

Holographic dictionary for Lifshitz and hyperscaling violating black holesWan Cong,^{1,*} David Kubizňák^{2,†} Robert B. Mann^{3,‡} and Manus R. Visser^{4,§}¹*University of Vienna, Faculty of Physics, Boltzmannngasse 5, A 1090 Vienna, Austria*²*Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic*³*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*⁴*Institute for Mathematics, Astrophysics and Particle Physics, and Radboud Center for Natural Philosophy, Radboud University, 6525 AJ Nijmegen, The Netherlands*

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We develop a novel holographic dictionary for the thermodynamics of black holes with Lifshitz and hyperscaling violating asymptotics, generalizing the dictionary for anti-de Sitter black holes. Using our dictionary we show that the holographic Euler equation is dual to a generalized Smarr formula for these black holes, and we find a precise match between the extended bulk and boundary first law. Notably, the dictionary for the central charge appearing in the Euler relation depends on the hyperscaling violating parameter, but not on the Lifshitz dynamical exponent.

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Introduction. Holography, or gauge/gravity duality, is the idea that a theory of quantum gravity in a $(d+1)$ -dimensional bulk spacetime is dual to a quantum theory without gravity living on the d -dimensional asymptotic boundary of the bulk spacetime [1,2]. This duality is best understood for gravitational theories in asymptotically anti-de Sitter (AdS) spacetime, for which the dual theory is a conformal field theory (CFT), also known as the “AdS/CFT correspondence” [3–5]. An important feature of this correspondence is that the thermodynamics of stationary AdS black holes [6] is dual to the thermodynamics of high-energy equilibrium states in a strongly coupled, large- N conformal gauge theory [7]. For instance, the black hole entropy $S = A/4G$ [8,9] is identified with the thermodynamic entropy $\tilde{S}$ of the CFT equilibrium state, and the internal energy $\tilde{E}$, temperature $\tilde{T}$, electric potential $\tilde{\Phi}$, and charge $\tilde{Q}$ of the CFT state are proportional to, respectively, the mass M , temperature T , electric potential Φ and charge Q of the black hole [10–13]

$$\tilde{E} = \frac{ML}{R}, \quad \tilde{T} = \frac{TL}{R}, \quad \tilde{S} = \frac{A}{4G}, \quad \tilde{\Phi} = \frac{\Phi}{R}, \quad \tilde{Q} = QL, \quad (1)$$

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where R is the curvature radius of the spatial geometry on which the CFT lives, L is the AdS curvature radius, A is the black hole horizon area, and G is Newton's constant. The proportionality factors in the holographic dictionary for $\tilde{E}$, $\tilde{T}$, and $\tilde{\Phi}$ appear due to a specific choice of CFT metric, for which the CFT time is R/L times the static AdS time and the curvature radius is R [4,5,14].

The general principles behind holography are expected to apply to nonasymptotically AdS spacetimes as well. Since the isometries of the bulk spacetime should match with the global symmetries of the dual boundary theory, field theories that are not conformally (or scale) invariant are not dual to AdS spacetime, but possibly to a different bulk geometry. For example, some quantum field theories exhibit the anisotropic scaling symmetry $\{t, x^i\} \rightarrow \{\alpha^z t, \alpha x^i\}$, parametrized by the dynamical critical exponent z . Together with time and space translations and space rotations, this scaling symmetry forms the Lifshitz symmetries. For $z \neq 1$, Lifshitz invariant theories describe nonrelativistic quantum critical systems. The anisotropic scaling symmetry can be geometrically realized in the bulk by the so-called Lifshitz spacetimes, which are a generalization of AdS spacetime [15–18] (see [19] for a review). Another generalization is given by hyperscaling violating geometries, which are dual to theories violating the hyperscaling property that entropy (or free energy) scales with its naive dimension. For hyperscaling violating Lifshitz (HVL) theories (on the plane), entropy scales with temperature as $S \propto T^{(d-1-\theta)/z}$, where θ is the hyperscaling violation exponent [20,21]. Various black hole solutions with Lifshitz and/or hyperscaling violating asymptotics have been obtained in [18,22–33].

