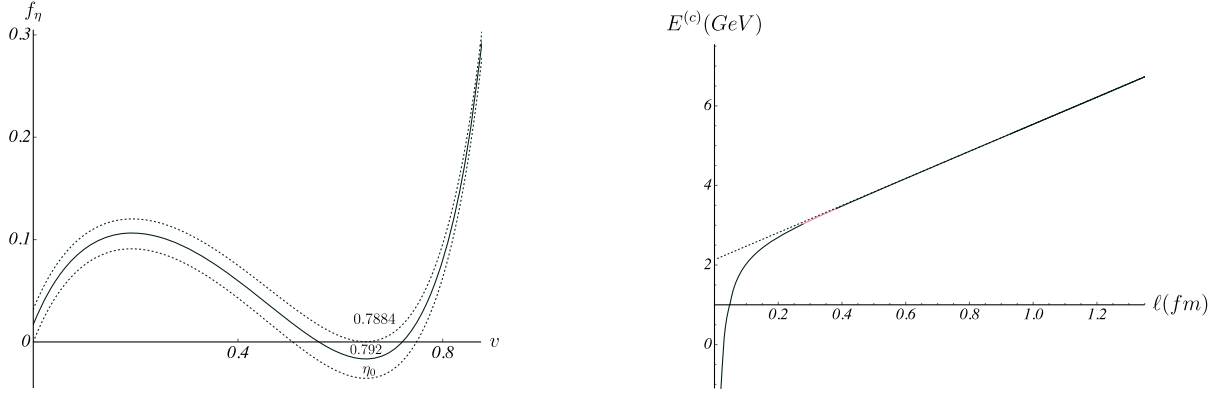


FIG. 25. Left: The function f_η for several values of η . Right: $E^{(c)}$ vs ℓ at $\eta = 0.792$. The magenta and black curves correspond respectively to configurations (I) and (II), here and below.

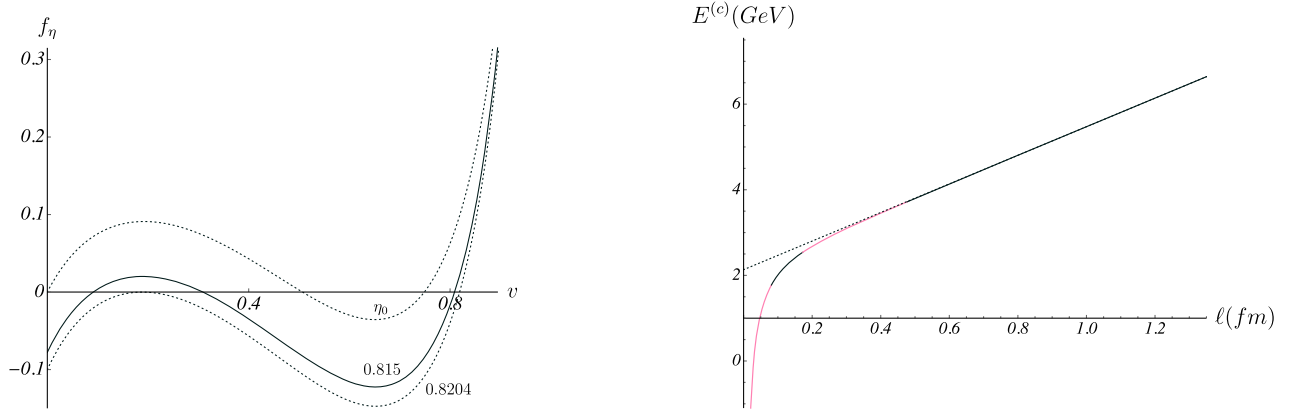


FIG. 26. Left: The function f_η for several values of η . Right: $E^{(c)}$ vs ℓ at $\eta = 0.815$.

5. The range $\eta_0 < \eta < 0.8204$

The function f_η now has three zeros, as illustrated in Fig. 26. This implies that $E^{(c)}$ is a piecewise function consisting of four segments. Explicitly,

$$\ell = \begin{cases} \ell^{(I)}(v), & 0 \leq v \leq v_{01}, \\ \ell^{(II)}(v), & v_{01} \leq v \leq v_{02}, \\ \ell^{(I)}(v), & v_{02} \leq v \leq v_{03}, \\ \ell^{(II)}(v), & v_{03} \leq v \leq v_1, \end{cases}$$

$$E^{(c)} = \begin{cases} E^{(I)}(v), & 0 \leq v \leq v_{01}, \\ E^{(II)}(v), & v_{01} \leq v \leq v_{02}, \\ E^{(I)}(v), & v_{02} \leq v \leq v_{03}, \\ E^{(II)}(v), & v_{03} \leq v \leq v_1. \end{cases} \quad (D7)$$

As an example, let us take $\eta = 0.815$. For this value, the zeros are $v_{01} = 0.0914$, $v_{02} = 0.3084$, and $v_{03} = 0.8088$. The corresponding functions f_η and $E^{(c)}$ are shown in Fig. 26. We see that $E^{(c)}$ also becomes near linear for $\ell \gtrsim 0.3$ fm.

