

Projective time, Cayley transformations and the Schwarzian geometry of the free particle-oscillator correspondence

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(Received 11 February 2026; accepted 18 March 2026; published 13 April 2026)

We investigate the relation between the one-dimensional free particle and the harmonic oscillator from a unified viewpoint based on projective geometry, Cayley transformations, and the Schwarzian derivative. Treating time as a projective coordinate on $\mathbb{RP}^1$ clarifies the $SL(2, \mathbb{R}) \cong Sp(2, \mathbb{R})$ conformal sector of the Schrödinger-Jacobi symmetry and provides a common framework for two seemingly different correspondences: the Cayley-Niederer (lens) map between the time-dependent Schrödinger equations and the conformal bridge transformation relating the stationary problems. We formulate these relations as canonical transformations on the extended phase space and as their metaplectic lifts, identifying the quantum Cayley map with the Bargmann transform. General time reparametrizations induce oscillator-type terms governed universally by the Schwarzian cocycle, connecting the present construction to broader appearances of Schwarzian dynamics.

DOI: [10.1103/ks12-b2v5](https://doi.org/10.1103/ks12-b2v5)

I. INTRODUCTION

The one-dimensional free particle and the harmonic oscillator are paradigmatic examples of classical and quantum dynamics. They sit at the opposite ends of the intuitive spectrum: *freedom* versus *confinement*. Yet precisely because of their simplicity, these systems recur—often implicitly—across theoretical physics and mathematics. The free particle underlies heat kernels and diffusion, semiclassical propagation, and scattering, and serves as a universal local model in perturbative constructions. It is also a natural “seed” for integrable hierarchies: through inverse scattering and Darboux-Crum dressing one generates reflectionless one-dimensional Schrödinger operators whose isospectral flows are governed by the Korteweg-de Vries (KdV) equation and its hierarchy [1–8]. Under Wick rotation the free Schrödinger equation turns into the heat equation, and the Cole-Hopf transformation (closely related, at the linear level, to Darboux-type dressing) linearizes the Burgers equation [9–11].

The harmonic oscillator is the canonical normal form near stable equilibria, the backbone of quantization around classical vacua, and the basic building block of quantum fields and strings. Both systems also appear in less direct guises: via coherent and squeezed states, metaplectic (Bogoliubov) transformations, Gaussian wave packets,

and the geometric structures that control their propagation and phase [12–17].

At the same time, the two systems differ in a fundamental way that is already visible at the spectral level. The free particle has a continuous spectrum, while the oscillator has a purely discrete one. This immediately suggests that no standard Darboux-Crum (isospectral or almost-isospectral) mechanism can relate them in a direct way. Nevertheless, it has long been known that they are tightly connected by symmetry. In the free particle, the (nonrelativistic) conformal symmetry acts as a *dynamical* symmetry, generated by the $\mathfrak{sl}(2, \mathbb{R})$ triple of time translation, dilation, and special conformal transformation, together with the Heisenberg algebra of space translations and Galilean boosts; together they form the Schrödinger (Jacobi) algebra $\mathfrak{h}_3 \ltimes \mathfrak{sl}(2, \mathbb{R})$ [18–23].

For the oscillator, the same algebraic skeleton reappears in a different realization, adapted to confinement: time evolution becomes naturally compact (Newton-Hooke viewpoint), and the $\mathfrak{sl}(2, \mathbb{R})$ sector is realized through quadratic phase-space generators and their metaplectic lift [24–29].

A central geometric theme is that the $\mathfrak{sl}(2, \mathbb{R})$ symmetry is inherently *projective*: it acts on time by Möbius transformations of $\mathbb{RP}^1$, and its global structure is best understood after *projectivizing* time. From this viewpoint, the Schwarzian derivative appears as the canonical projective object controlling reparametrizations, cocycles, and induced quadratic terms [30–32]. This is not merely a formal analogy. The Schwarzian derivative and its projective cocycle play a distinguished role in modern physics, most notably in the low-energy sector of the Sachdev-Ye-Kitaev (SYK) model and in two-dimensional Jackiw-Teitelboim gravity, where the effective boundary

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dynamics is governed by the Schwarzian action and its $\text{Diff}(S^1)/SL(2, \mathbb{R})$ structure [33–49].

Historically, the projective viewpoint was already present in classical works of Cayley on fractional linear transformations and projective invariants, where the Schwarzian derivative enters as a fundamental differential invariant [50,51]. Much later, Niederer discovered a remarkable map relating the free particle and the harmonic oscillator at the level of the time-dependent Schrödinger equation (TDSE) [19,28,52], and Arnol'd developed a broader geometric framework for such correspondences between differential equations [53,54].

In parallel, a different question arises if one looks at the stationary Schrödinger equation (SSE): can one relate *spectral* problems associated with freedom and confinement? Although a direct Darboux-Crum link is excluded by the radically different spectral types, a relation *does* exist. The conformal bridge transformation (CBT) establishes a map between the free particle and oscillator at the level of the stationary problem, and admits natural generalizations to a variety of backgrounds and symmetry-extended settings [55–65].

A priori, it is not obvious why a TDSE map (an evolution problem, valid for arbitrary initial data) and an SSE map (a spectral problem) should be governed by a common organizing principle. One expects a relationship only indirectly, through the fact that propagators and evolution kernels can be built from spectral data. The main purpose of this article is to show that, in the present context, there *is* a unifying structure, and to identify it explicitly. We argue that the appropriate “roof” is provided by a projective-geometric reformulation of the $\mathfrak{sl}(2, \mathbb{R})$ symmetry together with a complexified canonical transformation generated by a rotated Cayley matrix, and by its quantum (metaplectic) counterpart. This framework clarifies, in a single language, (i) the Cayley-Niederer map relating the free particle and oscillator at the level of the TDSE, (ii) the CBT relating their stationary problems, and (iii) the appearance of the Schwarzian derivative as the universal projective object that induces quadratic (oscillator-type) terms under general time reparametrizations. In addition, the same viewpoint yields a transparent interpretation of the Bargmann transform as a canonical (and, at the quantum level, metaplectic/unitary) passage between the Schrödinger and Bargmann-Fock realizations of the Heisenberg algebra [66–71].

The paper is organized as follows. In Sec. II we introduce the complexified canonical transformation in phase space defined by the Cayley matrix, and recall its interpretation in hyperbolic geometry as the map between the upper half-plane and unit disc models, together with the role of the Cayley parameter on $\mathbb{RP}^1$. In Sec. III we review the free particle dynamical integrals and the Schrödinger/Jacobi symmetry, emphasizing the projective (Möbius) action on time. In Sec. IV we construct the quantum Cayley map and discuss its metaplectic realization, relating it to the Bargmann

transform and to complex symplectic structures. In Sec. V we present the quantum Cayley transform underpinning the CBT between the free particle and oscillator stationary problems. In Sec. VI we discuss the classical symmetry structure of the oscillator and formulate the Cayley-Niederer map in a way adapted to the compact (Newton-Hooke) realization of the $\mathfrak{sl}(2, \mathbb{R})$ sector. In Sec. VII we develop a unified treatment of coordinate and time transformations of the TDSE and SSE, highlighting the natural appearance of $(\pm 1/2)$ -weight densities and the metaplectic (half-density) factors. In Sec. VIII we analyze the extended free particle and show how a Schwarzian term generated by time reparametrization induces an oscillator-type potential. In Sec. IX we treat general reparametrizations and present a factorized metaplectic form controlled by the Ermakov-Pinney amplitude. Several technical developments and complementary viewpoints are collected in the Appendixes: Appendix A discusses compatible complex structures on $(\mathbb{R}^2, \omega)$ and their parametrization by the upper half-plane; Appendix B reviews the metaplectic representation; Appendix C explains the role of Jordan states from analyticity in the energy; Appendix D connects $\text{Sp}(2, \mathbb{R})$ factorizations with ABCD optics; Appendix E provides a technical derivation of a squeezed-vacuum Gaussian formula; Appendix F summarizes Schrödinger-group Gaussian packets; Appendix G collects Schwarzian identities (including the cocycle property); and Appendix H comments on exact versus semiclassical quantization of canonical transformations.

II. CAYLEY TRANSFORM AND COMPLEXIFIED CANONICAL TRANSFORMATION

In this section we introduce the *complexified canonical transformation* in the dimensional phase space defined by the 2×2 unitary Cayley matrix C . This matrix preserves the symplectic form and, in this sense, provides the classical -space counterpart of the familiar Cayley map in complex analysis and hyperbolic geometry. In the following sections it will serve as the basic ingredient for the projectivization of time that underlies the $SL(2, \mathbb{R})$ (conformal) symmetry of the free particle and of the harmonic oscillator, and it will clarify the origin and structure of the conformal bridge and of the Cayley-Niederer maps between these systems at the level of the stationary and time-dependent Schrödinger equations. Finally, the same viewpoint will lead to a natural interpretation of the Bargmann transform as a canonical (and, at the quantum level, unitary) passage between the Schrödinger and Bargmann-Fock realizations of the Heisenberg algebra.

A. Real symplectic basis and linear canonical transformations

Consider a two-dimensional phase space $\mathbb{R}^2$ with a real basis

$$\xi_\alpha = (q, p)^\top, \quad \{\xi_\alpha, \xi_\beta\} = \Omega_{\alpha\beta}, \quad \Omega = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: J. \quad (2.1)$$

Here Ω is the standard Poisson tensor (the inverse symplectic matrix); with the convention $\omega = dp \wedge dq$ one has $\omega(u, v) = u^\top \check{\Omega} v$ and $\Omega = \check{\Omega}^{-1}$. Linear canonical transformations are precisely those real linear maps $\xi \mapsto \xi' = M\xi$ that preserve it, i.e.,

$$M \in Sp(2, \mathbb{R}) \Leftrightarrow M^\top J M = J \Leftrightarrow \det M = 1, \quad (2.2)$$

so that $Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$ in one degree of freedom.

A phase-space function $F(q, p)$ generates a one-parameter Hamiltonian canonical flow when applied to any function $f(q, p)$ by the adjoint action of the Poisson bracket,

$$\begin{aligned} \Phi(F; \tau) : f(q, p) &\mapsto f(q, p; \tau) = \exp(\tau X_F) f(q, p) \\ &= \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \{F, f\}_n, \end{aligned} \quad (2.3)$$

where X_F is the Hamiltonian vector field $X_F = \frac{\partial F}{\partial \xi_\alpha} \Omega_{\alpha\beta} \frac{\partial}{\partial \xi_\beta}$, and $\{F, f\}_0 := f$, $\{F, f\}_1 := \{F, f\}$, $\{F, f\}_n := \{F, \{F, f\}_{n-1}\}$. On the canonical coordinates themselves one has $\xi(\tau) = \exp(\tau X_F) \xi$, and for quadratic F the flow is linear,

$$\xi_\alpha(\tau) = (M_F(\tau) \xi)_\alpha, \quad M_F(\tau) = e^{\tau A_F} \in Sp(2, \mathbb{R}). \quad (2.4)$$

B. Quadratic functions

The quadratic functions

$$H_+ = \frac{1}{2}(p^2 + q^2), \quad D = qp, \quad H_- = \frac{1}{2}(p^2 - q^2) \quad (2.5)$$

generate the Lie algebra $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{sl}(2, \mathbb{R})$ via

$$\{D, H_\pm\} = 2H_\mp, \quad \{H_+, H_-\} = 2D. \quad (2.6)$$

Their one-parameter Hamiltonian flows are represented by the 2×2 matrices

$$\begin{aligned} M_{H_+}(\tau) &= \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix}, \quad M_D(\tau) = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^\tau \end{pmatrix}, \\ M_{H_-}(\tau) &= \begin{pmatrix} \cosh \tau & -\sinh \tau \\ -\sinh \tau & \cosh \tau \end{pmatrix}, \end{aligned}$$

which satisfy the symplectic condition (2.2). The corresponding real 2×2 generators A_F in (2.4), are

$$A_{H_+} = -J = -i\sigma_2, \quad A_D = -\sigma_3, \quad A_{H_-} = -\sigma_1, \quad (2.7)$$

in terms of standard Pauli matrices. These generators, when taken with a minus sign, obey the commutation relations of the form (2.6), and each A_F lies in $\mathfrak{sp}(2, \mathbb{R})$, $A_F^\top J + J A_F = 0$.

C. Parabolic (shear) basis

It is often convenient to switch to the alternative quadratic basis

$$H_0 := \frac{1}{2}p^2 = \frac{1}{2}(H_+ + H_-), \quad K := \frac{1}{2}q^2 = \frac{1}{2}(H_+ - H_-),$$

so that (H_0, K, D) also spans $\mathfrak{sp}(2, \mathbb{R})$. The Poisson algebra in this basis reads

$$\{D, H_0\} = 2H_0, \quad \{D, K\} = -2K, \quad \{H_0, K\} = -D,$$

and the corresponding Hamiltonian flows are the elementary shears

$$\begin{aligned} M_{H_0}(\tau) &= \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} = \exp(-\tau \sigma_+), \\ M_K(\tau) &= \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} = \exp(\tau \sigma_-), \end{aligned}$$

with the nilpotent generators $\sigma_\pm := \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. In particular, H_0 generates the classical “drift” (free propagation) $q(\tau) = q - \tau p$, $p(\tau) = p$, while K generates the “thin-lens” shear $q(\tau) = q$, $p(\tau) = p + \tau q$; these are precisely the two elementary parabolic subgroups that will be used later for factorizations in $Sp(2, \mathbb{R})$. At the quantum level, the $Sp(2, \mathbb{R})$ factorizations lift to the metaplectic double cover $Mp(2, \mathbb{R})$ and therefore to operator (propagator) factorizations; in particular, the harmonic and inverted-oscillator evolution operators, as well as $U(\pi/4) = \exp(\frac{\pi}{4} \hat{H}_-)$, see below, are metaplectic representatives of the corresponding classical matrices.

D. Complex canonical basis and the Cayley matrix

Let us introduce another, *complex* canonical basis

$$\zeta_\alpha = (a^+, -ia^-)^\top = (C\xi)_\alpha, \quad \{\zeta_\alpha, \zeta_\beta\} = \Omega_{\alpha\beta}, \quad (2.8)$$

where $a^\pm = \frac{1}{\sqrt{2}}(q \mp ip)$. The two symplectic bases are linearly related by the complex-valued matrix

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\mathbf{1} - i\sigma_1) = \exp\left(-i\frac{\pi}{4}\sigma_1\right). \quad (2.9)$$

This is a specific *complex* symplectic, $C \in Sp(2, \mathbb{C}) \cong SL(2, \mathbb{C})$, matrix: it is unitary, $C^\dagger C = \mathbf{1} \Rightarrow C \in SL(2, \mathbb{C}) \cap SU(2)$, symmetric, $C^\top = C$, and satisfies

the relation $CJC = J$. Being a particular case of the $M_{H_-}(\tau)$ flow,

$$C = M_{H_-}(i\pi/4), \quad (2.10)$$

it commutes with $M_{H_-}(\tau)$. It is the Cayley matrix, whose fourth power is the central element $-I$ of $SU(2)$,

$$C^4 = -\mathbf{1}, \quad C^8 = \mathbf{1}. \quad (2.11)$$

The one-parameter Hamiltonian $SL(2, \mathbb{R})$ flows $M_F(\tau)$ in the real basis ξ_α and their matrix generators A_F are transformed by C -conjugation into the $SU(1, 1)$ flows in the complex basis ζ_α , $\zeta(\tau) = \tilde{M}_F \zeta$, $\tilde{M}_F(\tau) = CM_F(\tau)C^{-1}$, $\tilde{A}_F = CA_FC^{-1}$, and one has

$$\tilde{M}_{H_+}(\tau) = M_D(i\tau), \quad \tilde{M}_D(\tau) = M_{H_+}(i\tau), \quad \tilde{M}_{H_-}(\tau) = M_{H_-}, \quad (2.12)$$

and $\tilde{A}_{H_+} = -i\sigma_3$, $\tilde{A}_D = \sigma_2$, $\tilde{A}_{H_-} = A_{H_-} = -\sigma_1$. In correspondence with (2.10), the flows $M_{H_-}(\tau)$ and $\tilde{M}_{H_-}(\tau)$ are the same in both bases, while $\tilde{M}_{H_+}(\tau)$ and $\tilde{M}_D(\tau)$ are related to $M_D(\tau)$ and $M_{H_+}(\tau)$ via the Wick rotation $\tau \rightarrow i\tau$. This corresponds to the map

$$iD = iqp = \frac{1}{2}(a^{-2} - a^{+2}) \mapsto \tilde{H}_+ = a^+a^- \quad (2.13)$$

under the complexified canonical transformation $\xi_\alpha \mapsto \zeta_\alpha$.

Note here that with any quadratic function $F(\xi)$, one can associate a symmetric matrix $K_F = K_F^\top$ by means of its Hessian, $F = \frac{1}{2}\xi^\top K_F \xi$, $(K_F)_{\alpha\beta} = \frac{\partial^2 F}{\partial \xi_\alpha \partial \xi_\beta}$. In terms of K_F , the $SL(2, \mathbb{R})$ matrix generators A_F are presented as $A_F = -JK_F$, and under the change of the basis (2.9), $K_F \mapsto \tilde{K}_F = C^{-1}K_FC^{-1}$.

E. Cayley conjugation: from $SL(2, \mathbb{R})$ to $SU(1, 1)$

A general element of the linear $SU(1, 1)$ transformation given by composition of transformations $\tilde{M}_{H_+}$, $\tilde{M}_{H_-}$, and $\tilde{M}_D$ is

$$\tilde{S} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = CSC^{-1} \in SU(1, 1), \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),$$

where $ad - bc = |\alpha|^2 - |\beta|^2 = 1$. Real a, b, c, d , and complex α, β , parameters are related as $\alpha = \frac{1}{2}((a+d) + i(b-c))$, $\beta = \frac{1}{2}((b+c) + i(a-d))$, $a = \text{Re } \alpha + \text{Im } \beta$, $d = \text{Re } \alpha - \text{Im } \beta$, $b = \text{Re } \beta + \text{Im } \alpha$, $c = \text{Re } \beta - \text{Im } \alpha$.

The $SU(1, 1)$ map $\zeta \mapsto \zeta' = \tilde{S}\zeta$ is a Bogoliubov transformation. In terms of $(a^-, a^+)^\top$, it takes a usual form

$$\begin{pmatrix} a^{-'} \\ a^{+'} \end{pmatrix} = B \begin{pmatrix} a^- \\ a^+ \end{pmatrix}, \quad B = \begin{pmatrix} u & v \\ \bar{u} & \bar{v} \end{pmatrix}, \quad u = \bar{\alpha}, \quad v = i\bar{\beta}. \quad (2.14)$$

In what follows, we will see how the described classical relations corresponding to the complexified canonical map underlie the CBT between the quantum free particle and harmonic oscillator systems, as well as the Bargmann transform that relates Schrödinger (coordinate) and Fock-Bargmann (holomorphic) representations of the Heisenberg group.

Matrix $M_{H_-}(\tau)$ of the hyperbolic flow can be presented as a rotated through $\pi/4$ dilation flow that is consistent with Cartan decomposition of $SL(2, \mathbb{R})$,

$$M_{H_-}(\tau) = M_{H_+}(\pi/4)M_D(\tau)M_{H_+}(-\pi/4). \quad (2.15)$$

This corresponds to the relation between D and H_- ,

$$H_- = \frac{1}{2}(p^2 - q^2) = -D' = -q'p',$$

$$\text{where } \xi'_\alpha = (M_{H_+}(\pi/4)\xi)_\alpha,$$

i.e., one of the generators of classical Bogoliubov transformations (squeezing), H_- , is the same as $-D$ (dilation) seen from the rotated by $\pi/4$ phase space coordinates. Relations (2.15) and (2.10) allow one to present the basis-changing Cayley matrix C in the form

$$C = M_{H_+}(\pi/4)M_D(i\pi/4)M_{H_+}(-\pi/4).$$

In this way, the complex-valued symplectic matrix $C \in SL(2, \mathbb{C}) \cap SU(2)$, being a particular element of the compact real form of $SL(2, \mathbb{C})$, establishes a unitary equivalence between the two noncompact real forms of $SL(2, \mathbb{C})$, $SL(2, \mathbb{R}) \cong Sp(2, \mathbb{R})$, and $SU(1, 1)$.

F. Hyperbolic geometry: $\mathbb{H}_+$, $\mathbb{D}$, and the Cayley map

As is well known, the groups $SL(2, \mathbb{R})$ and $SU(1, 1)$ ¹ are isometry groups of the upper half-plane, $\mathbb{H}_+$, and unit disc, $\mathbb{D}$, Poincaré models of a hyperbolic (Lobachevsky) plane,

$$\mathbb{H}_+ := \{z = x + iy \in \mathbb{C}; y > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad \partial\mathbb{H}_+ = \mathbb{RP}^1, \quad (2.16)$$

$$\mathbb{D} := \{w \in \mathbb{C}; |w| < 1\}, \quad ds^2 = 4 \frac{dw d\bar{w}}{(1 - |w|^2)^2}, \quad \partial\mathbb{D} = S^1. \quad (2.17)$$

¹More exactly, their projective versions $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathcal{Z}(SL(2, \mathbb{R})) = SL(2, \mathbb{R})/\{I, -I\}$ and $PSU(1, 1) = SU(1, 1)/\mathcal{Z}(SU(1, 1)) = SU(1, 1)/\{I, -I\}$.

The upper-half plane $\mathbb{H}_+$ (together with its ideal boundary $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$) is mapped conformally into the unit disc $\mathbb{D}$ (and its boundary S^1) by the same Cayley matrix (2.9), $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a = d = \frac{1}{\sqrt{2}}$, $b = c = -\frac{i}{\sqrt{2}}$, but that now acts as a fractional linear (Möbius) transformation, $C \cdot z := \frac{az+b}{cz+d}$,

$$z \mapsto C \cdot z = i \frac{z-i}{z+i} = \frac{z-i}{-iz+1} := w. \quad (2.18)$$

In accordance with this, the isometry groups $SL(2, \mathbb{R})$ and $SU(1, 1)$ act on $\mathbb{H}_+$ and $\mathbb{D}$ projectively, by fractional linear transformation, $ds^2(z') = ds^2(z)$, $z' = S \cdot z$, $S \in SL(2, \mathbb{R})$, $ds^2(w') = ds^2(w)$, $w' = \tilde{S} \cdot w$, $\tilde{S} \in SU(1, 1)$. Note that the two standard models of the hyperbolic plane, (2.16) and (2.17), also appear naturally in geometric quantization as the moduli of compatible complex structures (equivalently, holomorphic polarizations) on the symplectic plane; see Appendix A. The upper half-plane $\mathbb{H}_+$ and the unit disc $\mathbb{D}$ structures also appear in coherent squeezed states, see Sec. V.

Also note that the more common Cayley map

$$C_0 \cdot z = \frac{z-i}{z+i} := w_0, \quad C_0 = \exp\left(-i \frac{\pi}{3} (\mathbf{n} \cdot \boldsymbol{\sigma})\right) \in SU(2),$$

$$\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1), \quad (2.19)$$

differs from (2.18) only by a rigid rotation of the unit disc, $w = iw_0 = e^{i\pi/2} w_0$, i.e., by a $+\pi/2$ rotation. Unlike our matrix (2.9), C_0 produces a complexified linear canonical transformation $(q, p)^T \mapsto e^{-i\pi/4} (a^+, a^-)^T$ and satisfies $C_0^3 = -\mathbf{1}$, $C_0^6 = \mathbf{1}$, cf. (2.11).

G. Ideal boundary, Cayley parameter, and projective time

Restricting the Cayley map (2.18) to the ideal boundary of $\mathbb{H}_+$ (i.e. to $z = t \in \mathbb{R} \cup \{\infty\}$), one obtains a conformal identification of $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ with the unit circle S^1 ,

$$w = w(t) = C \cdot t = i \frac{t-i}{t+i} \in S^1. \quad (2.20)$$

In particular, $t \in \{-1, 0, 1, \infty\} \mapsto w \in \{-1, -i, 1, i\} \subset S^1$, so that $t = \infty$ corresponds to the point $w(\infty) = i$.