In this note we develop a novel holographic dictionary for black hole thermodynamics in HVL theories. We find a

generalization of the standard central charge dictionary $C \propto L^{d-1}/G$ for Einstein gravity, and obtain a dual relation between a generalized Smarr formula, relating the thermodynamic quantities in the bulk, and the holographic Euler relation on the boundary [11]

$$\tilde{E} = \tilde{T}\tilde{S} + \tilde{\Phi}\tilde{Q} + \mu C. \quad (2)$$

The central charge C is defined as the dimensionless proportionality factor of the energy, entropy and electric charge in the canonical ensemble, i.e., $\tilde{\chi} = Cf_{\tilde{\chi}}(\tilde{T}, \tilde{Q}, V)$ for $\tilde{\chi} \in \{\tilde{E}, \tilde{S}, \tilde{Q}\}$. Furthermore, the chemical potential is defined as the conjugate quantity to C in the first law, as $\mu \equiv (\partial\tilde{E}/\partial C)_{\tilde{S}, \tilde{Q}, V}$. We suppress the tildes on μ , C since they do not have an equivalent definition in the bulk. The Euler relation (2) follows from the nonextensive scaling relation $\tilde{E}(\alpha\tilde{S}, V, \alpha\tilde{Q}, \alpha C) = \alpha\tilde{E}(\tilde{S}, V, \tilde{Q}, C)$, where V is the spatial volume. This scaling relation only holds in the large- C limit, since there are $1/C$ corrections at finite central charge. More specifically, the Euler relation is valid for any large- C field theory at finite temperature in the deconfined phase. Moreover, it is the same for theories with different global symmetries, e.g., conformal and Lifshitz symmetries.

Furthermore, we show that an extended version of the first law for these black holes is dual to the thermodynamic first law on the boundary

$$d\tilde{E} = \tilde{T}d\tilde{S} + \tilde{\Phi}d\tilde{Q} - pdV + \mu dC, \quad (3)$$

provided that μ obeys the holographic Euler equation (2) and the boundary pressure p satisfies the equation of state for Lifshitz and hyperscaling violating theories, see Eq. (22) below. Although we work within the context of a large class of static, charged analytic black hole solutions to a generalized Einstein-Maxwell-Dilaton (EMD) theory [33], we expect that our results apply to HVL black holes in general, thus showing that the recently developed holographic description of extended black hole thermodynamics in [11,13] has a wide range of applicability.

Lifshitz and hyperscaling violating black holes. A line element of the general form¹

$$ds^2 = \chi \left[-\left(\frac{r}{L}\right)^{2z} f(r) dt^2 + \frac{L^2}{f(r)r^2} dr^2 + r^2 d\Omega_{k,d-1}^2 \right],$$

$$\chi(r) := \left(\frac{r}{r_F}\right)^{-\frac{2\theta}{d-1}}, \quad (4)$$

in $(d+1)$ dimensions, with $f(r) \rightarrow 1$ as $r \rightarrow \infty$, has been suggested to be dual to states in Lifshitz invariant field

theories with hyperscaling violation. For $z = 1$ and $\theta = 0$ we recover asymptotically AdS spacetime. Here, k parametrizes the topology of the $(t, r) = \text{constant}$ surfaces: $k = \{-1, 0, 1\}$ for hyperbolic, planar, and spherical topology respectively. When $f(r) = 1$ under the rescalings

$$t \rightarrow \alpha^z t, \quad \Omega \rightarrow \alpha \Omega, \quad r \rightarrow \alpha^{-1} r, \quad (5)$$

the line element (4) scales as

$$ds \rightarrow \alpha^{\theta/(d-1)} ds, \quad (6)$$

with α being a dimensionless scaling parameter. Leaving aside the scaling of the bulk radius r , this rescaling of the boundary metric is characteristic of hyperscaling violating field theories. Hence, we refer to (4) as asymptotically Lifshitz and hyperscaling violating geometries (although, strictly speaking, this is only true for $k = 0$, since geometries with $k = \pm 1$ are not invariant under spatial translations). As is standard for hyperscaling violating theories, the dual black hole geometry is considered to be valid up to a certain large radius r_F , which corresponds to a UV cutoff scale on the boundary, since the dual field theory could have a different UV completion [28,31,34].

