

Constrained Padé ensembles for thermal $\mathcal{N} = 4$ SYM: Quantified uncertainties and next-order predictions

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We quantify the transition between weak and strong coupling in thermal $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four space-time dimensions by constructing an *admissible ensemble* of log-aware Padé approximants that incorporate the weak- and strong-coupling expansions through $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^{-3/2})$ (λ is the 't Hooft coupling), including the nonanalytic $\lambda^{3/2}$ and $\lambda^2 \log \lambda$ terms. This replaces single-curve estimates with a reproducible uncertainty band and a well-defined central curve across the intermediate regime. The framework is *predictive*, setting testable benchmarks for forthcoming perturbative and holographic calculations.

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I. INTRODUCTION

The thermodynamics of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions ($\mathcal{N} = 4$ SYM) is a useful benchmark for interpolation across coupling. Conformality implies that the Stefan-Boltzmann-normalized ratios of pressure, energy density, and entropy density coincide,

$$\begin{aligned} p/p_0 = \varepsilon/\varepsilon_0 = S/S_0 =: f(\lambda), \\ \varepsilon - 3p = 0, \quad c_s^2 = \frac{1}{3}, \end{aligned} \quad (1)$$

so a single function exhausts equilibrium thermodynamics with $S_0 = \frac{2\pi^2}{3} d_A T^3$, $\mathcal{F}_0 = -p_0 = -\frac{\pi^2}{6} d_A T^4$, and with $d_A = N_c^2 - 1$ being the dimension of the adjoint representation. We work with the ratios $f(\lambda) = S/S_0 = p/p_0 = \varepsilon/\varepsilon_0$ throughout. On the weak side, the $\mathcal{O}(\lambda^2)$ expansion with its exact nonanalytic structure was obtained by direct/hard-thermal-loop (HTL) resummation and independently reproduced via an effective field theory (EFT) construction [1,2]; on the strong side, the large- λ expansion at large N_c follows from AdS/CFT [3,4].

Previous Padé studies provided *point estimates* without uncertainty bars, leaving the robustness of crossover predictions unclear. Earlier work used single near-diagonal Padés, which are sensitive to matching choices and to the

weak-side logarithm. Precisely in the intermediate regime where weak-coupling ceases to converge ($\lambda \gtrsim 1$) but strong-coupling corrections are still sizable ($\lambda \lesssim 10$), predictions become particularly dependent on interpolation choices. This window overlaps the phenomenologically relevant range for hot, strongly interacting matter.

Our contribution is to replace single-curve Padés with an *admissible ensemble* and to report a *model band*. We develop two independent *log-aware* routes: (i) a Hermite-Padé (HP) interpolant that matches the generalized two-point Padé of Ref. [1], including the exact $\lambda^2 \log \lambda$ term and the 4/3 factors enforcing $f \rightarrow 3/4$; and (ii) a log-subtracted two-point Padé (LSTP) that removes the known $\lambda^2 \log \lambda$ term before fitting a rational approximant to the remainder. Both satisfy standard *admissibility constraints*: no poles on $\lambda > 0$, bounded within $0.75 \leq f \leq 1$, and monotone in $\log \lambda$. The surviving ensemble quantifies interpolation uncertainty with a reproducible band and a well-defined central curve, enabling next-order predictions. The result is a quantitatively defensible crossover with transparent model dependence.

Motivated by suggestions raised in private correspondence (2021) after our $\mathcal{O}(\lambda^2)$ work [1], we asked whether a Padé construction could *predict* the next strong-coupling coefficient using only the weak-coupling expansion through $\mathcal{O}(\lambda^2)$ and the leading holographic correction $\mathcal{O}(\lambda^{-3/2})$. While we initially expected that one more weak-side order would be necessary, the *admissible ensemble* developed here (log-aware, pole-free, and bounded) shows that such predictions are feasible: we extract $A_{5/2}$ and constrain $\hat{S}_3(\lambda_*)$ with quantified model uncertainties.

Both our ensemble approach and Ref. [5] interpolate between the $\mathcal{O}(\lambda^2)$ weak-coupling expansion [1,2] and the $\mathcal{O}(\lambda^{-3/2})$ strong-coupling expansion [4], and both use a

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curvature-based diagnostic to identify a pseudocritical coupling λ_c . The methodological difference is that we replace a single interpolant with an *admissible ensemble*, yielding a reproducible uncertainty band (and central curve) rather than a single point estimate.

II. WEAK- AND STRONG-COUPLING EXPANSIONS

We use the 't Hooft coupling $\lambda = g^2 N_c$. At weak coupling the Stefan-Boltzmann-normalized entropy ratio is

$$f(\lambda) = 1 - \frac{3}{2\pi^2}\lambda + \frac{3 + \sqrt{2}}{\pi^3}\lambda^{3/2} + \frac{3}{2\pi^4}\lambda^2 \log \frac{\lambda}{\pi^2} + A_{20}\left(\frac{\lambda}{\pi^2}\right)^2 + \mathcal{O}(\lambda^{5/2}), \quad (2)$$

where

$$A_{20} = -\frac{21}{8} - \frac{9\sqrt{2}}{8} + \frac{3}{2}\gamma_E + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{25}{8}\log 2,$$

and the logarithmic coefficient multiplying $\lambda^2 \log(\lambda/\pi^2)$ is exactly $A_{2\log} = \frac{3}{2\pi^4}$. The constants A_{20} and $A_{2\log}$ were obtained and cross-checked by direct resummation [1] and by an EFT reconstruction [2].