A convenient parametrization of (a chart of) $\mathbb{RP}^1$ is given by the Cayley parameter²

²The coordinate $t = \tan \theta$ (equivalently, $t = \tan(\varphi/2)$ with $\varphi = 2\theta$) is classical. We call t a *Cayley parameter* because it is the real boundary coordinate induced by the Cayley fractional-linear map $w = \frac{t-i}{t+i}$ (up to a rigid rotation), which identifies $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ with the unit circle S^1 . This terminology is consistent with the standard (matrix) Cayley transform introduced by Cayley [50]. For historical overviews of the Schwarzian in conformal/projective settings see, e.g., [30–32].

$$t = \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} (U_0), \quad (2.21)$$

which yields the simple angular form

$$w(\theta) = i \frac{\tan \theta - i}{\tan \theta + i} = -ie^{2i\theta} = -ie^{i\varphi}, \quad \varphi := 2\theta \pmod{2\pi}. \quad (2.22)$$

In this description, the limits $\theta \rightarrow \pm \frac{\pi}{2}$ both correspond to $t \rightarrow \pm \infty$, i.e. to the same point $\infty \in \mathbb{RP}^1$. One may therefore either identify the end points $\theta = \pm \frac{\pi}{2}$, or (as we will prefer later) cover $\mathbb{RP}^1$ by two standard charts. Concretely, passing from U_0 to the complementary chart U_∞ amounts to using the coordinate $t \mapsto \tilde{t} = -\frac{1}{t}$ ($0 \mapsto \infty$), which corresponds to the shifts $\theta \mapsto \theta + \frac{\pi}{2}$, hence $\varphi \mapsto \varphi + \pi$, and therefore $w \mapsto -w$. So, here $U_0 = \mathbb{RP}^1 \setminus \{\infty\}$, and $U_\infty = \mathbb{RP}^1 \setminus \{0\}$. Thus, on the circle S^1 the chart transition induces the antipodal map $w \mapsto -w$. Equivalently, viewing S^1 as the space of *unit vectors* in $\mathbb{R}^2$, the natural projection $S^1 \rightarrow \mathbb{RP}^1$ identifies antipodal points $\pm w$ as the same (unoriented) line, and $w \mapsto -w$ is the corresponding deck involution. This is consistent with the geometric picture of $\mathbb{RP}^1$ as the space of (unoriented) lines through the origin in $\mathbb{R}^2$. In this sense, $\mathbb{RP}^1$ is the moduli space (parameter space) of such lines, i.e., of one-dimensional linear subspaces of $\mathbb{R}^2$. Choosing a unit vector on each line produces a double cover $S^1 \rightarrow \mathbb{RP}^1$, with the two antipodal points $\pm w$ corresponding to the same underlying line.

This Cayley reparametrization will be used below in the discussion of the conformal symmetry of the free particle and harmonic-oscillator systems. Introducing a new time variable τ by the same Cayley parameter $t := \tan \tau$, a translation in τ becomes a fractional-linear transformation of t ,

$$\tau \mapsto \tau + \alpha \Rightarrow t \mapsto t' = \tan(\tau + \alpha) = \frac{\cos \alpha t + \sin \alpha}{-\sin \alpha t + \cos \alpha}. \quad (2.23)$$

Equivalently, on the boundary circle S^1 one has $w = -ie^{2i\tau}$, and therefore $\tau \mapsto \tau + \alpha$ acts as a rigid rotation, $w \mapsto w' = -ie^{2i(\tau+\alpha)} = e^{2i\alpha} w$.

Thus translations in τ correspond to the compact rotation subgroup acting on S^1 . In this language, the Niederer transformation [28] is precisely the Cayley transform of the time variable accompanied by the corresponding transformation of the coordinate variable treated as a $(-\frac{1}{2})$ -weight density (cf. the Liouville transformation law for wave functions in the stationary Schrödinger equation, Sec. VII). We stress that the global implementation of conformal transformations requires some care; this will be addressed below by working on $\mathbb{RP}^1$ (two charts) and,

when appropriate, promoting time to a dynamical variable in a reparametrization-invariant formulation.

III. FREE PARTICLE: MÖBIUS AND SCHRÖDINGER SYMMETRIES

In this section we recall the symmetry structure of the one-dimensional free particle and fix our notation. We first list its continuous and discrete classical symmetries and identify the corresponding integrals of motion, distinguishing the ordinary (kinematical) ones from the dynamical generators. We then pass to an extended, reparametrization-invariant formulation in which time is promoted to a dynamical variable and the theory is described by a first-class constraint. This extended viewpoint makes transparent how the $\text{SL}(2, \mathbb{R})$ (Möbius) action on time and the associated conformal boost can be realized by well-defined canonical generators. Finally, we projectivize time, $t \in \mathbb{R} \rightarrow \mathbb{RP}^1$, discussing several equivalent ways to do this; this resolves the global (noncompleteness) problem of conformal boosts and naturally leads to the density/half-density transformation laws that will be needed in the quantum theory.

A. Classical continuous symmetries

A nonrelativistic free particle described by the action

$$S[x(t)] = \frac{1}{2}m \int_{t_i}^{t_f} \dot{x}^2 dt = \int L dt, \quad (3.1)$$

possesses continuous symmetries,

$$T_a^t: t \mapsto t' = t + a, \quad x \mapsto x' = x, \quad \text{time translation;} \quad (3.2a)$$

$$T_b^x: x \mapsto x' = x + b, \quad t \mapsto t' = t, \quad \text{space translation;} \quad (3.2b)$$

$$\mathcal{G}_v: x \mapsto x' = x + vt, \quad t \mapsto t' = t, \quad \text{Galilean boost;} \quad (3.2c)$$

$$\mathcal{D}_a: x \mapsto x' = e^a x, \quad t \mapsto t' = e^{2a} t, \quad \text{dilation;} \quad (3.2d)$$

$$\mathcal{B}_c: t \mapsto t' = \frac{t}{1+ct}, \quad x \mapsto x' = \frac{x}{1+ct}, \quad \text{conformal boost.} \quad (3.2e)$$

Under transformations (3.2a), (3.2b), (3.2d) action does not change: $S \mapsto S' = S$. Under (3.2c), one has $S \mapsto S' = S + \frac{m}{2}(2vx + v^2t)|_{t_i}^{t_f}$, while under (3.2e), $S \mapsto S' = S - \frac{m}{2}c \frac{x^2}{1+ct}|_{t_i}^{t_f}$.

Conformal boost transformations (3.2e) are not globally well defined; they require a compactification (projectivization) of time: $t \in \mathbb{R} \rightarrow \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$, see below.

B. Discrete symmetries and their relation to boosts

Action (3.1) is also characterized by discrete symmetries:

• $\mathcal{R}^t: t \mapsto -t$, time reflection, • $\mathcal{R}^x: x \mapsto -x$, space reflection, • Time inversion³:

$$t \mapsto t' = \mathcal{S}(t) = -\frac{1}{t}, \quad x' = \mathcal{S}(x) = -\frac{x}{t} \\ \Rightarrow \quad S \mapsto S' = S + \frac{m}{2} \left(-\frac{x^2}{t} \right) \Big|_{t_i}^{t_f}.$$

The last discrete symmetry also requires the projectivization of time, $t \in \mathbb{R} \rightarrow \mathbb{RP}^1$; then $\mathcal{S}: t = 0 \leftrightarrow t = \infty$, and $\mathcal{S}^2(t) = t$, $\mathcal{S}^2(x) = -x = \mathcal{R}^x(x) \Rightarrow \mathcal{S}^4 = \text{Id}$.

Galilean and conformal boosts can be generated by the space and time translations composed with discrete symmetries $\mathcal{S}$ and $\mathcal{R}^x$,

$$\mathcal{G}_v = \mathcal{R}^x \circ \mathcal{S} \circ T_{-v}^x \circ \mathcal{S}, \quad \mathcal{B}_c = \mathcal{R}^x \circ \mathcal{S} \circ T_{-c}^t \circ \mathcal{S}.$$

C. Integrals of motion: Kinematical and dynamical generators

From solution of equation of motion, $x(t) = x(0) + \dot{x}(0)t = x(0) + \frac{1}{m}pt$, we identify the integrals,

$$P = m\dot{x}(0) = p, \quad G = mx(0) = mx - pt, \quad (3.3)$$

$$H_0 = \frac{1}{2m}p^2, \quad D = \frac{1}{m}Gp = xp - 2H_0t, \\ K = \frac{1}{2m}G^2 = \frac{1}{2}mx^2 - Dt - H_0t^2. \quad (3.4)$$

The integrals G, D, K are dynamical, explicitly depending on time, $\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, H_0\} = 0$.

D. Schrödinger (Jacobi) algebra and Poisson brackets

Dynamical integrals transform solutions of equations of motion into solutions, and generate Lie algebra of the (centrally extended) Schrödinger group (also known as the Jacobi group) $\mathfrak{h}_3 \rtimes \mathfrak{sp}(2, \mathbb{R})$ given by nonzero Poisson brackets

$$\{D, H_0\} = 2H_0, \quad \{D, K\} = -2K, \quad \{K, H_0\} = D, \\ \{D, P\} = P, \quad \{D, G\} = -G, \\ \{H_0, G\} = -P, \quad \{K, P\} = G, \quad (3.5)$$

with mass m playing the role of the central charge of $1D$ Heisenberg algebra $\mathfrak{h}_3$, $\{G, P\} = m$.

³Although $t \mapsto -1/t$ is an element of the continuous $\text{SL}(2, \mathbb{R})$ symmetry, it is often useful to single it out as a discrete involution with special geometric meaning (exchange of charts on $\mathbb{RP}^1$, see below).

The linear fractional (Möbius) transformations $PSL(2, \mathbb{R})$ act on $t \in \mathbb{R}\mathbb{P}^1$,

$$t \rightarrow t' = \frac{at + b}{ct + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

Their infinitesimal form and corresponding generators are

$$\begin{aligned} \delta t = \delta\alpha, \quad l_{-1} = \partial_t; \quad \delta t = \delta\beta t, \\ l_0 = t\partial_t; \quad \delta t = \delta\gamma t^2, \quad l_{+1} = t^2\partial_t. \end{aligned}$$

Vector fields $l_0, l_{\pm 1}$ generate the $\mathfrak{sl}(2, \mathbb{R})$ algebra in the

standard form (3.6) [equivalent to (3.5) under the identification $l_{-1} \equiv H_0$, $l_0 \equiv -\frac{1}{2}D$, $l_{+1} \equiv K$],

$$\{l_0, l_{\pm 1}\} = \pm l_{\pm 1}, \quad \{l_{-1}, l_{+1}\} = 2l_0, \quad (3.6)$$

and represent the $(n = -1, 0, 1)$ slice of the Witt algebra

$$\{l_n, l_m\} = (m - n)l_{n+m}, \quad n, m \in \mathbb{Z}. \quad (3.7)$$

Infinitesimal transformations of x under continuous symmetries are $\delta x = \bar{\delta}x + \dot{x}\delta t$, where $\bar{\delta}x = \varepsilon\{x, F\}$, $F = (P, G, H_0, D, K)$. Explicitly,

$$\begin{aligned} F = P: \mathcal{T}_\varepsilon^x(x) = \bar{\delta}x = \varepsilon, \quad \delta t = 0, \\ F = G: \mathcal{G}_\varepsilon(x) = -\varepsilon t, \quad \varepsilon = -\delta v, \quad \delta t = 0, \\ F = H_0: \mathcal{T}_\varepsilon^t(x) = 0, \quad \delta t = \delta\alpha = -\varepsilon, \\ F = D: \mathcal{D}_\varepsilon(x) = \bar{\delta}x + \delta\beta\dot{x}t = \varepsilon(x - 2\dot{x}t) + \delta\beta\dot{x}t = \varepsilon x, \quad \varepsilon = \frac{1}{2}\delta\beta, \quad \delta t = \delta\beta t, \\ F = K: \mathcal{K}_\varepsilon(x) = -\varepsilon(x - \dot{x}t)t + \delta\gamma\dot{x}t^2 = \delta\gamma x t, \quad \varepsilon = -\delta\gamma, \delta t = \delta\gamma t^2. \end{aligned}$$

E. Extended formulation: Reparametrization invariance and constraint

Let us promote t to be a dynamical variable, $t \rightarrow t(s)$, $x \rightarrow x(s)$, and present action (3.1) in the reparametrization-invariant form,

$$S[x(s), t(s)] = \int \frac{1}{2} m \frac{(\dot{x})^2}{\dot{t}} ds, \quad (3.8)$$

where $\dot{x} = dx/ds$, $\dot{t} = dt/ds$, $\dot{t} > 0$. Canonical momenta are $p_x = \frac{\partial L}{\partial \dot{x}} = m \frac{\dot{x}}{\dot{t}}$, $p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2} \left(\frac{\dot{x}}{\dot{t}}\right)^2$. Reparametrization invariance of S corresponds to the change $s \rightarrow \alpha(s)$, $\dot{\alpha} > 0$, and is generated by the first-class constraint

$$\mathcal{C}_{\text{rep}} := p_t + \frac{1}{2m} p_x^2 \approx 0, \quad (3.9)$$

which, according to Dirac, at the quantum level transforms into the time-dependent Schrödinger equation for a 1D free particle.

The first order action corresponding to (3.8) is $S = \int (ip_t + \dot{x}p_x - e\mathcal{C}_{\text{rep}})ds$, where $e = e(s)$ is the Lagrange multiplier (einbein, lapse function). The integrals have the same form (3.3), (3.4) with $p = p_x$, and they weakly Poisson commute with constraint (3.9) and Hamiltonian $\mathcal{H} = e\mathcal{C}_{\text{rep}} \approx 0$.

F. $sl(2, \mathbb{R})$ generators on extended phase space and projectability

Vector fields generating conformal symmetry transformations of t and x are

$$J_- = \partial_t, \quad \delta t = \varepsilon, \quad \delta x = 0, \quad (3.10)$$

$$J_0 = t\partial_t + \frac{1}{2}x\partial_x, \quad \delta t = \varepsilon t, \quad \delta x = \frac{1}{2}\varepsilon x, \quad (3.11)$$

$$J_+ = t^2\partial_t + tx\partial_x, \quad \delta t = \varepsilon t^2, \quad \delta x = \varepsilon tx. \quad (3.12)$$

They satisfy the $sl(2, \mathbb{R})$ algebra in the form (3.6). The phase space functions

$$\mathcal{J}_- = p_t = \mathcal{C}_{\text{rep}} - H_0, \quad (3.13)$$

$$\mathcal{J}_0 = tp_t + \frac{1}{2}xp_x = \frac{1}{2}D + t\mathcal{C}_{\text{rep}}, \quad (3.14)$$

$$\mathcal{J}_+ = t^2p_t + txp_x - \frac{1}{2}mx^2 = -K + t^2\mathcal{C}_{\text{rep}} \quad (3.15)$$

“corrected” by the constraint (3.9) correspond to (3.10)–(3.12) realized as the Hamiltonian vector fields $X_F = \partial_\mu F \Omega_{\mu\nu} \frac{\partial}{\partial z_\nu}$ in the extended phase space $z_\mu = (x, t, p_x, p_t)$, $\{z_\mu, z_\nu\} = \Omega_{\mu\nu}$. The presence of the terms proportional to the constraint $\mathcal{C}_{\text{rep}}$ in (3.13)–(3.15) reflects the fact that any two phase space functions that differ by the term $\lambda(t, x)\mathcal{C}_{\text{rep}}$ generate the same transformation on the physical phase space; they belong to the same equivalence class.

Reparametrizing the world-line ($s \rightarrow s + \varepsilon(s)$) is a pure gauge transformation, it moves along the orbits of $\mathcal{C}_{\text{rep}}$. The pieces $t\mathcal{C}_{\text{rep}}$ and $t^2\mathcal{C}_{\text{rep}}$ in (3.14), (3.15) subtract precisely those gauge directions that would otherwise contaminate action of D or K on (t, x) . Hence the “mixing” is not an *ad hoc* trick but a standard feature of gauge systems: to realize a space-time symmetry on the extended configuration/phase space, one must pick a generator that is projectable onto the physical phase space, and that inevitably differs from the naïve Noether charge by a term proportional to the first-class constraint. The term $-\frac{1}{2}mx^2$ in (3.15) is harmless; it Poisson commutes with both x and t .

To have a truly global (everywhere-defined, complete) action of the conformal boost as a diffeomorphism generated by $J_+ = t^2\partial_t + tx\partial_x$ on the extended configuration/phase space, we have to do the projective compactification of the t variable.

G. Möbius action, conformal boosts, and half-density weight

Under the linear fractional transformation, $t \rightarrow T = \frac{at+b}{ct+d}$, $ad - bc = 1$, the spatial coordinate transforms as $x \rightarrow X = \frac{x}{ct+d}$ in accordance with (3.2a), (3.2d), (3.2e). So, the quantity $x(t)(dt)^{-1/2}$ is invariant under Möbius $SL(2, \mathbb{R})$ transformations,

$$x(t)(dt)^{-1/2} = X(T)(dT)^{-1/2}, \quad \Leftrightarrow \quad X(T) = x(t) \left(\frac{dT}{dt} \right)^{1/2}. \quad (3.16)$$

Geometrically, $(dt)^{1/2}$ is a half-density of weight $+\frac{1}{2}$, and $x(t)$ transforms as the time-density of weight $-\frac{1}{2}$, i.e., as a section of the $(-\frac{1}{2})$ power of the density bundle over projective time, or, equivalently, as a $(-\frac{1}{2})$ -density under time reparametrization.

H. Projectivization of time: Two-chart description of $\mathbb{RP}^1$

To projectivize (compactify) time, we introduce two charts.

(i) Chart U_0 (affine, “standard,” $t = 0 \in U_0$),

$$t = \frac{T_1}{T_2}, \quad x = \frac{X}{T_2}, \quad \text{with } T_2 \neq 0.$$

A fiber coordinate X is assumed to be a scalar under linear $SL(2, \mathbb{R})$ transformations, while $T_1 \rightarrow aT_1 + bT_2$, $T_2 \rightarrow cT_1 + dT_2$, and so, t undergoes a linear fractional transformation.

Naïvely, the change $t \rightarrow 1/t$ would correspond to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\det = -1$, which is orientation reversing, an element of $PGL(2, \mathbb{R})$ but not $PSL(2, \mathbb{R})$. To preserve orientation and stay in $PSL(2, \mathbb{R})$, we take instead.

(ii) Chart U_∞ ($t = \infty \in U_\infty$),

$$\tilde{t} = -\frac{1}{t}, \quad \tilde{x} = -\frac{x}{t},$$

obtained from U_0 by the Möbius map with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{R})$. In homogeneous variables, the transition from chart U_0 to U_∞ is the linear change, $(\tilde{T}_1, \tilde{T}_2, \tilde{X}) = (-T_2, T_1, -X)$, and so,

$$\tilde{t} = \frac{\tilde{T}_1}{\tilde{T}_2} = -\frac{T_2}{T_1} = -\frac{1}{t}, \quad \tilde{x} = \frac{\tilde{X}}{\tilde{T}_2} = -\frac{X}{T_1} = -\frac{x}{t},$$

with $T_1 \neq 0$.

Working in two charts makes the finite $SL(2, \mathbb{R})$ flows of conformal symmetry global: the “blow-up” is just a chart switch.

In chart U_0 , the Lagrangian takes the form

$$\frac{m}{2} \frac{\dot{x}^2}{\dot{t}} = \frac{m}{2} \frac{\mathcal{A}_2^2}{\mathcal{B}} \cdot \frac{1}{T_2^2}, \quad \text{where } \mathcal{A}_2 = \dot{X}T_2 - X\dot{T}_2, \\ \mathcal{B} = \dot{T}_1T_2 - T_1\dot{T}_2.$$

With conjugate momenta P_{T_1} , P_{T_2} , P_X , we obtain two first-class constraints

$$\Phi_{\mathbb{RP}^1} := T_1P_{T_1} + T_2P_{T_2} + XP_X \approx 0, \\ \Phi_{\text{rep}} := T_2P_{T_1} + \frac{1}{2m}T_2^2P_X^2 \approx 0, \quad \{\Phi_{\mathbb{RP}^1}, \Phi_{\text{rep}}\} = 0.$$

Constraint $\Phi_{\mathbb{RP}^1}$ generates gauge transformation $\Phi_{\mathbb{RP}^1}: (T_1, T_2, X) \rightarrow \lambda(T_1, T_2, X)$, $\lambda \neq 0$. Hence, simultaneously rescaled by the same constant λ coordinates (T_1, T_2, X) belong to the same gauge orbit. The phase space functions

$$t = \frac{T_1}{T_2}, \quad x = \frac{X}{T_2}, \quad p_t := T_2P_{T_1}, \quad p_x := T_2P_X \quad (3.17)$$

Poisson commute with $\Phi_{\mathbb{RP}^1}$, hence are gauge invariant under this constraint, and we have the canonical transformation

$$(T_1, P_{T_1}; T_2, P_{T_2}; X, P_X) \\ \rightarrow (t = T_1/T_2, p_t; \ln T_2, \Phi_{\mathbb{RP}^1}; x = X/T_2, p_x).$$

Constraint Φ_{rep} generates reparametrizations and with (3.17) equivalently is presented in the form (3.9), $\Phi_{\text{rep}} = p_t + \frac{1}{2m}p_x^2 \approx 0$. Generators of Galilean boosts, dilations, and conformal boosts expressed in terms of x , p_x , and t given by (3.17), $G = mx - tp_x$, $D = \frac{1}{m}p_xG$, $K = \frac{1}{2m}G^2$, Poisson commute with both constraints.

Introducing the gauge $\chi_{\mathbb{RP}^1} := T_2 - 1 \approx 0$ for $\Phi_{\mathbb{RP}^1} \approx 0$, and reducing the system to the surface of the second-class constraints, $(\Phi_{\mathbb{RP}^1} \approx 0, \chi_{\mathbb{RP}^1} \approx 0)$, we return to the reparametrization-invariant formulation based on action (3.8) with $t = T_1$, $x = X$.

In chart U_∞ , we have $\frac{\tilde{x}}{\tilde{t}} = \frac{\mathcal{A}_1^2}{\mathcal{B}} \cdot \frac{1}{T_1^2}$, where $\mathcal{B}$ has the same form as in chart U_0 , while $\mathcal{A}_1$ is obtained from $\mathcal{A}_2$ by changing T_2 for T_1 . The difference of Lagrangians on intersection of the charts $U_0 \cap U_\infty$ is a total derivative, $L_0 - L_\infty = \frac{d}{ds}(\frac{m}{2} \frac{X^2}{T_1 T_2})$, confirming that the transition from one chart to another is a canonical transformation.

In chart U_∞ , the two first-class constraints take the same form with the change of phase space variables $T_1, P_{T_1}, T_2, P_{T_2}, X, P_X$ for their tilted analogs, with $\tilde{P}_{T_1} = -P_{T_2}, \tilde{P}_{T_2} = P_{T_1}, \tilde{P}_X = -P_X$. Fixing in chart U_∞ the gauge $\tilde{T}_2 = 1$, gives local coordinates $\tilde{t} = \tilde{T}_1, \tilde{x} = \tilde{X}$.

I. Projectivisation as a circle: Angular coordinate and Cayley parameter

Alternatively, the projectivization of time can be achieved based on an equivalent interpretation of $\mathbb{RP}^1$ as a circle S^1 with antipodal identification of the points on it. For this, we assign $t = \tan \theta$ with $-\pi/2 < \theta < \pi/2$, cf. (2.21). This corresponds to the domain spanned by all lines through the origin in the (T_1, T_2) plane except the vertical one (chart U_0). Chart U_∞ is obtained by $\theta \mapsto \theta + \pi/2$ and $\tilde{t} = -1/t = -\cot \theta$ with the domain spanned in the (T_1, T_2) plane by all lines through the origin except the horizontal one. Setting $\phi := 2\theta \in (-\pi, \pi)$, then passing from U_0 to U_∞ is $\phi \mapsto \phi + \pi$, i.e., antipodal identification on S^1 . Thus, $\mathbb{RP}^1 \simeq S^1 / \{\phi \sim \phi + \pi\}$. Equivalently, one could consider $\phi \in (-\pi, \pi]$, i.e., including $+\pi$ and excluding $-\pi$ values, and declaring $\phi = \pi \leftrightarrow t = \infty$. This is purely a convention to avoid double counting at the seam. However, we prefer to use the stereographic/Cayley coordinate

$$t = \tan \frac{\phi}{2}, \quad \phi \in (-\pi, \pi), \quad (3.18)$$

by implying the antipodal identification on S^1 realized via the usage of the second chart $\phi \mapsto \phi + \pi$. Note that

$$\frac{dt}{d\phi} = \frac{1}{2\cos^2(\phi/2)}, \quad \frac{d\phi}{dt} = 2\cos^2(\phi/2) = \frac{2}{1+t^2}. \quad (3.19)$$

In this way, translation in ϕ , $\phi \mapsto \phi + \alpha$ corresponds to a linear fractional transform of t ,

$$t \mapsto \frac{\cos \frac{\alpha}{2} t + \sin \frac{\alpha}{2}}{-\sin \frac{\alpha}{2} t + \cos \frac{\alpha}{2}},$$

cf. (2.23), that is an element of the compact (elliptic) one-parameter subgroup of $SL(2, \mathbb{R})$. The translation for $\alpha = \pi$ corresponds to the transition between the charts U_0 and U_∞ , $t \mapsto -1/t$.