Static, charged black holes with line element (4) were found as analytic solutions to a generalized Einstein-Maxwell-Dilaton theory [33], consisting of Einstein gravity minimally coupled to a real scalar field ϕ , called the “dilaton,” and three Abelian gauge fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. The action of the generalized EMD theory is given by

$$I = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\phi)^2 + V(\phi) - \frac{1}{4}X(\phi)F^2 - \frac{1}{4}Y(\phi)H^2 - \frac{1}{4}Z(\phi)K^2 \right], \quad (7)$$

with field strengths $F = d\mathbf{A}$, $H = d\mathbf{B}$ and $K = d\mathbf{C}$ and

$$\mathcal{V} = -2\Lambda_0 e^{\lambda_0 \phi}, \quad X = X_0 e^{\lambda_1 \phi}, \quad Y = Y_0 e^{\lambda_2 \phi}, \quad Z = Z_0 e^{\lambda_3 \phi},$$

where the λ_i are given in terms of (d, θ, z) ; namely,

$$\lambda_0 = \frac{2\theta}{\gamma(d-1)}, \quad \lambda_1 = -\frac{2(d-\theta-1+\theta/(d-1))}{\gamma},$$

$$\lambda_2 = -\frac{2(d-2)(d-\theta-1)}{\gamma(d-1)}, \quad \lambda_3 = \frac{\gamma}{d-\theta-1},$$

$$\gamma = \sqrt{2(d-\theta-1)\left(z-1-\frac{\theta}{d-1}\right)}, \quad (8)$$

allowing for a nonzero hyperscaling violating parameter θ . The gauge field $\mathbf{A}$ is responsible for the anisotropic scaling symmetry, $\mathbf{B}$ supports nontrivial (spherical or hyperbolic) horizon topology, and $\mathbf{C}$ allows for nonzero electric charge. The analytic solution for these black holes is given by (4)

¹Note that we are working with $d+1$ bulk spacetime dimensions, whereas [33] has $d+2$ bulk dimensions. Moreover, we redefined V_0 in [33] as $-2\Lambda_0$, and ϕ_0 as $-\gamma \log r_0$, see (11).

with [33]

$$f(r) = 1 + k \frac{(d-2)^2}{(d-\theta+z-3)^2} \frac{L^2}{r^2} - \frac{m}{r^{d-\theta+z-1}} + \frac{q^2}{r^{2(d-\theta+z-2)}}, \quad (9)$$

$$\mathbf{C} = -\frac{\rho_3}{d-\theta+z-3} r_0^{\gamma_{\lambda_3}} r^{-(d-\theta+z-3)} dt, \quad (10)$$

$$\phi = \gamma \log(r/r_0), \quad (11)$$

generalizing previously discovered solutions for charged spherical black holes with Lifshitz asymptotics [27] (to nonzero θ) and charged black branes with arbitrary Lifshitz and hyperscaling violation parameters [29] (to nontrivial topology). The respective mass and charge parameters are m and q , and

$$L^2 = -\frac{(d-\theta+z-2)(d-\theta+z-1)}{2\Lambda_0 r_F^{2\theta/(d-1)}} r_0^{2\theta/(d-1)}, \quad (12)$$

$$\rho_3^2 = \frac{2q^2(d-\theta-1)(d-\theta+z-3)}{Z_0 L^{2z} r_0^{\gamma_{\lambda_3}}} r_F^{2\theta/(d-1)}, \quad (13)$$

with L the bulk curvature radius (analogous to the AdS curvature radius), and r_0 an arbitrary length scale. This solution is only valid for $d-\theta+z-3 > 0$ and $\theta < d-1$.