At strong coupling (large N_c) [4],

$$f(\lambda) = \frac{3}{4} \left[1 + \frac{15}{8}\zeta(3)\lambda^{-3/2} + \mathcal{O}(\lambda^{-3}) \right], \quad (3)$$

with no $\lambda^{-1/2}$ or λ^{-1} terms.

III. LOG-AWARE CONFORMAL PADÉ METHODOLOGY

Let $y = \sqrt{\lambda}$ and map the positive axis via

$$z = \frac{y}{1 + \alpha y + \beta y^2}, \quad \alpha > 0, \quad \beta \geq 0, \quad (4)$$

which includes one-parameter maps at $\beta = 0$. Rational/conformal mappings are routinely used to suppress spurious poles and compactify semi-infinite domains [6,7]. We use two complementary routes.

A. Route A: LSTP

Define

$$g(\lambda) = f(\lambda) - \frac{3}{2\pi^4}\lambda^2 \log \frac{\lambda}{\pi^2} \chi(\lambda; \Lambda_0, p),$$

$$\chi(\lambda; \Lambda_0, p) = \frac{1}{1 + (\lambda/\Lambda_0)^p}, \quad (5)$$

with integer $p \geq 1$ so the subtraction is exact at small λ but dies off at large λ . Subtracting the logarithm and rationally approximating the residual is standard in series analysis (Dlog/Padé-type preprocessing) [6,8]. We employ a smooth cutoff χ to decouple the weak-side logarithm from the strong-side constraints [7]. We then approximate

$$g(\lambda) \approx \frac{P_m(z)}{Q_n(z)}, \quad (6)$$

where

$$P_m(z) = \sum_{k=0}^m p_k z^k, \quad Q_n(z) = 1 + \sum_{k=1}^n q_k z^k, \quad (7)$$

and set

$$f(\lambda) \approx \frac{P_m(z)}{Q_n(z)} + \frac{3}{2\pi^4}\lambda^2 \log \frac{\lambda}{\pi^2} \chi(\lambda; \Lambda_0, p). \quad (8)$$

Coefficients are fixed by expanding about $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$ and matching Eqs. (2) and (3). We scan near-diagonal orders $[m/n] = [4/4]$ and mapping parameters (α, β) , with (Λ_0, p) controlling the cutoff.

1. Choice of p and strong-coupling artifact

The cutoff $\chi(\lambda) = 1/(1 + (\lambda/\Lambda_0)^p)$ leaves the weak-coupling subtraction exact through $\mathcal{O}(\lambda^2 \log \frac{\lambda}{\pi^2})$ for any $p \geq 1$ [deviations first appear at $\mathcal{O}(\lambda^{2+p} \log \frac{\lambda}{\pi^2})$]. At large λ , however, the cutoff tail $\chi \sim (\Lambda_0/\lambda)^p$ induces a leakage

$$\Delta f_{\text{leak}} = \frac{3}{2\pi^4} \Lambda_0^p \lambda^{2-p} \log \frac{\lambda}{\pi^2}. \quad (9)$$

For $p = 1$ or 2 this term does not decay ($\sim \lambda \log \frac{\lambda}{\pi^2}$ or $\sim \log \frac{\lambda}{\pi^2}$, respectively); $p = 3$ is the *minimal integer* for which the leakage decays at large λ . Setting $p = 3$ in Eq. (9) gives

$$|\Delta f_{\text{leak}}| \leq \frac{3}{2\pi^4} \frac{\Lambda_0^3}{\lambda} \left| \log \frac{\lambda}{\pi^2} \right| \lesssim \begin{cases} 1.7 \times 10^{-2} & (\lambda = 5), \\ 5 \times 10^{-3} & (\lambda \geq 10), \end{cases} \quad (10)$$

for $\Lambda_0 \leq 2$, well within the admissible-band width. Increasing p suppresses the leakage faster (e.g., $\lambda^{-2} \log \frac{\lambda}{\pi^2}$ for $p = 4$) but makes the cutoff steeper near Λ_0 . Repeating the full admissible scan with $p = 4$ yields an envelope that differs by $\sup |\Delta f| \approx 0.010$ from the $p = 3$ band ($\lesssim 1.1\%$ in $\mathcal{S}/\mathcal{S}_0$). For $p = 5$ the sharper cutoff narrows the band (fewer candidates pass the admissibility filters), but the central tendency is consistent. We adopt $p = 3$ as the minimal well-motivated choice and make the resulting artifact explicit rather than hidden.

