

Ladders and rainbows in the minimal subtraction scheme

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In dimensional regularization with $D = D_0 - 2\epsilon$, the minimal subtraction (MS) scheme is characterized by counterterms that only consist of singular terms in ϵ . We develop a general method to compute the infinite sums of massless ladder or rainbow Feynman integrals in MS at D_0 . Our method is based on relating the MS-solution to a kinematic solution at a coupling-dependent renormalization point. If the ϵ -dependent Mellin transform of the kernel diagram of the insertions can be computed in closed form, we typically obtain a closed form expression for the all-order solution in MS. As examples, we consider Yukawa theory and ϕ^4 theory in $D_0 = 4$, and ϕ^3 theory in $D_0 = 6$.

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I. INTRODUCTION

Many quantum field theories contain infinite families of Feynman diagrams, which arise from repeatedly inserting subdiagrams into the same kernel diagram. Two particular such cases are *ladders* and *rainbows*. Almost 30 years ago, the exact sum of ladders and rainbows for ϕ^3 and Yukawa theory has been computed in kinematic (MOM) renormalization conditions [1,2].

From the perspective of Hopf algebra theory of renormalization [3–5], the renormalized amplitudes of such sums of diagrams are the solutions to *linear* single-scale Dyson-Schwinger equations. By now, there is a systematic procedure to construct the solution in the MOM scheme from the Mellin transform of the kernel diagram, developed by Broadhurst, Kreimer, Yeats, and collaborators [6–11]. Conversely, solutions in the minimal subtraction (MS) scheme have so far only been computed numerically [6,12], where in the latter publication, some closed formulas have been discovered empirically by matching their series expansion.

In the present work, we present a general method to compute the closed-form solution of linear single-scale single-kernel Dyson-Schwinger equations in the MS scheme. We confirm the formulas found in [12], and compute the exact solution for some further examples.

A. Linear Dyson-Schwinger equations

Ladders and rainbows are infinite families of Feynman diagrams that are characterized by recursively inserting an already existing nested diagram into some fixed *kernel* diagram at every new order in perturbation theory. In the present article, we restrict ourselves to the case where the kernel diagram is ultraviolet divergent, free of ultraviolet subdivergences, and free of infrared divergences. This ensures that the class of diagrams obtained this way is closed under perturbative renormalization, that is, they form a sub-Hopf algebra in the Hopf algebra of renormalization [13–15]. The decisive feature of ladders/rainbows is that only one copy of the existing nested diagram is inserted at each iteration, and it is always inserted into the same position in the kernel diagram. This implies that the procedure can be described by a *linear* Dyson-Schwinger equation (DSE), schematically of the form

$$G_{\mathcal{R}} = 1 + \alpha(1 - \mathcal{R})B_+[G_{\mathcal{R}}]. \quad (1)$$

Here, $G_{\mathcal{R}}$ is a renormalized 1PI Green's function, an infinite formal series of Feynman diagrams. $G_{\mathcal{R}}$ is a scalar. If the theory in question has nontrivial tensor structures, $G_{\mathcal{R}}$ is understood to be a projection onto a suitable basis tensor, also called form factor. α is the coupling, the operator B_+ denotes insertion into the kernel diagram, and $\mathcal{R}$ is a renormalization operator. We shall clarify the precise meaning of Eq. (1) in the following, but first, to have a concrete example at hand, we consider the case of rainbows for the 1PI propagator of ϕ^3 theory.

In this case, the kernel diagram is the 1-loop multiedge shown in Fig. 1, and the operator $B_+[G]$ denotes insertion of G (which may be a single diagram or a sum of diagrams, in which case B_+ acts linearly) into the lower one of the two internal edges. In particular, $B_+[1]$ is the kernel itself

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FIG. 1. 1-loop kernel for the propagator of ϕ^3 theory. The operator $B_+[G]$ means to compute the Feynman integral of this diagram, where a subdiagram G has been inserted into the lower edge. Notice that a propagator-type (amputated) subdiagram cancels one of its adjacent edges since it is proportional to an inverse propagator.

without any insertions,

$$B_+[1](p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{(k-p)^2},$$

$$B_+[G](p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} G(k) \frac{1}{(k-p)^2}.$$

Here, we have left the spacetime dimension $D = D_0 - 2\epsilon$ arbitrary in order to use dimensional regularization. For the renormalization operator $\mathcal{R}$, we may then choose *minimal subtraction* renormalization conditions, which means that $\mathcal{R}$ projects onto the pole terms in ϵ , such that $(1 - \mathcal{R})$ subtracts pole terms. Alternatively, we can choose *kinematic renormalization*, still with $D = D_0 - 2\epsilon$, then $\mathcal{R}$ projects onto a fixed value of external kinematic parameters, such that $(1 - \mathcal{R})$ vanishes at that value. Repeatedly inserting the already existing sum $G_{\mathcal{R}}$ into the kernel diagram, as described by the DSE in Eq. (1), gives rise to the sum of rainbow diagrams shown in Fig. 2.

Obviously, the sum of rainbows only represents a small subset of the diagrams in full ϕ^3 theory. Conceptually, a linear DSE such as Eq. (1) is a simplification in multiple ways; a full quantum field theory typically has multiple coupled DSEs (in particular, in a nonscalar theory, a single Green's function can have multiple basis tensor structures), each of them has multiple kernel diagrams, and insertions can happen in more than one place in each kernel. We comment on these generalizations after describing how to solve the single-scale single-kernel equation in MOM.

B. Solution in kinematic renormalization

All-order solutions to ladder and rainbow DSEs in MOM were first derived by Delbourgo and collaborators in position space and for arbitrary dimension D in [1,2].

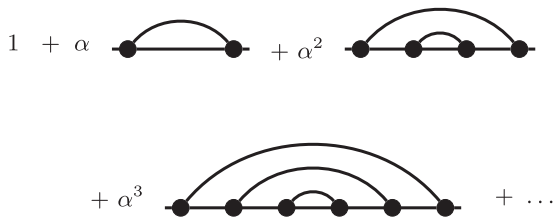


FIG. 2. Sum of rainbows in ϕ^3 theory.

However, for our solution in minimal subtraction, a different method, developed by Broadhurst, Kreimer, Yeats, and collaborators [6–11,16], is more suitable. This method has the additional benefits that it has straightforward relations to conventional momentum-space Feynman integral calculations, it allows for various generalizations beyond the linear case, and it is fundamentally related to the Hopf algebra theory of renormalization. We briefly review this method in the following as far as we need it, comprehensive accounts are, for example, the books/theses [17–20].

We consider massless *single-scale* Dyson-Schwinger equations, this means that there is only one kinematic variable. For a propagator-type Green's function, this is the external momentum p , for other Green's functions, all but one scale need to be fixed. Under this condition, the solution of every Feynman integral is itself proportional to the kinematic variable. A Feynman diagram in spacetime dimension D with ℓ loops and propagator powers a_e has superficial degree of convergence

$$\omega_G := \sum_{e \in E_G} a_e - \ell \frac{D}{2}, \quad (2)$$

and its integral is proportional to $(p^2)^{-\omega_G}$. This means that the operation $B_+[G]$ of inserting a subdiagram G into an edge e , where k_e is the edge momentum, is the same as replacing the exponent of the propagator of e by an appropriate noninteger value,

$$\frac{1}{(k_e^2)} \mapsto \frac{1}{(k_e^2)^{1+\omega_G}}.$$

Consequently, we can compute *all* of the Feynman diagrams if we can compute the kernel diagram for arbitrary powers of the propagator where the insertion happens. Let $k_1, \dots, k_{|E_G|}$ be the edge momenta of a kernel diagram G , and assume we insert into k_1 . The *Mellin transform* of G is defined by

$$F_G(\epsilon, \rho) := (p^2)^{\omega_G} \int \frac{d^{\ell(D_0-2\epsilon)}}{(2\pi)^{\ell(D_0-2\epsilon)}} \frac{1}{(k_1^2)^{1-\rho}} \frac{1}{(k_2^2)} \cdots \frac{1}{(k_{|E_G|}^2)^2}. \quad (3)$$

This function depends on the integer spacetime dimension D_0 , but we do not write this dependence since within one theory, D_0 is fixed. The prefactor in Eq. (3) ensures that F_G is independent of the kinematic variable p . Since we assume that the kernel diagrams are superficially UV divergent, the Mellin transform has a simple pole of the form

$$F(\epsilon, \rho) = \frac{\mathcal{P}}{\ell\epsilon - \rho} + \text{regular terms}, \quad (4)$$

where ℓ is the loop number and $\mathcal{P}$ is the *period* [21–23].