Extended bulk thermodynamics. The thermodynamic quantities of the hyperscaling violating Lifshitz black hole solution were computed in [33] (see also, e.g., [27,29,35–37]). The Hawking temperature T , Bekenstein-Hawking entropy S , ADM mass M , electric charge Q , and electric potential Φ are²

$$T = \frac{f'(r_h)}{4\pi} \left(\frac{r_h}{L}\right)^{z+1} = \frac{r_h^z}{4\pi L^{z+1}} \left[d-\theta+z-1 + \frac{k(d-2)^2}{d-\theta+z-3} \frac{L^2}{r_h^2} - \frac{(d-\theta+z-3)q^2}{r_h^{2(d-\theta+z-2)}} \right], \quad S = \frac{\omega_{k,d-1}}{4G} r_h^{d-\theta-1} r_F^\theta, \\ M = \frac{\omega_{k,d-1}}{16\pi G} (d-\theta-1) m L^{-z-1} r_F^\theta, \quad Q = \frac{\omega_{k,d-1}}{16\pi G} Z_0 \rho_3 L^{z-1} r_F^{\theta-2\theta/(d-1)}, \quad \Phi = \frac{q r_F^{\theta/(d-1)} r_0^{\lambda_3 \gamma/2}}{\sqrt{\frac{Z_0(d-\theta+z-3)}{2(d-\theta-1)}} L^z r_h^{d-\theta+z-3}}, \quad (14)$$

where r_h is horizon radius, given by the largest positive root of $f(r) = 0$. The mass M depends on the parameters (m, L, r_F) ; m can be expressed, in turn, in terms of r_h , L , and q by solving $f(r_h, L, m, q) = 0$ for m . To derive a generalized Smarr relation and first law, we express M as a function of the thermodynamic variables S and Q , and the analog of the “bulk pressure” $P := -\frac{\Lambda_0}{8\pi G}$ for EMD theory [38], where Λ_0 is the “bare” cosmological constant appearing in the Lagrangian. Due to the coupling of Λ_0 to the dilaton, this does not correspond to the pressure of a bulk perfect fluid as usual for the cosmological constant, but we will continue to use P as a useful theory parameter. The result (see Supplemental Material [39]) is that $M = M(S, Q, P)$. The mass also depends on the length scales r_F and r_0 , but we treat them here as fixed parameters, since they do not have a clear thermodynamic interpretation from the bulk and boundary perspective.

From the mass function (A1) in the Supplemental Material [39] the following scaling relation may be inferred:

$$M(S, Q, P) = \alpha^{2+\theta-d} M(\alpha^{d-\theta-1} S, \alpha^{d-\theta+z-3} Q, \alpha^{-2} P), \quad (15)$$

which arises from the rescaling $r_h \rightarrow \alpha r_h$, $L \rightarrow \alpha L$, and $q \rightarrow \alpha^{d-\theta+z-2} q$. A generalized Smarr formula can now be derived from this scaling relation by applying Euler’s theorem for homogeneous functions [38]

$$(d-\theta-2)M = (d-\theta-1)TS + 2\Theta P + (d-\theta+z-3)\Phi Q, \quad (16)$$

with

$$\Theta := -\frac{\partial M}{\partial P} \Big|_{S,Q} \quad (17)$$

the conjugate quantity to P . If $\theta = 0$ and $z = 1$, Θ reduces to the expected expression $-r^d \omega_{k,d-1}/d$. Hence Θ [see Eq. (A8) in the Supplemental Material [39]] is the HVL generalization of (minus) the “thermodynamic volume,” see e.g. [40].

The generalized Smarr relation (16) is new in the literature, as the scaling relation (15) was not properly derived before. In fact, in [33,35,41] it was assumed that the standard Smarr formula for AdS black holes [given by (16) with $z = 1$ and $\theta = 0$] also holds for HVL black holes.

²The Arnowitt-Deser-Misner (ADM) mass was obtained in [33] through the background subtraction method, so that the solution with $m = 0$ has zero total gravitational energy. For hyperbolic black holes, however, this solution does not correspond to the ground state of the grand canonical ensemble, which is given by the extremal solution with zero charge and in our conventions hence has negative energy (whereas in [33] the ADM mass for $k = -1$ was redefined such that the extremal solution has zero energy).