B. Route B: Two-point (Hermite-Padé) rational approximant (HP)

We use the generalized two-point Padé form introduced in Appendix G of Ref. [1] (see also Appendix C of Ref. [2]),

$$f(\lambda) = \frac{1 + a\lambda^{1/2} + b(\lambda)\lambda + c\lambda^{3/2} + d\lambda^2 + e(\lambda)\lambda^{5/2}}{1 + a\lambda^{1/2} + \bar{b}(\lambda)\lambda + \frac{4}{3}c\lambda^{3/2} + \frac{4}{3}d\lambda^2 + \frac{4}{3}e(\lambda)\lambda^{5/2}}. \quad (11)$$

The factor $4/3$ in the denominator enforces $f \rightarrow 3/4$ at large λ while eliminating $\lambda^{-1/2}$ and λ^{-1} terms in the strong-coupling expansion. The shifted coefficient $\bar{b}(\lambda) = b(\lambda) + \frac{3}{2\pi^2}$ ensures the $\mathcal{O}(\lambda)$ term matches Eq. (2). Constants a, c, d and the nonlog parts of b, e are uniquely fixed by matching Eqs. (2) and (3); this is a two-point (Hermite) Padé approximant in the sense of Refs. [6,9]. The coefficients are

$$\begin{aligned} a &= \frac{4\pi^2}{135\zeta(3)} + \frac{2(3 + \sqrt{2})}{3\pi} \approx 1.1800, \\ b(\lambda) &= b_0 + \frac{1}{\pi^2} \log \frac{\lambda}{\pi^2}, \\ b_0 &= \frac{16\pi[45(3 + \sqrt{2})\zeta(3) + \pi^3]}{18225\zeta(3)^2} \\ &\quad + \frac{36[\frac{\zeta'(-1)}{\zeta(-1)} + \gamma_E] + 69\sqrt{2} + 59 - 75 \log 2}{36\pi^2} \\ &\approx 1.0689, \\ \bar{b}(\lambda) &= b(\lambda) + \frac{3}{2\pi^2}, \\ c &= \frac{2}{15\zeta(3)} \approx 0.1109, \\ d &= \frac{180(3 + \sqrt{2})\zeta(3) + 8\pi^3}{2025\pi\zeta(3)^2} \approx 0.1309, \\ e(\lambda) &= \frac{2b(\lambda)}{15\zeta(3)} - \frac{3}{5\pi^2\zeta(3)}. \end{aligned} \quad (12)$$

The $\log(\lambda/\pi^2)$ in $b(\lambda)$ reproduces the exact $\lambda^2 \log \lambda$ coefficient. The coefficient $e(\lambda)$ inherits a logarithmic part from $b(\lambda)$; this is not fixed by the current perturbative expansion and is a consequence of the rational structure of the ansatz.

1. Strong-coupling structure of the HP

At large λ , all three matched strong-coupling constraints are reproduced *exactly* by Eq. (11), not merely asymptotically: (i) $f \rightarrow 3/4$, (ii) the coefficients of $\lambda^{-1/2}$ and λ^{-1} vanish identically for any value of $\log \lambda$ (enforced by the $4/3$ factors in the denominator), and (iii) the coefficient of $\lambda^{-3/2}$ equals $S_{3/2} = \frac{15}{8}\zeta(3)$ exactly. The last identity

follows because the log-dependent parts of $b(\lambda)$ and $e(\lambda)$ cancel in the $\lambda^{-3/2}$ coefficient: defining $e_0 = 2b_0/[15\zeta(3)] - 3/[5\pi^2\zeta(3)]$ (the nonlog part of e), one verifies $\pi^2 b_0 - \frac{9}{2} = \frac{15}{2}\pi^2\zeta(3)e_0$ analytically. Moreover, the coefficients of λ^{-2} and $\lambda^{-5/2}$ vanish exactly—one verifies algebraically that $2a = 15d\zeta(3)$ and $15c\zeta(3) = 2$ —so the next nonvanishing strong-coupling contribution appears at $\mathcal{O}(\lambda^{-3})$, consistent with the large- N_c structure in which the expansion proceeds in powers of $\lambda^{-3/2}$ [4]. Log artifacts enter only at this order: the λ^{-3} coefficient (the HP prediction for S_3) carries an $\mathcal{O}(1/\log \frac{\lambda}{\pi^2})$ correction from the rational structure. Crucially, no multiplicative $\lambda^{-n} \log \frac{\lambda}{\pi^2}$ artifact is generated through the orders we have checked; the contamination is $1/\log$ suppressed and decays monotonically for $\lambda \gg 1$, so it does not spoil the admissibility filters.

C. Admissibility filters and band

Candidates must satisfy, on $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ evaluated on a logarithmic grid:

- (1) Bounds: $0.75 \leq f(\lambda) \leq 1$.
- (2) Monotonicity in log space: $\frac{df}{d(\log \lambda)} \leq 0$.
- (3) *Pole exclusion*: compute all roots of $Q_n(z)$ and (for HP) the full denominator, map them to the λ plane using (4), and reject any pole on the positive real axis. We also reject near-canceling Froissart doublets (root-pole pairs whose separation is numerically indistinguishable on the grid).