In our formalism, a Green's function is a formal power series in the coupling α , a logarithmic kinematic variable $L = \ln(p^2/\mu^2)$, and the dimensional regulator ϵ . At each finite order in perturbation theory, we are therefore working with polynomials. The operator $B_+[G]$, which in Eq. (1) acts on Feynman diagrams by insertion, algebraically is a Hochschild 1-cocycle in the Hopf algebra of renormalization. Translated to polynomials, it gets replaced by a differential operator acting on the Mellin transform $F(0, \rho)$ of the kernel graph by a polynomial $f(\alpha, L)$ [24],

$$B_+[f](L) \mapsto f(\alpha, \partial_\rho) e^{L\rho} F(0, \rho)|_{\rho=0}.$$

The notation $f(\alpha, \partial_\rho)$ means that the parameter L in $f(\alpha, L)$ is to be replaced by a differential operator which acts on all terms to the right. We see why MOM renormalization conditions are preferred: In MOM, an expression is renormalized by subtracting the same expression at the kinematic renormalization point, which one can choose to be $L = 0$ (i.e., $p^2 = \mu^2$). Since the L dependence of the cocycle is very simple, MOM conditions can be realized simply by replacing $e^{L\rho} \mapsto (e^{L\rho} - 1)$.

To better understand the effects of minimal subtraction later on, it will prove beneficial to generalize Eq. (1) to a *nonlinear* DSE, which allows an arbitrary, but fixed, insertion exponent $s \in \mathbb{R}$,

$$G_{\mathcal{R}} = 1 + \alpha(1 - \mathcal{R})B_+[G_{\mathcal{R}}^{1+s}]. \quad (5)$$

The linear DSE is recovered with $s = 0$. The analytic version of this DSE, including the ϵ dependence, reads

$$G_{\mathcal{R}}(\alpha, \epsilon, L) = 1 + \alpha G_{\mathcal{R}}^{1+s}(\alpha, \epsilon, \partial_\rho) (e^{L(\rho - \ell\epsilon)} - 1) F(\epsilon, \rho)|_{\rho=0}. \quad (6)$$

Since the power series $G_{\mathcal{R}}(\alpha, \epsilon, L)$ starts with 1, it can be computed recursively order by order from Eq. (6). The parenthesis $e^{L(\rho - \ell\epsilon)} - 1 = L(\rho - \ell\epsilon) + \mathcal{O}(\rho^2, \epsilon^2)$ cancels the simple pole of $F(\epsilon, \rho)$ from Eq. (4), and the right-hand side is regular at $\epsilon = 0$, as it should be for a renormalized Green's function. To compute the solution in MOM at the physical dimension $\epsilon = 0$, it is sufficient to work with $F(0, \rho)$.

The Green's function in MOM is unity at $L = 0$ (recall that $G_{\mathcal{R}}$ denotes the projection onto suitable tensors, such that the tree level term is indeed 1), it therefore has a power series expansion of the form

$$G_{\mathcal{R}}(\alpha, \epsilon, L) = 1 + \sum_{j=1}^{\infty} \gamma_j(\alpha, \epsilon) L^j. \quad (7)$$

Here, $\gamma_1(\alpha, \epsilon) =: \gamma(\alpha, \epsilon)$ is the anomalous dimension. In our setting, where there are no other Green's functions, the beta function of the theory is $\beta(\alpha, \epsilon) = s\alpha\gamma(\alpha, \epsilon) - \alpha\epsilon$, where s

is the exponent from Eq. (5). The extra factor $\alpha\epsilon$ represents the explicit scale dependence of the coupling constant in a noninteger spacetime dimension, to be discussed in Sec. II A. The Callan-Symanzik equation [25,26],

$$\partial_L G_{\mathcal{R}}(\alpha, \epsilon, L) = (\gamma(\alpha, \epsilon) + \beta(\alpha, \epsilon)\partial_\alpha) G_{\mathcal{R}}(\alpha, \epsilon, L), \quad (8)$$

implies that all higher functions $\gamma_j(\alpha, \epsilon)$ in Eq. (7) are determined from γ_1 according to

$$\gamma_j(\alpha, \epsilon) = \frac{1}{j} (\gamma(\alpha, \epsilon) + (s\alpha\gamma(\alpha, \epsilon) - \epsilon\alpha)\partial_\alpha) \gamma_{j-1}(\alpha, \epsilon). \quad (9)$$

Hence, solving the DSE is equivalent to determining $\gamma_1(\alpha, \epsilon)$, which at the same time is the anomalous dimension and the seed for the recurrence for γ_j . Inserting Eq. (7) into the DSE Eq. (6), and expanding with respect to L , produces a pseudodifferential equation for $\gamma(\alpha, \epsilon)$ [17],

$$\left. \frac{1}{F(\epsilon, \rho + \ell\epsilon)} \right|_{\rho \mapsto \gamma(\alpha, \epsilon) + (s\alpha\gamma(\alpha, \epsilon) - \epsilon\alpha)\partial_\alpha} = \alpha. \quad (10)$$

The differential operator on the lhs appears to “act on nothing,” in fact it acts on $\gamma(\alpha, \epsilon)$ itself. One may equivalently include another factor $\frac{1}{\rho}$, this makes the argument explicit,

$$\left. \frac{1}{\rho \cdot F(\epsilon, \rho + \ell\epsilon)} \right|_{\rho \mapsto \gamma(\alpha, \epsilon) + (s\alpha\gamma(\alpha, \epsilon) - \epsilon\alpha)\partial_\alpha} \gamma(\alpha, \epsilon) = \alpha.$$

For later use, we introduce the series expansion

$$\frac{1}{F(\epsilon, \rho + \ell\epsilon)} =: T_0(\rho) \cdot \left(1 + \sum_{j=1}^{\infty} \epsilon^j T_j(\rho) \right), \quad (11)$$

so that the limit $\epsilon \rightarrow 0$ involves only $T_0(\rho) = \frac{1}{F(0, \rho)}$ and produces the pseudodifferential equation

$$T_0(\gamma + s\alpha\gamma\partial_\alpha) = \alpha. \quad (12)$$

The discovery of this differential equation for the cases of Yukawa and ϕ^3 theory by Broadhurst and Kreimer in [7], and its exact solution, was the starting point for the development of the present formalism. It draws its power from the fact that in MOM, one works with power series in only one parameter, α . Versions of Eqs. (6) and (12) have allowed for symbolic and numerical calculations to very high loop order [27–29] and the study of their asymptotic and resurgent features [30–37], where the series coefficients have a combinatorial interpretation in terms of chord diagrams [38–41] and, more recently, tubings of rooted trees [37,42,43]. These references also contain generalizations to multiple kernels, to multiple insertion places, and to systems of coupled DSEs.