However this is clearly not the case, since the Smarr relation (16) explicitly involves z and θ .

Next, from the function $M = M(S, Q, P)$ a variational relation follows straightforwardly. Viewing r_F and r_0 as fixed, the extended first law for charged HVL black holes reads

$$dM = TdS + \Phi dQ - \Theta dP. \quad (18)$$

Since P depends via Λ_0 on L due to (12), using (16) we can write the extended first law as

$$dM = TdS + \Phi dQ + ((d - \theta - 2)M - (d - \theta - 1)TS - (d - \theta + z - 3)\Phi Q) \frac{dL}{L}. \quad (19)$$

This expression is useful for finding a precise match between the bulk and boundary first laws.

Holographic dictionary. We would now like to find a gauge/gravity dictionary matching (19) to the thermodynamic first law of the dual field theory, given by

$$d\tilde{E} = \tilde{T}d\tilde{S} + \tilde{\Phi}d\tilde{Q} - p dV + \mu dC \quad (20)$$

where $\tilde{E}$ is the internal energy of the field theory, $\tilde{S} = S$ and $\tilde{T}$ its entropy and temperature, $\tilde{\Phi}$ and $\tilde{Q}$ its electric potential and charge, p and V its pressure and volume, and μ and C its chemical potential and central charge [11]. We assume that the gauge theory lives on a spatial geometry with line element $R^2 d\Omega_{k,d-1}^2$, where R is the boundary curvature scale, so that the spatial volume is proportional to

$$V \propto R^{d-1}. \quad (21)$$

The holographic dictionary follows from requiring (i) that the extended bulk first law (19) exactly matches with the boundary first law (20); (ii) that the chemical potential μ obeys the holographic Euler equation (2); and (iii) that the pressure p satisfies the equation of state for HVL theories [19,36]

$$p = \frac{\tilde{E}}{V} \frac{z - \theta/(d-1)}{d-1-\theta}. \quad (22)$$

These three conditions yield the dictionary given below in (24)–(28)³ We note that this equation of state is a consequence of the scaling relation

$$\tilde{E}(\tilde{S}, \alpha^{d-1-\theta} V, \tilde{Q}, C) = \alpha^{-z+\frac{\theta}{d-1}} \tilde{E}(\tilde{S}, V, \tilde{Q}, C), \quad (23)$$

which follows from the anisotropic scaling symmetry and the hyperscaling violating property. Differentiating (23)

³We also checked that the matching between the bulk and boundary extended first laws continues to hold if r_F and r_0 are allowed to vary in the bulk (see Supplemental Material [39]).

with respect to α and setting $\alpha = 1$ yields (22), where the pressure is defined as $p := -(\partial \tilde{E} / \partial V)_{\tilde{S}, \tilde{Q}, C}$. We now discuss the dictionary for the various thermodynamic parameters in turn.

First, we find the following dictionary for the central charge

$$C \propto \frac{L^{d-\theta-1} r_F^\theta}{G}. \quad (24)$$

For $\theta = 0$ this dictionary reduces to the well-known result for Einstein gravity. The central charge is dimensionless and should only depend on theory parameters, not on the specific details of the state, such as its temperature. Hence, the dictionary for C does not involve the horizon radius r_h , which is related to the temperature (14). Further, it is interesting to note that the right-hand side of (24) depends on θ but not z . It is well known that θ effectively lowers the number of dimensions, which is also apparent from (24). Moreover, it is quite natural that (24) involves r_F , since that corresponds to a UV cutoff in the boundary theory, on which the number of field degrees of freedom depends.

We note the dictionary (24) is consistent with the definition of C as the proportionality constant of the energy, entropy, and electric charge, i.e., $\tilde{E}, S, \tilde{Q} \propto C$. For instance, the Bekenstein-Hawking entropy (14) is proportional to the central charge,

$$S \propto C x^{d-\theta-1}, \quad (25)$$

where we defined $x \equiv r_h/L$. Since $S \propto C$ and the entropy does not depend on r_0 , the entry (24) for C likewise does not.