6. The range $0.8204 \leq \eta$

Finally, consider the range $0.8204 \leq \eta$. As seen from Fig. 27, the function f_η has a single zero, except at $\eta = 0.8404$, where it has two. However, this does not affect the tetraquark configuration. So, the energy can be written in parametric form as

$$\ell = \begin{cases} \ell^{(I)}(v), & 0 \leq v \leq v_0, \\ \ell^{(II)}(v), & v_0 \leq v \leq v_1, \end{cases}$$

$$E^{(c)} = \begin{cases} E^{(I)}(v), & 0 \leq v \leq v_0, \\ E^{(II)}(v), & v_0 \leq v \leq v_1. \end{cases} \quad (D8)$$

As a concrete example, consider $\eta = 1$. In this case, the zero is given by $v_0 = 0.9102$. The functions f_η and $E^{(c)}$ are plotted in Fig. 27. A notable feature is that $E^{(c)}$ becomes near linear for slightly larger values of ℓ , namely for $\ell \gtrsim 0.4$ fm. This has a natural explanation: as $\eta \rightarrow \infty$, the function $E^{(c)}$ tends to $E_{Q\bar{Q}}$, where such a behavior occurs for $\ell \gtrsim 0.6$ fm (see Fig. 5).

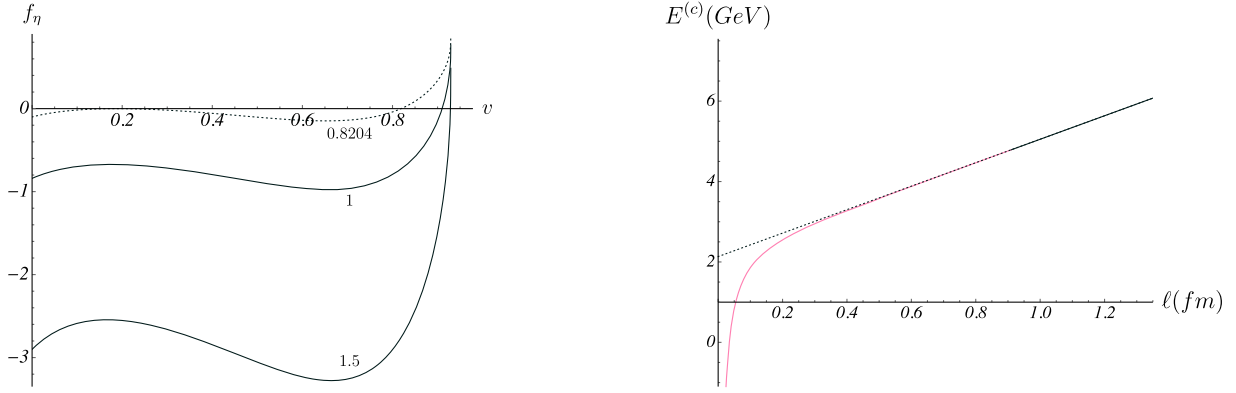


FIG. 27. Left: The function f for several values of η . Right: $E^{(c)}$ vs ℓ at $\eta = 1$.

7. The Coulomb and quark-quark couplings

Assuming that at sufficiently small separations the energy of the tetraquark configuration is due to pairwise quark interactions, the leading approximation to the energy takes the form,

$$E^{(c)} = -\frac{\alpha^{(c)}}{\ell} + O(1) \\ = -2\eta \frac{\alpha_{QQ}}{\ell} - \frac{1}{2} \left(1 + \frac{1}{\sqrt{1+\eta^{-2}}} \right) \frac{\alpha_{Q\bar{Q}}}{\ell} + O(1). \quad (D9)$$

In general, both $\alpha_{Q\bar{Q}}$ and α_{QQ} may depend on η and differ from their counterparts in the $Q\bar{Q}$ and QQQ systems.²⁶

