

Nonperturbative corrections to the Casimir energy for a scalar field theory with nonlinear boundary conditions

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We consider the case of a free real massive bulk scalar in $D = 4$ dimensions, and embed two parallel plates as interfaces on which we impose nonlinear boundary conditions, either Dirichlet- or Neumann-like, parametrized by a new coupling constant g . The system mimics a non-Abelian gauge theory supplemented with boundary conditions on surfaces embedded in the bulk. We present the first evidence for a nonperturbative $\frac{1}{g^2}$ boundary mass generation and its ensuing correction to the standard Casimir energy. This result becomes possible by incorporating dynamical corrections to the effective boundary fields, which are used to build in the boundary conditions directly at the action level.

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I. INTRODUCTION

Recently, the Casimir effect [1–4], with its origins dating back to Ref. [5], saw a revived interest in Yang-Mills (YM) gauge theories,¹ thanks to nonperturbative lattice QCD Monte Carlo simulations of $D = (3 + 1)$ or $D = (2 + 1)$ setups, with various geometries of interfaces on which standard perfect color-electric boundary conditions were applied [6–9]. For simplicity, here we focus on the plate geometry. An interesting feature was observed, namely, the

exponential decay in terms of the plate separation. This suggests the emergence of a new, nonperturbative mass scale of a few hundred MeV from the self-interaction of the gluon. This mass scale is considerably smaller than the standard lowest mass scale in gluodynamics, the scalar glueball having a mass of around 1.5 GeV [10]. In Ref. [7], this new scale was attributed to a novel type of edge (boundary) modes, consisting of a (colorless) bound state of a gluon and its mirror image relative to the plate(s), suggestively named “glueton” as analog of the familiar excitons in solid state physics.² This does not, however, answer the question where the mass of these gluetons would come from, as *a priori*, its constituent gluons are massless. Similar to plateless QCD (or YM), the pertinent issue still is: how to obtain massive glueballs/gluetons from *a priori* massless gluons?

Interestingly, in the recent works Refs. [12,13], see also Ref. [14], it was shown that very reasonable analytical estimates for the YM Casimir energies/forces of Refs. [6–9] were possible to obtain from incorporating a gluon mass

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¹To be more precise, $SU(2)$ and $SU(3)$ gluodynamics, without quark degrees of freedom.

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²The analogous “quarkiton” was studied in more detail in Ref. [11].

scale into the quadratic approximation. This gluon mass could be introduced either based on $D = (2 + 1)$ works of Refs. [15,16] or on the Curci-Ferrari model [17] in Landau gauge ($D = (3 + 1)$ or $D = (2 + 1)$). The latter saw a revived interest over the years as an easy-to-compute model with an effective description of certain nonperturbative aspects of QCD, see Ref. [18] for an extensive review. The boundary conditions were introduced in those works [12,13] via auxiliary fields restricted to the plates, see also Refs. [19–26] for more about this approach. Afterwards an effective boundary theory can be computed from which the Casimir energy follows from a computation in one less dimension.

Here, we will analyze another intriguing novel source of nonperturbative Casimir physics: the interactions induced by the boundary conditions. For this, we introduce a scalar toy model to mimic the nonlinear YM boundary conditions. Indeed, due to gauge invariance, the boundary conditions on the plates must be expressed in terms of the non-Abelian field strength which possesses both a linear term in the gauge potential and a quadratic term coming from the commutator. Therefore, the proper implementation of the non-Abelian version of the common perfect electric/magnetic boundary conditions requires the analysis of boundary conditions which are nonlinear in the gauge potential. The nonlinear boundary conditions for the scalar field in the present manuscript have been exactly designed to shed preliminary novel light on the nonperturbative contributions to the Casimir energy arising from the non-Abelian nature of the gauge fields.