The surviving set $\{f_i\}$ defines the *admissible band* $[f_{\min}(\lambda), f_{\max}(\lambda)]$ and a central curve

$$f_{\text{cent}} = \arg \min_i \int_{\log \lambda_{\min}}^{\log \lambda_{\max}} \left(\frac{d^2 f_i}{d(\log \lambda)^2} \right)^2 d(\log \lambda). \quad (13)$$

We define the crossover as the inflection in log space,

$$\left. \frac{d^2 f}{d(\log \lambda)^2} \right|_{\lambda=\lambda_c} = 0, \quad (14)$$

choosing the zero nearest to the largest curvature peak if multiple inflections exist.

IV. EQUILIBRIUM THERMODYNAMICS: ADMISSIBLE BAND, CROSSOVER, AND HIGHER-ORDER PREDICTIONS

A. Ensemble and central solution

The HP generalized Padé passes all filters and minimizes the curvature functional in Eq. (13); we take it as the central curve. For the LSTP route with near-diagonal $[m/n] = [4/4]$ and $p = 3$, the scan yields *nine* admissible survivors when $\beta = 0$, at

$$\alpha \in \{0.5, 1.0, 2.0\}, \quad \Lambda_0 \in \{0.5, 1.0, 2.0\}, \quad (15)$$

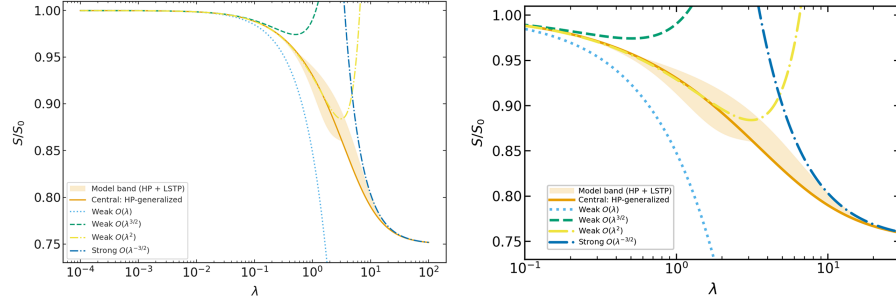


FIG. 1. Admissible Padé band for $f(\lambda) = S/S_0$ in $\mathcal{N} = 4$ SYM. Left: full-range view. Right: enlarged view emphasizing the intermediate region $0.2 \lesssim \lambda \lesssim 10$ where most of the variation occurs. Shaded: band; solid: central curve. Also shown are the weak truncations $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^{3/2})$, $\mathcal{O}(\lambda^2)$ (including the exact $\lambda^2 \log \lambda$ term) and the strong truncation $\mathcal{O}(\lambda^{-3/2})$.

with no survivors at $\Lambda_0 = 4.0$. Cases with $\beta > 0$ typically develop positive-axis poles or violate the bounds and are rejected. For $\beta = 0$ the Möbius map $z = y/(1 + \alpha y)$ reparametrizes the same $[4/4]$ rational function class, so the nine parameter sets yield *three distinct curves* (one per Λ_0); the α independence serves as an internal consistency check (cf. Tables II and III).

1. Crossover scales

For the central HP curve,

$$\lambda_c^{\text{center}} \simeq 3.5223, \quad f(\lambda_c^{\text{center}}) \simeq 0.8539. \quad (16)$$

Across the admissible ensemble,

$$\lambda_c \in [2.9520, 6.7321], \quad f(\lambda_c) \in [0.8345, 0.8609]. \quad (17)$$

2. Pole safety

All LSTP survivors are free of poles on $\lambda > 0$ [mapping in Eq. (4)]. The nearest mapped poles lie well away from the positive axis; the minimal imaginary part satisfies $\text{Im } \lambda \geq 7.38$ across survivors.

3. Physical interpretation

The central crossover, defined by the inflection in $\log \lambda$ [Eq. (14)], occurs at $\lambda_c \simeq 3.52$, where $f(\lambda_c) \simeq 0.854$ —i.e., the entropy density is $\sim 85\%$ of the ideal value, indicating substantial interaction effects already at moderate coupling. The admissible *range* $\lambda_c \in [2.95, 6.73]$ is not statistical noise; it reflects genuine model dependence given the present expansion (no $\mathcal{O}(\lambda^{5/2})$ term on the weak side and no $\mathcal{O}(\lambda^{-3})$ correction on the strong side). Interpreting any single-curve Padé without an uncertainty band would therefore overstate precision precisely in this intermediate-coupling regime; the admissible ensemble renders this uncertainty explicit and reproducible.

The full admissible band with the central curve is shown in Fig. 1; all individual survivors (HP and LSTP) are overlaid in Fig. 2. Figure 2 is a diagnostic of admissible interpolation freedom; localized wiggles and double-peak structures in $dS/d \ln \lambda$ for some survivors are artifacts of the rational family and are not interpreted as physical.

B. Predictions for unmeasured coefficients

Beyond quantifying interpolation uncertainty, the ensemble *predicts* higher-order series data on both sides.