Notice that the entire procedure never requires us to explicitly work with counterterms. We are free to include the ϵ dependence of renormalized quantities, but we can also work at $\epsilon = 0$ throughout. Nevertheless, the formalism is entirely consistent with ordinary multiplicative renormalization, namely with

$$G_{\mathcal{R}}(\alpha, \epsilon, L) = Z_2 \cdot G_0(Z_\alpha \mu^{2\epsilon} \cdot \alpha, \epsilon, L). \quad (13)$$

The first argument of the bare Green's function G_0 , $Z_\alpha \mu^{2\epsilon} \alpha =: \alpha_0$, is the bare coupling. It has mass dimension 2ϵ , whereas the renormalized coupling α is dimensionless. The counterterms are related to the renormalization group functions by the Gross t'Hooft relations [44,45], and the equation $\beta(\alpha, \epsilon) = s\alpha\gamma(\alpha, \epsilon) - \alpha\epsilon$ is equivalent to $Z_\alpha = Z_2^s$, concretely

$$\begin{aligned} \beta(\alpha, \epsilon) &= \frac{-\epsilon\alpha}{1 + \alpha\partial_\alpha \ln(Z_\alpha(\alpha, \epsilon))}, \\ \gamma(\alpha, \epsilon) &= -\beta(\alpha, \epsilon)\partial_\alpha (\ln Z_2(\alpha, \epsilon)), \\ Z_2 &= \exp\left(-\int_0^\alpha \frac{du}{u} \frac{\gamma(u, \epsilon)}{s\gamma(u, \epsilon) - \epsilon}\right), \\ Z_\alpha &= \exp\left(-\int_0^\alpha \frac{du}{u} \frac{s\gamma(u, \epsilon)}{s\gamma(u, \epsilon) - \epsilon}\right) = Z_2^s. \end{aligned} \quad (14)$$

C. The linear DSE in MOM

If the DSE is linear, that is, $s = 0$, Eq. (10) becomes

$$\left. \frac{1}{\rho F(\epsilon, \rho + \ell\epsilon)} \right|_{\rho \rightarrow \gamma - \epsilon\alpha\partial_\alpha} \gamma(\alpha, \epsilon) = \alpha. \quad (15)$$

At this point it becomes clear why we generalized to the nonlinear DSE Eq. (5): As long as $\epsilon \neq 0$, even the linear DSE still gives rise to a differential equation, similar to the nonlinear DSE at $\epsilon = 0$ [Eq. (12)]. However, in Eq. (15) the ϵ and α dependence of $\gamma(\alpha, \epsilon)$ is essentially decoupled: We can first set $\epsilon = 0$ to obtain the algebraic equation

$$T_0(\gamma(\alpha)) = \alpha. \quad (16)$$

Its solution $\gamma(\alpha) := \gamma(\alpha, 0)$ is the exact anomalous dimension of the linear DSE at the physical dimension D_0 . We can then construct a power-series expansion in ϵ order by order from Eq. (15). Knowing $\gamma(\alpha, \epsilon)$, we get the full Green's function from the recurrence Eq. (9). Equivalently, we solve the Callan-Symanzik equation

$$\partial_L G_{\mathcal{R}}(\alpha, \epsilon, L) = (\gamma(\alpha, \epsilon) - \epsilon\alpha\partial_\alpha) G_{\mathcal{R}}(\alpha, \epsilon, L) \quad (17)$$

by separation of variables, imposing MOM conditions $G_{\mathcal{R}}(\alpha, \epsilon, 0) = 1$. This leads to

$$G_{\mathcal{R}}(\alpha, \epsilon, L) = \exp\left(\int_{\alpha e^{-\epsilon L}}^\alpha du \frac{\gamma(u, \epsilon)}{u\epsilon}\right). \quad (18)$$

Both approaches produce, in the physical limit $\epsilon \rightarrow 0$, the scaling solution

$$G_{\mathcal{R}}(\alpha, 0, L) = \exp(L \cdot \gamma(\alpha)) = \left(\frac{p^2}{\mu^2}\right)^{\gamma(\alpha)}. \quad (19)$$

II. MINIMAL SUBTRACTION

A. Scheme-dependent renormalization group

We now consider an arbitrary nonkinematic renormalization scheme, which will later be specialized to MS. Our strategy is to relate such a scheme to kinematic renormalization, in order to be able to use the machinery of Sec. IB. Concretely, we are constructing a power series $\delta(\alpha, \epsilon)$ such that the MS Green's function equals a MOM Green's function, but with renormalization point $L = -\delta$ instead of $L = 0$. A crucial insight from Secs. IB and IC is that for $\epsilon \neq 0$, the theory has a nonvanishing beta function, even in the linear case, arising from the fact that we have tacitly absorbed a scale $\mu^{2\epsilon}$ into $\alpha = \alpha_0 \mu^{-2\epsilon} Z_\alpha^{-1}$ [Eq. (13)]. An analogous phenomenon occurs when we express a MS solution through a shifted MOM solution: In MS, we are working with an expansion parameter $\bar{\alpha}$, which is related to the MOM expansion parameter α by $\bar{\alpha}(\alpha) = \alpha e^{-\epsilon\delta(\alpha, \epsilon)}$. Another way to see this is that the logarithmic scale $\delta(\alpha, \epsilon) = \ln \frac{\Delta^2}{\mu^2}$ amounts to some nonlogarithmic momentum scale $\Delta(\alpha, \epsilon)$, and the MS-coupling $\bar{\alpha}$ is expressed relative to *that* scale,

$$\bar{\alpha} = \alpha e^{-\epsilon\delta} = \alpha_0 Z_\alpha^{-1} \mu^{-2\epsilon} \left(\frac{\Delta^2}{\mu^2}\right)^{-\epsilon} = \alpha_0 Z_\alpha^{-1} \Delta^{-2\epsilon}.$$

The distinction $\alpha \neq \bar{\alpha}$ only appears since we explicitly relate MS to MOM. If one works in MS throughout, one would always be using $\bar{\alpha}$ (and simply call it α).

We thus demand that the MS-renormalized $\bar{G}_{\mathcal{R}}$ should be related to the MOM-renormalized $G_{\mathcal{R}}$ by

$$\bar{G}_{\mathcal{R}}(\bar{\alpha}(\alpha), \epsilon, L) = G_{\mathcal{R}}(\alpha, \epsilon, L + \delta(\alpha, \epsilon)). \quad (20)$$

This relation constitutes the definition of $\delta(\alpha, \epsilon)$. The fact that it is possible to find such $\delta(\alpha, \epsilon)$ is obvious in perturbation theory and discussed at length in [12,17]: At every new order in α , the renormalized Green's function is a polynomial in α and L , and one can introduce an arbitrary finite shift of its value by adding an offset, of the same order in α , to L . In fact, this mechanism is completely analogous to the mechanism that allows the order-by-order construction of counterterms, just that it involves only finite shifts. Conversely, a choice of $\delta(\alpha, \epsilon)$ amounts to a choice of perturbative renormalization scheme, and the kinematic

schemes are exactly those where δ is independent of α and ϵ .

For the MS solution, too, we can write a generic expansion of the form Eq. (7), but this time there is an additional nontrivial function $\bar{\gamma}_0(\bar{\alpha}, \epsilon)$,

$$\bar{G}_{\mathcal{R}}(\bar{\alpha}, \epsilon, L) = \sum_{j=0}^{\infty} \bar{\gamma}_j(\bar{\alpha}, \epsilon) L^j. \quad (21)$$

With our definitions, the Callan-Symanzik equation (8) takes the same form in all schemes,

$$\partial_L \bar{G}_{\mathcal{R}}(\bar{\alpha}, \epsilon, L) = (\bar{\gamma}(\bar{\alpha}, \epsilon) + \bar{\beta}(\bar{\alpha}, \epsilon) \partial_{\bar{\alpha}}) \bar{G}_{\mathcal{R}}(\bar{\alpha}, \epsilon, L). \quad (22)$$

Since we are still working with a single DSE [Eq. (5)], the renormalization group functions are still related by

$$\bar{\beta}(\alpha, \epsilon) = s\bar{\alpha} \bar{\gamma}(\bar{\alpha}, \epsilon) - \bar{\alpha} \epsilon. \quad (23)$$

Recall that in MOM, the first expansion function $\gamma_1(\alpha, \epsilon)$ coincides with the anomalous dimension $\gamma(\alpha, \epsilon)$. This is not true in general renormalization schemes. Instead, from inserting Eq. (21) into Eq. (22), we have for all schemes

$$\bar{\gamma}(\bar{\alpha}, \epsilon) = \frac{\bar{\gamma}_1(\bar{\alpha}, \epsilon) - \epsilon \bar{\alpha} \partial_{\bar{\alpha}} \bar{\gamma}_0(\bar{\alpha}, \epsilon)}{(1 + s\bar{\alpha} \partial_{\bar{\alpha}}) \bar{\gamma}_0(\bar{\alpha}, \epsilon)}. \quad (24)$$