The fact that interactions restricted to a lower-dimensional subspace can lead to interesting new physics can also be appreciated from works such as Refs. [27] or [28]. In our case, the effective boundary field method allows us to take these nonlinear boundary interactions into explicit account and explore their possible consequences for what concerns the Casimir energy. We will provide evidence for a non-perturbative correction to the Casimir energy/force (which may even change sign, depending on the size of the coupling constant).

II. NONLINEAR NEUMANN-LIKE BOUNDARY CONDITIONS

As it has already been emphasized, eventually we wish to explore the possibility of novel, genuinely non-Abelian effects entering the Casimir energy of Yang-Mills theory. Even restricting to leading order, while the gauge field then satisfies a free field equation in the bulk, the boundary conditions on the plates possess both a term which is linear in the derivative of the gauge potential, as well as a quadratic one. To be more precise, e.g., the gauge invariant non-Abelian perfect magnetic boundary conditions read

$$n_\mu F_{\mu\nu}^a \Big|_{z=\pm L/2} = \partial_z A_i^a - \partial_i A_z^a - g f^{abc} A_z^b A_i^c \Big|_{z=\pm L/2} = 0, \quad (1)$$

where the index i refers to the (t, x, y) coordinates. Therefore, one can appreciate that the Neumann boundary conditions, see Eq. (3) below, are mimicking the Yang-Mills case. The analysis, when extrapolated to the Yang-Mills case, will then strongly suggest the interesting possibility of the coexistence of two ground states (a trivial and a nonperturbative one) with different energies but separated from one another via infinite energy walls.

Concretely, let us start from the following scalar toy model:

$$S = \int d^4x \left[-\frac{1}{2} \varphi (\partial^2 - m^2) \varphi - b \delta \left(z + \frac{L}{2} \right) \left(\partial_z \varphi + \frac{g}{2} \varphi^2 \right) - \bar{b} \delta \left(z - \frac{L}{2} \right) \left(\partial_z \varphi + \frac{g}{2} \varphi^2 \right) \right], \quad (2)$$

where the $b(\bar{x})$ and $\bar{b}(\bar{x})$ auxiliary fields impose the boundary conditions on the idealized (i.e., zero thickness) plates, as follows:

$$\partial_z \varphi + \frac{g}{2} \varphi^2 \Big|_{z=\pm L/2} = 0. \quad (3)$$

We took the same g on both plates for simplicity. We introduced the notation $\bar{x} = (t, x, y)$ for the $\mathbb{R}^3$ space orthogonal to the planes, and we are working in Euclidean space. The mass dimension of the fields or coupling g is determined via $\dim(b) = \dim(\bar{b}) = 1$, $\dim(g) = 0$.

Let us now look in detail at the quantum consequences of such nonlinear boundary conditions. The idea is to allow for a dynamically generated boundary mass, so we will construct the effective potential for b_0 and $\bar{b}_0$ with

$$b = b_0 + \beta; \quad b_0 = \langle b | b \rangle, \\ \bar{b} = \bar{b}_0 + \bar{\beta}; \quad \bar{b}_0 = \langle \bar{b} | \bar{b} \rangle. \quad (4)$$

Let us first construct the two-point Green's function for the φ -field, but not yet enforcing the boundary conditions, i.e., not yet including the Lagrange multiplier fields $\beta, \bar{\beta}$. The relevant quadratic part of the action reads

$$S_2 = \int d^4x \left[-\frac{1}{2} \varphi (\partial^2 - m^2) \varphi + \Delta(z) \varphi^2 \right]. \quad (5)$$

We set $\Delta(z) = -\frac{g}{2} (b_0 \delta_+ + \bar{b}_0 \delta_-)$ with $\delta_\pm \equiv \delta(z \pm L/2)$. Notice that this $\Delta(z)$ indeed acts as a boundary mass (per plate).