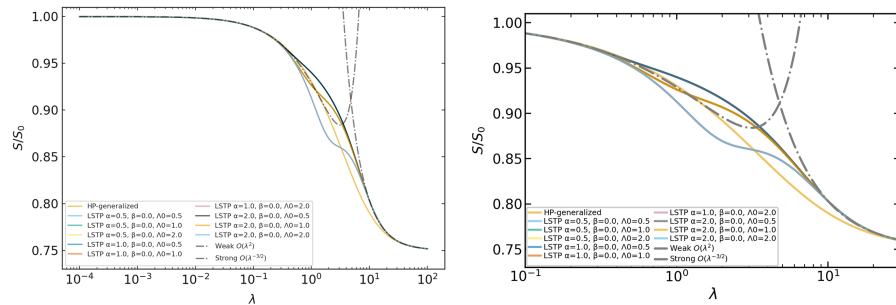


FIG. 2. All admissible individual curves (HP and LSTP) overlaid, together with the weak $\mathcal{O}(\lambda^2)$ and strong $\mathcal{O}(\lambda^{-3/2})$ truncations. Left: full-range view. Right: enlarged view emphasizing the region where most of the variation occurs. The spread defines the admissible band.

1. Weak-coupling $A_{5/2}$

Let $f_{\text{known}}(\lambda)$ denote the weak-coupling truncation in Eq. (2) through $\mathcal{O}(\lambda^2)$, so that

$$f(\lambda) = f_{\text{known}}(\lambda) + A_{5/2} \frac{\lambda^{5/2}}{\pi^5} + \dots \quad (18)$$

Because $e(\lambda)$ in Eq. (11) inherits a logarithmic dependence from $b(\lambda)$, the HP approximant generates a *spurious* $\lambda^{5/2} \log \frac{\lambda}{\pi}$ contribution at the first unmatched order. Expanding f_{HP} through $\mathcal{O}(\lambda^{5/2})$ gives

$$f_{\text{HP}} = f_{\text{known}} - \frac{3a}{\pi^4} \lambda^{5/2} \log \frac{\lambda}{\pi^2} + r_{5,\text{reg}}^{\text{HP}} \lambda^{5/2} + \dots, \quad (19)$$

where $r_{5,\text{reg}}^{\text{HP}}$ denotes the nonlogarithmic $\lambda^{5/2}$ coefficient after separating the $\lambda^{5/2} \log \frac{\lambda}{\pi}$ term, and $-3a/\pi^4 \approx -0.0363$ is fully determined by the rational structure [a is given in Eq. (12)]. The HP-generated $\lambda^{5/2} \log \frac{\lambda}{\pi}$ term is an artifact of the rational ansatz and should not be identified with the perturbative weak-coupling structure. A naïve estimator $Q(\lambda) \equiv \pi^5 \lambda^{-5/2} [f_{\text{HP}} - f_{\text{known}}]$ therefore diverges logarithmically as $\lambda \rightarrow 0$. After subtracting the analytically known spurious log, the corrected estimator

$$Q_{\text{corr}}(\lambda) = \frac{\pi^5}{\lambda^{5/2}} \left[f_{\text{HP}} - f_{\text{known}} + \frac{3a}{\pi^4} \lambda^{5/2} \log \frac{\lambda}{\pi^2} \right] \quad (20)$$

converges as $\lambda \rightarrow 0$, yielding $A_{5/2}^{\text{HP}} = \pi^5 r_{5,\text{reg}}^{\text{HP}}$. We evaluate Q_{corr} in extended-precision arithmetic at 11 points $\lambda = 10^{-3}, 10^{-4}, \dots, 10^{-13}$ and extrapolate to $\lambda \rightarrow 0$ by fitting $Q_{\text{corr}} = A_{5/2}^{\text{HP}} + B\sqrt{\lambda}$ (the value is in principle obtainable in closed form from the rational structure; the numerical evaluation serves as a convergence check on the residual $\mathcal{O}(\sqrt{\lambda})$ remainder, which includes both $\sqrt{\lambda}$ and $\sqrt{\lambda} \log \frac{\lambda}{\pi^2}$ terms from the rational structure; both vanish as $\lambda \rightarrow 0$, and the fit-window stability confirms they do not bias the intercept at the stated precision). The fit is stable: restricting the window to $\lambda \leq 10^{-4}$ or widening it to $\lambda = 10^{-3}$ shifts the intercept by less than 0.1. We find

$$A_{5/2}^{\text{HP}} = -43.8 \pm 0.1, \quad (21)$$

where the uncertainty reflects fit-window variation. We stress that $A_{5/2}^{\text{HP}}$ is a prediction of the HP rational ansatz and is therefore model dependent; the true perturbative value of $A_{5/2}$ will be settled by an explicit $\mathcal{O}(\lambda^{5/2})$ EFT or diagrammatic calculation. The LSTP construction is not constrained beyond $\mathcal{O}(\lambda^2 \log \lambda)$ on the weak side and therefore does not yield a reliable independent $A_{5/2}$.