We insert both series expansions [Eqs. (7) and (21)] into Eq. (20) to obtain

$$\begin{aligned} \bar{\gamma}_k(\bar{\alpha}(\alpha), \epsilon) &= \bar{\gamma}_k(\alpha e^{-\epsilon \delta(\alpha, \epsilon)}, \epsilon) \\ &= \sum_{j=k}^{\infty} \binom{j}{k} \gamma_j(\alpha, \epsilon) \delta^{j-k}(\alpha, \epsilon). \end{aligned} \quad (25)$$

This reveals that of the three functions $\{\bar{\gamma}(\bar{\alpha}, \epsilon), \gamma(\alpha, \epsilon), \delta(\alpha, \epsilon)\}$, only two are independent. In Eq. (25), it is natural to use the variable α of the MOM solution since we have defined $\delta = \delta(\alpha, \epsilon)$ as a function of α . However, we can revert these series,

$$\begin{aligned} \bar{\delta}(\bar{\alpha}, \epsilon) &:= \delta(\alpha(\bar{\alpha}, \epsilon), \epsilon), \quad \bar{\alpha} = \alpha e^{-\epsilon \delta(\alpha, \epsilon)} \\ \Leftrightarrow \alpha &= \bar{\alpha} e^{+\epsilon \delta(\alpha(\bar{\alpha}, \epsilon), \epsilon)} = \bar{\alpha} e^{\epsilon \bar{\delta}(\bar{\alpha}, \epsilon)}. \end{aligned} \quad (26)$$

From now on we leave out the arguments; $\bar{\delta}$ is a function of $\bar{\alpha}$ and δ is a function of α . The chain rule implies

$$\begin{aligned} \frac{\partial \bar{\alpha}}{\partial \alpha} &= \frac{\bar{\alpha}}{\alpha} - \epsilon \bar{\alpha} \partial_{\alpha} \delta \\ \Rightarrow \alpha \partial_{\alpha} \delta &= \frac{\bar{\alpha} \partial_{\bar{\alpha}} \bar{\delta}}{1 + \epsilon \bar{\alpha} \partial_{\bar{\alpha}} \bar{\delta}}, \quad \bar{\alpha} \partial_{\bar{\alpha}} \bar{\delta} = \frac{\alpha \partial_{\alpha} \delta}{1 - \epsilon \alpha \partial_{\alpha} \delta}. \end{aligned} \quad (27)$$

Theorem 1. With the shift δ resp. $\bar{\delta}$ from Eq. (26), the shifted anomalous dimension $\bar{\gamma}(\bar{\alpha}, \epsilon)$ is related to the MOM anomalous dimension $\gamma(\alpha, \epsilon)$ via

$$\begin{aligned} \bar{\gamma}(\bar{\alpha}(\alpha), \epsilon) &= \frac{\gamma(\alpha, \epsilon)}{1 + (s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha) \partial_{\alpha} \delta}, \\ \text{equivalently } \bar{\gamma}(\bar{\alpha}, \epsilon) &= \frac{\gamma(\alpha(\bar{\alpha}), \epsilon) (1 + \epsilon \bar{\alpha} \partial_{\bar{\alpha}} \bar{\delta})}{1 + s\gamma(\alpha(\bar{\alpha}), \epsilon) \bar{\alpha} \partial_{\bar{\alpha}} \bar{\delta}}. \end{aligned}$$

Proof. Equation (9) implies that

$$\frac{(j+1)\gamma_{j+1}(\alpha, \epsilon) - \gamma(\alpha, \epsilon)\gamma_j(\alpha, \epsilon)}{s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha} = \partial_{\alpha} \gamma_j(\alpha, \epsilon).$$

Derive Eq. (25) with respect to α and insert the previous equation.

$$\begin{aligned} \frac{\partial \bar{\alpha}}{\partial \alpha} \partial_{\bar{\alpha}} \bar{\gamma}_k(\bar{\alpha}, \epsilon) &= \sum_{j=k}^{\infty} \binom{j}{k} \partial_{\alpha} \gamma_j(\alpha, \epsilon) \delta^{j-k}(\alpha, \epsilon) + \sum_{j=k}^{\infty} \binom{j}{k} (j-k) \gamma_j(\alpha, \epsilon) \delta^{j-k-1}(\alpha, \epsilon) \partial_{\alpha} \delta \\ &= \sum_{j=k}^{\infty} \binom{j}{k} \frac{(j+1)\gamma_{j+1}(\alpha, \epsilon) - \gamma(\alpha, \epsilon)\gamma_j(\alpha, \epsilon)}{s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha} \delta^{j-k}(\alpha, \epsilon) + \sum_{j=k}^{\infty} \binom{j}{k+1} (k+1) \gamma_j(\alpha, \epsilon) \delta^{j-k-1}(\alpha, \epsilon) \partial_{\alpha} \delta \\ &= \frac{(k+1)\bar{\gamma}_{k+1}(\bar{\alpha}, \epsilon)}{s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha} - \frac{\gamma(\alpha, \epsilon) \bar{\gamma}_k(\bar{\alpha}, \epsilon)}{s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha} + \partial_{\alpha} \delta \cdot (k+1) \bar{\gamma}_{k+1}(\alpha, \epsilon). \end{aligned}$$

Solve the equation for $\bar{\gamma}_{k+1}$. The chain rule of Eq. (27) cancels a denominator, and ensures that the resulting equation,

$$\begin{aligned} (k+1) \bar{\gamma}_{k+1}(\bar{\alpha}, \epsilon) &= \left(\frac{s\bar{\alpha} \gamma(\alpha, \epsilon)}{1 + (s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha) \partial_{\alpha} \delta} - \epsilon \bar{\alpha} \right) \partial_{\bar{\alpha}} \bar{\gamma}_k(\bar{\alpha}, \epsilon) \\ &\quad + \frac{\gamma(\alpha, \epsilon) \bar{\gamma}_k(\bar{\alpha}, \epsilon)}{1 + (s\alpha \gamma(\alpha, \epsilon) - \epsilon \alpha) \partial_{\alpha} \delta}, \end{aligned}$$

has precisely the expected form

$$\begin{aligned} (k+1) \bar{\gamma}_{k+1}(\bar{\alpha}, \epsilon) &= (s\bar{\alpha} \bar{\gamma}(\bar{\alpha}, \epsilon) - \epsilon \bar{\alpha}) \partial_{\bar{\alpha}} \bar{\gamma}_k(\bar{\alpha}, \epsilon) + \bar{\gamma}(\bar{\alpha}, \epsilon) \bar{\gamma}_k(\bar{\alpha}, \epsilon). \end{aligned}$$

The second formula then follows from Eq. (27). ■

The special case $\epsilon = 0$ of Theorem 1 had been given already in [12].

B. Beta function in minimal subtraction

The construction in Sec. II A has been for an arbitrary renormalization scheme obtained through a shift $\delta(\alpha, \epsilon)$. We now specialize to MS.