The Green's function $G(x, x')$ will thence obey

$$(-\partial^2 + m^2 + \Delta(z)) G(x, x') = \delta(x - x'), \quad (6)$$

reducing to, after a partial 3D Fourier transformation $\bar{x} \mapsto \tilde{k}$ with $k = |\tilde{k}|$, as follows:

$$(-\partial_z^2 + k^2 + m^2 + \Delta(z))G(\vec{k}, z, z') \equiv \mathcal{M}(z)G(\vec{k}, z, z') \quad \text{given by} \\ = \delta(z - z'). \quad (7)$$

For $\Delta(z) = 0$, we have the standard free Green's function,

$$G_0(\vec{k}, z, z') = \frac{1}{2\sqrt{k^2 + m^2}} e^{-|z-z'|\sqrt{k^2+m^2}}. \quad (8)$$

To avoid notational clutter, from here on we will omit the Fourier coordinate $\vec{k}$ from the Green's functions. Continuing, the complete Green's function is formally

or, concretely

$$G(z, z') = G_0(z, z') - \int dx G_0(z, x) \Delta(x) G(x, z'), \quad (9) \\ G(z, z') = G_0(z, z') + gb_0 G_0(z, -L/2) G(-L/2, z') \\ + g\bar{b}_0 G_0(z, L/2) G(L/2, z'). \quad (10)$$

From this equation, we can infer

$$G(-L/2, z') = G_0(-L/2, z') + gb_0 G_0(-L/2, -L/2) G(-L/2, z') + g\bar{b}_0 G_0(-L/2, +L/2) G(+L/2, z') \\ G(+L/2, z') = G_0(+L/2, z') + gb_0 G_0(+L/2, -L/2) G(-L/2, z') + g\bar{b}_0 G_0(+L/2, +L/2) G(+L/2, z'), \quad (11)$$

and solve these for

$$G(-L/2, z') = -\frac{g\bar{b}_0 e^{-\sqrt{k^2+m^2}(L+|L/2-z'|)} + e^{-\sqrt{k^2+m^2}|L/2+z'|}(-g\bar{b}_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})} \\ G(+L/2, z') = -\frac{gb_0 e^{-\sqrt{k^2+m^2}(L+|L/2+z'|)} + e^{-\sqrt{k^2+m^2}|L/2-z'|}(-gb_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})}, \quad (12)$$

to end up with a closed analytical expression for the S_2 Green's function $G(z, z')$ in Eq. (10), as follows:

$$G(z, z') = \frac{1}{2\sqrt{k^2+m^2}} e^{-|z-z'|\sqrt{k^2+m^2}} \\ - gb_0 \frac{1}{2\sqrt{k^2+m^2}} e^{-|z+L/2|\sqrt{k^2+m^2}} \frac{g\bar{b}_0 e^{-\sqrt{k^2+m^2}(L+|L/2-z'|)} + e^{-\sqrt{k^2+m^2}|L/2+z'|}(-g\bar{b}_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})} \\ - g\bar{b}_0 \frac{1}{2\sqrt{k^2+m^2}} e^{-|z-L/2|\sqrt{k^2+m^2}} \frac{gb_0 e^{-\sqrt{k^2+m^2}(L+|L/2+z'|)} + e^{-\sqrt{k^2+m^2}|L/2-z'|}(-gb_0 + 2\sqrt{k^2+m^2})}{g^2 b_0 \bar{b}_0 e^{-2L\sqrt{k^2+m^2}} - (gb_0 - 2\sqrt{k^2+m^2})(g\bar{b}_0 - 2\sqrt{k^2+m^2})}. \quad (13)$$

We are now armed to construct the effective potential at leading order for the full action S in Eq. (2).³ We start from

$$Z = \int [\mathcal{D}\varphi][\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \exp \left[-S_2 - \int d^4x \partial_z \varphi ((b_0 + \beta)\delta_+ + (\bar{b}_0 + \bar{\beta})\delta_-) \right] \\ = \int [\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \exp \left[\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} dz dz' (J(\vec{k}, z) G(z, z') (2\pi)^3 \delta^3(\vec{k} - \vec{k}') J(\vec{k}', z')) \right] \\ \times \exp \left(-\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln \det \mathcal{M} \right), \quad (14)$$

where the operator $\mathcal{M}(z)$ was defined in Eq. (7). $J(\vec{k}, z)$ is the 3D Fourier transform of the combination of terms in Z coupling to φ , that is,