J. Quantum free particle and Schrödinger covariance

With (3.18), the Lagrangian from (3.8) becomes

$$L_\phi = m \frac{\cos^2(\phi/2) \dot{x}^2}{\dot{\phi}}, \quad (3.20)$$

and the reparametrization (Hamiltonian) constraint takes the form

$$\mathcal{C}_\phi \equiv p_\phi + \frac{p_x^2}{4m \cos^2(\phi/2)} \approx 0. \quad (3.21)$$

This is the free-particle constraint written in a compact time coordinate ϕ . Dirac quantization amounts to imposing $\hat{\mathcal{C}}_\phi \Psi = 0$ on physical states $\Psi(x, \phi)$, with $\hat{p}_\phi = -i\hbar \partial_\phi$ and $\hat{p}_x = -i\hbar \partial_x$. Hence

$$\begin{aligned} i\hbar \partial_\phi \Psi(x, \phi) &= \frac{\hat{p}_x^2}{4m \cos^2(\phi/2)} \Psi(x, \phi) \\ &= -\frac{\hbar^2}{4m \cos^2(\phi/2)} \partial_x^2 \Psi(x, \phi). \end{aligned} \quad (3.22)$$

Returning to the affine time t via (3.18), i.e., using (3.19) and $\Psi(x, \phi) = \psi(x, t(\phi))$ so that $\partial_\phi \Psi = (dt/d\phi) \partial_t \psi$, one obtains after cancelling the common factor $dt/d\phi$ the standard free-particle time-dependent Schrödinger equation

$$i\hbar \partial_t \psi(x, t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x, t). \quad (3.23)$$

In particular, $\phi \rightarrow \pm\pi$ corresponds to $t \rightarrow \pm\infty$.

IV. QUANTUM CAYLEY MAP: BARGMANN TRANSFORM AND THE COMPLEX-METAPLECTIC OPERATOR

In this section we describe the quantum counterpart of the complexified canonical transformation generated by the Cayley matrix C introduced in Sec. II. On the one hand, this transformation is implemented (in a canonical choice of normalization) by the unitary Bargmann transform, which may be viewed as a unitary change of polarization from the Schrödinger realization on $L^2(\mathbb{R}, dq)$ to the Bargmann-Fock space of holomorphic functions. On the other hand, the same canonical map admits an operator realization inside the (complexified) metaplectic group as an

exponential of a quadratic generator; this makes transparent the relation between the Cayley matrix and a metaplectic similarity action on $(\hat{q}, \hat{p})$. We also explain how the remaining overall scalar factor is fixed once the operator is interpreted as the intertwiner between the Schrödinger and Bargmann-Fock realizations of the Heisenberg algebra.

A. Bargmann transform as a unitary change of polarization

In what follows we work in units $\hbar = 1$. On $L^2(\mathbb{R}, dq)$ we use $\hat{q}\psi(q) = q\psi(q)$, $\hat{p}\psi(q) = -i\partial_q\psi(q)$, so that $i\hat{p} = \partial_q$, and assume the choice of units with dimensionless q .

Consider the Cayley matrix (2.9), $C = \frac{1}{\sqrt{2}}(1 - i\sigma_1) \in SL(2, \mathbb{C}) \cap SU(2)$. It generates the complex linear canonical transformation $(q, p)^\top \mapsto (u, v)^\top$,

$$(u, v)^\top = C(q, p)^\top, \quad u = \frac{1}{\sqrt{2}}(q - ip) = a^+,$$

$$v = \frac{1}{\sqrt{2}}(p - iq) = -ia^-, \quad (4.1)$$

where $a^\pm = \frac{1}{\sqrt{2}}(q \mp ip)$ is the standard complex canonical pair.

We look for the generating function of the first type for this canonical transformation in the form $F(q, u) = \frac{1}{2}\alpha q^2 + \beta qu + \frac{1}{2}\gamma u^2$. From the relations $p = \partial_q F$ and $v = -\partial_u F$ we obtain $\alpha = -i$, $\beta = i\sqrt{2}$, $\gamma = -i$, and hence $F(q, u) = i(-\frac{1}{2}q^2 + \sqrt{2}qu - \frac{1}{2}u^2)$.

For a Schwartz function $f(q)$, we define the quantum analog of this classical (complex) canonical transformation by the integral operator

$$(\mathcal{B}f)(z) = k \int_{\mathbb{R}} \exp(-iF(q, z))f(q)dq,$$

where we identify the new canonical coordinate u with the complex variable $z \in \mathbb{C}$. The constant k is fixed by requiring that the lowest Hermite function $\psi_0(q) = \pi^{-1/4} \exp(-q^2/2)$ is mapped to the constant function 1. This gives $k = \pi^{-1/4}$, and therefore

$$(\mathcal{B}f)(z) = \int_{\mathbb{R}} K(z, q)f(q)dq,$$

$$K(z, q) = \pi^{-1/4} \exp\left(-\frac{1}{2}z^2 + \sqrt{2}zq - \frac{1}{2}q^2\right). \quad (4.2)$$

B. Bargmann-Fock Hilbert space and unitarity

The function $K(z, q)$ is nothing but the integral kernel of the (unitary) Bargmann transform from the Hilbert space $L^2(\mathbb{R})$ to the Bargmann-Fock Hilbert space $\mathcal{F}$ of entire holomorphic functions with

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty, \quad d\mu(z) = \pi^{-1} e^{-|z|^2} d^2z,$$

$$d^2z = d(\operatorname{Re} z) d(\operatorname{Im} z),$$

and inner product

$$\langle f|g \rangle = \int_{\mathbb{C}} \overline{f(z)} g(z) d\mu(z).$$

Since $\mathcal{B}: L^2(\mathbb{R}) \rightarrow \mathcal{F}$ is unitary, its inverse is the adjoint with respect to $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, and is given by

$$(\mathcal{B}^{-1}F)(q) = \int_{\mathbb{C}} \overline{K(z, q)} F(z) d\mu(z),$$

$$\overline{K(z, q)} = \pi^{-1/4} \exp\left(-\frac{1}{2}\bar{z}^2 + \sqrt{2}\bar{z}q - \frac{1}{2}q^2\right).$$

Thus, the complexified linear canonical transformation defined by the Cayley matrix C is implemented at the quantum level by the unitary Bargmann transform between the Schrödinger (coordinate) and Fock-Bargmann representations, i.e., the unitary isomorphism $\mathcal{B}: L^2(\mathbb{R}) \rightarrow \mathcal{F}$ [70,71].

C. Intertwining of the Heisenberg generators

In the Bargmann-Fock representation one has the standard holomorphic realization $\mathcal{B}\hat{a}^+\mathcal{B}^{-1} = z$, $\mathcal{B}\hat{a}^-\mathcal{B}^{-1} = \partial_z$, $[\partial_z, z] = 1$, so that

$$\mathcal{B}\hat{q}\mathcal{B}^{-1} = \frac{1}{\sqrt{2}}(z + \partial_z), \quad \mathcal{B}(i\hat{p})\mathcal{B}^{-1} = \frac{1}{\sqrt{2}}(\partial_z - z).$$

Equivalently, the canonical pair $(u, v) = (a^+, -ia^-)$ is represented as $u \mapsto z$ and $v \mapsto -i\partial_z$ on $\mathcal{F}$.

Note that if in (4.2) one restricts $z \in \mathbb{C}$ to a real variable $y \in \mathbb{R}$ (that would contradict the complex nature of the u variable), the resulting integral operator is no longer unitary: unitarity here is tied to the holomorphic (complex) polarization and the Gaussian measure $d\mu(z)$.

The quantum analog of (4.1) can also be realized by a (nonunitary) complex-metaplectic (see Appendix B) operator, acting by similarity on $(\hat{q}, \hat{p})$; this operator also provides, after pairing with the holomorphic polarization, the unitary Bargmann transform described above.

D. Complex-metaplectic operator realization

We now provide an operator realization of the same complexified canonical transformation (4.1) generated by the Cayley matrix C , and explain how its overall normalization is fixed once it is interpreted as an intertwiner between the Schrödinger and Bargmann-Fock realizations of the Heisenberg algebra.

E. From the inverted oscillator flow to the Cayley matrix

Consider the quadratic Hamiltonian (of the inverted harmonic oscillator)

$$\begin{aligned}\hat{H}_- &= \frac{1}{2}(\hat{p}^2 - \hat{q}^2) = -\frac{1}{2}((\hat{a}^+)^2 + (\hat{a}^-)^2), \\ \hat{a}^\pm &= \frac{1}{\sqrt{2}}(\hat{q} \mp i\hat{p}).\end{aligned}\quad (4.3)$$

In correspondence with classical relation (2.3) for $F = H_-$, define the one-parameter (generally nonunitary) similarity flow

$$U(s) = \exp(s\hat{H}_-). \quad (4.4)$$

Using $[\hat{q}, \hat{p}] = i$, one finds $[\hat{H}_-, \hat{q}] = -i\hat{p}$, $[\hat{H}_-, \hat{p}] = -i\hat{q}$, and therefore the Heisenberg-picture operators $\hat{q}(s) = U(s)\hat{q}U(s)^{-1}$, $\hat{p}(s) = U(s)\hat{p}U(s)^{-1}$ satisfy

$$\frac{d}{ds} \begin{pmatrix} \hat{q}(s) \\ \hat{p}(s) \end{pmatrix} = -i\sigma_1 \begin{pmatrix} \hat{q}(s) \\ \hat{p}(s) \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{q}(s) \\ \hat{p}(s) \end{pmatrix} = \exp(-is\sigma_1) \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}.$$

At the special value $s = \pi/4$ one gets $\exp(-i\frac{\pi}{4}\sigma_1) = \frac{1}{\sqrt{2}}(1 - i\sigma_1) = C$, cf. classical relation (2.9), and hence

$$U(\pi/4) \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} U(\pi/4)^{-1} = C \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}. \quad (4.5)$$

Equivalently, $U(\pi/4)\hat{q}U(\pi/4)^{-1} = \hat{a}^+$, $U(\pi/4)\hat{p}U(\pi/4)^{-1} = -i\hat{a}^-$, which is precisely the quantum counterpart of (4.1). In the usual “time-evolution” notation $e^{-it\hat{H}_-}$, the choice $U(\pi/4) = e^{(\pi/4)\hat{H}_-}$ corresponds to the pure imaginary time $t = i\pi/4$.

F. Disentangling (factorized) form

Introduce $\hat{H}_+ = \hat{a}^+\hat{a}^- + \frac{1}{2}$ and the $\mathfrak{su}(1, 1)$ generators

$$K_+ = \frac{1}{2}(\hat{a}^+)^2, \quad K_- = \frac{1}{2}(\hat{a}^-)^2, \quad K_0 = \frac{1}{2}\hat{H}_+, \quad (4.6)$$

so that $\hat{H}_- = -(K_+ + K_-)$. Then one has the standard (analytic continuation of the) $SU(1, 1)$ disentangling identity, and at $s = \pi/4$ it yields the factorization, see (D12),

$$U(\pi/4) = \exp\left(-\frac{1}{2}(\hat{a}^+)^2\right) \exp\left(\frac{1}{2}\ln 2 \hat{H}_+\right) \exp\left(-\frac{1}{2}(\hat{a}^-)^2\right). \quad (4.7)$$

G. Overall constant and its fixation by the Bargmann-Fock normalization

Let us now introduce

$$U_C = cU(\pi/4) = c \exp\left(\frac{\pi}{4}\hat{H}_-\right), \quad c \in \mathbb{C}^\times. \quad (4.8)$$

As an operator acting by similarity on $(\hat{q}, \hat{p})$, the scalar $c \neq 0$ is invisible: $U_C \hat{A} U_C^{-1} = U(\pi/4) \hat{A} U(\pi/4)^{-1}$ for any operator $\hat{A}$. Thus, the canonical transformation alone does not fix c . The scalar is fixed only after one specifies an intertwining problem and a corresponding normalization convention. In particular, the unitary Bargmann transform $\mathcal{B}: L^2(\mathbb{R}, dq) \rightarrow \mathcal{F}$ is fixed by the Bargmann-Fock Hilbert-space normalization, while the nonunitary Schrödinger-space operator $U(\pi/4)$ admits different convenient scalings on polynomial (rigged) cores.

H. Bargmann-Fock normalization ($\mathcal{B}\psi_0 = 1$)

The Bargmann-Fock space $\mathcal{F}$ is the holomorphic realization in which $\hat{a}^+ \mapsto z$, $\hat{a}^- \mapsto \partial_z$, $[\partial_z, z] = 1$, and therefore the corresponding generalized eigenbra $\langle z|$ of $\hat{a}^+$, $\langle z|\hat{a}^+ = z\langle z|$, $\langle z|0\rangle = 1$, produces a holomorphic wave function from a Schrödinger state $|\psi\rangle$ by

$$(\mathcal{B}\psi)(z) := \langle z|\psi\rangle. \quad (4.9)$$

I. Dilations and the half-density factor

In the coordinate representation this becomes the integral transform (4.2), since the kernel $K(z, q)$ is the solution of the eigenvalue equation $\hat{a}^+ K(z, q) = zK(z, q)$, $\hat{a}^+ = \frac{1}{\sqrt{2}}(q - \partial_q)$, namely

$$K(z, q) = A(z) \exp\left(-\frac{1}{2}q^2 + \sqrt{2}zq\right).$$

Holomorphicity in z and compatibility with the Fock-Bargmann normalization fix $A(z) = A_0 e^{-z^2/2}$, which reproduces the Gaussian factor in (4.2). Equivalently, the full exponent in $K(z, q)$ equals $-iF(q, z)$, where $F(q, u)$ is the type-I generating function of the classical complexified canonical transformation (4.1) (with the identification $u \equiv z$).

The exponential phase alone does *not* determine a unitary quantum operator. A correct quantization of a linear canonical transformation (metaplectic operator) requires a square-root Jacobian prefactor (equivalently, an action on half-densities), which is the quantum remnant of ordering choices (and of the metaplectic correction). In the Bargmann case, this prefactor is fixed by holomorphicity and by Hilbert-space normalization. With the standard convention that the oscillator vacuum $\psi_0(q) = \pi^{-1/4} e^{-q^2/2}$ is mapped to the constant function 1, one gets $A_0 = \pi^{-1/4}$. This fixes the usual *unitary* Bargmann transform $\mathcal{B}: L^2(\mathbb{R}, dq) \rightarrow \mathcal{F}$.

J. Schrödinger-side normalization ($U_C \chi_n = \psi_n$) and the factor $2^{1/4}$

We now adopt an alternative, Schrödinger-side normalization for U_C , which in general differs from the Bargmann-Fock (unitary) convention above. The operator $U(\pi/4) = \exp(\frac{i}{2} \hat{H}_-)$ acting on q space is *nonunitary* on $L^2(\mathbb{R}, dq)$, and its overall scalar is not fixed by the similarity action on $(\hat{q}, \hat{p})$. A different and very natural choice is to fix the Schrödinger-side normalization by requiring that, when U_C is applied to the polynomial (Jordan) states $\chi_n(q) := q^n / \sqrt{n!}$, see (C8), one has

$$U_C \chi_0 = \psi_0, \quad \text{equivalently} \quad U_C \chi_n = \psi_n \quad (n \geq 0),$$

where $\{\psi_n\}$ are the normalized harmonic-oscillator eigenfunctions.

To see what this implies for the scalar, one may use the factorization

$$U(\pi/4) = \exp\left(-\frac{1}{2} \hat{q}^2\right) \exp\left(\frac{1}{2} \hat{H}_0\right) \exp\left(\frac{i}{2} \ln 2 \hat{D}\right),$$

$$\hat{H}_0 = \frac{1}{2} \hat{p}^2, \quad \hat{D} = \frac{1}{2} (\hat{q} \hat{p} + \hat{p} \hat{q}),$$

see Appendix D. In the coordinate representation, $\hat{p} = -i\partial_q$ and $i\hat{D} = q\partial_q + \frac{1}{2}$. Consequently, for any real α one has the exact dilation formula

$$\exp(i\alpha \hat{D}) f(q) = \exp\left(\alpha \left(q\partial_q + \frac{1}{2}\right)\right) f(q) = e^{\alpha/2} f(e^\alpha q), \quad (4.10)$$

and the presence of the constant $+\frac{1}{2}$ term is precisely what produces the half-density (Jacobian) factor $e^{\alpha/2}$. Moreover, (4.10) implies unitarity of dilations on $L^2(\mathbb{R}, dq)$,

$$\int_{\mathbb{R}} |\exp(i\alpha \hat{D}) f(q)|^2 dq = \int_{\mathbb{R}} e^\alpha |f(e^\alpha q)|^2 dq = \int_{\mathbb{R}} |f(u)|^2 du.$$

Applying (4.10) to the last factor with $\alpha = \frac{1}{2} \ln 2$ yields

$$\exp\left(\frac{i}{2} \ln 2 \hat{D}\right) f(q) = 2^{1/4} f(\sqrt{2}q), \quad \text{in particular}$$

$$\exp\left(\frac{i}{2} \ln 2 \hat{D}\right) 1 = 2^{1/4}.$$

Since $\exp(\frac{1}{2} \hat{H}_0) 1 = 1$, we obtain $U(\pi/4) 1 = 2^{1/4} e^{-q^2/2} = (2\pi)^{1/4} \psi_0(q)$, and therefore the condition $U_C \chi_0 = \psi_0$ fixes uniquely

$$U_C := (2\pi)^{-1/4} U(\pi/4) = (2\pi)^{-1/4} \exp\left(\frac{\pi}{4} \hat{H}_-\right).$$

In this way, the discrepancy between the Bargmann-kernel constant $\pi^{-1/4}$ and the Schrödinger-side constant $(2\pi)^{-1/4}$ is completely explained: $(2\pi)^{-1/4} = 2^{-1/4} \pi^{-1/4}$, and the factor $2^{\pm 1/4}$ is traced back directly to the constant $+\frac{1}{2}$ term in $i\hat{D} = q\partial_q + \frac{1}{2}$ (the metaplectic “half-form” contribution). If one were to omit this term (i.e., replace $i\hat{D}$ by $q\partial_q$), then the prefactor $e^{\alpha/2}$ in (4.10) would disappear, but dilations would no longer be unitary on $L^2(\mathbb{R})$: $\|\exp(\alpha q\partial_q) f\|_{L^2}^2 = e^{-\alpha} \|f\|_{L^2}^2$.

K. A commutative diagram on the polynomial core

On the polynomial core, set $e_n(z) := z^n / \sqrt{n!}$ and define the coefficient-wise “analytic continuation” map $\mathcal{A}: \chi_n \mapsto e_n$. [Since $\chi_n(q) = q^n / \sqrt{n!}$ are polynomials, this identification is unambiguous on $\text{span}\{\chi_n\}$, but it does *not* define a canonical map on all of $L^2(\mathbb{R})$.] Since $(\mathcal{B}\psi_n)(z) = e_n(z)$ and $U_C \chi_n = \psi_n$, one has on $\text{span}\{\chi_n\}$ the algebraic identity

$$\mathcal{B} \circ U_C = \mathcal{A}, \quad \text{equivalently} \quad U_C = \mathcal{B}^\dagger \circ \mathcal{A},$$

summarized by the commutative diagram

$$\begin{array}{ccc} \text{span}\{\chi_n\} & \xrightarrow{U_C} & \text{span}\{\psi_n\} \subset L^2(\mathbb{R}, dq) \\ \downarrow \mathcal{A} & & \downarrow \mathcal{B} \\ \text{span}\{e_n\} \subset \mathcal{F} & \xrightarrow{\text{id}} & \text{span}\{e_n\} \subset \mathcal{F} \end{array} \quad (4.11)$$

(which is to be understood on this dense common core).

L. Conformal-bridge interpretation

The same operator $U(\pi/4) = e^{(\pi/4)\hat{H}_-}$ can be interpreted as the evolution generated by the inverted harmonic oscillator at the imaginary time $t = i\pi/4$, and it serves as the quantum “conformal bridge”: it maps polynomial Jordan states of the free particle at $E = 0$ to harmonic-oscillator eigenstates, plane waves to oscillator coherent states, and ψ_0 to a squeezed state. We record this here only for completeness; a detailed discussion is presented in the next section.

Note also that the conformal-bridge transformation is underpinned by the parallel realisations of the generators of the $\mathfrak{sl}(2, \mathbb{R})$ conformal algebra in the Schrödinger representation on $L^2(\mathbb{R}, dq)$ and in the Bargmann-Fock representation on $\mathcal{F}$. This parallels the classical relation (2.12) between the corresponding flows in real and complex symplectic bases, and is summarized by the following scheme:

$L^2(\mathbb{R}, dq)$	$\mathcal{F}$
$i\hat{D} = q\partial_q + \frac{1}{2}$	$\hat{H}_+ = z\partial_z + \frac{1}{2}$
$\hat{H}_+ = \frac{1}{2}(-\partial_q^2 + q^2)$	$i\hat{D} = \frac{1}{2}(-\partial_z^2 + z^2)$
$\hat{H}_- = -\frac{1}{2}(\partial_q^2 + q^2)$	$\hat{H}_- = -\frac{1}{2}(\partial_z^2 + z^2)$

(4.12)

V. QUANTUM CAYLEY TRANSFORM AND CONFORMAL BRIDGE TRANSFORMATION

In the previous section we constructed the quantum Cayley operator

$$U(\pi/4) = \exp\left(\frac{\pi}{4}\hat{H}_-\right),$$

$$\hat{H}_- = \frac{1}{2}(\hat{p}^2 - \hat{q}^2) = -\frac{1}{2}((\hat{a}^+)^2 + (\hat{a}^-)^2),$$

and fixed the overall scalar by the Schrödinger-side normalization condition

$$U_C := (2\pi)^{-1/4}U(\pi/4), \quad U_C\chi_n = \psi_n \quad (5.1)$$

(which determines U_C uniquely up to an irrelevant global phase). In the present section we use U_C as the basic intertwiner implementing, at the quantum level, the complexified canonical transformation generated by the Cayley matrix C . This provides a convenient and conceptually transparent realization of the conformal bridge viewpoint [55,59,62], and it allows one to relate oscillator coherent/squeezed states to their free-particle images (or preimages) in a uniform language.

A. From $E=0$ Jordan states to oscillator eigenstates

The free-particle Hamiltonian is $\hat{H}_0 = \frac{1}{2}\hat{p}^2$. At $E=0$ one has the polynomial Jordan chain $\{\chi_n\}_{n \geq 0}$, see Appendix C. In particular, the dilation generator $\hat{D} := \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) \Rightarrow i\hat{D} = q\partial_q + \frac{1}{2}$ acts diagonally on χ_n ,

$$i\hat{D}\chi_n = \left(n + \frac{1}{2}\right)\chi_n. \quad (5.2)$$

The key conformal-bridge statement is the similarity transformation

$$U_C(i\hat{D})U_C^{-1} = \hat{H}_+, \quad (5.3)$$

where $\hat{H}_+ = \hat{a}^+\hat{a}^- + \frac{1}{2}$ is the harmonic-oscillator Hamiltonian which that follows from $U_C(\hat{q}, i\hat{p})U_C^{-1} = (\hat{a}^+, \hat{a}^-)$. Combining (5.2) with (5.3) yields

$$\hat{H}_+(U_C\chi_n) = U_C(i\hat{D})\chi_n = \left(n + \frac{1}{2}\right)U_C\chi_n,$$

so $U_C\chi_n$ is an eigenstate of $\hat{H}_+$ with eigenvalue $n + \frac{1}{2}$. With the normalization (5.1) this becomes the explicit bridge (4.11),

$$U_C\chi_n = \psi_n, \quad n = 0, 1, 2, \dots \quad (5.4)$$

Thus, at the level of stationary problems, the Cayley operator

trades the noncompact spectral problem for $i\hat{D}$ (on the free side at $E=0$) for the compact oscillator spectrum of $\hat{H}_+$.