The defining property of MS is that the counterterms only include pole terms in ϵ . We need to translate this to a statement about the renormalization group functions, because our formalism does not involve the counterterms explicitly. To this end, we use that the counterterm is related to the beta function by Eq. (14). Introduce the function $B(\bar{\alpha}, \epsilon) := \bar{\beta}(\bar{\alpha}, \epsilon) + \bar{\alpha}\epsilon$, then

$$\begin{aligned} Z_\alpha &= \exp \left(\int_0^{\bar{\alpha}} \frac{du}{u} \frac{B(u, \epsilon)}{u\epsilon - B(u, \epsilon)} \right) \\ &= \exp \left(\int_0^{\bar{\alpha}} \frac{du}{u} \frac{B(u, \epsilon)}{u\epsilon} \sum_{j=0}^{\infty} \left(\frac{B(u, \epsilon)}{u\epsilon} \right)^j \right). \end{aligned}$$

The right-hand side should be viewed as a power series in $\bar{\alpha}$. In MS, it is required to consist of pole terms in ϵ exclusively. This implies that $\frac{B(u, \epsilon)}{u\epsilon}$ consists of poles only. On the other hand, the beta function itself is regular in ϵ , therefore $B(\bar{\alpha}, \epsilon)$ does not contain poles in ϵ . The only remaining possibility is that $B(\bar{\alpha}, \epsilon)$ does not depend on ϵ at all. We obtain a well-known alternative definition of the MS scheme: The beta function $\bar{\beta}(\bar{\alpha}, \epsilon)$ in MS depends on ϵ only through a single term,

$$\bar{\beta}(\bar{\alpha}, \epsilon) = \bar{\beta}(\bar{\alpha}) - \bar{\alpha}\epsilon. \quad (28)$$

One can repeat the same argument for the counterterm Z_2 and its relation to the anomalous dimension $\gamma(\alpha, \epsilon)$, or one uses Eq. (23). In either case, one finds that in MS the anomalous dimension $\bar{\gamma}(\bar{\alpha}, \epsilon) = \bar{\gamma}(\bar{\alpha})$ is entirely independent of ϵ . This restricts the dependence of the expansion functions $\gamma_j(\alpha, \epsilon)$ of Eq. (21) on ϵ , but it does not imply that they, too, are independent. Namely Eq. (9) reads

$$\bar{\gamma}_j(\bar{\alpha}, \epsilon) = \frac{1}{j} (\bar{\gamma}(\bar{\alpha}) + (s\bar{\alpha}\bar{\gamma}(\bar{\alpha}) - \epsilon\bar{\alpha})\partial_{\bar{\alpha}})\bar{\gamma}_{j-1}(\bar{\alpha}, \epsilon). \quad (29)$$

C. The linear DSE in MS

In the linear case, $s = 0$, the second formula in Theorem 1 simplifies and the anomalous dimensions of MOM and MS are related via

$$\bar{\gamma}(\bar{\alpha}, \epsilon) = \gamma(\alpha(\bar{\alpha}), \epsilon) \cdot (1 + \epsilon\bar{\alpha}\partial_{\bar{\alpha}}\bar{\delta}(\bar{\alpha}, \epsilon)). \quad (30)$$

The anomalous dimension in MS, as a function of $\bar{\alpha}$, is independent of ϵ . Consequently, it coincides with the limit $\epsilon \rightarrow 0$ of the anomalous dimension in MOM, using $\bar{\alpha} = \alpha + \mathcal{O}(\epsilon)$,

$$\bar{\gamma}(\bar{\alpha}, \epsilon) = \bar{\gamma}(\bar{\alpha}) = \gamma(\bar{\alpha}, 0) = \gamma(\alpha). \quad (31)$$

All our definitions have been engineered such that the Callan-Symanzik equation [Eq. (17)] holds, in exactly the same form, in all schemes. Consequently, the solution of this equation in MS has the same form as in MOM, namely Eq. (18), with γ from Eq. (31). The only difference is that in MS, we do not have the boundary condition at $L = 0$, and therefore, the solution is multiplied by the undetermined factor $\bar{\gamma}_0(\bar{\alpha}, \epsilon)$ of Eq. (21),

$$\begin{aligned} \bar{G}_{\mathcal{R}}(\bar{\alpha}, \epsilon, L) &= \bar{\gamma}_0(\bar{\alpha}, \epsilon) \cdot \exp \left(\int_{\bar{\alpha}e^{-\epsilon L}}^{\bar{\alpha}} du \frac{\bar{\gamma}(u)}{u\epsilon} \right) \\ &= \bar{\gamma}_0(\bar{\alpha}, \epsilon) \cdot G_{\mathcal{R}}(\bar{\alpha}, \epsilon, L). \end{aligned} \quad (32)$$

Setting $L = 0$ in Eq. (20), one obtains

$$\bar{\gamma}_0(\bar{\alpha}(\alpha), \epsilon) = G_{\mathcal{R}}(\alpha, \epsilon, \delta(\alpha, \epsilon)). \quad (33)$$

We remark that this construction is consistent, in the sense that one can insert Eq. (30) into the integral of Eq. (32), do a change of variables $\bar{\alpha} = \alpha e^{-\epsilon\delta}$ with Eq. (27), and identify the integral Eq. (18), to recover the MOM solution written in terms of α ,

$$\exp \left(\int_{\bar{\alpha}e^{-\epsilon L}}^{\bar{\alpha}} d\bar{u} \frac{\bar{\gamma}(\bar{u})}{\bar{u}\epsilon} \right) = \frac{G_{\mathcal{R}}(\alpha, \epsilon, L + \delta(\alpha, \epsilon))}{G_{\mathcal{R}}(\alpha, \epsilon, \delta)}.$$

Another useful perspective on the quantity $\bar{\gamma}_0$ is to view an overall multiplicative scaling of $G_{\mathcal{R}}$ as a scaling of the counterterm Z_2 according to Eq. (13), namely $\bar{\gamma}_0(\bar{\alpha}, \epsilon)Z_2(\bar{\alpha}, \epsilon) = \bar{Z}_2(\bar{\alpha}, \epsilon)$. By Eq. (14), the counterterm Z_2 determines the anomalous dimension,

$$\begin{aligned} \bar{\gamma}(\bar{\alpha}, \epsilon) &= \bar{\alpha}\epsilon\partial_{\bar{\alpha}} \ln(\bar{Z}_2(\bar{\alpha}, \epsilon)) \\ &= \epsilon\bar{\alpha}\partial_{\bar{\alpha}} \ln(\bar{\gamma}_0(\bar{\alpha}, \epsilon)) + \gamma(\bar{\alpha}, \epsilon). \end{aligned} \quad (34)$$

This formula relates the MOM and MS anomalous dimensions similar to Eq. (30), but in terms of $\bar{\gamma}_0$ instead of δ . It has the additional benefit that it does not involve a change of variables $\alpha \leftrightarrow \bar{\alpha}$. We can integrate it, using Eq. (31), to compute $\bar{\gamma}_0$ explicitly,

$$\bar{\gamma}_0(\bar{\alpha}, \epsilon) = \exp \left(- \int_0^{\bar{\alpha}} \frac{du}{u} \frac{\gamma(u, \epsilon) - \gamma(u)}{\epsilon} \right). \quad (35)$$

D. Physical spacetime dimension

We still consider the linear DSE, $s = 0$, in MS. For the physical limit $\epsilon = 0$, where $\alpha = \bar{\alpha}$, the situation simplifies further. The MOM Green's function is then the scaling solution of Eq. (19), $G_{\mathcal{R}}(\alpha, 0, \delta) = e^{\delta(\alpha)\gamma(\alpha)}$. Consequently, Eq. (33) implies $\bar{\gamma}_0(\alpha) = e^{\delta(\alpha)\gamma(\alpha)}$.

Theorem 2. The solution of a linear DSE, in the physical dimension $\epsilon = 0$, is given by

$$\bar{G}_{\mathcal{R}}(\alpha, L) = \bar{\gamma}_0(\alpha) e^{L\gamma(\alpha)} = \exp((L + \delta(\alpha))\gamma(\alpha)),$$

where $\gamma(\alpha)$ is the anomalous dimension in MOM, computed from Eq. (16), and

$$\bar{\gamma}_0(\bar{\alpha}, 0) = \exp\left(-\int_0^{\bar{\alpha}} \frac{du}{u} g(u)\right),$$

$$\delta(\alpha, 0) = \frac{\ln \bar{\gamma}_0(\alpha)}{\gamma(\alpha)}, \quad \text{using } \bar{\alpha} = \alpha + \mathcal{O}(\epsilon).$$

The function $g(\alpha)$ will be determined in Theorem 3.