³Since we are working at leading order (mean field approximation), we do not need to include the higher order vertex terms $\propto \beta\varphi^2$ and $\propto \bar{\beta}\varphi^2$.

$$J(\vec{k}, z) = -(b_0(2\pi)^3 \delta^{(3)}(\vec{k}) + \beta(\vec{k}))\delta'_+ - (\bar{b}_0(2\pi)^3 \delta^{(3)}(\vec{k}) + \bar{\beta}(\vec{k}))\delta'_-. \quad (15)$$

First, we shall deal with the $\det \mathcal{M}$. We shall proceed in a two-step way. First we will compute the $\det$ per Fourier mode $\vec{k}$, afterwards we will trace over these Fourier modes themselves.

Let us call $u_n(z)$ an eigenfunction of $\mathcal{M}$ with corresponding eigenvalue λ_n , as follows:

$$\mathcal{M}u_n = \lambda_n u_n. \quad (16)$$

It will be sufficient to focus on the zero mode $\lambda_0 = 0$. For the normalizable solution, we find

$$u_0(z) = \begin{cases} ae^{\sqrt{k^2+m^2}z} & \text{for } z < -L/2, \\ be^{-\sqrt{k^2+m^2}z} + ce^{\sqrt{k^2+m^2}z} & \text{for } -L/2 < z < L/2, \\ de^{-\sqrt{k^2+m^2}z} & \text{for } L/2 < z. \end{cases} \quad (17)$$

The “gluing” at the boundaries $z = \pm L/2$ easily happens by imposing $u_0(z)$ to be continuous, while integrating Eq. (16) over either $\pm\epsilon - L/2$ or $\pm\epsilon + L/2$, respectively, yielding

$$\begin{aligned} u_0(-L/2 - \epsilon) - u_0(-L/2 + \epsilon) &= 0, \\ u_0(+L/2 + \epsilon) - u_0(+L/2 - \epsilon) &= 0, \\ u'_0(-L/2 + \epsilon) - u'_0(-L/2 - \epsilon) + gb_0 u(-L/2) &= 0, \\ u'_0(+L/2 + \epsilon) - u'_0(+L/2 - \epsilon) + g\bar{b}_0 u(+L/2) &= 0, \end{aligned} \quad (18)$$

always assuming $\epsilon \rightarrow 0^+$. Denoting the matrix of the linear set of equations (18) by $\mathcal{N}$, it follows that $\det \mathcal{N} = 0$ for a nontrivial solution to survive.

In fact, even if $\lambda_n \neq 0$, a very similar reasoning can be made, leading to the requirement that normalizable eigen-solutions will exist iff $\det \mathcal{N}(\lambda) = 0$ for $\lambda = \lambda_n$. Said otherwise, we have found a function $\det \mathcal{N}(\lambda)$ (an ordinary rather than a functional determinant) that vanishes at the eigenvalues of the original problem (16).

As such, we can invoke the Gel'fand-Yaglom approach [29] to extract $\ln \det \mathcal{M}$, following the line of reasoning in⁴ Ref. [30]. (An alternative approach would consist of using the identity of Jacobi for computing $\frac{d}{dL} \ln \det \mathcal{M}$.) In short, using ζ -function regularization, one has

⁴A more detailed version can be accessed via <https://saalburg.aei.mpg.de/wp-content/uploads/sites/25/2017/03/dunne.pdf>, including more references.

$$\det \mathcal{M} = e^{-\zeta'(0)}, \quad \zeta(s) = \sum_n \frac{1}{\lambda_n^s}. \quad (19)$$