Having computed the Coulomb coefficient $\alpha^{(c)}$, we would like to estimate the quark-quark coupling that appears in (D9). This, however, is impossible without making an additional assumption on $\alpha_{Q\bar{Q}}$. There are two ways to do so. The first way is to assume the relation $\alpha_{Q\bar{Q}} = 2\alpha_{QQ}$.²⁷ With this, one gets

$$\alpha_{QQ} = \frac{\alpha^{(c)}}{1 + 2\eta + \frac{1}{\sqrt{1+\eta^{-2}}}}. \quad (D10)$$

The second way is to assume that $\alpha_{Q\bar{Q}}$ is the same as in the $Q\bar{Q}$ (meson) system. This yields

$$\alpha_{QQ}^m = \frac{1}{4\eta} \left(2\alpha^{(c)} - \left(1 + \frac{1}{\sqrt{1+\eta^{-2}}} \right) \alpha_{Q\bar{Q}} \right), \quad (D11)$$

where $\alpha_{Q\bar{Q}}$ is defined in (C2).

The results of our estimates are summarized in Table I. Importantly, for the considered values of η , the small- ℓ behavior of the potential V_0 is determined by configuration (c) that makes our estimates also applicable to the ground-

²⁶In the QQQ system, α_{QQ} is not constant but depends weakly on the angles of a triangle formed by the quarks [15,16].

²⁷In the literature, it is sometimes referred to as $\frac{1}{2}$ rule.

TABLE I. Estimates for the couplings in the small- ℓ limit. Here $\alpha_{Q\bar{Q}}$ is the Coulomb coefficient of the quark-antiquark potential (see Appendix C).

η	$\alpha^{(c)}$	$\alpha^{(c)}/\alpha^{(a)}$	$\alpha_{QQ}/\alpha_{Q\bar{Q}}$	$\alpha_{QQ}^m/\alpha_{Q\bar{Q}}$
1.1794	0.5053	1	0.4853	0.4743
$\sqrt{3}$	0.6505	1.2873	0.4830	0.4739
3	0.9570	1.8939	0.4765	0.4689
6	1.6655	3.2960	0.4713	0.4666
15	3.7854	7.4912	0.4679	0.4658
20	4.9630	9.8217	0.4677	0.4635

state potential as well. As seen from the Table, both quark-quark couplings are close to $\frac{1}{2}$ and depend weakly on η . While the former behavior was observed in lattice QCD using the approximation (D10) [9], the latter is our prediction, which will hopefully be testable in future numerical simulations. The expressions (D10) and (D11), in turn, provide reasonable approximations for the η -dependent Coulomb coefficient $\alpha^{(c)}$.

APPENDIX E: THE TETRAQUARK CONFIGURATION (c') FOR RHOMBUS GEOMETRY

Our goal here is to clarify that the tetraquark configuration exists for type-B ordering. To proceed, instead of a rectangle, we consider a rhombus in the xy -plane with its center at the origin, as shown in Fig. 28. We also impose the geometrical constraint,

$$|\bar{Q}_1 Y| = d|Q_1 Y|, \quad (E1)$$

which fixes one of the rhombus angles. A simple but useful relation is $L = \sqrt{1+d^2}|Q_1 Y|$.

The tetraquark configuration (c') can be analyzed along the lines of Sec. IV, with the only modification arising from the above constraint. We begin by describing the basic

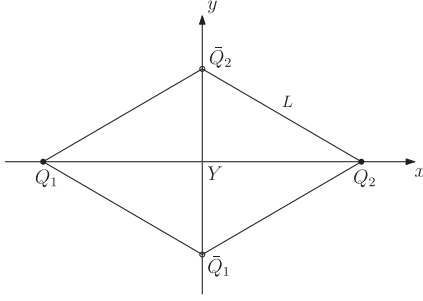


FIG. 28. A rhombus of length L . The quarks and antiquarks are placed at its vertices according to type-B ordering, with point Y at the origin.

configurations and next illustrate the construction with a concrete example.