The ϵ -independent function $\bar{\gamma}_0(\bar{\alpha}) = \bar{\gamma}_0(\bar{\alpha}, 0)$ is, by Eq. (35), entirely determined by the order ϵ^1 term of the MOM anomalous dimension, which we call $g(\alpha) := [\epsilon^1]\gamma(\alpha, \epsilon)$. This, in turn, can be computed from the ϵ -dependent version of the Dyson-Schwinger equation given in Eq. (15).

Theorem 3. With the expansion functions of Eq. (11), $\frac{1}{F(\epsilon, \rho + \epsilon)} = T_0(\rho) + \epsilon T_0(\rho) T_1(\rho) + \dots$, the order $[\epsilon^1]$ of the MOM anomalous dimension $\gamma(\alpha, \epsilon)$ of a linear DSE is given by

$$g(\alpha) = \alpha \frac{\frac{1}{2} \partial_\rho^2 T_0|_{\rho=\gamma(\alpha)} \cdot \partial_\alpha \gamma(\alpha) - T_1(\gamma(\alpha))}{\partial_\rho T_0|_{\rho=\gamma(\alpha)}}.$$

Proof. In the proof, we write γ for $\gamma(\alpha) = \gamma(\alpha, 0)$ and g for $g(\alpha)$. The ODE Eq. (15) with Eq. (11) is

$$(T_0(\rho) + \epsilon T_0(\rho) T_1(\rho) + \dots)_{\rho \mapsto \gamma(\alpha, \epsilon) - \epsilon \alpha \partial_\alpha} = \alpha. \quad (36)$$

We take the order $[\epsilon^1]$ of this equation, the ... terms are of higher order and do not contribute. The second summand is already at order ϵ^1 , hence we merely replace $\rho \mapsto \gamma$ and use Eq. (16), $T_0(\gamma) = \alpha$, and the summand becomes

$$(T_0(\rho) T_1(\rho))_{\rho \mapsto \gamma} = \alpha T_1(\gamma).$$

For the first summand in Eq. (36), it is clear that we need at most the linear order in ϵ of the argument, $\gamma(\alpha, \epsilon) - \epsilon \alpha \partial_\alpha = \gamma + \epsilon(g - \alpha \partial_\alpha) + \mathcal{O}(\epsilon^2)$. We write $T_0(\rho) = \sum_{j=1}^{\infty} t_j \rho^j$ and consider a fixed order j .

$$[\epsilon^1](\gamma + \epsilon(g - \alpha \partial_\alpha))^j = \sum_{k=0}^{j-1} \gamma^k (g - \alpha \partial_\alpha) \gamma^{j-k-1}.$$

The factor g can be pulled out, giving $\gamma^{j-k-1} \cdot g$. With the chain rule, the derivative term becomes

$$\sum_{k=0}^{j-1} \gamma^k (j-k-1) \gamma^{j-k-2} \partial_\alpha \gamma = \frac{j(j-1)}{2} \gamma^{j-2} \cdot \partial_\alpha \gamma.$$

In both cases, the summand can be interpreted as a summand of a derivative of T_0 , therefore

$$[\epsilon^1]T_0(\dots) = (\partial_\rho T_0)_{\rho \mapsto \gamma} \cdot g - \frac{\alpha}{2} (\partial^2 \rho T_0)_{\rho \mapsto \gamma} \cdot \partial_\alpha \gamma.$$

We see that in the order $[\epsilon^1]$ of Eq. (36), the sought-after g appears as a factor,

$$(\partial_\rho T_0)_{\rho \mapsto \gamma} \cdot g - \frac{\alpha}{2} (\partial^2 \rho T_0)_{\rho \mapsto \gamma} \cdot \partial_\alpha \gamma + \alpha T_1(\gamma) = 0.$$

As long as we are able to compute the Mellin transform T_0 , T_1 in closed form, we obtain $g(\alpha)$ in closed form. Furthermore, notice that

$$\partial_\rho^2 T_0|_{\rho=\gamma(\alpha)} \partial_\alpha \gamma(\alpha) = \partial_\alpha (\partial_\rho T_0|_{\rho=\gamma(\alpha)})$$

and therefore the integral

$$\begin{aligned} & \int_0^{\bar{\alpha}} \frac{du}{u} u \frac{\frac{1}{2} \partial_\rho^2 T_0|_{\rho=\gamma(u)} \partial_u \gamma(u)}{\partial_\rho T_0|_{\rho=\gamma(u)}} \\ &= \int_0^{\bar{\alpha}} \frac{1}{2} du \partial_u \ln(\partial_\rho T_0|_{\rho=\gamma(u)}) = \frac{1}{2} \ln(\partial_\rho T_0|_{\rho=\gamma(\bar{\alpha})}) \end{aligned} \quad (37)$$

can always be solved analytically. The only potentially nontrivial integration in the computation of $\bar{\gamma}_0$ in Theorem 2 is that of the second summand, $\frac{T_1}{\partial_\rho T_0}$ in Theorem 3, so that we have a good chance of finding a closed-form solution for $\bar{\gamma}_0$ provided we know T_1 and T_0 in closed form, as claimed in the abstract. In all examples considered below, the integration can be done analytically.

III. EXAMPLES

In the remainder of the article, we demonstrate how to use the formalism for some examples of linear DSEs. As will be explained around Eq. (38), we will actually be using the MS-bar scheme in order to streamline the notation. All computations are implemented in a *Mathematica* notebook that is available as an attachment to the electronic version of the article and from the author's website [46]. To keep the paper short, we restrict ourselves to rather simple examples. Recall that the Green's function $G_{\mathcal{R}}$ is a projection onto a tree level tensor, therefore, it is a scalar quantity regardless of whether the fields are scalars themselves. The formalism allows for the kernel diagram to have arbitrary loop number, however, since we are considering only one kernel, it is physically sensible to choose the one that has lowest loop number.

In $D = D_0 - 2\epsilon$ dimensions, with propagator powers 1 and $1 - \rho$, the 1-loop multiedge diagram of Fig. 1 has superficial degree of convergence

$$\omega = 2 - \frac{D_0}{2} - \rho + \epsilon.$$

The multiedge has two vertices, each of which has a Feynman rule $(-i\lambda)$. Its Minkowski-space integral evaluates to

$$\begin{aligned} \tilde{F}(\epsilon, \rho) &= (p^2)^{2-\frac{D_0}{2}-\rho+\epsilon} \int \frac{d^D k}{(2\pi)^D} \frac{i}{(k+p)^2} \frac{i}{(k^2)^{1-\rho}} (-i\lambda)^2 \\ &= \frac{i\lambda^2}{(4\pi)^{\frac{D_0}{2}-\epsilon}} \frac{\Gamma(2-\frac{D_0}{2}-\rho+\epsilon)\Gamma(\frac{D_0}{2}-\epsilon-1+\rho)\Gamma(\frac{D_0}{2}-\epsilon-1)}{\Gamma(D_0-2-2\epsilon+\rho)\Gamma(1-\rho)}. \end{aligned}$$

We define the coupling α such that the power in α coincides with the loop number. Hence, $\alpha \propto \lambda^2$. Moreover, we switch to MS bar instead of MS renormalization conditions in order to absorb powers of (4π) and the Euler Mascheroni constant γ_E into the coupling, so that

$$\alpha := \frac{\lambda^2}{(4\pi)^{\frac{D_0}{2}}} \left(\frac{4\pi}{e^{\gamma_E}} \right)^\epsilon. \quad (38)$$

Notice that if we want to recover the MS solution from MS bar, the ϵ expansion in Eq. (11) acquires additional powers of γ_E and $\ln(4\pi)$, which are absent in MS bar. The order $[\epsilon^0]$ is unchanged, consistent with our above result (Theorem 1) that for a linear DSE, the anomalous dimension at $\epsilon = 0$ is the same in all schemes. The transformation MS $\leftrightarrow$ MS bar has been spelled out in more detail in [12].