2. Finite-coupling estimator $\hat{S}_3(\lambda_*)$ (admissibility bound)

At large λ ,

$$f(\lambda) = \frac{3}{4} [1 + S_{3/2} \lambda^{-3/2} + S_3 \lambda^{-3} + \mathcal{O}(\lambda^{-9/2})],$$

$$S_{3/2} = \frac{15}{8} \zeta(3). \quad (22)$$

Define the estimator

$$\hat{S}_3(\lambda) = \lambda^3 \left[\frac{4}{3} f(\lambda) - 1 - S_{3/2} \lambda^{-3/2} \right],$$

$$S_3 = \lim_{\lambda \rightarrow \infty} \hat{S}_3(\lambda), \quad (23)$$

so that $\hat{S}_3(\lambda) = S_3 + \mathcal{O}(\lambda^{-3/2})$. Using only $f_{\min} \leq f(\lambda) \leq f_{\max}$ with $f_{\min} = 0.75$ and $f_{\max} = 1$, we obtain for any fixed λ_* ,

$$\hat{S}_3(\lambda_*) \in \left[\lambda_*^3 \left(\frac{4}{3} f_{\min} - 1 - S_{3/2} \lambda_*^{-3/2} \right), \right. \\ \left. \lambda_*^3 \left(\frac{4}{3} f_{\max} - 1 - S_{3/2} \lambda_*^{-3/2} \right) \right], \quad (24)$$

Evaluated at $\lambda_* = 10$, this yields

$$\hat{S}_3(10) \in [-71.27, 262.06]. \quad (25)$$

This interval is a direct consequence of the physical bounds $0.75 \leq f \leq 1$, the known $S_{3/2}$, and the definition of the estimator, and therefore provides a conservative, falsifiable benchmark for future holographic computations at the reference point $\lambda_* = 10$. The admissibility bound in Eq. (24) is formulated at a finite reference coupling λ_* : it bounds the estimator $\hat{S}_3(\lambda_*)$ using only these inputs. With only this information, taking λ_* larger does not strengthen the constraint. In fact, Eq. (24) implies a bound width $\Delta \hat{S}_3(\lambda_*) = \frac{1}{3} \lambda_*^3$, so the interval rapidly becomes weaker as λ_* increases (e.g., increasing λ_* from 10 to 20 widens the interval by a factor of 8). Taking λ_* much smaller is also unhelpful, because one then leaves the regime where the strong-coupling asymptotic expansion and the very notion of an S_3 coefficient are under perturbative control. We therefore quote the interval at $\lambda_* = 10$ as a representative strong-coupling reference point where the constraint remains nontrivial and our interpolants have already entered the asymptotic regime; other choices follow from Eq. (24) by the same scaling.

This also answers a 2021 private query on whether Padé methods can anticipate the next strong-coupling correction using existing information: applying the estimator in Eq. (23) together with the admissibility window $0.75 \leq f \leq 1$ yields Eq. (25), i.e., $\hat{S}_3(10) \in [-71.27, 262.06]$. On the weak side, the log-aware HP central extrapolation, after accounting for a spurious $\lambda^{5/2} \log \lambda$ artifact [see below

TABLE I. Crossover scale from curvature diagnostics. $\lambda_{\pm}$ denote the admissible-ensemble bounds; $F(\lambda_c)$ is the observable evaluated at λ_c .

Observable	λ_c	λ_-	λ_+	$F(\lambda_c)$
S/S_0	3.52	2.95	6.73	0.854

Eq. (19)], yields $A_{5/2}^{\text{HP}} = -43.8 \pm 0.1$ in Eq. (21). Together, these provide concrete benchmarks to be tested and sharpened by future perturbative or holographic results.

C. Summary of crossover scales

Table I summarizes the curvature-defined crossover scale for S/S_0 . Our entropy crossover $\lambda_c \simeq 3.52$ is consistent with Ref. [5] ($\lambda_c = 3.14$); the difference reflects the admissible-ensemble construction, which shifts the inflection relative to a single-curve estimate.

V. DISCUSSION

We upgrade Padé interpolation from a single-curve estimate to a *controlled, admissible band*. Both routes are explicitly *log aware* and reproduce the *full* weak-coupling expansion through $\mathcal{O}(\lambda^2)$ *exactly*—namely the coefficients of λ , $\lambda^{3/2}$, $\lambda^2 \log \lambda$, and the finite λ^2 term A_{20} . On the strong side, the HP enforces $f \rightarrow 3/4$, the exact vanishing of $\lambda^{-1/2}$ and λ^{-1} terms, and the known $\lambda^{-3/2}$ correction $S_{3/2} = \frac{15}{8}\zeta(3)$. The LSTP matching conditions enforce $f \rightarrow 3/4$ and $S_{3/2}$, but the cutoff tail introduces a controlled $\lambda^{-1} \log \frac{\lambda}{\pi}$ artifact [Eq. (9)]. In addition, we exclude poles on the positive λ axis, impose $0.75 \leq f \leq 1$ and monotonicity in $\log \lambda$, and select the central curve by minimal curvature. The HP and LSTP constructions use identical inputs but different architectures. Their agreement, within a narrow band after admissibility filtering, is a strong internal consistency check. Where they differ defines the admissible ensemble uncertainty that any single-curve approach conceals. On the weak side we used the $\mathcal{O}(\lambda^2)$ series obtained via *direct resummation* in the Arnold-Zhai framework [10] (extended by us to $\mathcal{N} = 4$ SYM) and then rederived via EFT reconstruction in the Braaten-Nieto approach [11] (with our $\mathcal{N} = 4$ implementation and corrected normalization of Ref. [2]). Relative to earlier Padé analyses, we replace point estimates with a reproducible admissible band. For entropy, the central crossover $\lambda_c \simeq 3.52$ agrees with previous single-curve values, while we now quantify a realistic range, $\lambda_c \in [2.95, 6.73]$, arising from admissible choices of mapping and rational order.