B. From plane waves ($E > 0$) to Glauber coherent states

Consider free-particle momentum eigenfunctions (as distributions)

$$\phi_k(q) := \frac{1}{\sqrt{2\pi}}e^{ikq}, \quad \hat{p}\phi_k = k\phi_k, \quad E = \frac{k^2}{2} > 0.$$

The Cayley similarity implements the complexified canonical transformation

$$U(\pi/4)\hat{q}U(\pi/4)^{-1} = \hat{a}^+, \quad U(\pi/4)\hat{p}U(\pi/4)^{-1} = -i\hat{a}^-,$$

hence the same holds for U_C . Applying this to $\hat{p}\phi_k = k\phi_k$ gives $\hat{a}^-(U_C\phi_k) = ik(U_C\phi_k)$. Therefore $U_C\phi_k$ is (up to an overall k -dependent phase/constant) the harmonic-oscillator *Glauber coherent state* with eigenvalue $\alpha = ik$,

$$U_C\phi_k \propto \psi_\alpha(q)|_{\alpha=ik}, \quad \hat{a}^-\psi_\alpha = \alpha\psi_\alpha. \quad (5.5)$$

For convenience we recall the standard normalized coordinate-space form

$$\psi_\alpha(q) = \pi^{-1/4} \exp\left(-\frac{1}{2}(q - \sqrt{2}\operatorname{Re}\alpha)^2 + i\sqrt{2}(\operatorname{Im}\alpha)q - \frac{i}{2}\operatorname{Re}\alpha\operatorname{Im}\alpha\right), \quad (5.6)$$

which for purely imaginary $\alpha = ik$ reduces to a Gaussian with a plane-wave phase.

C. Free-particle preimages of oscillator coherent squeezed states

We use the $\mathfrak{su}(1,1)$ generators (K_+, K_-, K_0) defined in Eq. (4.6). For $\alpha, \zeta \in \mathbb{C}$ we introduce the standard displacement and single-mode squeezing operators

$$D(\alpha) := \exp(\alpha\hat{a}^+ - \bar{\alpha}\hat{a}^-), \quad S(\zeta) := \exp(\zeta K_+ - \bar{\zeta} K_-), \quad (5.7)$$

and the normalized oscillator coherent squeezed states (Perelomov/Jacobi-group coherent states)

$$|\alpha, \zeta\rangle := D(\alpha)S(\zeta)|0\rangle. \quad (5.8)$$

Writing $\zeta = re^{i\phi}$, we set

$$\kappa := e^{i\phi} \tanh r, \quad |\kappa| < 1 (r \in \mathbb{R}), \quad (5.9)$$

and employ the standard $\mathfrak{su}(1,1)$ disentangling (valid for $|\kappa| < 1$),

$$S(\zeta) = \exp(\kappa K_+) \exp(\ln(1 - |\kappa|^2) K_0) \exp(-\bar{\kappa} K_-), \quad (5.10)$$

see Appendix D. Since $K_-|0\rangle = 0$ and $K_0|0\rangle = \frac{1}{4}|0\rangle$, one gets the squeezed vacuum in the number basis as

$$S(\zeta)|0\rangle = (1 - |\kappa|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \kappa^n |2n\rangle. \quad (5.11)$$

In the coordinate representation this is a Gaussian wave function, see Appendix E,

$$\begin{aligned} \psi_{0,\zeta}(q) &:= \langle q|S(\zeta)|0\rangle \\ &= \pi^{-1/4} (1 - |\kappa|^2)^{1/4} (1 + \kappa)^{-1/2} \exp\left(-\frac{1}{2} \frac{1 - \kappa}{1 + \kappa} q^2\right), \end{aligned} \quad (5.12)$$

where the square root is taken on a fixed branch (different branch choices change only an overall phase). For $|\kappa| < 1$ one has $\text{Re}(\frac{1-\kappa}{1+\kappa}) > 0$, so $\psi_{0,\zeta} \in L^2(\mathbb{R}, dq)$.

D. Free-particle preimages under U_C^{-1}

Define the free-particle preimages by

$$\Psi_{\alpha,\zeta}(q) := (U_C^{-1}|\alpha, \zeta\rangle)(q). \quad (5.13)$$

A convenient parametrization of normalized free Gaussians is given in Appendix F (Robertson-Schrödinger saturation): for $\tau \in \mathbb{H}_+$ and real (q_0, p_0) ,

$$\begin{aligned} \psi_{q_0, p_0; \tau}(q) &= \left(\frac{\text{Im } \tau}{\pi |\tau|^2}\right)^{1/4} \exp\left(i p_0 q - \frac{i}{2\tau} (q - q_0)^2\right), \\ \tau &\in \mathbb{H}_+. \end{aligned} \quad (5.14)$$

The inverse Cayley map generated by U_C^{-1} identifies the squeezing parameter κ with the upper-half-plane parameter τ by the same Cayley transform used in (2.18),

$$\kappa = i \frac{\tau - i}{\tau + i}, \quad \Leftrightarrow \quad \tau = i \frac{1 - i\kappa}{1 + i\kappa}. \quad (5.15)$$

Moreover, the displacement parameter α fixes the real center (q_0, p_0) in the usual way (with $\hat{a}^- = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$): $q_0 = \sqrt{2}\text{Re } \alpha$, $p_0 = \sqrt{2}\text{Im } \alpha$. With these identifications one has, up to an overall (physically irrelevant) phase,

$$\Psi_{\alpha,\zeta}(q) = e^{i\Theta(\alpha,\zeta)} \psi_{q_0, p_0; \tau}(q), \quad \tau = i \frac{1 - i\kappa}{1 + i\kappa}. \quad (5.16)$$

E. L^2 condition and analytic continuation

From (5.14) the preimage belongs to $L^2(\mathbb{R}, dq)$ if and only if

$$\text{Im } \tau > 0 \quad \Leftrightarrow \quad |\kappa| < 1. \quad (5.17)$$

Thus all *unitary* oscillator squeezed coherent states [real r in (5.9)] have square-integrable free preimages. If one analytically continues the squeezing parameter beyond the unit disk (so that $|\kappa| \geq 1$), then τ leaves $\mathbb{H}_+$ and the Gaussian in (5.14) ceases to be normalizable; this reflects the fact that U_C^{-1} is an unbounded (nonunitary) similarity transform on $L^2(\mathbb{R})$.

F. Bogoliubov viewpoint

Conjugation by $S(\zeta)$ implements the Bogoliubov transformation (2.14),

$$S(\zeta)\hat{a}^\mp S(\zeta)^{-1} = \hat{a}^\mp \cosh r - e^{\pm i\phi} \hat{a}^\pm \sinh r, \quad (5.18)$$

i.e., the $SU(1,1)$ action generated by (5.10). Under the inverse Cayley intertwiner U_C^{-1} this becomes the corresponding linear canonical transformation on the free-particle side, encoded by the Möbius action on τ [equivalently, by the Cayley relation (5.15)].

G. cos and sin as free-particle images of cat states

Define the (unnormalized) even/odd cat combinations [72]

$$|\text{cat}_\pm(\alpha)\rangle := |\alpha\rangle \pm |-\alpha\rangle.$$

Choosing $\alpha = ik$ with $k \in \mathbb{R}$ and using (5.5), one finds $U_C^{-1}|\text{cat}_\pm(ik)\rangle \propto \phi_k \pm \phi_{-k} \propto e^{ikq} \pm e^{-ikq}$. Thus, up to constants and phases,

$$U_C^{-1}|\text{cat}_+(ik)\rangle \propto \cos(kq), \quad U_C^{-1}|\text{cat}_-(ik)\rangle \propto \sin(kq). \quad (5.19)$$

In this way the standard even/odd superpositions of oscillator coherent states have natural free-particle counterparts given by the trigonometric eigenfunctions.

VI. CLASSICAL SYMMETRIES OF THE HARMONIC OSCILLATOR AND CAYLEY-NIEDERER TRANSFORM

In this section we turn to the one-dimensional harmonic oscillator and recall its classical symmetry structure in a form adapted to later comparison with the free particle. Besides time translations generated by H_+ , the oscillator possesses Newton-Hooke analogs of spatial translations and Galilean boosts, realized as dynamical integrals of motion with explicit $(2\pi/\omega)$ -periodic time dependence. Together with a distinguished set of quadratic dynamical integrals they generate a $\mathfrak{h}_3 \rtimes \mathfrak{sp}(2, \mathbb{R})$ Lie-algebraic structure, and we also indicate the $\omega \rightarrow 0$ limit in which the free-particle generators are recovered. We then rewrite the quadratic dynamical integrals in an explicit trigonometric form, emphasizing in particular the π/ω periodicity and the natural compactification of time. This prepares the ground for the Cayley-Niederer map: via an appropriate

reparametrization (chart-wise on $\mathbb{RP}^1$) the oscillator evolution is related to the free particle, and the corresponding time reparametrizations acquire a manifest Möbius form. We conclude with the discrete symmetries of the oscillator.

A. Equations of motion and their solution

Consider the one-dimensional harmonic oscillator ($\omega > 0$),

$$L_{\text{osc}} = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2), \quad H_+ = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \quad \{x, p\} = 1. \quad (6.1)$$

The canonical equations are $\dot{x} = \{x, H_+\} = \frac{p}{m}$, $\dot{p} = \{p, H_+\} = -m\omega^2 x$, and therefore $\ddot{x} + \omega^2 x = 0$. With initial data $x(0) = x_0$, $p(0) = p_0$,

$$x(t) = x_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t, \quad p(t) = p_0 \cos \omega t - m\omega x_0 \sin \omega t. \quad (6.2)$$

B. Newton-Hooke translations and boosts: Dynamical integrals $P(t)$ and $B(t)$

Solving (6.2) for (x_0, p_0) yields the dynamical integrals

$$P(t) = p \cos \omega t + m\omega x \sin \omega t, \\ B(t) = mx \cos \omega t - \frac{p}{\omega} \sin \omega t, \quad (6.3)$$

which satisfy $\frac{dN(t)}{dt} = \frac{\partial N(t)}{\partial t} + \{N(t), H_+\} = 0$, $N(t) = P(t), B(t)$. Their Poisson bracket reproduces the Heisenberg algebra (with the mass m as the central charge), $\{B(t), P(t)\} = m$. The Newton-Hooke relations with H_+ for any t read

$$\{H_+, P(t)\} = -\omega^2 B(t), \quad \{H_+, B(t)\} = P(t). \quad (6.4)$$

Thus, time translations generated by H_+ rotate the translation/boost doublet.

C. $sl(2, \mathbb{R})$ dynamical integrals and the $\omega \rightarrow 0$ limit

Guided by the free particle case, define

$$H_0^{\text{NH}}(t) = \frac{1}{2m} P(t)^2, \quad D^{\text{NH}}(t) = \frac{1}{m} B(t) P(t), \quad K^{\text{NH}}(t) = \frac{1}{2m} B(t)^2, \quad (6.5)$$

which are dynamical integrals. Their Poisson bracket algebra is $sl(2, \mathbb{R})$,

$$\{D^{\text{NH}}, H_0^{\text{NH}}\} = 2H_0^{\text{NH}}, \quad \{D^{\text{NH}}, K^{\text{NH}}\} = -2K^{\text{NH}}, \quad \{K^{\text{NH}}, H_0^{\text{NH}}\} = D^{\text{NH}}, \quad (6.6)$$

together with the action on (P, B) ,

$$\{D^{\text{NH}}, P\} = P, \quad \{D^{\text{NH}}, B\} = -B, \quad \{H_0^{\text{NH}}, B\} = -P, \quad \{K^{\text{NH}}, P\} = B. \quad (6.7)$$

The oscillator Hamiltonian H_+ is the compact (Newton-Hooke) combination

$$H_+ = H_0^{\text{NH}} + \omega^2 K^{\text{NH}}, \quad (6.8)$$

which indeed reduces to $H_+ = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \text{const}$ for any t .

D. Explicit trigonometric form; π/ω periodicity and $\mathbb{RP}^1$

On a fixed time slice $t=0$, $B(0) = mx_0$, $P(0) = p_0$, and one has $H_0 = \frac{p_0^2}{2m}$, $D^{\text{NH}}(0) = x_0 p_0 := D_0$, $K^{\text{NH}}(0) = \frac{1}{2} m x_0^2 := K_0$. From (6.3) and (6.5) one obtains

$$H_0^{\text{NH}}(t) = H_0 \cos^2 \omega t + \frac{\omega}{2} D_0 \sin(2\omega t) + \omega^2 K_0 \sin^2 \omega t, \quad (6.9)$$

$$D^{\text{NH}}(t) = D_0 \cos(2\omega t) + \left(\omega K_0 - \frac{1}{\omega} H_0 \right) \sin(2\omega t), \quad (6.10)$$

$$K^{\text{NH}}(t) = K_0 \cos^2 \omega t - \frac{1}{2\omega} D_0 \sin(2\omega t) + \frac{1}{\omega^2} H_0 \sin^2 \omega t. \quad (6.11)$$

In the limit $\omega \rightarrow 0$, $P(t) \rightarrow p$, $B(t) \rightarrow mx - pt$, $H_0 = \frac{p^2}{2m}$, $D^{\text{NH}}(t) \rightarrow xp - 2H_0 t$, and $K^{\text{NH}}(t) \rightarrow K_0 - D_0 t + H_0 t^2$, cf. (3.3) and (3.4).

Although $(x(t), p(t))$ are $2\pi/\omega$ -periodic, the dynamical integrals $D^{\text{NH}}(t)$ and $K^{\text{NH}}(t)$ depend on $2\omega t$ and therefore have period π/ω . From the viewpoint of the $sl(2, \mathbb{R})$ generators, time is naturally a coordinate on a circle with identified antipodes, i.e., on $\mathbb{RP}^1$.

A convenient Cayley parameter on this $\mathbb{RP}^1$ is

$$z = e^{2i\omega t}, \quad (6.12)$$

cf. (2.22), which is invariant under $t \mapsto t + \pi/\omega$ and makes the double-angle periodicity manifest.

E. Compact $sl(2, \mathbb{R})$ evolution and ladder variables

Define the complementary combination

$$J^{\text{NH}} := H_0^{\text{NH}} - \omega^2 K^{\text{NH}}, \quad (6.13)$$

so that $H_+ = H_0^{\text{NH}} + \omega^2 K^{\text{NH}}$ and J^{NH} are the two linear combinations of $(H_0^{\text{NH}}, K^{\text{NH}})$, with J^{NH} corresponding to H_- considered above. Using (6.6) one finds

$$\begin{aligned} \dot{D}^{\text{NH}} &= \{D^{\text{NH}}, H_+\} = 2J^{\text{NH}}, \\ \dot{J}^{\text{NH}} &= \{J^{\text{NH}}, H_+\} = -2\omega^2 D^{\text{NH}}, \end{aligned} \quad (6.14)$$

and hence $\ddot{D}^{\text{NH}} + (2\omega)^2 D^{\text{NH}} = 0$. Introduce the ladder combinations

$$J_{\pm}^{\text{NH}} := J^{\text{NH}} \pm i\omega D^{\text{NH}}. \quad (6.15)$$

Then (6.14) gives $\dot{J}_{\pm}^{\text{NH}} = \{J_{\pm}^{\text{NH}}, H_+\} = \pm 2i\omega J_{\pm}^{\text{NH}}$, and therefore

$$J_{\pm}^{\text{NH}}(t) = e^{\pm 2i\omega t} J_{\pm}^{\text{NH}}(0) = z^{\pm 1} J_{\pm}^{\text{NH}}(0), \quad (6.16)$$

with $z = e^{2i\omega t}$ from (6.12).

F. Cayley-Niederer map and the emergence of Möbius transformations

To reveal the fractional-linear structure, introduce, coherently with (2.21) and (3.18), the projective time variable

$$\tau = \frac{1}{\omega} \tan(\omega t), \quad \Leftrightarrow \quad t = \frac{1}{\omega} \arctan(\omega \tau) \text{ (chart-wise on } \mathbb{RP}^1), \quad (6.17)$$

and the rescaled coordinate

$$y = \frac{x}{\cos(\omega t)} = x \sqrt{\frac{d\tau}{dt}}. \quad (6.18)$$

The factor $\sqrt{d\tau/dt}$ shows that x behaves as a half-density of weight $-1/2$ under time reparametrizations in this construction, cf. (3.16). A direct computation shows that $y(\tau)$ satisfies the free equation $d^2 y/d\tau^2 = 0$. Equivalently, the free and oscillator Lagrangians are related up to a total derivative: $\frac{m}{2} \left(\frac{dy}{d\tau}\right)^2 d\tau = \left[\frac{m}{2} (\dot{x}^2 - \omega^2 x^2) + \frac{d}{dt} \left(\frac{m\omega}{2} x^2 \tan(\omega t)\right)\right] dt$, cf. (8.4) below. The map (6.17)–(6.18) is the (classical)

Cayley-Niederer transformation relating the harmonic oscillator to the free particle.

In the (τ, y) variables, the $sl(2, \mathbb{R})$ transformations act in the standard projective way,

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad y' = \frac{y}{c\tau + d}, \quad ad - bc = 1, \quad (6.19)$$

so that in terms of the physical time t one obtains the fractional-linear law for $\tan(\omega t)$,

$$\tan(\omega t') = \frac{a \tan(\omega t) + b\omega}{c \tan(\omega t) + d\omega}, \quad (6.20)$$

cf. (2.23). This explains why, in contrast to the free particle case, the $sl(2, \mathbb{R})$ transformations are not manifestly Möbius in t : the physical time t is an angular coordinate on $\mathbb{RP}^1$, while $\tau = \tan(\omega t)/\omega$ is an affine coordinate on the projective line.

For completeness, the induced transformation of x can be written as

$$x' = y' \cos(\omega t') = x \frac{\cos(\omega t')}{\cos(\omega t)} \frac{1}{d + \frac{c}{\omega} \tan(\omega t)}. \quad (6.21)$$

A more detailed discussion of the Cayley-Niederer transformation including its quantum implementation will be given below in Sec. VIII.

G. Discrete symmetries

Besides the continuous symmetries above, the oscillator admits the standard discrete symmetries:

- (i) Parity: $x \mapsto -x$, $p \mapsto -p$, $t \mapsto t$; and
- (ii) Time reversal: $t \mapsto -t$, $x \mapsto x$, $p \mapsto -p$.

In addition, the free-particle conformal inversion is naturally realized for the oscillator as an inversion of the *projective* time variable,

$$\omega \tau \mapsto \omega \tau' = -\frac{1}{\omega \tau}. \quad (6.22)$$

Using $\omega \tau = \tan(\omega t)$, this corresponds (on $\mathbb{RP}^1$) to a shift by a quarter period in time t ,

$$\begin{aligned} \tan(\omega t') &= -\cot(\omega t) = \tan\left(\omega t - \frac{\pi}{2}\right) \\ \Rightarrow \quad t' &\equiv t - \frac{\pi}{2\omega} \left(\text{mod } \frac{\pi}{\omega}\right), \end{aligned} \quad (6.23)$$

equivalently, on the Cayley circle (6.12) it is the antipodal map $z \mapsto z' = -z$. On phase space this can be represented as the canonical quarter-rotation $(x, p) \mapsto (\frac{p}{m\omega}, -m\omega x)$, under which

$$H_0^{\text{NH}} \leftrightarrow \omega^2 K^{\text{NH}}, \quad D^{\text{NH}} \mapsto -D^{\text{NH}}, \quad (6.24)$$

$J^{\text{NH}} \mapsto -J^{\text{NH}}$, while the compact combination $H_+ = H_0^{\text{NH}} + \omega^2 K^{\text{NH}}$ is preserved.

VII. STATIONARY AND TIME-DEPENDENT SCHRÖDINGER EQUATIONS, AND $(\pm 1/2)$ -WEIGHT DENSITIES

Before we discuss the quantum version of the Cayley-Niederer transform for the free-particle-harmonic oscillator correspondence, we recall here a few structural facts about the stationary and time-dependent Schrödinger equations that are naturally formulated in projective terms. For the stationary equation in Liouville (normal) form, the unique reparametrization that preserves the absence of the first-derivative term is the $(-\frac{1}{2})$ (inverse half-density) law, and the potential term $Q(x)$ transforms as a projective connection with an inhomogeneous Schwarzian contribution. For the TDSE, by contrast, the natural requirement is unitarity of the Hilbert-space identification under spatial changes of variables, which leads to a $(+\frac{1}{2})$ (half-density) law; a nontrivial map preserving the standard flat kinetic term necessarily involves a correlated time reparametrization (lens/Cayley-Niederer-type transform), in which the Schwarzian term is the *time* Schwarzian of $t = t(\tau)$. We conclude by reconciling these two weights, which arise from different goals (Liouville form versus unitarity).

A. Liouville transformation and the $(-\frac{1}{2})$ (inverse half-density) law

Consider the stationary 1D Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x), \quad (7.1)$$

which we rewrite in Liouville (normal) form

$$\psi''(x) + Q(x)\psi(x) = 0, \quad Q(x) := \frac{2m}{\hbar^2}(E - V(x)). \quad (7.2)$$

In the discussion below one may regard $Q(x)$ as an arbitrary function.

Let $y = y(x)$ be locally monotone, with inverse $x = x(y)$. A mere change of variable produces a first-derivative term; the unique rescaling that keeps the *Liouville form* (no first derivative) is the *Liouville rescaling*

$$\psi(x) = (y'(x))^{-1/2}\phi(y), \quad y' = \frac{dy}{dx}. \quad (7.3)$$

Equivalently, $\phi(y) = (x'(y))^{-1/2}\psi(x(y))$. This is precisely the invariance law for an *inverse half-density*,

$$\psi(x)(dx)^{-1/2} = \phi(y)(dy)^{-1/2}. \quad (7.4)$$

Indeed, since $dy = y'(x)dx$, one has $(dy)^{-1/2} = (y')^{-1/2}(dx)^{-1/2}$, and (7.4) is equivalent to (7.3). This $(-\frac{1}{2})$ weight should be understood as the density weight relevant to the *Liouville-normal-form problem* for the ordinary differential equation (ODE), cf. (3.16) and (6.18).

B. Projective-connection transformation of Q : Two equivalent forms

Substituting (7.3) into (7.2) yields again the Liouville form

$$\phi_{yy}(y) + \tilde{Q}(y)\phi(y) = 0, \quad (7.5)$$

with the projective-connection transformation law

$$\tilde{Q}(y) = (x'(y))^2 Q(x(y)) + \frac{1}{2}\{x; y\}, \quad (7.6)$$

where $\{x; y\}$ is the Schwarzian derivative [30–32, 73], see Appendix G. Using the inverse-map identity (G7), the same formula can be rewritten in terms of $\{y; x\}$ with a common prefactor,

$$\tilde{Q}(y) = (x'(y))^2 \left[Q(x(y)) - \frac{1}{2}\{y; x\} \Big|_{x=x(y)} \right]. \quad (7.7)$$

Thus Q is not a quadratic differential [which would transform as $\tilde{Q} = (x')^2 Q$], but an affine connectionlike object whose differences are quadratic differentials, with the inhomogeneous Schwarzian term encoding its projective-connection (cocycle) nature, see Appendix G.

C. Projective coordinate and the intrinsic meaning of Q

Let ψ_1, ψ_2 be two linearly independent solutions of (7.2). Their ratio

$$w(x) := \frac{\psi_1(x)}{\psi_2(x)} \quad (7.8)$$

is a projective coordinate [defined up to Möbius transformations $w \mapsto (aw + b)/(cw + d)$], and one has the identity

$$\{w; x\} = 2Q(x). \quad (7.9)$$

Equivalently, the reduction-of-order integral

$$u(x) := \int^x \frac{dt}{\psi(t)^2} \quad (7.10)$$

is invariant under reparametrizations when ψ obeys the inverse half-density law (7.4), because

$$\psi(x)^2 = (y'(x))^{-1} \phi(y)^2 \Rightarrow \frac{dx}{\psi(x)^2} = \frac{dy}{\phi(y)^2}, \quad (7.11)$$

and therefore $\int^x dt/\psi(t)^2 = \int^{y(x)} d\eta/\phi(\eta)^2$. In this language the identity (7.9) can be viewed as $\{u; x\} = 2Q(x)$.

D. TDSE: Spatial reparametrizations at fixed time and unitarity

Consider the TDSE

$$i\hbar\partial_t\Psi(t, x) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V(t, x)\right)\Psi(t, x). \quad (7.12)$$

Let $x = x(y)$ be a time-independent diffeomorphism with Jacobian $J(y) := \frac{dx}{dy} > 0$.