The overall factor i in the Mellin transform gets absorbed by the definition of the 1PI self energy $i\Sigma$. Since the full propagator is a geometric series in 1PI propagators, the DSE in Eq. (5) for a propagator 1PI Green's function $G_{\mathcal{R}}$ should have a negative sign, $G_{\mathcal{R}} = 1 - \alpha(1 - \mathcal{R})B_+[G_{\mathcal{R}}^{1+s}]$. We will continue using our original definition Eq. (5), which means that formally the physical value of α is negative for propagator DSEs. We leave out the prefactor $i\alpha$ from the Mellin transform by setting $i\alpha\tilde{F}(\epsilon, \rho) = F(\epsilon, \rho)$, so that now

$$\begin{aligned} F(\epsilon, \rho) &= \frac{\Gamma(2-\frac{D_0}{2}-\rho+\epsilon)\Gamma(\frac{D_0}{2}-\epsilon-1+\rho)\Gamma(\frac{D_0}{2}-\epsilon-1)}{e^{-\gamma_E\epsilon} \cdot \Gamma(D_0-2-2\epsilon+\rho)\Gamma(1-\rho)}. \end{aligned} \quad (39)$$

A. Yukawa rainbows

Massless Yukawa theory contains fermions ψ and mesons ϕ with an interaction vertex $\lambda\bar{\psi}\psi\phi$, and is perturbatively renormalizable at $D_0 = 4$. After projection to the tree level tensor structure, see [1,7,32], the Feynman integral for the fermion propagator coincides with the scalar multiedge. Setting $D_0 = 4$, the Mellin transform

Eq. (39) is

$$F(\epsilon; \rho + \epsilon) = \frac{-e^{\gamma_E\epsilon}\pi\Gamma(1-\epsilon)}{\sin(\pi\rho)\Gamma(1-\epsilon-\rho)\Gamma(2-\epsilon+\rho)}.$$

The functions of Eq. (11) are

$$\begin{aligned} T_0(\rho) &= -\rho \cdot (1 + \rho) \\ T_1(\rho) &= -H_{-\rho} - H_{1+\rho}. \end{aligned}$$

Here, $H_n = \sum_{k=0}^n \frac{1}{k}$ is the harmonic number, whose analytic continuation is the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ according to $H_z = \gamma_E + \psi(z+1)$.

The anomalous dimension at $\epsilon = 0$, in all schemes, is computed from Eq. (16),

$$\alpha = T_0(\gamma) = -\gamma(1 + \gamma),$$

hence we reproduce the result of [1,9] for the fermion propagator in MOM,

$$\begin{aligned} G_{\mathcal{R}} &= e^{L\gamma(\alpha)}, \\ \gamma(\alpha) &= \frac{-1 + \sqrt{1-4\alpha}}{2} = -\alpha - \alpha^2 - 2\alpha^3 - 5\alpha^4 - \dots \end{aligned} \quad (40)$$

The series coefficients are Catalan numbers. We determine the function $g(\alpha)$ from Theorem 3, where

$$\partial_\rho T_0 = -1 - 2\rho, \quad \frac{1}{2}\partial_\rho^2 T_0 = -1, \quad \partial_\alpha \gamma = \frac{-1}{\sqrt{1-4\alpha}}.$$

This leads to an expression with digamma functions ψ ,

$$\begin{aligned} g &= \frac{-\alpha}{1-4\alpha} - \alpha \frac{2\gamma_E + \psi\left(\frac{3-\sqrt{1-4\alpha}}{2}\right) + \psi\left(\frac{3+\sqrt{1-4\alpha}}{2}\right)}{\sqrt{1-4\alpha}} \\ &= -2\alpha - 7\alpha^2 + (2\zeta(3) - 26)\alpha^3 + (8\zeta(3) - 99)\alpha^4 + \dots \end{aligned}$$

To find the offset function $\tilde{\gamma}_0(\alpha, 0)$ according to Theorem 2, we need to integrate $g(\alpha)/\alpha$. This integral is easier than it

looks because ψ is the derivative of the Euler gamma function. One obtains

$$\begin{aligned}\bar{\gamma}_0(\alpha) &= \frac{e^{\gamma_E(1-\sqrt{1-4\alpha})}\Gamma\left(\frac{3-\sqrt{1-4\alpha}}{2}\right)}{(1-4\alpha)^{\frac{1}{4}}\Gamma\left(\frac{3+\sqrt{1-4\alpha}}{2}\right)} = \frac{e^{-2\gamma_E\gamma}\Gamma(1-\gamma)}{\sqrt{1+2\gamma}\Gamma(2+\gamma)} \\ &= 1 + 2\alpha + \frac{11}{2}\alpha^2 + \left(17 - \frac{2}{3}\zeta(3)\right)\alpha^3 + \dots\end{aligned}\quad (41)$$

This confirms the formula [12, Eq. (4.15)], which had been discovered experimentally from matching the first 20 terms of the series expansion. In particular, the general result of Theorem 3 explains the empirical observation that $\bar{\gamma}_0$ and δ contain the anomalous dimension $\gamma(\alpha)$ as “building blocks.” From Theorem 2, we find the closed formula

$$\begin{aligned}\delta(\alpha) &= -2\gamma_E + \frac{2\ln\left((1-4\alpha)^{\frac{1}{4}}\frac{\Gamma\left(\frac{3+\sqrt{1-4\alpha}}{2}\right)}{\Gamma\left(\frac{3-\sqrt{1-4\alpha}}{2}\right)}\right)}{1-\sqrt{1-4\alpha}} \\ &= -2 - \frac{3}{2}\alpha + \left(\frac{2}{3}\zeta(3) - \frac{19}{6}\right)\alpha^2 + \left(\frac{4}{3}\zeta(3) - \frac{103}{12}\right)\alpha^3 \\ &\quad + \dots\end{aligned}\quad (42)$$

With these functions, the exact solution of the Yukawa rainbow DSE in minimal subtraction is

$$\bar{G}_R(\alpha, L) = \bar{\gamma}_0(\alpha)e^{L\gamma(\alpha)} = \exp((L + \delta(\alpha))\gamma(\alpha)).$$

B. ϕ^3 rainbows

In $D_0 = 6$, the one-loop multiedge, and its corresponding ladder solution shown in Fig. 3, appear as the propagator correction in ϕ^3 theory. We can immediately apply the formalism. The Mellin transform is the specialization of Eq. (39) to $D_0 = 6$,

$$\begin{aligned}F(\epsilon; \rho + \epsilon) &= \frac{e^{\gamma_E\epsilon}\pi\Gamma(2-\epsilon)}{\sin(\pi\rho)\Gamma(1-\epsilon-\rho)\Gamma(4-\epsilon+\rho)}, \\ T_0 &= \rho(\rho+1)(\rho+2)(\rho+3), \\ T_1 &= 1 - H_{-\rho} - H_{\rho+3}.\end{aligned}\quad (43)$$

The anomalous dimension is

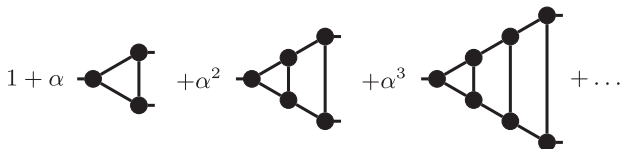


FIG. 3. Sum of ladders in ϕ^3 theory.

$$\gamma(\alpha) = \frac{-3 + \sqrt{5 + 4\sqrt{1+\alpha}}}{2}, \quad (44)$$

which again reproduces [2]. To compute the MS solution, Theorem 3 results in

$$\begin{aligned}g &= \alpha \frac{5 + \sqrt{1+\alpha}(6 + (3+2\gamma)(-1 + H_{-\gamma} + H_{3+\gamma}))}{4(1+\alpha)(3+2\gamma)^2} \\ &= \frac{4}{9}\alpha - \frac{535}{1296}\alpha^2 + \left(\frac{9077}{23328} - \frac{\zeta(3)}{108}\right)\alpha^3 + \dots\end{aligned}$$