1. Configuration (i)

The basic configuration (i) is similar to that shown in Fig. 8. Thus, we have $v > \bar{v}$, $\alpha_1 = \alpha_2 = \alpha$, and $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}$, as before. The force balance equations (4.4) remain valid, as does the expression for the energy given by the second equation in (4.5). The expression for the length is easily obtained using Eq. (B14) for $|Q_1 Y|$. So,

$$L^{(i)} = \frac{\sqrt{1+d^2}}{\sqrt{s}} \mathcal{L}^-(\lambda, v),$$

$$E^{(i)} = g\sqrt{s} \left(2\mathcal{E}^-(\lambda, v) + 2\mathcal{E}^+(\bar{\alpha}, \bar{v}) + \mathcal{Q}(v) - \mathcal{Q}(\bar{v}) + 3k \frac{e^{-2v}}{\sqrt{v}} + 3k \frac{e^{-2\bar{v}}}{\sqrt{\bar{v}}} \right) + 4c. \quad (\text{E2})$$

Similarly, for the geometric constraint we get

$$\mathcal{L}^+(\bar{\alpha}, \bar{v}) = d\mathcal{L}^-(\lambda, v), \quad (\text{E3})$$

which is a light modification of (4.6). The tangent angles and parameter $\bar{v}$ can be expressed in terms of v using the force balance equation and the geometrical constraint. As a result, the energy as a function of L can be written parametrically as $L = L^{(i)}(v)$ and $E = E^{(i)}(v)$.

It is instructive to consider the behavior of $E^{(i)}$ at small L , which corresponds to the limit $v \rightarrow 0$. Taking this limit in (E3), with the help of Eqs. (A2) and (A5), one finds $\bar{v} = (d \frac{\mathcal{L}^-(\alpha)}{\mathcal{L}^+(\bar{\alpha})})^2 v$. This restricts the allowed values of d to

$$d \leq \sqrt{\frac{\cos \alpha}{\cos \bar{\alpha}}} \frac{I(\cos^2 \bar{\alpha}, \frac{3}{4}, \frac{1}{2})}{1 + I(\sin^2 \alpha, \frac{1}{2}, \frac{3}{4})}, \quad (\text{E4})$$

as $v \geq \bar{v}$ by construction. To make a simple estimate of the upper bound, note that the values of the tangent angles are determined by Eq. (4.4) at $v = \bar{v} = 0$. Using those, we get

$d \leq 0.407$.²⁸ The energy behaves for $L \rightarrow 0$ as

$$E^{(i)} = -\frac{\alpha^{(i)}}{L} + 4c + o(1), \quad \text{with}$$

$$\alpha^{(i)} = -\sqrt{1+d^{-2}} [d\mathcal{L}_0^-(\alpha)(2\mathcal{E}_0^-(\alpha) - 1 + 3k) + \mathcal{L}_0^+(\bar{\alpha})(2\mathcal{E}_0^+(\bar{\alpha}) + 1 + 3k)]g, \quad (\text{E5})$$

exhibiting the leading Coulomb term.

2. Configuration (ii)

This basic configuration is similar to that shown on the right in Fig. 8, with $\bar{\alpha} \geq 0$. The force balance equation (4.8) remains unchanged. The expressions for the length and energy follow from (E2) and (4.9):

$$L^{(ii)} = \frac{\sqrt{1+d^2}}{\sqrt{s}} \mathcal{L}^-(\lambda, v),$$

$$E^{(ii)} = 2g\sqrt{s} \left(\mathcal{E}^-(\lambda, v) + \mathcal{E}^+(\bar{\alpha}, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c. \quad (\text{E6})$$

The geometrical constraint now reads,

$$\mathcal{L}^+(\bar{\alpha}, v) = d\mathcal{L}^-(\lambda, v), \quad (\text{E7})$$

which is obtained from (E3) by setting $\bar{v} = v$. After extracting the tangent angles from the force balance equation and the geometrical constraint, the energy can again be written in parametric form as $L = L^{(ii)}(v)$ and $E = E^{(ii)}(v)$.

3. Configuration (iii)

The only difference from configuration (ii) is that $\bar{\alpha}$ is negative. Therefore, all the formulas can be obtained from those of configuration (ii) by replacing $\mathcal{L}^+$ and $\mathcal{E}^+$ with $\mathcal{L}^-$ and $\mathcal{E}^-$. In this way, from (E6) we obtain

$$L^{(iii)} = \frac{\sqrt{1+d^2}}{\sqrt{s}} \mathcal{L}^-(\lambda, v),$$

$$E^{(iii)} = 2g\sqrt{s} \left(\mathcal{E}^-(\lambda, v) + \mathcal{E}^-(\bar{\lambda}, v) + 3k \frac{e^{-2v}}{\sqrt{v}} \right) + 4c, \quad (\text{E8})$$

and from (E7)

$$\mathcal{L}^-(\bar{\lambda}, v) = d\mathcal{L}^-(\lambda, v). \quad (\text{E9})$$

The force balance equation (4.8) remains valid. Again, the angles can be expressed in terms of v , and the energy can be written parametrically as $L = L^{(iii)}(v)$ and $E = E^{(iii)}(v)$.