(a) Pure coordinate substitution. With $\tilde{\Psi}(t, y) := \Psi(t, x(y))$ one has

$$\partial_x = \frac{1}{J}\partial_y, \quad \partial_x^2 = \frac{1}{J^2}\partial_y^2 - \frac{J'}{J^3}\partial_y, \quad J(y) := \frac{dx}{dy} > 0, \quad (7.13)$$

and (7.12) becomes a TDSE with a first-derivative term in the spatial operator. Thus a general spatial reparametrization does *not* preserve the standard flat form $-(\hbar^2/2m)\partial_y^2 + \tilde{V}(t, y)$.

(b) Unitary half-density transform and the spatial Schwarzian term. The natural requirement for quantum mechanics is unitarity, i.e., preservation of $\int |\Psi(t, x)|^2 dx$. Since $dx = J(y)dy$, the unitary identification $L^2(dx) \cong L^2(dy)$ is implemented by

$$\Phi(t, y) := \sqrt{J(y)}\tilde{\Psi}(t, y) = \sqrt{\frac{dx}{dy}}\Psi(t, x(y)). \quad (7.14)$$

Equivalently, $|\Psi(t, x)|^2 dx = |\Phi(t, y)|^2 dy$, $\Psi(t, x)(dx)^{1/2} = \Phi(t, y)(dy)^{1/2}$. Thus, in the TDSE/Hilbert-space context, Ψ behaves as a *spatial* $(+1/2)$ -density.

A direct substitution $\tilde{\Psi} = J^{-1/2}\Phi$ into the pulled-back equation yields

$$i\hbar\partial_t\Phi = -\frac{\hbar^2}{2mJ(y)^2}\partial_y^2\Phi + \left(V(t, x(y)) - \frac{\hbar^2}{4mJ(y)^2}\{x; y\}\right)\Phi. \quad (7.15)$$

Two key features appear: (i) the kinetic term acquires the variable coefficient J^{-2} (a variable, “position-dependent mass” [74]); (ii) a spatial Schwarzian correction contributes to the effective potential. Therefore, preserving the *flat* kinetic term with constant coefficient requires $J = \text{const}$, i.e., x linear in y . This is why the nontrivial free $\leftrightarrow$ oscillator map cannot be achieved by a purely spatial diffeomorphism: it must involve time.

E. Lens (Cayley-Niederer-type) transform: Time reparametrization and time-Schwarzian term

To preserve the *standard TDSE form* with a flat kinetic term, one uses a correlated transformation of time, space, and wave function.

Let

$$t = t(\tau), \quad \rho(\tau) := \frac{dt}{d\tau} > 0, \quad (7.16)$$

and scale space linearly,

$$x = \sqrt{\rho(\tau)}y. \quad (7.17)$$

(The spatial map is linear in y , hence its *spatial* Schwarzian vanishes identically.)

Define the transformed wave function by

$$\Psi(t(\tau), x) = \rho(\tau)^{-1/4} \exp\left(\frac{i}{\hbar}S(\tau, y)\right)\Phi(\tau, y), \quad x = \sqrt{\rho(\tau)}y, \quad (7.18)$$

with quadratic phase

$$S(\tau, y) := \frac{m\rho'(\tau)}{4\rho(\tau)}y^2, \quad ('= d/d\tau). \quad (7.19)$$

The prefactor $\rho^{-1/4}$ is exactly the unitary half-density factor corresponding to $dx = \sqrt{\rho}dy$,

$$\int_{\mathbb{R}} |\Psi|^2 dx = \int_{\mathbb{R}} |\Phi|^2 dy. \quad (7.20)$$

F. Result: transformed TDSE and the *time* Schwarzian

Substituting (7.18)–(7.19) into (7.12) yields again a standard TDSE,

$$i\hbar\partial_\tau\Phi(\tau, y) = \left(-\frac{\hbar^2}{2m}\partial_y^2 + \tilde{V}(\tau, y)\right)\Phi(\tau, y), \quad (7.21)$$

with transformed potential

$$\tilde{V}(\tau, y) = \rho(\tau)V(t(\tau), \sqrt{\rho(\tau)}y) + \frac{m}{4}\{t; \tau\}y^2. \quad (7.22)$$

Here $\{t; \tau\}$ is the Schwarzian derivative of $t = t(\tau)$,

$$\{t; \tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2 = \frac{\rho''(\tau)}{\rho(\tau)} - \frac{3}{2}\left(\frac{\rho'(\tau)}{\rho(\tau)}\right)^2. \quad (7.23)$$

This is the crucial structural point: in the lens/Cayley-Niederer transform the Schwarzian term is the Schwarzian of the time reparametrization, not of the spatial change (which is linear in y and has zero spatial Schwarzian).

Using (G7), one may also express this in terms of the inverse map $\tau = \tau(t)$: $\{t; \tau\} = -\rho(\tau)^2 \{\tau; t\}|_{t=\tau(\tau)}$.

G. Pulled-back TDSE and the origin of the chirp and half-density factors

To make explicit what the TDSE becomes *before* the chirp/half-density rescaling in (7.18), define the pulled-back wave function

$$\tilde{\Psi}(\tau, y) := \Psi(t(\tau), x(\tau, y)), \quad x(\tau, y) := \sqrt{\rho(\tau)}y,$$

$$\rho(\tau) = \frac{dt}{d\tau} > 0.$$

A direct chain-rule computation gives $\partial_t \Psi = \frac{1}{\rho}(\partial_\tau - \frac{\rho'}{2\rho}y\partial_y)\tilde{\Psi}$, $\partial_x = \frac{1}{\sqrt{\rho}}\partial_y$, $\partial_x^2 = \frac{1}{\rho}\partial_y^2$. Substituting into the general TDSE (7.12) [with arbitrary $V(t, x)$] yields, equivalently,

$$\frac{1}{\rho(\tau)} \left(i\hbar\partial_\tau + \frac{\hbar^2}{2m}\partial_y^2 - i\hbar\frac{\rho'}{2\rho}y\partial_y - \rho(\tau)V(t(\tau), \sqrt{\rho(\tau)}y) \right) \tilde{\Psi}(\tau, y) = 0,$$

or, after completing the square,

$$\left(i\hbar\partial_\tau + \frac{\hbar^2}{2m} \left(\partial_y - \frac{i}{\hbar} \frac{m\rho'}{2\rho}y \right)^2 + \frac{i\hbar\rho'}{4\rho} + \frac{m}{8} \left(\frac{\rho'}{\rho} \right)^2 y^2 - \rho V(t(\tau), \sqrt{\rho}y) \right) \tilde{\Psi} = 0.$$

The unitary chirp factor $U_S(\tau) = \exp(\frac{i}{\hbar}S(\tau, y))$ with S as in (7.19) removes the elongated derivative by conjugation, $U_S^{-1}(\partial_y - \frac{i}{\hbar} \frac{m\rho'}{2\rho}y)U_S = \partial_y$, $U_S^{-1}(i\hbar\partial_\tau)U_S = i\hbar\partial_\tau - \partial_\tau S$, so that the induced $-\partial_\tau S$ term combines with the remaining quadratic contribution to produce precisely the time-Schwarzian potential term in (7.22), (7.23). Finally, the half-density factor $\rho^{-1/4}$ in (7.18) cancels the residual pure-imaginary term, since $\rho^{1/4}(i\hbar\partial_\tau)\rho^{-1/4} = i\hbar\partial_\tau - \frac{i\hbar\rho'}{4\rho}$. This gives a concrete mechanism behind the required normalization and the associated Maslov/Fresnel phase bookkeeping when solutions are continued globally across caustics.

H. Cayley-Niederer map (free $\leftrightarrow$ oscillator)

The classical Cayley–Niederer choice

$$t = \tan \tau, \quad \rho(\tau) = \frac{dt}{d\tau} = \sec^2 \tau, \quad x = \sec \tau y, \quad (7.24)$$

cf. (6.17), gives $\{t; \tau\} = 2$, so for $V \equiv 0$ one obtains a harmonic oscillator term $\tilde{V} = (m/2)y^2$ in (7.21) (frequency $\omega = 1$ in τ time).

I. Action principle viewpoint and unitarity of the lens transform

The TDSE (7.12) follows from the action functional

$$\mathcal{S}[\Psi, \Psi^*] = \int dt \int dx \left[\frac{i\hbar}{2} (\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^*) - \frac{\hbar^2}{2m} |\partial_x \Psi|^2 - V(t, x) |\Psi|^2 \right]. \quad (7.25)$$

Under the transformation (7.18) with $t = t(\tau)$ and $x = \sqrt{\rho}y$, one has $dt dx = \rho(\tau) d\tau \sqrt{\rho(\tau)} dy$, while the

prefactor $\rho^{-1/4}$ and the phase (7.19) are precisely chosen so that: (i) the kinetic term retains the flat form $-(\hbar^2/2m)\partial_y^2$, (ii) the term $\propto y\partial_y$ cancels, and (iii) the potential transforms as in (7.22). As a result, $\mathcal{S}[\Psi, \Psi^*]$ is mapped to the same functional of (Φ, Φ^*) (with $\tilde{V}$), up to a total τ derivative coming from the quadratic phase (a boundary term in the action). Consequently, the map (7.18) is implemented by a (time-dependent) unitary operator between Hilbert spaces, which is also reflected in the exact norm preservation (7.20).

J. No contradiction between $(-\frac{1}{2})$ and $(+\frac{1}{2})$ weights

Let us stress here that the different “weights” arise from different *goals*:

- (i) For the stationary ODE (7.2), the goal is to preserve the *Liouville normal form* (no first derivative). This selects the inverse half-density law (7.3), equivalently (7.4), and produces the projective-connection law (7.6)–(7.7) with the *spatial* Schwarzian.
- (ii) For the TDSE as a Hilbert-space evolution, the natural requirement is *unitarity* (preservation of $\int |\Psi|^2 dx$). This yields the half-density law (7.14). To preserve the standard flat kinetic term in a nontrivial way (free $\leftrightarrow$ oscillator), one must also reparametrize time and scale space as in (7.16)–(7.17), leading to the *time* Schwarzian term $\{t; \tau\}$ in (7.22).

VIII. EXTENDED FREE PARTICLE AND THE SCHWARZIAN-INDUCED OSCILLATOR

In this section we relate the free particle and the harmonic oscillator (at both the classical and quantum levels) in the *extended* formulation, where time is promoted to a dynamical variable and the dynamics is encoded by a

first-class constraint. A general time reparametrization, combined with the accompanying space rescaling, induces a canonical transformation on the extended phase space that maps the free constraint to an oscillator-type constraint with a *time-dependent* frequency. The latter is governed by the *time Schwarzian* of the reparametrization, $\Omega^2(\tau) = \frac{1}{2}\omega^2\{F;\eta\}$, so that the oscillator potential is “Schwarzian induced.”

We then quantize this canonical map in the metaplectic/Fourier-integral framework. In particular, we show how the most economical choice of generating function implements the point transformation by elementary Fourier integrals (producing delta distributions), and how the quantum free constraint is intertwined into the time-dependent oscillator Schrödinger equation with potential $\tilde{V}(\tau, y) = \frac{m}{2}\Omega^2(\tau)y^2$. Finally, we single out the constant-frequency case (the Cayley-Niederer map) as the projective-time choice $t(\tau) = \omega^{-1}\tan(\omega\tau)$ and relate the free and oscillator propagators; in this way the Mehler kernel and its Maslov/branch structure emerge directly from the free kernel.

A. Classical canonical equivalence in the extended formulation

Let us return to the classical reparametrization-invariant formulation of the free particle described by action (3.8),

$$S_{\text{free}}[x(s), t(s)] = \int ds \frac{m}{2} \frac{\dot{x}^2}{\dot{t}}, \quad \cdot \equiv \frac{d}{ds}, \quad (8.1)$$

in which time parameter t is promoted to be the dynamical variable. As it was discussed in Sec. III, Hamiltonian formulation on the extended phase space (x, t, p_x, p_t) is characterized by the first-class constraint

$$C_{\text{free}} = p_t + \frac{p_x^2}{2m} \approx 0. \quad (8.2)$$

Let us perform the same time reparametrization and space rescaling as in (7.16)–(7.17),

$$t = \frac{1}{\omega}F(\omega\tau), \quad \rho(\tau) := \frac{dt}{d\tau} = F'(\eta) > 0, \quad x = \sqrt{\rho(\tau)}y, \quad (8.3)$$

where $\omega > 0$ is a constant of dimension of frequency, and so, $\eta = \omega\tau$ and $\omega t := \lambda$ are dimensionless. Action (8.1) takes then the form

$$S_{\Omega}[y, \tau] = \int ds \left(\frac{1}{2}m \frac{\dot{y}^2}{\dot{\tau}} - \frac{1}{2}m\Omega^2(\tau)y^2 + \frac{d}{ds}S(\tau, y) \right), \quad (8.4)$$

where

$$\Omega^2(\tau) = \frac{1}{2}\omega^2\{F;\eta\}, \quad \text{and} \quad S(\tau, y) = \frac{m\omega}{4} \frac{F''(\eta)}{F'(\eta)} y^2. \quad (8.5)$$

The first two terms in this reparametrization-invariant action correspond to Lagrangian of harmonic oscillator with time-dependent frequency. Free particle constraint (8.2) transforms now into

$$\Phi_{\Omega} = P_{\tau} + H_{\Omega} \approx 0, \quad H_{\Omega} = \frac{1}{2m}P_y^2 + \frac{1}{2}m\Omega^2(\tau)y^2. \quad (8.6)$$

Relations

$$p_x = \frac{1}{\sqrt{F'(\eta)}} \left(P_y + \frac{m\omega}{2} \mathcal{P}_F(\eta)y \right), \quad (8.7)$$

$$p_t = \frac{1}{F'(\eta)} \left(P_{\tau} - \frac{\omega}{2} \mathcal{P}_F(\eta)y P_y + \frac{m\omega^2}{4} \left(\{F;\eta\} - \frac{1}{2}(\mathcal{P}_F(\eta))^2 \right) y^2 \right) \quad (8.8)$$

provide a canonical lift of the point transformation (8.3), where $\mathcal{P}_F(\eta) = \frac{F''(\eta)}{F'(\eta)}$ is pre-Schwarzian (G3). The inverse relations that present P_y and P_{τ} in terms of (t, p_t, x, p_x) can be obtained easily from (8.7) and (8.8) via the exchange

$$(t, p_t, x, p_x) \leftrightarrow (\tau, P_{\tau}, y, P_y), \quad F \leftrightarrow R = F^{-1}, \quad \eta \leftrightarrow \lambda, \quad (8.9)$$

and using identities

$$F'(\eta) = \frac{1}{R'(\lambda)}, \quad \mathcal{P}_F(\eta) = -\frac{1}{R'(\lambda)} \mathcal{P}_R(\lambda),$$

$$\{F;\eta\} = -\frac{1}{(R'(\lambda))^2} \{R;\lambda\},$$

with F' and R' denoting the derivatives in η and λ , respectively, and $\mathcal{P}_R(\lambda) = \frac{R''(\lambda)}{R'(\lambda)}$.

The relation between corresponding Liouville 1-forms is

$$p_t dt + p_x dx = P_{\tau} d\tau + P_y dy + dS, \quad (8.10)$$

where $S = S(\tau, y)$ is exactly the function from (8.5), which appears under the sign of total derivative in the transformed action (8.4) as well as in quantum transformation (7.18).

A generating function of the described canonical transformation (8.3), (8.7), (8.8) is

$$F_{22}(t, P_{\tau}; x, P_y) = \frac{1}{\omega} R(\lambda) P_{\tau} + \sqrt{R'(\lambda)} x P_y - \frac{m\omega}{4} \mathcal{P}_R(\lambda) x^2, \quad (8.11)$$

from which $p_t = \frac{\partial F_{22}}{\partial t}$, $\tau = \frac{\partial F_{22}}{\partial P_{\tau}}$, $p_x = \frac{\partial F_{22}}{\partial x}$, $y = \frac{\partial F_{22}}{\partial P_y}$. Three other possible forms of generating functions can be obtained from (8.11) and (8.10) by applying a corresponding Legendre transformation. Particularly,

$$\begin{aligned}
F_{23}(t, P_\tau; p_x, y) &= F_{22} - x p_x - y P_y \\
&= \frac{1}{\omega} R(\lambda) P_\tau - \frac{y}{\sqrt{R'(\lambda)}} p_x \\
&\quad - \frac{m\omega}{4} \frac{R''(\lambda)}{(R'(\lambda))^2} y^2.
\end{aligned} \tag{8.12}$$

In analogous way, one obtains $F_{32}(p_t, \tau; x, P_y)$ and $F_{33}(p_t, \tau; p_x, y)$, whose explicit form can be written directly from (8.11) and (8.12) by using the transformation law

$$F_{22} \leftrightarrow -F_{33}, \quad F_{23} \leftrightarrow -F_{32},$$

under exchange (8.9). Note that the last term in (8.11) and (8.12), and its analogs in F_{32} and F_{33} , is the function S from (8.5) expressed in corresponding phase space variables.

Under canonical transformation (8.3), (8.7), (8.8), constraint (8.2) takes the form $\frac{1}{F'(\eta)} \Phi_\Omega \approx 0$, which is equivalent to the first class constraint (8.6).

So, at the classical level, the extended formulation of the free particle described by reparametrization-invariant action (8.1) is *canonically equivalent* to analogous extended formulation (8.4) of the harmonic oscillator with time-dependent frequency in general case of transformation (8.3).

B. Quantum implementation and distributional collapse

Each of the four classical generating functions F_{ij} gives rise, upon quantization, to a Fourier-integral (metaplectic) operator with phase F_{ij} . We fix the normalization so that each integration over a conjugate pair carries the unitary factor $(2\pi\hbar)^{-1/2}$; in particular, a double integral carries $(2\pi\hbar)^{-1}$. For the choice $F_{22}(t, P_\tau; x, P_y)$ given in (8.11) we define $\mathcal{U}_{22}$ by

$$\begin{aligned}
(\mathcal{U}_{22}\Psi)(P_\tau, P_y) &= \int_{\mathbb{R}} dt \int_{\mathbb{R}} dx \frac{1}{2\pi\hbar} \\
&\quad \times \exp\left(\frac{i}{\hbar} F_{22}(t, P_\tau; x, P_y)\right) \Psi(t, x).
\end{aligned} \tag{8.13}$$

The unitary Fourier transform in a conjugate pair is taken as

$$(\mathcal{F}_{x \rightarrow p_x} \phi)(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} e^{-\frac{i}{\hbar} p x} \phi(x) dx, \quad \mathcal{F}_{x \rightarrow p_x}^{-1} = \mathcal{F}_{p_x \rightarrow x}, \tag{8.14}$$

and similarly for the pairs (t, p_t) , (τ, P_τ) and (y, P_y) .

Passing to the (τ, y) representation by inverse Fourier transforms in P_τ and P_y , we write

$$\begin{aligned}
(\tilde{\mathcal{U}}_{22}\Psi)(\tau, y) &:= (\mathcal{F}_{P_\tau \rightarrow \tau}^{-1} \mathcal{F}_{P_y \rightarrow y}^{-1} \mathcal{U}_{22}\Psi)(\tau, y) \\
&= \int_{\mathbb{R}} dt \int_{\mathbb{R}} dx K_{22}(\tau, y; t, x) \Psi(t, x),
\end{aligned} \tag{8.15}$$

with kernel

$$\begin{aligned}
K_{22}(\tau, y; t, x) &= \int_{\mathbb{R}} dP_\tau \int_{\mathbb{R}} dP_y \frac{1}{(2\pi\hbar)^2} \\
&\quad \times \exp\left[\frac{i}{\hbar} (F_{22}(t, P_\tau; x, P_y) - P_\tau \tau - P_y y)\right].
\end{aligned} \tag{8.16}$$

Here the phase is fixed by the classical generating function F_{22} , while the overall amplitude (up to a constant normalization and a possible constant Maslov phase) is fixed by the metaplectic requirement. Since F_{22} is linear in P_τ and P_y , the P_τ - and P_y -integrations in (8.16) are elementary and collapse to delta distributions,

$$\begin{aligned}
K_{22}(\tau, y; t, x) &= \delta\left(\tau - \frac{1}{\omega} R(\omega t)\right) \delta(y - \sqrt{R'(\omega t)} x) \\
&\quad \times \exp\left[-\frac{i}{\hbar} \frac{m\omega}{4} \mathcal{P}_R(\omega t) x^2\right].
\end{aligned} \tag{8.17}$$

Thus the kernel is localized on the classical point transformation,

$$\begin{aligned}
\tau = \frac{1}{\omega} R(\omega t) &\Leftrightarrow t = t(\tau), \\
y = \sqrt{R'(\omega t)} x = \frac{x}{\sqrt{\rho(\tau)}} &\Leftrightarrow x = \sqrt{\rho(\tau)} y,
\end{aligned}$$

and the remaining quadratic term in F_{22} produces the phase factor $\exp(-\frac{i}{\hbar} S(\tau, y))$ after substituting $t = t(\tau)$ and $x = \sqrt{\rho(\tau)} y$ [using the identities relating $\mathcal{P}_R$ to ρ'/ρ and the definition of $S(\tau, y)$ given earlier].

To pass from this distributional collapse to a *unitary* map between configuration-space wave functions, one must still fix the overall amplitude (metaplectic half-density). Writing the most general pointwise form compatible with the canonical map as

$$\Phi(\tau, y) = A(\tau) \exp\left(-\frac{i}{\hbar} S(\tau, y)\right) \Psi(t(\tau), x), \quad x = \sqrt{\rho(\tau)} y,$$

the metaplectic (unitarity) requirement on spatial slices, (7.20), together with $dx = \sqrt{\rho(\tau)} dy$, yields $|A(\tau)|^2 = \sqrt{\rho(\tau)}$. Hence $A(\tau) = \rho(\tau)^{1/4}$ (up to an irrelevant constant phase), and one obtains the pointwise transformation

$$\Phi(\tau, y) = \rho(\tau)^{1/4} \exp\left(-\frac{i}{\hbar} S(\tau, y)\right) \Psi(t(\tau), x),$$

$$x = \sqrt{\rho(\tau)} y, \quad (8.18)$$

which is equivalent to (7.18).

The remaining generating functions F_{23} , F_{32} , and F_{33} describe the same classical canonical map, but their quantum operators are obtained from the F_{22} realization only after additional unitary Fourier transforms in the corresponding conjugate pairs, and are therefore less economical for recovering the configuration-space point transformation.

The quantum analog of the classical first-class constraint is obtained by the substitutions $p_t \mapsto -i\hbar\partial_t$ and $p_x \mapsto -i\hbar\partial_x$. For the free particle,

$$\hat{C}_{\text{free}} \Psi = 0, \quad \hat{C}_{\text{free}} := \hat{p}_t + \frac{\hat{p}_x^2}{2m}$$

$$\Leftrightarrow i\hbar\partial_t \Psi(t, x) = -\frac{\hbar^2}{2m} \partial_x^2 \Psi(t, x). \quad (8.19)$$

Under the unitary map (7.18) this is transformed into the constrained Schrödinger equation associated with the oscillator-type Hamiltonian $H_\Omega = \frac{1}{2m} P_y^2 + \frac{m}{2} \Omega^2(\tau) y^2$,

$$\hat{\Phi}_\Omega \Phi = 0, \quad \hat{\Phi}_\Omega := \hat{p}_\tau + \frac{\hat{p}_y^2}{2m} + \frac{m}{2} \Omega^2(\tau) y^2$$

$$\Leftrightarrow i\hbar\partial_\tau \Phi(\tau, y) = \left[-\frac{\hbar^2}{2m} \partial_y^2 + \tilde{V}(\tau, y) \right] \Phi(\tau, y),$$

with the time-dependent harmonic-oscillator potential

$$\tilde{V}(\tau, y) = \frac{m}{2} \Omega^2(\tau) y^2, \quad \Omega^2(\tau) = \frac{1}{2} \omega^2 \{F; \eta\}, \quad \eta = \omega\tau,$$

cf. (8.5). Thus the unitary operator induced by F_{22} intertwines the quantum constraints (equivalently, the corresponding TDSEs) in complete agreement with the classical canonical equivalence established above. The boundary term in (8.4) is the classical precursor of the quantum “chirp” factor in (7.18): it is precisely the total derivative responsible for the metaplectic phase (Maslov/Fresnel branch) bookkeeping when one crosses caustics.