Second, the dictionary for the internal energy and temperature follows partly from symmetry arguments. We can make the internal energy and temperature dimensionless by simply multiplying them with the bulk curvature radius L . On the field theory side, this should match with products of boundary quantities that are [from (5)] invariant under the anisotropic scaling symmetry, $\{t, x^i\} \rightarrow \{\alpha^z t, \alpha x^i\}$, and the simultaneous hyperscaling transformation $\{t, x^i\} \rightarrow \alpha^{-\theta/(d-1)} \{t, x^i\}$ that undoes the rescaling (6) of the metric at the boundary. The invariant combinations are

$$\tilde{E} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} = ML, \quad \tilde{T} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} = TL. \quad (26)$$

Note that the scaling dimension of energy and temperature is $[\tilde{E}] = [\tilde{T}] = -[t] = z - \theta/(d-1)$, and the scaling dimension of the boundary curvature scale and spatial coordinate is $[R] = [x^i] = -1 + \theta/(d-1)$ (see Appendix D in [36]), so that the lhs of (26) is indeed dimensionless.

Next, the dictionary entries for the electric potential and charge are more complicated, but also follow for a large part from dimensional analysis. We take the boundary electric charge $\tilde{Q}$ to be dimensionless, as for the entropy

and central charge in (20); the boundary potential $\tilde{\Phi}$ then has the same dimensions as $\tilde{E}$ and $\tilde{T}$, so that $\tilde{\Phi} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}}$ is scale invariant. The bulk charge scales as $Q \propto q L^{-1} r_F^{\theta/(d-1)} r_0^{1-z+\theta/(d-1)}/G$ and the bulk potential as $\Phi \propto q r_F^{\theta/(d-1)} r_0^{z-1-\theta/(d-1)}/(L^z r_h^{d-\theta+z-3})$, where the charge parameter and Newton's constant have mass dimension $[q] = -d + \theta - z + 2$ and $[G] = -d + 1$, respectively. We can thus make the boundary charge $\tilde{Q}$ and the product $\tilde{\Phi} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}}$ dimensionless with the following dictionary

$$\tilde{\Phi} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} = L^{-z-1} r_F^{\frac{\theta}{d-1}} r_0^{1-z+\frac{\theta}{d-1}} \Phi, \quad (27)$$

$$\tilde{Q} = L^{2-z} r_F^{\frac{\theta}{d-1}} r_0^{z-1-\frac{\theta}{d-1}} Q, \quad (28)$$

which satisfies

$$\tilde{\Phi} \tilde{Q} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} = \Phi Q L. \quad (29)$$

Note this product is independent of r_0 and r_F .

For $z = 1$ and $\theta = 0$ this agrees with the standard holographic dictionary [11]. There are no $\sqrt{G}$ factors due to a different normalization of the action compared to that in [13] (in our conventions the gauge coupling constant depends on G). We further comment that this corrects an earlier dictionary (Appendix D in [11]) for the electric charge and potential of Lifshitz black holes, where it was proposed that $\tilde{\Phi} = \Phi/R^z$ and $\tilde{Q} = QL$, which is different from the dictionary above for $\theta = 0$, i.e. $\tilde{\Phi} = \Phi L^z/R^z$ and $\tilde{Q} = QL^{2-z}$. This difference ultimately arises from the fact that in [11] (following [33,35]) it was assumed that the AdS Smarr formula also holds for Lifshitz black holes, whereas here we find that a more careful study of the scaling relation for the mass shows that the correct Smarr formula is (16).

We emphasize that the left-hand side of the holographic dictionary (26)–(28) follows from invariance under Lifshitz and hyperscaling transformations. However, in order to fix the right-hand side of these equations, we imposed that the boundary and bulk extended thermodynamics should be consistent with each other. Specifically, we imposed the three conditions stated above (22).