Inserting this into Theorem 2, we obtain closed-form expressions that confirm the experimental finding of [12, Eq. (5.1)],

$$\begin{aligned}\bar{\gamma}_0 &= \frac{6\sqrt{3}e^{-\gamma(2\gamma_E-1)}\Gamma(1-\gamma)}{(1+\alpha)^{\frac{1}{4}}\sqrt{2\gamma+3}\Gamma(4+\gamma)} \\ &= 1 - \frac{4}{9}\alpha + \frac{791}{2592}\alpha^2 + \left(-\frac{5507}{23328} + \frac{\zeta(3)}{324}\right)\alpha^3 + \dots,\end{aligned}\quad (45)$$

$$\begin{aligned}\delta &= 1 - 2\gamma_E + \frac{\ln\frac{6\sqrt{3}\Gamma(1-\gamma)}{(1+\alpha)^{\frac{1}{4}}\sqrt{2\gamma+3}\Gamma(4+\gamma)}}{\gamma} \\ &= -\frac{8}{3} + \frac{61}{144}\alpha + \left(-\frac{10493}{46656} + \frac{\zeta(3)}{54}\right)\alpha^2 + \dots\end{aligned}\quad (46)$$

C. ϕ^3 null ladders

In the present article, we restrict ourselves to Green's functions, which depend on only one kinematic variable. For the vertex in ϕ^3 theory, we achieve this by fixing one of the external momenta to zero, and inserting subdiagrams into the corresponding vertex as shown in Fig. 3.

A vertex insertion with zero momentum is equivalent to what would be a mass insertion, that is, a two-valent vertex that effectively squares the propagator it resides in, see Fig. 4. The kernel then amounts to a 1-loop multiedge with propagator powers 1 and $2-\rho$. We obtain its formula by replacing $\rho \mapsto \rho-1$ in Eq. (39) for $D_0 = 6$. Notice that the multiedge with one squared propagator is not infrared divergent in six dimensions.

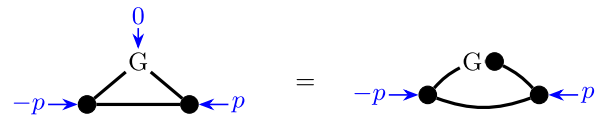


FIG. 4. Triangle diagram with zero momentum transfer is equivalent to a multiedge. Unlike Fig. 1, the inserted subdiagram does not cancel an adjacent edge, we indicate this by an extra dot.

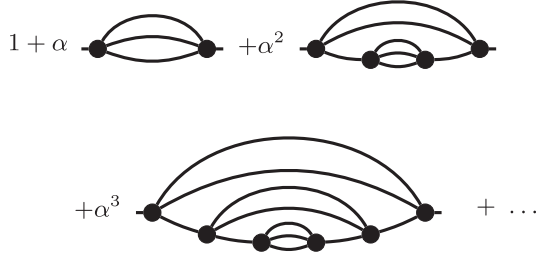


FIG. 5. Rainbows in ϕ^4 theory. The kernel is a 2-loop diagram, which is primitive since massless tadpoles vanish.

The resulting Mellin transform is very similar to Eq. (43), namely

$$\begin{aligned} T_0 &= (\rho - 1)\rho(\rho + 1)(\rho + 2), \\ T_1 &= 1 - H_{1-\rho} - H_{2+\rho}. \end{aligned}$$

The anomalous dimension is

$$\gamma(\alpha) = \frac{-1 + \sqrt{5 - 4\sqrt{1 + \alpha}}}{2}. \quad (47)$$

This, again, coincides with the MOM result of [2]. The remaining analysis proceeds as above, one finds

$$\begin{aligned} \bar{\gamma}_0 &= \frac{2e^{-\gamma(2\gamma_E-1)}\Gamma(2-\gamma)}{(1+\alpha)^{\frac{1}{4}}\sqrt{2\gamma+1}\Gamma(3+\gamma)} \\ &= 1 + \alpha + \frac{39}{32}\alpha^2 + \left(\frac{5}{3} - \frac{\zeta(3)}{12}\right)\alpha^3 + \dots, \end{aligned} \quad (48)$$

$$\begin{aligned} \delta &= 1 - 2\gamma_E + \frac{\ln\left(\frac{2}{(1+\alpha)^{\frac{1}{4}}\sqrt{2\gamma+1}\Gamma(3+\gamma)}\Gamma(2-\gamma)\right)}{\gamma} \\ &= -2 - \frac{15}{16}\alpha + \left(\frac{\zeta(3)}{6} - \frac{37}{64}\right)\alpha^2 + \left(-\frac{453}{512} + \frac{\zeta(3)}{12}\right)\alpha^3 \\ &\quad + \dots \end{aligned} \quad (49)$$

Unlike previous examples, this $\bar{\gamma}_0$ contains squares of Euler gamma functions. Taking the logarithm, as in all other cases, gives a closed formula for $\delta = \frac{\ln \bar{\gamma}_0}{\gamma}$, which can readily be obtained with a computer algebra system, its series starts with

$$\begin{aligned} \bar{\gamma}_0 &= \frac{2^{\frac{7}{4}}((1+4\alpha)(2(1+\gamma)^2 + 2\alpha(1+(1+\gamma)^2)))^{-\frac{1}{4}}\Gamma(1-\gamma)^2}{e^{4\gamma\gamma_E}\sqrt{6+4\gamma+2(\gamma+1)^2}\Gamma(1+\gamma)^2} \\ &= 1 - 2\alpha + \frac{175}{32}\alpha^2 + \left(-\frac{539}{32} + \frac{\zeta(3)}{6}\right)\alpha^3 + \left(\frac{113127}{2048} - \frac{23\zeta(3)}{24}\right)\alpha^4 + \dots \end{aligned}$$

D. ϕ^4 rainbows

The kernel diagram for the DSE may have arbitrary loop order, but it must not have subdivergences. For the propagator in ϕ^4 theory, the leading diagram is a 2-loop multiedge (“sunrise”), whose 1-loop multiedge subdiagrams are UV divergent. However, when the subdivergence is replaced by a counterterm, one obtains a 1-loop tadpole, which vanishes in a massless theory. Therefore, the 2-loop multiedge is indeed primitive and we can use our formalism for the version of rainbows shown in Fig. 5. The Mellin transform of the 2-loop multiedge is quite similar to that of the 1-loop multiedge in Eq. (39), namely

$$\frac{\Gamma(3-\rho-D_0+2\epsilon)\Gamma(\frac{D_0}{2}-\epsilon-1)^2\Gamma(\frac{D_0}{2}-\epsilon-1+\rho)}{e^{-2\gamma_E\epsilon}\Gamma(\frac{3}{2}D_0-3\epsilon-3+\rho)\Gamma(1-\rho)}.$$

We are interested in $D_0 = 4$. Since this is a 2-loop diagram, we now need $F(\epsilon, \rho - 2\epsilon)$ to compute T_1 of Eq. (11), the result is

$$\begin{aligned} T_0 &= \rho(\rho + 1)^2(\rho + 2), \\ T_1 &= -\frac{1}{\rho + 1} - \frac{1}{\rho + 2} - 2H_{-\rho} - 2H_{\rho}. \end{aligned}$$

Notice that T_0 contains a squared factor, which was not the case in any other model considered. This difference leads to interesting consequences for a resurgence analysis of a nonlinear DSEs in [37]. In our case of a linear DSE, the anomalous dimension is

$$\gamma(\alpha) = \sqrt{\frac{1 + \sqrt{1 + 4\alpha}}{2}} - 1. \quad (50)$$

Application of our formalism delivers

$$\begin{aligned} \delta &= -4 + \frac{31}{16}\alpha + \left(-\frac{811}{192} + \frac{\zeta(3)}{3}\right)\alpha^2 \\ &\quad + \left(\frac{18071}{1536} - \frac{5\zeta(3)}{6}\right)\alpha^3 + \dots \end{aligned}$$

Our list of examples is not exhaustive, further single-kernel DSEs that can be solved with this formalism have

appeared, e.g., in [6]. We merely remark that certain models that immediately come to mind, such as null boxes in ϕ^4 theory, require extra care because of potential infrared divergences. We leave the treatment of IR divergences for future work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [46].

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