Beyond uncertainty bands, the framework is *predictive* without any new loop or higher-order calculations. On the weak side, after analytically subtracting a spurious $\lambda^{5/2} \log \lambda$ artifact of the HP rational structure [coefficient $-3a/\pi^4 \approx -0.036$, see Eq. (19)], we infer $A_{5/2}^{\text{HP}} = -43.8 \pm 0.1$ in the normalization of Eq. (2). On the strong side,

using only the known $S_{3/2}$ and the admissibility window $0.75 \leq f \leq 1$, we obtain the model-independent interval

$$\hat{S}_3(10) \in [-71.27, 262.06],$$

which directly addresses a 2021 query (private communication) on whether Padé methods can anticipate the next strong-coupling correction. The framework is modular: additional weak-side information [e.g., $\mathcal{O}(\lambda^{5/2})$] or strong-side string corrections that fix S_3 [and beyond, $\mathcal{O}(\lambda^{-9/2})$] will automatically shrink the band with no change in methodology.

VI. OUTLOOK

A natural next step is to tighten and test the admissible ensemble band by incorporating higher-order terms in the weak-/strong-coupling expansions. Using EFT, the $\mathcal{O}(\lambda^{5/2})$ contribution to the free energy arises entirely from the soft scale $\sqrt{\lambda}T$ and is determined by three-loop vacuum diagrams in the electric effective theory, together with two-loop matching for the mass parameters m_E^2 and m_S^2 (cf. the QCD analysis in Ref. [11]). In parallel, one can extend the direct-resummation approach of Ref. [1]. Either route will fix $A_{5/2}^{\text{HP}}$ and provide a sharp test of our HP prediction $A_{5/2}^{\text{HP}} = -43.8 \pm 0.1$.

On the holographic side, the next unknown coefficient S_3 arises from stringy α' corrections beyond the known $\lambda^{-3/2}$ term. A computation of S_3 would turn our model-independent admissibility interval $\hat{S}_3(10) \in [-71.27, 262.06]$ into a definitive check of the ensemble at strong coupling.

Extension to transport observables (η/s , $\hat{q}/T^3$, $2\pi T D_s$), where admissible ensembles can systematically replace single-curve interpolations, is left to future work.

After validation in $\mathcal{N} = 4$ SYM, the method can be applied to QCD, where the running coupling and trace anomaly provide additional admissibility constraints, and to other gauge theories with accessible weak and strong limits.

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DATA AVAILABILITY

The code and data supporting the findings of this article are openly available [12].

APPENDIX: ANALYSIS AND REPRODUCIBILITY

1. Numerical domain and grids

We use $\lambda \in [10^{-4}, 10^2]$ on a uniform grid in $\log \lambda$ with at least 600 points. Weak- and strong-coupling series are evaluated as in Sec. II. All derivatives are taken with respect to $\log \lambda$ using centered finite differences on the log grid.

2. Route A (LSTP) admissible set

For LSTP we take near diagonal $[m/n] = [4/4]$ and scan

$$\begin{aligned} \alpha &\in \{0.5, 1.0, 2.0\}, & \beta &\in \{0, 0.05, 0.1\}, \\ \Lambda_0 &\in \{0.5, 1.0, 2.0, 4.0\}, & p &= 3. \end{aligned} \quad (\text{A1})$$