C. Constant frequency

If we choose the projective time parametrization

$$t(\tau) = \frac{1}{\omega} \tan(\omega\tau), \quad -\frac{\pi}{2\omega} < \tau < \frac{\pi}{2\omega}, \quad (8.20)$$

cf. Cayley map (2.21), then

$$F(\eta) = \tan \eta, \quad \rho(\tau) = \frac{dt}{d\tau} = \frac{1}{\cos^2(\omega\tau)}, \quad \{t; \tau\} = 2\omega^2, \quad (8.21)$$

so that $\Omega(\tau) = \omega$, and we obtain the ordinary harmonic oscillator. In this case (7.18) becomes the standard Niederer unitary transformation,

$$\Psi(t(\tau), x) = (\cos \omega\tau)^{1/2} \exp\left(\frac{im\omega}{2\hbar} \tan(\omega\tau) y^2\right) \Phi(\tau, y),$$

$$x = \frac{y}{\cos \omega\tau}. \quad (8.22)$$

The choice

$$F(\eta) = \tanh \eta, \quad \{F; \eta\} = -2, \quad (8.23)$$

results in the inverted harmonic oscillator, which can be obtained from the case (8.20) by the obvious change $\omega \rightarrow i\omega$.

D. Factorized unitary operator in the Cayley case

In the extended (reparametrization-invariant) formulation, where kinematical wave functions depend on (t, x) and the inner product is $\int_{\mathbb{R}} dt \int_{\mathbb{R}} dx |\Psi(t, x)|^2$, the Cayley-Niederer choice $t(\tau) = \omega^{-1} \tan(\omega\tau)$, $x = \sec(\omega\tau)y$ admits an *exact* unitary implementer on $L^2(\mathbb{R}_t \times \mathbb{R}_x)$ in a closed product form. Writing $\eta := \omega\tau$, one may express the corresponding unitary as

$$\mathcal{U}_{\text{tan}} = \mathcal{U}_{\text{tan}}^{(y)}(\tau) (\mathcal{U}_{\text{tan}}^{(t)}(\tau) \otimes \text{Id}), \quad (8.24)$$

where the *time-sector* unitary can be written as an exponential in $(t, \hat{p}_t)$,

$$\mathcal{U}_{\text{tan}}^{(t)}(\tau) = \exp\left(-\frac{i}{\hbar} \eta \hat{G}_{\text{tan}}\right),$$

$$\hat{G}_{\text{tan}} = \frac{1}{2\omega} ((1 + \omega^2 t^2) \hat{p}_t + \hat{p}_t (1 + \omega^2 t^2)), \quad \hat{p}_t = -i\hbar\partial_t,$$

and implements the reparametrization together with the half-density factor on $L^2(\mathbb{R}, dt)$,

$$(\mathcal{U}_{\text{tan}}^{(t)} \varphi)(\tau) = \sqrt{\frac{dt}{d\tau}} \varphi(t(\tau)) = \sec(\eta) \varphi\left(\frac{1}{\omega} \tan \eta\right). \quad (8.25)$$

The *spatial* factor is the product of a quadratic “chirp” and a metaplectic dilation,

$$\mathcal{U}_{\text{tan}}^{(y)}(\tau) = \exp\left(-\frac{im\omega}{2\hbar} \tan \eta \hat{y}^2\right)$$

$$\times \exp\left(-\frac{i}{2\hbar} \ln(\sec \eta) (\hat{y} \hat{p}_y + \hat{p}_y \hat{y})\right), \quad \hat{p}_y = -i\hbar\partial_y. \quad (8.26)$$

E. Remark (sign of the chirp)

Our metaplectic convention is fixed by the projective implementation $U(S)W(\xi)U(S)^{-1} = W(S\xi)$ (Appendix B), hence equivalently $U(S)\hat{\xi}U(S)^{-1} = S\hat{\xi}$ with $\hat{\xi} = (\hat{q}, \hat{p})^T$. In particular, a lower shear $S = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ is implemented by $U(S) = \exp(-\frac{i}{2}\gamma\hat{q}^2)$, cf. (B13). Accordingly, the quadratic chirp in (8.26) appears with a minus sign. When one writes instead the inverse pointwise relation for wave functions [as in (7.18)], the corresponding quadratic phase appears with the opposite sign.

This closure on a finite set of quadratic generators (chirp and dilation, together with the finite-dimensional projective time flow) is special to the constant-Schwarzian (projective) reparametrizations; for a generic reparametrization $t = \omega^{-1}F(\omega\tau)$ one typically loses such a closed factorization (time ordering enters), and recovering an operator product form requires additional structure (to be discussed in Sec. IX).

F. Propagators: Free kernel, Maslov phases, and the Mehler kernel

The relation between propagators of the free particle and harmonic oscillator (with time-dependent frequency in general case) is obtained by inserting (7.18) into the free

evolution kernel,

$$K_{\text{osc}}(y_2, \tau_2; y_1, \tau_1) = \rho_2^{1/4} \rho_1^{1/4} \exp\left(-\frac{i}{\hbar} S_2\right) \times K_{\text{free}}(x_2, t_2; x_1, t_1) \exp\left(\frac{i}{\hbar} S_1\right), \quad (8.27)$$

where $t_j = t(\tau_j)$, $\rho_j = \rho(\tau_j)$, $x_j = \sqrt{\rho_j} y_j$, $S_j = S(\tau_j, y_j)$. For the free particle,

$$K_{\text{free}}(x_2, t_2; x_1, t_1) = \left(\frac{m}{2\pi i \hbar (t_2 - t_1)}\right)^{1/2} \exp\left[\frac{im}{2\hbar} \frac{(x_2 - x_1)^2}{t_2 - t_1}\right], \quad (8.28)$$

where the square root is understood in the metaplectic sense (a choice of branch of the Fresnel factor, see Appendix B). In the oscillator variables this choice becomes the usual Maslov rule: at a caustic (where the Van Vleck determinant changes sign), the kernel picks up the universal phase jump $\exp(-i\pi/2)$.

Combining (8.27) with (8.28) and (8.20) yields the Mehler kernel,

$$K_{\text{osc}}(y_2, \tau_2; y_1, \tau_1) = \left(\frac{m\omega}{2\pi i \hbar \sin(\omega T)}\right)^{1/2} \exp\left[\frac{im\omega}{2\hbar \sin(\omega T)} ((y_2^2 + y_1^2) \cos(\omega T) - 2y_2 y_1)\right]. \quad (8.29)$$

The square-root prefactor is understood in the metaplectic sense, and the caustics $\sin(\omega T) = 0$, $T := \tau_2 - \tau_1$, produce the standard Maslov phase jumps. The inverted oscillator follows by $\omega \rightarrow i\omega$.

G. Mehler kernel from the ABCD shear factorization

In conclusion of this section we note that an alternative (and rather direct) way to recover the oscillator propagator uses the $\text{Sp}(2, \mathbb{R})$ shear factorization of the rotation matrix and its metaplectic lift [Appendix D, cf. (D4)]. Writing $T := \tau_2 - \tau_1$ and using that the classical oscillator flow over time T is the phase-space rotation by angle ωT , one obtains the operator identity

$$e^{-\frac{i}{\hbar} T \hat{H}_+} = \exp\left(-\frac{i}{\hbar} \tan \frac{\omega T}{2} \frac{m\omega}{2} \hat{y}^2\right) \exp\left(-\frac{i}{\hbar} \Delta t \hat{H}_0\right) \exp\left(-\frac{i}{\hbar} \tan \frac{\omega T}{2} \frac{m\omega}{2} \hat{y}^2\right), \quad (8.30)$$

where $\Delta t := \frac{\sin(\omega T)}{\omega}$, and $\hat{H}_0 = \hat{p}_y^2/(2m)$ is the free Hamiltonian (in the y -coordinate). Taking the (y_2, y_1) matrix element and inserting a resolution of the identity between the three factors gives

$$\begin{aligned} K_{\text{osc}}(y_2, \tau_2; y_1, \tau_1) &= \langle y_2 | e^{-\frac{i}{\hbar} T \hat{H}_+} | y_1 \rangle \\ &= \exp\left(-\frac{im\omega}{2\hbar} \tan \frac{\omega T}{2} y_2^2\right) K_{\text{free}}(y_2, \Delta t; y_1, 0) \exp\left(-\frac{im\omega}{2\hbar} \tan \frac{\omega T}{2} y_1^2\right). \end{aligned}$$

Using the explicit free kernel (8.28) with $x_j = y_j$ and $t_2 - t_1 = \Delta t$, and the elementary identity

$$\frac{(y_2 - y_1)^2}{\Delta t} - \omega \tan \frac{\omega T}{2} (y_2^2 + y_1^2) = \frac{\omega}{\sin(\omega T)} ((y_2^2 + y_1^2) \cos(\omega T) - 2y_2 y_1),$$

one recovers exactly the Mehler kernel (8.29) [with the same metaplectic branch choice and the same Maslov phase jumps at $\sin(\omega T) = 0$].

IX. GENERAL REPARAMETRIZATIONS F AND A FACTORIZED METAPLECTIC FORM: THE ERMAKOV-PINNEY AMPLITUDE

In the special constant-Schwarzian cases $F(\eta) = \eta$, $F(\eta) = \tan \eta$, and $F(\eta) = \tanh \eta$ (so $\{F; \eta\} = 0, 2, -2$, respectively), the time reparametrization is fractional-linear (projective), i.e., it lies in $\text{PSL}(2, \mathbb{R})$ acting on $\mathbb{RP}^1$; equivalently, on any interval I where it is smooth and strictly monotone it may be viewed as an element of $\text{Diff}_+(I)$. In these projective cases the quantum implementation closes on a finite set of quadratic generators, and one can write an exact closed product of exponentials for the corresponding unitary operator (including the time sector). In this section we turn to a general strictly monotone reparametrization $t = \omega^{-1}F(\omega\tau)$ with $F'(\eta) > 0$ and explain what survives of this picture in the extended (reparametrization-invariant) formulation considered in the previous section. As will be seen, one can still write an explicit factorization into a time-reparametrization unitary, a metaplectic dilation, and a quadratic phase (chirp), with the remaining dynamical content encoded in Ermakov-Pinney data. This provides a convenient framework for the generic case; for complementary remarks on the exact versus semiclassical implementation of canonical maps, see Appendix H.

A. Ermakov-Pinney amplitude and a closed product formula

As shown in the previous sections, a strictly monotone reparametrization $t = t(\tau)$ induces, for the free particle, a time-dependent oscillator with frequency $\Omega(\tau)^2 = \frac{1}{2}\omega^2\{F; \eta\}$, where $\eta = \omega\tau$. Here $\rho(\tau) = dt/d\tau > 0$ is the Jacobian of the time change; in the lens/Cayley-Niederer map the associated spatial scale factor is $b(\tau) = \sqrt{\rho(\tau)}$ (so $x = b(\tau)y$). This should not be confused with the Ermakov-Pinney amplitude $r(\tau) > 0$ defined below.

Introduce the standard $\mathfrak{sl}(2, \mathbb{R})$ quadratic generators on $L^2(\mathbb{R}, dy)$,

$$\begin{aligned}\hat{H} &:= \frac{1}{2m}\hat{p}_y^2, & \hat{K} &:= \frac{m}{2}\hat{y}^2, \\ \hat{D} &:= \frac{1}{2}(\hat{y}\hat{p}_y + \hat{p}_y\hat{y}), & \hat{p}_y &= -i\hbar\partial_y,\end{aligned}\quad (9.1)$$

and the time-dependent oscillator Hamiltonian

$$\hat{H}_\Omega(\tau) := \hat{H} + \Omega(\tau)^2\hat{K} = \frac{\hat{p}_y^2}{2m} + \frac{m}{2}\Omega(\tau)^2\hat{y}^2. \quad (9.2)$$

The exact quantum evolution is the time-ordered exponential

$$U_\Omega(\tau, \tau_0) = \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{\tau_0}^{\tau} \hat{H}_\Omega(\sigma) d\sigma \right).$$

Let $r(\tau) > 0$ solve the Ermakov-Pinney equation [75]

$$r''(\tau) + \Omega(\tau)^2 r(\tau) = \frac{\omega_0^2}{r(\tau)^3}, \quad \omega_0 > 0 \text{ constant}, \quad (9.3)$$

and define the reparametrized time

$$\theta(\tau) := \int_{\tau_0}^{\tau} \frac{\omega_0}{r(\sigma)^2} d\sigma. \quad (9.4)$$

Then one has the exact metaplectic factorization (up to an overall phase fixed by metaplectic normalization),

$$\begin{aligned}U_\Omega(\tau, \tau_0) &= \exp \left(\frac{i}{\hbar} \frac{r'(\tau)}{r(\tau)} \hat{K} \right) \exp \left(-\frac{i}{\hbar} \ln r(\tau) \hat{D} \right) \\ &\quad \times \exp \left(-\frac{i}{\hbar} \theta(\tau) \hat{H}_0 \right), \\ \hat{H}_0 &= \frac{1}{2m} \hat{p}_y^2 + \frac{m}{2} \omega_0^2 \hat{y}^2.\end{aligned}$$

Thus, once $r(\tau)$ is known, the time-dependent frequency problem is reduced to a constant-frequency evolution in the new time θ , together with an explicit chirp and a dilation. In the Cayley case $F(\eta) = \tan \eta$ one may choose $\omega_0 = \omega$ and the constant solution $r(\tau) = 1$, so that $\theta(\tau) = \omega(\tau - \tau_0)$ and the above reduces to a product with constant coefficients.

B. Including the time sector in the extended formulation

In the extended (reparametrization-invariant) setting, the canonical map includes the time change $t \mapsto \tau$. On $L^2(\mathbb{R}, dt)$, the corresponding unitary is the pullback with the Jacobian,

$$\begin{aligned}(U_F^{(t)} \varphi)(\tau) &= \sqrt{\frac{dt}{d\tau}} \varphi(t(\tau)) = \sqrt{\rho(\tau)} \varphi \left(\frac{1}{\omega} F(\omega\tau) \right), \\ \rho(\tau) &:= \frac{dt}{d\tau} > 0.\end{aligned}\quad (9.5)$$

Equivalently, if one represents the reparametrization as a (possibly nonautonomous) flow on the t line, its unitary action on half-densities is generated by the standard symmetrized operator built from $\hat{p}_t = -i\hbar\partial_t$,

$$\hat{G}_{\xi(\tau)} = \frac{1}{2}(\xi(\tau, t)\hat{p}_t + \hat{p}_t\xi(\tau, t)), \quad \hat{p}_t = -i\hbar\partial_t, \quad (9.6)$$

so that the same time-sector unitary may be written (along the chosen flow) as the time-ordered exponential

$$U_F^{(t)}(\tau, \tau_0) = \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{\tau_0}^{\tau} \hat{G}_{\xi(\sigma)} d\sigma \right). \quad (9.7)$$

In the projective (constant-Schwarzian) cases the relevant time flow is generated within the finite-dimensional span of $1, t, t^2$, and the corresponding exponential can be written without ordering complications; for a generic F one understands the exponential in the usual time-ordered sense.

Combining the time-sector unitary with the spatial factors (the slice identification from the previous section and/or the dynamical factorization U_Ω above, depending on the context) yields a compact factorized operatorial description of the quantum implementer of the extended canonical map.

X. SUMMARY, DISCUSSION, AND OUTLOOK

In this work we revisited the relation between the one-dimensional free particle (FP) and the harmonic oscillator (HO) from a unified viewpoint that combines projective geometry, Cayley transformations (in both phase space and time), and the Schwarzian derivative. Although FP and HO represent the opposite intuitive extremes of *freedom* and *confinement*, they share a common symmetry backbone: the semidirect product of the Heisenberg group with the conformal group $SL(2, \mathbb{R})$ [or, equivalently, $Sp(2, \mathbb{R})$]. While the FP-HO correspondence has a long history (see, e.g., [19,28,52]), our aim was to make explicit a single organizing mechanism behind its various incarnations. Our main point is that the *projective* nature of the $SL(2, \mathbb{R})$ action on time, together with its metaplectic lift, provides an organizing principle that simultaneously explains the FP-HO correspondence at the level of the TDSE (Cayley-Niederer/lens transforms) and at the level of the SSE (conformal bridge transformation, CBT), and clarifies the role of the Schwarzian as the universal cocycle inducing oscillator-type terms under time reparametrizations.

The main results can be summarized as follows.

- (i) In Sec. II we emphasized the dual role of the (rotated) Cayley matrix: it acts (i) as a linear canonical transformation between real and complexified symplectic bases in phase space, and (ii) as a Möbius transformation on the projectivized time line $\mathbb{RP}^1$, linking naturally the upper half-plane and unit disc models of hyperbolic geometry. This provides a convenient geometric language for complex structures compatible with $\omega = dp \wedge dq$ (see Appendix A) and for the appearance of the Cayley parameter as a global coordinate on $\mathbb{RP}^1$.
- (ii) In Secs. III and VI we reviewed how the same $\mathfrak{sl}(2, \mathbb{R})$ skeleton is realized as a *dynamical* symmetry of FP, and as the natural symmetry algebra underlying HO evolution. In particular, the global implementation of conformal boosts and time inversion is most naturally formulated after projectivizing

time, $t \in \mathbb{R} \mapsto \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$, and using the corresponding Cayley parametrization.

- (iii) In Sec. VII we separated two closely related, but conceptually distinct, covariance principles: the projective-connection transformation law for the SSE potential (the Liouville picture), and the half-density (unitary) transformation law for the TDSE. This separation explains why the Schwarzian derivative appears in both settings, but with different physical roles: as the projective cocycle controlling the SSE potential, and as the universal term induced by *time* reparametrizations in the TDSE.
- (iv) In Sec. VIII we formulated the FP-HO relation in the *extended*, reparametrization-invariant setting, where time is promoted to a dynamical variable and the dynamics is encoded by a first-class constraint. In this formulation, a time reparametrization combined with the accompanying space scaling becomes a canonical transformation on the extended phase space that maps the free constraint to an oscillator-type constraint with a *Schwarzian-induced* (generally time-dependent) frequency. The standard Cayley-Niederer (lens) transform arises as a distinguished choice of projective time map.
- (v) In Sec. IV we identified the quantum analogue of the complexified Cayley map with the Bargmann transform (metaplectic implementation of the same symplectic data).
- (vi) In Sec. V we showed how the same Cayley logic yields the CBT relating FP and HO at the level of the stationary problem. In particular, the CBT appears naturally as a (nonunitary) similarity transform generated by the inverted oscillator evolution at a purely complex time, making manifest the algebraic unity between the TDSE and SSE correspondences.
- (vii) Finally, Sec. IX exhibited a factorized/metaplectic form for general time maps, controlled by Ermakov-Pinney data. This clarifies how the projective reparametrization geometry governs not only the basic $FP \leftrightarrow HO$ map but also its generalizations to time-dependent oscillator profiles.

A recurring theme is that the conformal sector $SL(2, \mathbb{R})$ acts on time by Möbius transformations, hence the natural arena is $\mathbb{RP}^1$ rather than $\mathbb{R}$. For the HO, this projective viewpoint dovetails with periodic evolution: the $SL(2, \mathbb{R})$ rotation generator effectively sees the half-period π/ω as the fundamental cycle. In the extended FP formulation, the projective time map $t = \omega^{-1} \tan(\omega\tau)$ implements precisely this global completion of conformal boosts, while the requirement of maintaining the standard kinetic form (and hence unitarity in the TDSE) forces the accompanying space scaling and produces the quadratic term governed by the time Schwarzian $\{t; \tau\}$. The Schwarzian thus plays the role of the canonical projective cocycle measuring the departure from Möbius reparametrizations (Appendix G; see also the original sources [51,83]).

From the phase-space viewpoint, the same $Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$ can be generated by elementary canonical transformations (shears and “thin-lens” chirps). Their Lie algebra generators span $\mathfrak{sl}(2, \mathbb{R})$ and interpolate continuously between free evolution, oscillator evolution (compact rotation), and inverted-oscillator evolution (hyperbolic rotation). This algebraic fact underlies the various manifestations of the FP-HO correspondence: in classical Hamiltonian flows, in the TDSE lens transform, and in the quantum metaplectic operators that implement the same symplectic data.

The framework developed here suggests several natural extensions.

- (i) Inverted oscillator, resonances, and analyticity. Since HO and inverted HO are related by $\omega \mapsto i\omega$, the present constructions have a direct analytic continuation to the inverted case. In particular, it would be interesting to sharpen the relation between FP Jordan states (Appendix C) and resonant (Gamow) states of the inverted oscillator, and to clarify how the corresponding ladder structures are transported by the Cayley/CBT operators.
- (ii) Linear potential and mixed generators. Another natural test bed is the FP versus a particle in a linear potential, where the Hamiltonian mixes quadratic and linear generators of the Schrödinger/Jacobi symmetry. Understanding its projective-geometric organization may illuminate new “bridge” maps beyond the purely quadratic realm (cf. the Schrödinger-group perspective in [20,21,24,29]).
- (iii) Integrable hierarchies and projective connections. The projective-connection viewpoint on the SSE operator is closely related to integrable isospectral deformations: KdV and its hierarchy can be viewed as flows on Schrödinger operators/potentials preserving spectral data (Lax/Gelfand-Dikii picture). It would be valuable to explore systematically how the Schwarzian/projective cocycle language used here interfaces with KdV-type evolutions, Miura maps, and dressing/Darboux-Crum constructions (see, e.g., [2,3,5,6]).
- (iv) AdS₂, SYK, and hyperbolic geometry. Because our central mechanism is the projective action of $SL(2, \mathbb{R})$ and the Schwarzian cocycle, it is natural to connect it more directly with the low-energy dynamics of SYK and Jackiw-Teitelboim gravity, governed by $\text{Diff}(S^1)/SL(2, \mathbb{R})$ and the Schwarzian action. In particular, the hyperbolic-geometric interpretation of the Cayley map suggests revisiting Schwarzian boundary dynamics from the perspective of motion on (or moduli of) hyperbolic structures (see, e.g., [33,35,36,42,46,47]).

ACKNOWLEDGMENTS

The work was partially supported by the FONDECYT Project No. 1242046.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: COMPATIBLE COMPLEX STRUCTURES ON $(\mathbb{R}^2, \omega)$ AND THE UPPER HALF-PLANE

1. Symplectic conventions

We consider the real symplectic plane $V \cong \mathbb{R}^2$ with coordinates $\xi_\alpha = (q, p)$, $\alpha = 1, 2$, endowed with the symplectic form $\omega = dp \wedge dq$. In matrix notation, $\omega(u, v) = u^\top \check{\Omega} v$, $\check{\Omega} := -J$, $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that the Poisson brackets are $\{\xi_\alpha, \xi_\beta\} = J_{\alpha\beta}$. The linear symplectic group is $Sp(2, \mathbb{R}) = \{S \in GL(2, \mathbb{R}) | S^\top \check{\Omega} S = \check{\Omega}\}$, and in dimension 2 this coincides with $SL(2, \mathbb{R})$.

2. Compatible complex structures and $\mathbb{H}_+$

A (linear) complex structure on V is an endomorphism $\mathcal{J}: V \rightarrow V$ with $\mathcal{J}^2 = -\mathbf{1}$. It is called *compatible* with ω if

$$\mathcal{J}^\top \check{\Omega} \mathcal{J} = \check{\Omega} \quad \text{and} \quad g_{\mathcal{J}}(u, v) := \omega(u, \mathcal{J}v) \text{ is positive definite.} \quad (\text{A1})$$

In real dimension 2 the space of compatible complex structures is naturally parametrized by the upper half-plane $\mathbb{H}_+ := \{\tau = x + iy \in \mathbb{C} | y > 0\}$. A convenient explicit representative is

$$\mathcal{J}_\tau = \frac{1}{\text{Im } \tau} \begin{pmatrix} \text{Re } \tau & |\tau|^2 \\ -1 & -\text{Re } \tau \end{pmatrix} = \frac{1}{y} \begin{pmatrix} x & x^2 + y^2 \\ -1 & -x \end{pmatrix}. \quad (\text{A2})$$

One checks directly that $\mathcal{J}_\tau^2 = -\mathbf{1}$ and $\mathcal{J}_\tau^\top \check{\Omega} \mathcal{J}_\tau = \check{\Omega}$. Moreover, the associated Kähler metric is

$$\begin{aligned} g_\tau(u, u) &= \omega(u, \mathcal{J}_\tau u) = \frac{1}{\text{Im } \tau} |q + \tau p|^2 \\ &= \frac{(q + xp)^2 + y^2 p^2}{y} > 0 \quad (u = (q, p) \neq 0), \end{aligned} \quad (\text{A3})$$

which is positive definite precisely for $\text{Im } \tau > 0$. Consequently,

$$\{\text{compatible } \mathcal{J}\} \cong Sp(2, \mathbb{R})/U(1) \cong SL(2, \mathbb{R})/SO(2) \cong \mathbb{H}_+. \quad (\text{A4})$$

The induced invariant metric on this parameter space (often loosely called a moduli space) is the Poincaré metric (2.16), $ds_{\mathbb{H}_+}^2 = (dx^2 + dy^2)/y^2$. Indeed, denote this space by $\mathcal{M}$; by (A4) it is the Riemannian symmetric space $Sp(2, \mathbb{R})/U(1)$, hence it carries a canonical $Sp(2, \mathbb{R})$ -invariant Riemannian metric, unique up to an overall constant factor. One convenient way to write it is via the

trace pairing on endomorphisms: for a smooth curve $s \mapsto \mathcal{J}(s)$ in $\mathcal{M}$ (so $\mathcal{J}(s)^2 = -\mathbf{1}$), its tangent vector $\dot{\mathcal{J}} = \frac{d}{ds}\mathcal{J}(s)$ satisfies $\dot{\mathcal{J}}\mathcal{J} + \mathcal{J}\dot{\mathcal{J}} = 0$, and one sets $\langle \dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2 \rangle_{\mathcal{J}} := \frac{1}{2}\text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2)$, $ds^2 = \frac{1}{2}\text{tr}(d\mathcal{J}d\mathcal{J})$. This expression is invariant under the natural $Sp(2, \mathbb{R})$ action $\mathcal{J} \mapsto S\mathcal{J}S^{-1}$ because $\text{tr}((SX S^{-1})(SY S^{-1})) = \text{tr}(XY)$. Using the explicit parametrization $\tau = x + iy \in \mathbb{H}_+$ and (A2), one checks directly that

$$\frac{1}{2}\text{tr}(d\mathcal{J}_\tau d\mathcal{J}_\tau) = \frac{dx^2 + dy^2}{y^2},$$

i.e., the Poincaré metric on the upper half-plane.