On a *planar* boundary geometry (for $k = 0$) there is an additional relation between the thermodynamic quantities, which is dual to an extra Smarr formula in the bulk. This allows us to replace the μC term in the Euler relation with a pV term, but this comes at the expense of additional (z, θ, d) -dependent prefactors in front of the pV and $\tilde{\Phi} \tilde{Q}$ terms (that are absent for $z = 1$, $\theta = 0$), namely,

$$\tilde{E} = \tilde{T} \tilde{S} + \frac{d - \theta + z - 2}{d - \theta - 1} \tilde{\Phi} \tilde{Q} - \frac{z}{z - \theta/(d-1)} pV. \quad (30)$$

This implies that the grand canonical free energy $\tilde{F} = \tilde{E} - \tilde{T} \tilde{S} - \tilde{\Phi} \tilde{Q}$ is not equal to $-pV$ for generic z, θ . We refer to the Supplemental Material [39] for more details.

Finally, we demonstrate that the large- C Euler equation (2) is dual to the generalized Smarr relation (16). First, we express the ΘP term in the Smarr formula in terms of the boundary energy $\tilde{E}$

$$\begin{aligned} 2\Theta P &= -2P \left. \frac{\partial M}{\partial P} \right|_{S,Q} = L \left. \frac{\partial M}{\partial L} \right|_{S,Q} \\ &= R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} \left. \frac{\partial \tilde{E}}{\partial L} \right|_{S,Q} - \frac{\tilde{E}}{L} R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}}. \end{aligned} \quad (31)$$

In the first equality we used the definition of Θ (17), and in the second equality we employed the fact that Λ_0 depends on L, r_F and r_0 via (12), but the latter two length scales are fixed. In the final equality we inserted the dictionary that relates M and $\tilde{E}$. Subsequently, the partial derivative can be computed by noting that the boundary energy depends on the bulk quantities as follows:

$$\tilde{E} = \tilde{E}(S, \tilde{Q}(L, Q), V(R), C(L)), \quad (32)$$

yielding

$$\left. \frac{\partial \tilde{E}}{\partial L} \right|_{S,Q} = \frac{1}{L} ((2-z)\tilde{\Phi} \tilde{Q} + (d-\theta-1)\mu C). \quad (33)$$

To obtain this we used the dictionary for $\tilde{Q}$ (28) and C (24). Inserting this back into (31) gives

$$2\Theta P = R^{\frac{z-\theta/(d-1)}{1-\theta/(d-1)}} \frac{1}{L} ((2-z)\tilde{\Phi} \tilde{Q} + (d-\theta-1)\mu C - \tilde{E}). \quad (34)$$

This precisely agrees with the generalized Smarr formula (16) if we insert the large- C Euler equation (2) and employ the holographic dictionary (26)–(28). This extends previous derivations [10,11,13] of the Smarr formula from the Euler equation to HVL black holes, and it corrects Appendix D in [11] (see [42] for an alternative holographic interpretation of the Smarr formula). In the Supplemental Material [39] we give further evidence for the large- C Euler equation by deriving it from a generalized Cardy formula for $d = 2$, without invoking insights from holography.

Conclusion. Using the extended black hole thermodynamics as a guiding principle, we developed a novel holographic dictionary for thermal equilibrium states dual to HVL black holes, and checked that the first law and large- C Euler equation are still valid for these states. It is remarkable that the Euler equation has such a wide range of applicability.

For future work, it would be interesting to derive the dictionary for the thermodynamic black hole quantities using holographic renormalization [43–45] and, in particular, check our formula for the central charge (which can be read off from the Casimir energy for even d and $k \neq 0$). Moreover, we would like to understand whether there exists a thermodynamic interpretation of the length parameters r_F

and r_0 , perhaps relating these to “Lifshitz charges,” as proposed in a simplified setting in [36]. Finally, it is likely that our new setup will lead to a discovery of so far unseen phase transitions and critical phenomena arising for Lifshitz and hyperscaling violating theories.

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- [39] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kx5q-xk8y>. In Appendix A we state a formula for the ADM mass in terms of the other black hole parameters. Using scaling relations for the mass we derive three independent Smarr-type formulas for HVL black holes and we find an extended first law. In Appendix B we derive a Smarr formula for planar HVL black holes and we provide its holographically dual Euler relation. In Appendix C we derive the holographic Euler relation from a generalized Cardy formula for microcanonical entropy.
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