We subtract the weak-side logarithm with $\chi(\lambda; \Lambda_0, p) = 1/(1 + (\lambda/\Lambda_0)^p)$ and approximate the residual by $P_4(z)/Q_4(z)$ with z from Eq. (4). Coefficients are fixed by collocation at very small and very large λ , then candidates are tested against the filters in Sec. III C. All admissible survivors have $\beta = 0$, with $(\alpha, \Lambda_0) \in \{0.5, 1.0, 2.0\} \times \{0.5, 1.0, 2.0\}$ (*none* at $\Lambda_0 = 4.0$); crossover data and pole margins are collected in Tables II and III. We tested sensitivity to the conformal-map parameter by repeating the LSTP scan with $\alpha \in \{1, 2, 4\}$ (instead of $\{0.5, 1, 2\}$), holding all other scan ranges fixed. The admissible set and the resulting HP + LSTP envelope are unchanged within $\mathcal{O}(10^{-6})$ over $\lambda \in [10^{-4}, 10^2]$, and the crossover diagnostics are stable at the quoted precision. This invariance is expected for $\beta = 0$: since $z = y/(1 + \alpha y)$ is a Möbius reparametrization, varying α does not alter the $[4/4]$ rational function class, and the matching conditions fix a unique curve for each Λ_0 . At large λ , the HP-generalized curve lies slightly below the LSTP survivors. This reflects the fact that the HP construction enforces the strong-coupling structure globally in the mapped variable, whereas each LSTP interpolant is only anchored to the strong expansion at a finite set of large- λ matching points. The resulting offset is small compared to the admissible-band width and decreases rapidly with λ . The LSTP cutoff does introduce a residual $\lambda^{-1} \log \frac{\lambda}{\pi}$ artifact at large λ [quantified in Eq. (9)], but its prefactor is numerically negligible within the admissible band on our working domain $\lambda \leq 10^2$.

3. Route B (HP) central curve

The HP generalized Padé in Eq. (11) matches the *full* weak-coupling expansion through $\mathcal{O}(\lambda^2)$ exactly (i.e., the coefficients of λ , $\lambda^{3/2}$, $\lambda^2 \log \lambda$, and the finite λ^2 term A_{20}). At strong coupling it reproduces $f \rightarrow 3/4$ and the $\lambda^{-3/2}$ correction $S_{3/2} = \frac{15}{8} \zeta(3)$, with the absence of $\lambda^{-1/2}$ and λ^{-1} enforced. The HP curve passes all admissibility checks and minimizes the curvature functional in Eq. (13); we therefore use it as the central solution.

4. Admissibility diagnostics

a. Bounds and monotonicity

We require $0.75 \leq f(\lambda) \leq 1$ and $\frac{df}{d(\log \lambda)} \leq 0$ on the interior window $[10^{-3}, 10^2]$, while also checking the full domain for diagnostics.

b. Pole exclusion

We compute all roots of $Q_n(z)$ (and the HP denominator), map them to the λ plane via Eq. (4), and exclude any poles on the positive real axis. Near canceling Froissart doublets are rejected. We report the minimal imaginary part among mapped poles (in the λ plane) as a safety margin.

5. Crossover extraction

We locate λ_c by the log-space inflection condition, Eq. (14), using the zero of $d^2 f/d(\log \lambda)^2$. For S/S_0 we also quote an *ensemble* crossover window using the pointwise envelope of the admissible set.

6. Manual summary tables

TABLE II. Admissible curves with crossover and value at crossover.

Curve	α	Λ_0	λ_c	$f(\lambda_c)$
HP-generalized	...	...	3.52	0.854
LSTP survivors (all with $\beta = 0$)				
LSTP	0.5	0.5	6.45	0.839
LSTP	0.5	1.0	6.73	0.834
LSTP	0.5	2.0	2.95	0.861
LSTP	1.0	0.5	6.45	0.839
LSTP	1.0	1.0	6.73	0.834
LSTP	1.0	2.0	2.95	0.861
LSTP	2.0	0.5	6.45	0.839
LSTP	2.0	1.0	6.73	0.834
LSTP	2.0	2.0	2.95	0.861

TABLE III. Minimal imaginary part of mapped poles for LSTP survivors (all with $\beta = 0$). Larger values indicate greater separation from the positive real λ axis.

α	Λ_0	Min. Im. part
0.5	0.5	7.38
0.5	1.0	7.48
0.5	2.0	8.24
1.0	0.5	7.38
1.0	1.0	7.48
1.0	2.0	8.24
2.0	0.5	7.38
2.0	1.0	7.48
2.0	2.0	8.24

- [1] Q. Du, M. Strickland, and U. Tantary, *J. High Energy Phys.* **08** (2021) 064; **02** (2022) 053(E).
- [2] J. O. Andersen, Q. Du, M. Strickland, and U. Tantary, *Phys. Rev. D* **105**, 015006 (2022).
- [3] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
- [4] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, *Nucl. Phys.* **B534**, 202 (1998).
- [5] B. Müller, *Phys. Rev. D* **112**, 054007 (2025).
- [6] G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximants*, 2nd ed. (Cambridge University Press, Cambridge, England, 1996).
- [7] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd ed. (Dover, New York, 2001), Chs. 7–9.
- [8] A. J. Guttmann, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1989), Vol. 13, p. 1.
- [9] J. S. R. Chisholm, *Math. Comp.* **27**, 841 (1973).
- [10] P. B. Arnold and C. X. Zhai, *Phys. Rev. D* **50**, 7603 (1994).
- [11] E. Braaten and A. Nieto, *Phys. Rev. D* **53**, 3421 (1996).
- [12] U. H. Tantary, Data and scripts for “Constrained Padé Ensembles for Thermal $\mathcal{N} = 4$ SYM,” (Zenodo, 2026), [10.5281/zenodo.17568719](https://zenodo.org/record/17568719).