A useful viewpoint is the homogeneous-space identification $\mathbb{H}_+ \cong PSL(2, \mathbb{R})/SO(2)$ [cf. (A4)]. The induced action is the standard Möbius transformation $\tau \mapsto \tau' = \frac{a\tau+b}{c\tau+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, whose stabilizer at $\tau = i$ is $SO(2)$.

3. Holomorphic polarization and complex coordinate

The complex structure $\mathcal{J}_\tau$ determines a holomorphic polarization. A convenient $(1, 0)$ one-form is $\theta_\tau := dq + \tau dp$, and the associated complex coordinate

$$z_\tau := \frac{q + \tau p}{\sqrt{2 \text{Im } \tau}}, \quad \bar{z}_\tau := \frac{q + \bar{\tau} p}{\sqrt{2 \text{Im } \tau}}. \quad (\text{A5})$$

With our convention $\omega = dp \wedge dq$, one verifies $\omega = i dz_\tau \wedge d\bar{z}_\tau$, $g_\tau = 2 dz_\tau d\bar{z}_\tau$. Thus the same upper half-plane label $\tau \in \mathbb{H}_+$ that appears in hyperbolic geometry also parametrizes complex polarizations in geometric quantization [76–78].

4. $SL(2, \mathbb{R})$ action and the Cayley map

For $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ acting linearly as $\xi \mapsto \xi' = S\xi$, the induced action on $\tau \in \mathbb{H}_+$ is the fractional linear transformation $\tau \mapsto \tau' = \frac{a\tau+b}{c\tau+d}$, $\mathcal{J}_{\tau'} = S\mathcal{J}_\tau S^{-1}$. The Cayley map $w = i \frac{\tau-i}{\tau+i}$ identifies $\mathbb{H}_+$ with the unit disk $\mathbb{D} := \{w \in \mathbb{C} | |w| < 1\}$ and conjugates $PSL(2, \mathbb{R})$ to $PSU(1, 1)$. In the quantum theory, this same Cayley change of polarization underlies the passage from the real (Schrödinger) representation to the holomorphic (Bargmann-Fock) one; in the holomorphic basis the induced linear action on creation/annihilation operators takes the Bogoliubov $[SU(1, 1)]$ form.

APPENDIX B: METAPLECTIC REPRESENTATION IN QUANTUM MECHANICS

1. Heisenberg group, central extension, and the Weyl system

We work with the symplectic plane $(\mathbb{R}^2, \omega)$, $\omega = dp \wedge dq$, and $\hat{q}\psi(q) = q\psi(q)$, $\hat{p}\psi(q) = -i\partial_q\psi(q)$ so that $[\hat{q}, \hat{p}] = i$.

A convenient “exponentiated” form of the Schrödinger representation is the Weyl system of unitary operators

$$W(\xi) := \exp(i(p_0\hat{q} - q_0\hat{p})), \quad \xi = (q_0, p_0) \in \mathbb{R}^2. \quad (\text{B1})$$

The map $\xi \mapsto W(\xi)$ is *projective* as a representation of the Abelian group $\mathbb{R}^2$ (phase-space translations). Indeed, using the Baker-Campbell-Hausdorff formula and the commutator $[p_0\hat{q} - q_0\hat{p}, p_1\hat{q} - q_1\hat{p}] = -i\omega(\xi_0, \xi_1)$ with $\omega(\xi_0, \xi_1) = p_0q_1 - q_0p_1$, one finds the Weyl relations

$$W(\xi_0)W(\xi_1) = e^{\frac{i}{2}\omega(\xi_0, \xi_1)}W(\xi_0 + \xi_1). \quad (\text{B2})$$

The phase factor in (B2) is a (nontrivial) 2-cocycle on $\mathbb{R}^2$; it obstructs $\xi \mapsto W(\xi)$ from being a true representation of $\mathbb{R}^2$.

This obstruction is removed by passing to the *Heisenberg group* Heis_3 , the central extension of $\mathbb{R}^2$ by $\mathbb{R}$,

$$\text{Heis}_3 \cong \mathbb{R}^2 \times \mathbb{R}, \quad (\xi, s) \cdot (\eta, t) = \left(\xi + \eta, s + t + \frac{1}{2}\omega(\xi, \eta) \right). \quad (\text{B3})$$

Then

$$U(\xi, s) := e^{is}W(\xi) \quad (\text{B4})$$

is a genuine unitary representation of Heis_3 . [Equivalently, (B3) is precisely engineered so that (B2) becomes associativity of $U(\xi, s)$.] The Stone–von Neumann theorem states that, once the action of the center is fixed, this irreducible unitary representation is unique up to unitary equivalence.

2. Symplectic automorphisms, projective implementation, and $\text{Mp}(2, \mathbb{R})$

The group $Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$ consists of real 2×2 matrices $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, preserving ω , i.e., $\omega(S\xi, S\eta) = \omega(\xi, \eta)$. Therefore S acts by automorphisms of Heis_3 ,

$$\alpha_S: (\xi, s) \mapsto (S\xi, s), \quad W(\xi) \mapsto W(S\xi). \quad (\text{B5})$$

By Stone–von Neumann, for each $S \in Sp(2, \mathbb{R})$ there exists a unitary operator $\mathcal{U}(S)$ on $L^2(\mathbb{R}, dq)$ (unique up to an overall phase) such that

$$\mathcal{U}(S)W(\xi)\mathcal{U}(S)^{-1} = W(S\xi), \quad \forall \xi \in \mathbb{R}^2. \quad (\text{B6})$$

Because $\mathcal{U}(S)$ is only determined up to phase, the map $S \mapsto \mathcal{U}(S)$ is generally only a *projective* representation,

$$\mathcal{U}(S_1)\mathcal{U}(S_2) = e^{i\sigma(S_1, S_2)}\mathcal{U}(S_1S_2), \quad (\text{B7})$$

where σ is a 2-cocycle (the “multiplier”).

The *metaplectic group* $\text{Mp}(2, \mathbb{R})$ is the double cover of $\text{Sp}(2, \mathbb{R})$ that resolves this cocycle. Concretely, there exists a canonical projection $\pi: \text{Mp}(2, \mathbb{R}) \rightarrow \text{Sp}(2, \mathbb{R})$ with $\ker \pi = \{\pm 1\}$, and one can choose a continuous lift $S \mapsto \tilde{S}$ together with operators $\tilde{S} \mapsto \mathcal{U}(\tilde{S})$ giving a *true* (not merely projective) representation,

$$\mathcal{U}(\tilde{S}_1)\mathcal{U}(\tilde{S}_2) = \mathcal{U}(\tilde{S}_1\tilde{S}_2), \quad \pi(\tilde{S}) = S \in \text{Sp}(2, \mathbb{R}). \quad (\text{B8})$$

For a fixed $S \in \text{Sp}(2, \mathbb{R})$ there are exactly two lifts $\tilde{S}$ and $-\tilde{S}$ in the fiber $\pi^{-1}(S)$, so $\mathcal{U}(\tilde{S})$ is defined only up to an overall sign. Imposing continuity along paths in $\text{Sp}(2, \mathbb{R})$ makes this sign jump in a controlled way when one crosses caustics (see below); equivalently, the relevant square-root branch changes by a phase quantized in multiples of $\pi/2$, encoded by the Maslov (Fresnel) index.

3. Generating function and the metaplectic kernel for $b \neq 0$

Consider the linear canonical transformation $(Q, P)^T = S(q, p)^T$, $Q = aq + bp$, $P = cq + dp$, with $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. For $b \neq 0$ one can solve $p = (Q - aq)/b$ and $P = (dQ - q)/b$, and a (type-1) generating function $G_S(q, Q)$ satisfying $p = \frac{\partial G_S}{\partial q}$, $P = -\frac{\partial G_S}{\partial Q}$, is

$$G_S(q, Q) = \frac{1}{2b}(2qQ - aq^2 - dQ^2), \quad (b \neq 0). \quad (\text{B9})$$

(Equivalently, $pdq - PdQ = dG_S$.)

Treating the phase $\exp(-iG_S)$ as an integral kernel leads to the standard metaplectic operator (one of the two lifts; the other differs by an overall minus sign),

$$\begin{aligned} (\mathcal{U}(S)\psi)(Q) &= \frac{1}{\sqrt{2\pi|b|}} e^{-i\frac{\pi}{4}\text{sgn}(b)} \int_{\mathbb{R}} \exp\left[\frac{i}{2b}(aq^2 - 2qQ + dQ^2)\right] \\ &\quad \times \psi(q) dq, \quad (b \neq 0). \end{aligned} \quad (\text{B10})$$

Here the *square-root factor* $|b|^{-1/2}$ and the Fresnel phase $e^{-i\pi\text{sgn}(b)/4}$ arise from enforcing unitarity and the correct composition law under Gaussian (Fresnel) integrals. The same phenomenon is already visible in the elementary Fresnel integral,

$$\int_{\mathbb{R}} \exp\left(\frac{i}{2}ax^2\right) dx = \sqrt{\frac{2\pi}{|a|}} \exp\left(i\frac{\pi}{4}\text{sgn}(a)\right), \quad a \in \mathbb{R} \setminus \{0\}.$$

It exhibits the square-root/branch effect: the factor $a^{-1/2}$ is not single-valued globally, and its continuation produces phase changes by multiples of $\pi/4$. Setting $a = 1/b$ shows that the prefactor in (B10) is precisely the corresponding normalization, i.e., a branch choice for $(ib)^{-1/2}$. They are

also the simplest manifestation of the Maslov-type ambiguity: the choice of branch for $\sqrt{b}$ (or $\sqrt{|b|}$ together with the Fresnel phase) cannot be made globally on $\text{Sp}(2, \mathbb{R})$ without passing to the double cover $\text{Mp}(2, \mathbb{R})$.

4. Half-density viewpoint

The Jacobian-type factor in (B10) is the one-dimensional version of the statement that, under changes of polarization, wave functions naturally transform as *half-densities*. In the present linear setting this shows up precisely through the square-root of the relevant determinant [here $|b|$; see also (B13) and the $\det B$ factor in the $\text{Sp}(2n, \mathbb{R})$ remark below].

5. The case $b = 0$ and caustics

When $b = 0$, the matrix has the form $S = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$, $a \neq 0$, and (B10) is not applicable (this is a “caustic” for the b chart). In this case one uses the factorization into a shear and a dilation,

$$\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad (\text{B11})$$

whose metaplectic lifts are elementary,

$$\begin{aligned} \mathcal{U}\left(\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\right) &= \exp\left(-i\frac{\gamma}{2}\hat{q}^2\right), \\ (\mathcal{U}(\text{diag}(a, a^{-1}))\psi)(Q) &= \sqrt{|a|}\psi(aQ). \end{aligned} \quad (\text{B12})$$

Combining them gives, up to the global sign $\pm$,

$$(\mathcal{U}(S)\psi)(Q) = \sqrt{|a|} \exp\left(-i\frac{c}{2a}Q^2\right) \psi(aQ), \quad (b=0). \quad (\text{B13})$$

Crossing from $b \neq 0$ to $b = 0$ [or, more generally, changing charts in $\text{Sp}(2, \mathbb{R})$] forces a choice of branch for the square root factor and produces a quantized phase jump. Tracking these jumps along a closed loop yields an integer Maslov index; the associated phase $e^{-i\pi\mu/2}$ is exactly the Fresnel/Maslov correction needed for a globally consistent metaplectic lift.

6. Metaplectic group as a double cover via square-root automorphy factors

A recurring theme in the main text and in Appendix D below is that linear canonical maps $S \in \text{Sp}(2, \mathbb{R})$ have a perfectly honest action on classical phase space, but their quantum implementation (as unitary operators on wave functions) is only *projective*: one meets a $\pm$ sign (more generally, a phase) in composition. The conceptual reason is always the same: unitarity forces a *square root* of a Jacobian (equivalently, an action on half-densities), and

square roots carry an unavoidable sign/branch ambiguity. The metaplectic group $\text{Mp}(2, \mathbb{R})$ is the canonical device that resolves this ambiguity by passing to a double cover.

(i) Automorphy factors and the double cover. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$, $\tau \in \mathbb{H}_+ := \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$, $S \cdot \tau = \frac{a\tau + b}{c\tau + d}$. The standard automorphy factor is

$$j(S, \tau) := c\tau + d.$$

It satisfies the chain rule

$$j(S_1 S_2, \tau) = j(S_1, S_2 \cdot \tau) j(S_2, \tau), \quad (\text{B14})$$

Indeed, writing $S_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$, one has

$$\begin{aligned} j(S_1, S_2 \cdot \tau) j(S_2, \tau) &= \left(c_1 \frac{a_2 \tau + b_2}{c_2 \tau + d_2} + d_1 \right) (c_2 \tau + d_2) \\ &= (c_{12} \tau + d_{12}) = j(S_1 S_2, \tau), \end{aligned}$$

where $\begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix} = S_1 S_2$.

Many constructions (metaplectic kernels, modular weight $1/2$, etc.) require a choice of $\sqrt{j(S, \tau)}$. If one tries to choose $\sqrt{j(S, \tau)}$ globally and multiplicatively in S , one encounters the $\{\pm 1\}$ multiplier

$$\sigma(S_1, S_2; \tau) := \frac{\sqrt{j(S_1, S_2 \cdot \tau)} \sqrt{j(S_2, \tau)}}{\sqrt{j(S_1 S_2, \tau)}} \in \{\pm 1\}, \quad (\text{B15})$$

which is a two cocycle (it satisfies the cocycle identity because it is built from the chain rule for j). The metaplectic double cover can be described as the set of pairs

$$\text{Mp}(2, \mathbb{R}) = \{(S, \phi(\tau)) : S \in \text{SL}(2, \mathbb{R}), \phi(\tau)^2 = j(S, \tau)\},$$

with multiplication

$$(S_1, \phi_1)(S_2, \phi_2) = (S_1 S_2, \phi_1(S_2 \cdot \tau) \phi_2(\tau)). \quad (\text{B16})$$

Projectively, ϕ is precisely the missing square root: once the choice is made in $\text{Mp}(2, \mathbb{R})$, compositions become strictly consistent.

(ii) Half-densities and Liouville transformations. The same “square-root mechanism” underlies the ubiquitous $\pm \frac{1}{2}$ exponent in Liouville transformations of second-order equations, as used in (7.2) (stationary Schrödinger) and (7.12) (time-dependent case). Concretely, under a reparametrization $x = x(y)$ a wave function treated as a *half-density* transforms as $\psi(x)|dx|^{1/2} = \tilde{\psi}(y)|dy|^{1/2}$, $\Leftrightarrow \tilde{\psi}(y) = \left(\frac{dx}{dy}\right)^{1/2} \psi(x(y))$. Exactly this $(dx/dy)^{1/2}$ factor removes the first-derivative term that otherwise appears when rewriting a second-order differential operator in the new variable; in the Schrödinger context it is the familiar Liouville rescaling. The sign/phase ambiguity of the square root is the same phenomenon that, for linear canonical maps, manifests itself as the metaplectic sign (Maslov-type phases along paths).

(iii) Brief remark on $\text{Sp}(2n, \mathbb{R})$. For $n > 1$ the same structure persists. On the open chart where B is invertible for $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the metaplectic operator is an oscillatory integral whose prefactor contains the half-density factor $|\det B|^{-1/2}$; choosing a continuous branch of $\sqrt{\det B}$ along paths produces the familiar Maslov-type phase and is globally resolved by the double cover $\text{Mp}(2n, \mathbb{R})$. Equivalently, $\text{Sp}(2n, \mathbb{R})$ acts on the Siegel upper half space $\mathfrak{H}_n$ by $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$ with automorphy factor $j(S, \Omega) = \det(C\Omega + D)$, and unitarity requires a consistent choice of the square root of $j(S, \Omega)$, again leading to $\text{Mp}(2n, \mathbb{R})$.

APPENDIX C: JORDAN STATES FROM ANALYTICITY IN THE ENERGY

1. General construction

Consider the one-dimensional stationary Schrödinger problem

$$\hat{H}\psi(x; E) = E\psi(x; E), \quad \hat{H} = -\frac{d^2}{dx^2} + V(x), \quad (\text{C1})$$

and fix an energy value E_0 . Assume that for E near E_0 there exists a (nontrivial) solution $\psi(x; E)$ analytic in E . Then

$$\psi(x; E) = \psi_0(x) + \sum_{n=1}^{\infty} \frac{(E - E_0)^n}{n!} \chi_n(x), \quad \psi_0(x) := \psi(x; E_0), \quad \chi_n(x) := \partial_E^n \psi(x; E)|_{E=E_0}. \quad (\text{C2})$$

Substituting (C2) into (C1) and comparing powers of $(E - E_0)$ gives

$$(\hat{H} - E_0)\psi_0 = 0, \quad (\hat{H} - E_0)\chi_1 = \psi_0, \quad (\hat{H} - E_0)\chi_n = n\chi_{n-1}, \quad n \geq 2, \quad (\text{C3})$$

hence

$$(\hat{H} - E_0)^n \chi_n = n! \psi_0 \neq 0, \quad (\hat{H} - E_0)^{n+1} \chi_n = 0, \quad n = 1, 2, \dots \quad (\text{C4})$$

so χ_n is a generalized eigenfunction (Jordan state) of rank $n+1$ at E_0 . (Jordan chains may also be generated by confluent Darboux/Darboux-Crum transformations [79,80], but in this paper we only use the analyticity construction above.)

2. Free particle at $E_0 = 0$

For the free Hamiltonian $\hat{H}_0 = -\frac{1}{2} \frac{d^2}{dq^2}$ the stationary equation is $\psi''(q) + 2E\psi(q) = 0$. At $E_0 = 0$ we may take the basis

$$\psi_0^{(+)}(q) = 1, \quad \psi_0^{(-)}(q) = q. \quad (\text{C5})$$

A convenient choice of analytic families extending these solutions away from $E = 0$ is

$$\begin{aligned} \psi^{(+)}(q; E) &:= \sum_{n=0}^{\infty} (-1)^n \frac{(2E)^n q^{2n}}{(2n)!}, \\ \psi^{(-)}(q; E) &:= \sum_{n=0}^{\infty} (-1)^n \frac{(2E)^n q^{2n+1}}{(2n+1)!}, \end{aligned} \quad (\text{C6})$$

[so $\psi^{(+)} = \cos(\sqrt{2E}q)$ and $\psi^{(-)} = \sin(\sqrt{2E}q)/\sqrt{2E}$ via their E -Taylor series]. Define Jordan states by energy derivatives, $\chi_n^{(\pm)}(q) := \partial_E^n \psi^{(\pm)}(q; E)|_{E=0}$. From (C6) one finds explicitly

$$\begin{aligned} \chi_n^{(+)}(q) &= (-1)^n 2^n n! \frac{q^{2n}}{(2n)!}, \\ \chi_n^{(-)}(q) &= (-1)^n 2^n n! \frac{q^{2n+1}}{(2n+1)!}. \end{aligned} \quad (\text{C7})$$

Remark. The two chains $\{\chi_n^{(+)}\}$ and $\{\chi_n^{(-)}\}$ span the even and odd polynomial sectors at $E = 0$. In the main text we work with the normalized monomials

$$\chi_n(q) := \frac{q^n}{\sqrt{n!}}, \quad (\text{C8})$$

which are obtained from (C7) by simple rescalings and reindexing.

APPENDIX D: ABCD OPTICS AND ABCD $\text{Sp}(2, \mathbb{R})$ FACTORIZATIONS: SHEARS, OSCILLATORS, AND METAPLECTIC LIFTS

A linear canonical map $\xi = (q, p)^\top \mapsto \xi' = S\xi$ with $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $AD - BC = 1$ lies in $\text{Sp}(2, \mathbb{R}) \cong \text{SL}(2, \mathbb{R})$ and preserves $\omega = dp \wedge dq$. Equivalently, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (the matrix Ω of Sec. II), $S^\top JS = J$, and the bilinear form $\Omega(\xi_1, \xi_2) := \xi_1^\top J \xi_2 = q_1 p_2 - q_2 p_1$ is invariant: $\Omega(\xi'_1, \xi'_2) = \Omega(\xi_1, \xi_2)$.

In paraxial optics [81] the same S is the $ABCD$ matrix. Writing $q := x$ and $p := \kappa\theta$ for transverse position x and

slope θ gives $(q', p')^\top = S(q, p)^\top$ and the Lagrange invariant $\mathcal{I} = \kappa(x_1\theta_2 - x_2\theta_1) = x_1 p_2 - x_2 p_1$ (independent of the choice of κ).

1. Elementary shears (drift and thin lens)

Introduce

$$\begin{aligned} \mathcal{T}(a) &:= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \exp(a\sigma_+), \\ \mathcal{L}(b) &:= \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \exp(b\sigma_-), \quad a, b \in \mathbb{R}, \end{aligned}$$

so that $\mathcal{T}(a): (q, p) \mapsto (q + ap, p)$ and $\mathcal{L}(b): (q, p) \mapsto (q, p + bq)$. In the conventions of Sec. II, $M_{H_0}(\tau) = \mathcal{T}(-\tau)$ for $H_0 = \frac{1}{2}p^2$ and $M_K(\tau) = \mathcal{L}(\tau)$ for $K = \frac{1}{2}q^2$; in optics, $\mathcal{T}(L)$ is free propagation and $\mathcal{L}(-1/f)$ a thin lens.