²⁸As we saw in Sec. IV, the tetraquark configuration (c') does not exist at $d = 1$.

Finally, let us briefly discuss the large- L behavior, which corresponds to the limit $\lambda, \bar{\lambda} \rightarrow 1$. It follows then from Eq. (B15) that $\cos \alpha = \cos \bar{\alpha} = v e^{1-v}$. Thus, in this limit configuration (c') approaches to configuration (d) of Secs. III and IV. This implies that the upper bound on v is v_p , defined by (3.41), and the expansions (3.44) can be rewritten as

$$E^{(iii)} = 2 \frac{1+d}{\sqrt{1+d^2}} \sigma L + C^{(d)} + o(1). \quad (E10)$$

Here the first term is proportional to the total length of the rhombus diagonals.

4. An example

We now describe a concrete example, namely $d = \frac{1}{4}$, in which one can explicitly construct the tetraquark configuration from the basic configurations. As it turns out, $E^{(c')}$ is a piecewise function of L , given by

$$L = \frac{\sqrt{1+d^2}}{\sqrt{s}} \mathcal{L}^-(\lambda, v), \quad E^{(c')} = \begin{cases} E^{(i)}(v), & 0 \leq v \leq \bar{v}, \\ E^{(ii)}(v), & \bar{v} \leq v \leq v_0, \\ E^{(iii)}(v), & v_0 \leq v \leq v_p. \end{cases} \quad (E11)$$

Here $\bar{v} = 0.399$ and $v_0 = 0.929$. The first value corresponds to the transition between configurations (i) and (ii),

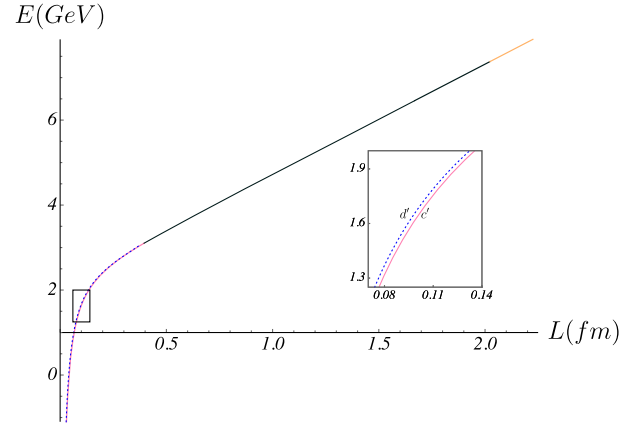


FIG. 29. $E^{(c')}$ and $E^{(d')}$ vs L at $d = \frac{1}{4}$. The magenta, black, and yellow curves correspond to the basic configurations (i)–(iii), respectively. In addition, the dotted blue curve represents configuration (ii) for $L < 0.395$ fm, which forms part of configuration (d') in this interval.

which occurs when the vertices collide, i.e., $v = \bar{v}$. The second value corresponds to the transition between configurations (ii) and (iii), which occurs at $\bar{\alpha} = 0$.

To complete the picture, we also present an analog of the pinched tetraquark configuration, constructed from the two basic configurations (ii) and (iii). Explicitly,

$$L = \frac{\sqrt{1+d^2}}{\sqrt{s}} \mathcal{L}^-(\lambda, v), \quad E^{(d')} = \begin{cases} E^{(ii)}(v), & 0 \leq v \leq v_0, \\ E^{(iii)}(v), & v_0 \leq v \leq v_p. \end{cases} \quad (E12)$$

Note that configuration (d') is antisymmetric in the sense that $\alpha \neq \bar{\alpha}$ unless $v = v_p$. Clearly, these configurations differ only for $L < L(\bar{v})$, due to the difference between the basic configurations (i) and (ii). In Fig. 29 we plot $E^{(c')}(L)$ and $E^{(d')}(L)$. For $L < 0.395$ fm, the plots corresponding to configurations (i) and (ii) are indistinguishable, as the difference between them is extremely small. However, a more detail analysis shows that configuration (c') has a lower energy than configuration (d'). Note that configuration (i) transforms into configuration (ii) at $L = 0.395$ fm when the baryon vertices collide. From the perspective of ten-dimensional string theory, this corresponds to the creation of a brane-antibrane bound state. A similar pinching effect has also been observed in the doubly heavy tetraquark systems [29].

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