2. Generation of $\text{Sp}(2, \mathbb{R})$ and Gauss factorization

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2, \mathbb{R})$. If $B \neq 0$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{L}\left(\frac{D-1}{B}\right) \mathcal{T}(B) \mathcal{L}\left(\frac{A-1}{B}\right), \quad (B \neq 0), \quad (\text{D1})$$

i.e., *lens-propagation-lens*. If $B = 0$, symplecticity forces the matrix to be of the form $\begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix}$ with $A \neq 0$. Introduce the dilation/squeeze matrix

$$\mathcal{D}(d) := \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} = \exp(-(\ln d)\sigma_3), \quad (d \neq 0), \quad (\text{D2})$$

so that, for $B = 0$,

$$\begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix} = \mathcal{D}(1/A) \mathcal{L}(AC), \quad \mathcal{D}(1/A) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}. \quad (\text{D3})$$

3. Key examples

(i) Harmonic oscillator rotation. Define

$$\mathcal{R}(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

For $\sin \theta \neq 0$ one has the shear decomposition

$$\mathcal{R}(\theta) = \mathcal{L}\left(\tan \frac{\theta}{2}\right) \mathcal{T}(-\sin \theta) \mathcal{L}\left(\tan \frac{\theta}{2}\right). \quad (\text{D4})$$

(ii) Quarter turn, Fourier transform, and inversion. The symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathcal{R}(-\pi/2) \in \text{Sp}(2, \mathbb{R})$$

acts as $(q, p) \mapsto (p, -q)$; in the metaplectic lift it is (up to phase) the Fourier transform and induces $\tau \mapsto -1/\tau$. Conjugation by J swaps the two shears,

$$JT(a)J^{-1} = \mathcal{L}(-a), \quad \text{equivalently} \\ \mathcal{L}(b) = JT(-b)J^{-1}. \quad (\text{D5})$$

(iii) Dilation/squeeze. A shear product is

$$\mathcal{D}(d) = T(1)\mathcal{L}(d^{-1}-1)T(-d)\mathcal{L}\left(\frac{d-1}{d^2}\right), \quad (d \neq 0). \quad (\text{D6})$$

(iv) Hyperbolic rotation (inverted oscillator). Define

$$\mathcal{H}(\tau) := \begin{pmatrix} \cosh \tau & -\sinh \tau \\ -\sinh \tau & \cosh \tau \end{pmatrix}.$$

For $\tau \neq 0$ it factorizes as

$$\mathcal{H}(\tau) = \mathcal{L}\left(-\tanh \frac{\tau}{2}\right)T(-\sinh \tau)\mathcal{L}\left(-\tanh \frac{\tau}{2}\right). \quad (\text{D7})$$

4. Metaplectic lifts and quadratic propagators

At the quantum level one uses the metaplectic cover $\text{Mp}(2, \mathbb{R})$: for each $S \in \text{Sp}(2, \mathbb{R})$ there are two lifts $\pm \hat{S}$ implementing $\hat{S} \hat{\xi} \hat{S}^{-1} = S \hat{\xi}$ on $\hat{\xi} = (\hat{q}, \hat{p})^T$ (products are defined up to the Maslov sign). For the elementary shears one may take the standard quadratic operators

$$\hat{T}(a) := \exp\left(+ia \frac{\hat{p}^2}{2}\right), \quad \hat{\mathcal{L}}(b) := \exp\left(-ib \frac{\hat{q}^2}{2}\right),$$

$$\hat{D}(d) := \exp(-i(\ln d) \hat{D}),$$

where $\hat{D} = \frac{1}{2}(\hat{q} \hat{p} + \hat{p} \hat{q})$. These implement the same linear actions on $(\hat{q}, \hat{p})$ as $T(a)$, $\mathcal{L}(b)$ and $\mathcal{D}(d)$ on (q, p) . Thus classical factorizations immediately yield quantum ones. For example, (D4) gives

$$\hat{\mathcal{R}}(\theta) = \hat{\mathcal{L}}\left(\tan \frac{\theta}{2}\right) \hat{T}(-\sin \theta) \hat{\mathcal{L}}\left(\tan \frac{\theta}{2}\right), \quad (\sin \theta \neq 0), \quad (\text{D8})$$

and (D7) yields

$$\hat{\mathcal{H}}(\tau) = \hat{\mathcal{L}}\left(-\tanh \frac{\tau}{2}\right) \hat{T}(-\sinh \tau) \hat{\mathcal{L}}\left(-\tanh \frac{\tau}{2}\right), \quad (\tau \neq 0). \quad (\text{D9})$$

In the oscillator normalization of Sec. II, $\hat{\mathcal{R}}(\theta) = \exp(-i\theta \hat{H}_+)$ with $\hat{H}_+ = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$, while $\hat{\mathcal{H}}(\tau) = \exp(-i\tau \hat{H}_-)$ with $\hat{H}_- = \frac{1}{2}(\hat{p}^2 - \hat{q}^2)$ (analytic continuation $\theta \mapsto i\tau$ relates the two).

Cayley operator. The bridge operator $U(\pi/4) = \exp(\frac{\pi}{4} \hat{H}_-)$ used in the main text is the metaplectic lift of the complex-parameter flow $M_{H_-}(\tau)$ at $\tau = i\pi/4$. Its factorization used in Sec. IV is

$$U(\pi/4) = \exp\left(-\frac{1}{2} \hat{q}^2\right) \exp\left(\frac{1}{2} \hat{H}_0\right) \exp\left(-\frac{i}{2} \ln 2 \hat{D}\right), \\ \hat{H}_0 = \frac{1}{2} \hat{p}^2, \quad \hat{D} = \frac{1}{2}(\hat{q} \hat{p} + \hat{p} \hat{q}). \quad (\text{D10})$$

At the level of matrices this corresponds to the Gauss decomposition (consistent with $\hat{S} \hat{\xi} \hat{S}^{-1} = S \hat{\xi}$)

$$C = \mathcal{L}(-i)T\left(-\frac{i}{2}\right)\mathcal{D}(2^{1/2}), \\ C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \exp\left(-i \frac{\pi}{4} \sigma_1\right) \in \text{Sp}(2, \mathbb{C}). \quad (\text{D11})$$

5. Link to $SU(1,1)$ disentangling

Under $\text{Sp}(2, \mathbb{R}) \cong SL(2, \mathbb{R}) \cong SU(1, 1)$, Gauss decompositions lift to the standard disentangling identity for the generators (4.6), $\hat{K}_{\pm} = \frac{1}{2}(\hat{a}^{\pm})^2$ and $\hat{K}_0 = \frac{1}{2} \hat{H}_+$:

$$\exp[-s(\hat{K}_+ + \hat{K}_-)] = \exp[-\tan s \hat{K}_+] \exp[-2 \ln(\cos s) \hat{K}_0] \exp[-\tan s \hat{K}_-], \quad (\cos s \neq 0), \quad (\text{D12})$$

whose special case $s = \pi/4$ yields the factorization (4.7), which is equivalent to (D10).

function $\psi_{0,\zeta}(q) = \langle q | S(\zeta) | 0 \rangle$. Starting from the number-basis expansion (5.11), we have

APPENDIX E: DEMONSTRATION OF EQ. (5.12)

In this appendix we demonstrate Eq. (5.12), i.e., the closed Gaussian form of the squeezed-vacuum wave

$$\psi_{0,\zeta}(q) = \langle q | S(\zeta) | 0 \rangle = (1 - |\kappa|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \kappa^n \psi_{2n}(q), \quad (\text{E1})$$

where $\kappa = e^{i\phi} \tanh r$ with $|\kappa| < 1$, and $\psi_n(q)$ are the normalized harmonic-oscillator eigenfunctions,

$$\psi_n(q) = \pi^{-1/4} \frac{1}{\sqrt{2^n n!}} e^{-q^2/2} H_n(q). \quad (\text{E2})$$

Substituting (E2) with $n \mapsto 2n$ into (E1) gives

$$\begin{aligned} \psi_{0,\zeta}(q) &= (1 - |\kappa|^2)^{1/4} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \kappa^n \\ &\quad \times \left[\pi^{-1/4} \frac{1}{\sqrt{2^{2n} (2n)!}} e^{-q^2/2} H_{2n}(q) \right] \\ &= \pi^{-1/4} (1 - |\kappa|^2)^{1/4} e^{-q^2/2} \sum_{n=0}^{\infty} \frac{\kappa^n}{4^n n!} H_{2n}(q). \quad (\text{E3}) \end{aligned}$$

Thus it remains to evaluate the even-Hermite series $\sum_{n \geq 0} \frac{\kappa^n}{4^n n!} H_{2n}(q)$.

We use the standard identity relating even Hermite polynomials to generalized Laguerre polynomials,

$$H_{2n}(q) = (-1)^n 2^{2n} n! L_n^{-1/2}(q^2) = (-1)^n 4^n n! L_n^{-1/2}(q^2). \quad (\text{E4})$$

Inserting (E4) into the series in (E3) yields

$$\sum_{n=0}^{\infty} \frac{\kappa^n}{4^n n!} H_{2n}(q) = \sum_{n=0}^{\infty} (-\kappa)^n L_n^{-1/2}(q^2). \quad (\text{E5})$$

Now apply the generating function for generalized Laguerre polynomials,

$$\sum_{n=0}^{\infty} L_n^\alpha(x) t^n = (1-t)^{-\alpha-1} \exp\left(-\frac{t}{1-t}x\right), \quad |t| < 1, \quad (\text{E6})$$

with $\alpha = -\frac{1}{2}$, $x = q^2$, $t = -\kappa$. Since $|\kappa| < 1$, the sum converges and we obtain

$$\sum_{n=0}^{\infty} (-\kappa)^n L_n^{-1/2}(q^2) = (1 + \kappa)^{-1/2} \exp\left(\frac{\kappa}{1 + \kappa} q^2\right). \quad (\text{E7})$$

Substituting (E7) into (E3) gives

$$\begin{aligned} \psi_{0,\zeta}(q) &= \pi^{-1/4} (1 - |\kappa|^2)^{1/4} e^{-q^2/2} (1 + \kappa)^{-1/2} \exp\left(\frac{\kappa}{1 + \kappa} q^2\right) \\ &= \pi^{-1/4} (1 - |\kappa|^2)^{1/4} (1 + \kappa)^{-1/2} \exp\left(-\frac{1 - \kappa}{2(1 + \kappa)} q^2\right), \quad (\text{E8}) \end{aligned}$$

which is exactly Eq. (5.12).

APPENDIX F: SCHRÖDINGER-GROUP GAUSSIAN PACKETS, COVARIANCE, AND THE UPPER HALF-PLANE

This appendix collects the basic facts about Gaussian wave packets of the free particle that are needed in the discussion of the Cayley/conformal bridge. The essential point is that the natural parameter space of (pure) Gaussian covariances is the upper half-plane $\mathbb{H}_+$, the same space that appears as the moduli of compatible complex structures in Appendix A.

1. Robertson-Schrödinger uncertainty relation and covariance matrix

We work with $[\hat{q}, \hat{p}] = i$ ($\hbar = 1$). For a normalized state define

$$\Delta\hat{q} := \hat{q} - \langle\hat{q}\rangle, \quad \Delta\hat{p} := \hat{p} - \langle\hat{p}\rangle,$$

and the real symmetric covariance matrix

$$\begin{aligned} \Sigma &:= \begin{pmatrix} \langle(\Delta\hat{q})^2\rangle & \frac{1}{2}\langle\Delta\hat{q}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{q}\rangle \\ \frac{1}{2}\langle\Delta\hat{q}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{q}\rangle & \langle(\Delta\hat{p})^2\rangle \end{pmatrix} \\ &= \begin{pmatrix} (\Delta q)^2 & \text{Cov}(q, p) \\ \text{Cov}(q, p) & (\Delta p)^2 \end{pmatrix}. \quad (\text{F1}) \end{aligned}$$

Remark: Meaning of $\text{Cov}(q, p)$. The off-diagonal entry

$$\text{Cov}(q, p) = \frac{1}{2} \langle \Delta\hat{q}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{q} \rangle = \text{Re} \langle \Delta\hat{q}\Delta\hat{p} \rangle \quad (\text{F2})$$

is the (symmetrized) covariance of the noncommuting observables q and p . It is real and quantifies the *linear correlation* between the fluctuations $\Delta\hat{q}$ and $\Delta\hat{p}$ in the given state. Introducing the dimensionless correlation coefficient

$$\rho_{qp} := \frac{\text{Cov}(q, p)}{\Delta q \Delta p}, \quad |\rho_{qp}| \leq 1, \quad (\text{F3})$$

one may regard $\text{Cov}(q, p) \neq 0$ as encoding a *tilt* of the uncertainty ellipse in phase space, while $\text{Cov}(q, p) = 0$ corresponds to an “untilted” (often called *unchirped*) situation. In particular, for a Gaussian wave packet this tilt is precisely the quadratic qp (“chirp”) term in the exponent and is preserved under linear canonical transformations.

The sharp Robertson-Schrödinger (RS) uncertainty relation [82] is equivalently

$$\det \Sigma \geq \frac{1}{4}, \quad \text{i.e.,} \quad (\Delta q)^2 (\Delta p)^2 - \text{Cov}(q, p)^2 \geq \frac{1}{4}. \quad (\text{F4})$$

Dropping the nonnegative $\text{Cov}(q, p)^2$ term yields the familiar Heisenberg product bound $\Delta q \Delta p \geq \frac{1}{2}$.

2. Phase-space ellipse and “minimal area” interpretation

The matrix Σ defines the uncertainty ellipse in the classical (Q, P) plane by

$$E_1 := \left\{ (Q, P) \in \mathbb{R}^2 \mid \begin{pmatrix} Q \\ P \end{pmatrix}^\top \Sigma^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} \leq 1 \right\}. \quad (\text{F5})$$

Its Euclidean area is

$$\text{Area}(E_1) = \pi \sqrt{\det \Sigma}. \quad (\text{F6})$$

Hence (F4) implies $\text{Area}(E_1) \geq \pi/2$. In this precise sense, saturating the RS inequality means *minimal phase-space area* of the covariance ellipse (for $\hbar = 1$).

3. A convenient Gaussian parametrization by $\tau \in \mathbb{H}_+$

Introduce the upper half-plane parameter $\tau = u + iv \in \mathbb{C}$, $v > 0$, and the normalized centered Gaussian, cf. (5.14),

$$\psi_\tau(q) := \left(\frac{\text{Im } \tau}{\pi |\tau|^2} \right)^{1/4} \exp \left(-\frac{i}{2\tau} q^2 \right). \quad (\text{F7})$$

More general packets are obtained by Heisenberg translation; up to an overall phase, a convenient explicit form is

$$\psi_{q_0, p_0; \tau}(q) := e^{ip_0(q - \frac{1}{2}q_0)} \psi_\tau(q - q_0), \quad (q_0, p_0) \in \mathbb{R}^2, \quad \tau \in \mathbb{H}_+. \quad (\text{F8})$$

For these Gaussians one finds the second moments

$$\begin{aligned} (\Delta q)^2 &= \frac{|\tau|^2}{2 \text{Im } \tau}, & (\Delta p)^2 &= \frac{1}{2 \text{Im } \tau}, \\ \text{Cov}(q, p) &= -\frac{\text{Re } \tau}{2 \text{Im } \tau}. \end{aligned} \quad (\text{F9})$$

Consequently,

$$(\Delta q)^2 (\Delta p)^2 - \text{Cov}(q, p)^2 = \frac{1}{4},$$

so the RS bound is saturated for all $\tau \in \mathbb{H}_+$.

The Heisenberg product is

$$\Delta q \Delta p = \frac{|\tau|}{2 \text{Im } \tau} \geq \frac{1}{2}, \quad (\text{F10})$$

and equality holds iff $\text{Re } \tau = 0$. Thus “unchirped” means $\text{Re } \tau = 0$, equivalently $\text{Cov}(q, p) = 0$.

4. Stability under free evolution and evolution of moments

For the free Hamiltonian $\hat{H}_0 = \frac{1}{2} \hat{p}^2$, Heisenberg evolution gives $\hat{q}(t) = e^{it\hat{H}_0} \hat{q} e^{-it\hat{H}_0} = \hat{q} + t\hat{p}$, $\hat{p}(t) = e^{it\hat{H}_0} \hat{p} e^{-it\hat{H}_0} = \hat{p}$. Hence the centered operators satisfy $\Delta \hat{q}(t) = \Delta \hat{q} + t\Delta \hat{p}$, $\Delta \hat{p}(t) = \Delta \hat{p}$, and the second moments evolve exactly as

$$\begin{aligned} (\Delta p(t))^2 &= (\Delta p(0))^2, & \text{Cov}(q, p)(t) &= \text{Cov}(q, p)(0) + t(\Delta p(0))^2, \\ (\Delta q(t))^2 &= (\Delta q(0))^2 + 2t\text{Cov}(q, p)(0) + t^2(\Delta p(0))^2. \end{aligned}$$

For pure Gaussians (F7)–(F8) one has $\det \Sigma = \frac{1}{4}$ at $t = 0$, and the RS-saturating character is preserved by the free evolution (the ellipse is sheared/tilted, but its symplectic area remains minimal).

Equivalently, on the parameter τ one has the simple flow (up to phase),

$$e^{-it\hat{H}_0} \psi_{q_0, p_0; \tau} \propto \psi_{q_0 + tp_0, p_0; \tau - t}. \quad (\text{F11})$$

This expresses the “stability under free time evolution” as closure of the Gaussian family under the one-parameter subgroup $e^{-it\hat{H}_0}$.

APPENDIX G: SCHWARZIAN DERIVATIVE: COCYCLE PROPERTY AND INVERSE-MAP IDENTITY

For a locally univalent function $f = f(x)$, its Schwarzian derivative [30–32] is

$$\{f; x\} := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2. \quad (\text{G1})$$

It vanishes iff f is a Möbius (projective) transformation,

$$\{f; x\} = 0 \Leftrightarrow f(x) = \frac{ax + b}{cx + d}, \quad ad - bc \neq 0. \quad (\text{G2})$$

In terms of the *pre-Schwarzian*

$$\mathcal{P}_f(x) := \frac{f''(x)}{f'(x)} = (\ln f'(x))', \quad (\text{G3})$$

one may also write

$$\{f; x\} = \mathcal{P}'_f(x) - \frac{1}{2} (\mathcal{P}_f(x))^2. \quad (\text{G4})$$

1. Cocycle (chain) property

As a quadratic differential $S(f) := \{f; x\}(dx)^2$, the Schwarzian is a (nontrivial) one cocycle on $\text{Diff}(I)$,

$$\{f \circ g; x\} = (g'(x))^2 \{f; g(x)\} + \{g; x\}, \quad (\text{G5})$$

equivalently $S(f \circ g) = g^* S(f) + S(g)$. In particular, $\{M; z\} = 0$ for any Möbius map M , hence

$$\{M \circ f; x\} = \{f; x\}. \quad (\text{G6})$$

[Under precomposition $x \mapsto M(x)$ one uses (G5).]

2. Inverse-map identity

If $y = y(x)$ is locally invertible with inverse $x = x(y)$, then

$$\{x; y\} = -(x'(y))^2 \{y; x\}|_{x=x(y)}. \quad (\text{G7})$$

This is the form used in the main text [cf. (7.6) and (7.7)].

Historical note. The Schwarzian derivative was isolated in conformal mapping by Schwarz (1869, 1873) [83,84] and the terminology became standard after Cayley (1880) [51]; related third-order expressions occur (implicitly) earlier in transformation theory [85–87]. As a simple corollary, for $Q \equiv k^2$ one has $\{\tan(kx); x\} = 2k^2$ (up to postcomposition by a Möbius map).

APPENDIX H: EXACT VERSUS SEMICLASSICAL QUANTIZATION OF CANONICAL TRANSFORMATIONS

This appendix clarifies the relation between classical canonical transformations and their quantum counterparts. The key point is that a generating function controls the *oscillatory phase* of an integral kernel, but in general does not determine a full unitary operator: one must also include a nontrivial amplitude (Van Vleck/half-density factor) and a Maslov phase. Exact quantization in the strong sense of an exact Egorov property [88] on a large algebra of observables is exceptional, although there are important exact classes strictly larger than the linear/metaplectic one.

1. Two notions of “quantum analog”

Let κ be a canonical transformation (symplectomorphism) on phase space, $\kappa: (q, p) \mapsto (Q, P)$, $dp \wedge dq = dP \wedge dQ$. There are two distinct (and often conflated) requirements:

(A) Exact implementation on observables (exact Egorov). One asks for a unitary U_κ such that for a large class of symbols a ,

$$U_\kappa^{-1} \text{Op}^W(a) U_\kappa = \text{Op}^W(a \circ \kappa), \quad (\text{H1})$$

where Op^W denotes Weyl quantization. This is a very

strong property; it holds, for example, for the metaplectic representation of linear symplectic maps.

(B) Exact intertwining of a *specific* differential operator. Often one only needs an exact intertwiner for one operator (or a small family), for instance a Schrödinger/constraint operator S ,

$$U S_{\text{old}} = S_{\text{new}} U. \quad (\text{H2})$$

This can hold exactly even when (H1) fails for the full observable algebra. In particular, the free $\leftrightarrow$ oscillator unitary map in the extended formulation is naturally viewed as an exact intertwiner of Schrödinger operators.

2. Generating functions determine the phase, not the full kernel

Assume that κ admits locally a generating function of type (q, Q) , $p = \frac{\partial S(q, Q)}{\partial q}$, $P = -\frac{\partial S(q, Q)}{\partial Q}$. A standard semiclassical ansatz for an implementer is a Fourier integral operator (FIO) with kernel

$$K(Q, q) \sim (2\pi i \hbar)^{-1/2} A(Q, q; \hbar) \exp\left(\frac{i}{\hbar} S(q, Q)\right), \quad (\text{H3})$$

where A is an amplitude. The classical function S supplies the eikonal phase $\exp(iS/\hbar)$, while A is fixed by unitarity and composition (transport equations). In 1D and away from caustics ($\partial^2 S / \partial q \partial Q \neq 0$), the leading amplitude is the Van Vleck factor

$$A_{\text{VV}}(Q, q) = \left| \frac{\partial^2 S(q, Q)}{\partial q \partial Q} \right|^{1/2} \quad (\text{H4})$$

(up to an overall phase); higher orders in $\hbar$ generally correct it.

3. Egorov theorem: Generic nonlinear symplectomorphisms are only asymptotically exact

For a general (nonlinear) symplectomorphism κ , suitable FIO implementers satisfy Egorov’s theorem in the form of an *asymptotic* expansion,

$$U_\kappa^{-1} \text{Op}^W(a) U_\kappa = \text{Op}^W(a \circ \kappa) + \hbar \text{Op}^W(a_1) + \hbar^2 \text{Op}^W(a_2) + \dots \quad (\text{H5})$$

The correction symbols a_j depend on the *full* quantization data (including amplitudes), not only on S ; hence exactness in the strong sense (H1) is not generic.

4. Exact classes beyond the linear/metaplectic case

- (i) Linear symplectic maps and the metaplectic representation. If κ is linear symplectic ($\kappa(z) = Mz$, $M \in \text{Sp}(2, \mathbb{R})$), then S is quadratic and the

metaplectic operator $U_M \in \text{Mp}(2, \mathbb{R})$ is exact. In this case the Van Vleck factor is constant (a determinant of a constant block), and the remaining global datum is the Maslov sign/phase (the $\text{Mp} \rightarrow \text{Sp}$ double cover).

- (ii) Polarization-preserving canonical transformations on $T^*\mathbb{R}$. A nonlinear exact class is given by the canonical maps that preserve the Schrödinger (vertical) polarization: cotangent lifts of diffeomorphisms of configuration space, possibly composed with an exact one-form shift. Given a diffeomorphism $Q = Q(q)$ and a function $G(q)$, consider

$$Q = Q(q), \quad P = \frac{p + G'(q)}{Q'(q)}. \quad (\text{H6})$$

One checks that $pdq - PdQ = dG(q)$, so (H6) is canonical. Its exact unitary action on wavefunctions is

$$(U\psi)(Q) = \left(\frac{dQ}{dq} \right)^{-1/2} \exp \left(\frac{i}{\hbar} G(q) \right) \psi(q) \Big|_{q=q(Q)}. \quad (\text{H7})$$

The Jacobian factor expresses that ψ is naturally a half-density: $\psi(q)(dq)^{1/2} = \Psi(Q)(dQ)^{1/2}$.

- (iii) Exactness by truncation of the Moyal commutator. A useful sufficient condition for exact Egorov on the Weyl/Moyal algebra is that the Moyal commutator truncates for the generator Hamiltonian. This includes quadratic Hamiltonians (metaplectic case), and also Hamiltonians at most affine in momentum,

$$H(q, p) = a(q)p + b(q), \quad (\text{H8})$$

whose flows are precisely of the polarization-preserving form (H6). In such cases the quantum evolution $U(t) = \exp(-i\hat{H}t/\hbar)$ produces an exact automorphism of the Weyl algebra (no infinite $\hbar$ series is generated).

5. Maslov phase and the metaplectic double cover

Whenever the kernel involves a square root of a determinant (or, in 1D, a square root of $\partial^2 S / \partial q \partial Q$), one must choose a branch. Along continuous families this branch can jump when passing through caustics (where the chosen projection becomes singular). The standard way to encode this is a Maslov phase. In 1D the leading kernel is

$$K(Q, q) \approx (2\pi i \hbar)^{-1/2} \left| \frac{\partial^2 S}{\partial q \partial Q} \right|^{1/2} \exp \left(\frac{i}{\hbar} S(q, Q) - \frac{i\pi}{2} \mu \right), \quad (\text{H9})$$

where $\mu \in \mathbb{Z}$ changes by ± 1 upon crossing a caustic. In the linear symplectic case, μ reduces to the discrete datum distinguishing the two lifts in $\text{Mp}(2, \mathbb{R})$ of a given element of $\text{Sp}(2, \mathbb{R})$.

Remark. The exact classes (H6)–(H7) are the natural nonlinear extensions of “phase from a generating function” in the Schrödinger polarization: the phase is supplied by the exact differential dG in the Liouville form, while the half-density Jacobian is forced by unitarity (and is the same mechanism behind the familiar Schwarzian terms appearing when one transforms second-order operators).

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