It can then be shown that

$$-\zeta'(0) = \ln \det \mathcal{N}(0) - \ln \det \mathcal{N}(-\infty). \quad (20)$$

For $\lambda \rightarrow -\infty$, $\mathcal{M}(z) \rightarrow \mathcal{M}_{\text{free}} = -\partial_z^2$, so we can write

$$\ln \det \mathcal{M} - \ln \det \mathcal{M}_{\text{free}} = \ln \det \mathcal{N}(0) - \ln \det \mathcal{N}_{\text{free}}(0). \quad (21)$$

In our case, $\ln \det \mathcal{M}_{\text{free}}$ and $\ln \det \mathcal{N}_{\text{free}}(0)$ will merely be infinite constants that are irrelevant for the eventual Casimir energy/force, as an equal infinite constant will be subtracted when considering $E_{\text{Cas,N}}(L) - E_{\text{Cas,N}}(\infty)$. Or, alternatively, in, e.g., dimensional regularization, these infinite constants are replaced by zero, leading to the same end result. So,

$$\ln \det \mathcal{M} = \ln \det \mathcal{N}(0). \quad (22)$$

After some manipulations, we find, for a given $\vec{k}$, as follows:

$$\begin{aligned} \ln \det \mathcal{M} = \ln \left(g^2 b_0 \bar{b}_0 \left(1 - e^{-2L\sqrt{k^2+m^2}} \right) + 4k^2 + 4m^2 - 2g \right. \\ \left. \times (b_0 + \bar{b}_0) \sqrt{k^2 + m^2} \right). \end{aligned} \quad (23)$$

It then remains to compute

$$\begin{aligned} \int [\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \exp \left(\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} dz dz' [J(\vec{k}, z) G(z, z') \right. \\ \left. \times (2\pi)^3 \delta^3(\vec{k} - \vec{k}') J(\vec{k}', z')] \right). \end{aligned} \quad (24)$$

The terms linear in β or $\bar{\beta}$ will vanish upon proper minimization of the effective potential. Working out the various $\delta_{\pm}$ s, we get

$$\begin{aligned} (24) = \exp \left(\frac{V_3}{2} \lim_{\vec{k} \rightarrow 0} (b_0 \quad \bar{b}_0) \cdot P \cdot \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix} \right) \int [\mathcal{D}\beta][\mathcal{D}\bar{\beta}] \\ \times \exp \left(\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} (\beta(\vec{k}) \quad \bar{\beta}(\vec{k})) \cdot \mathcal{P} \cdot \begin{pmatrix} \beta(-\vec{k}) \\ \bar{\beta}(-\vec{k}) \end{pmatrix} \right), \end{aligned} \quad (25)$$

where we defined

$$\mathcal{P} = \begin{pmatrix} \partial_z \partial_{z'} G(z, z')|_{z=-L/2, z'=-L/2} & \partial_z \partial_{z'} G(z, z')|_{z=-L/2, z'=+L/2} \\ \partial_z \partial_{z'} G(z, z')|_{z=+L/2, z'=-L/2} & \partial_z \partial_{z'} G(z, z')|_{z=+L/2, z'=+L/2} \end{pmatrix}. \quad (26)$$

We also identified here $V_3 = (2\pi)^3 \delta^{(3)}(\vec{k} = 0)$ for the 3D space volume.

For completeness, let us quote the matrix elements of $\mathcal{P}$, as follows:

$$\begin{aligned} \mathcal{P}_{--} &= -\frac{(k^2 + m^2)(b_0 g(-e^{4L\sqrt{k^2+m^2}}(\bar{b}_0^2 g^2 + 4(k^2 + m^2)) + \bar{b}_0^2 g^2 + 4\bar{b}_0 g\sqrt{k^2+m^2}e^{4L\sqrt{k^2+m^2}}))}{(b_0 g(2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}} - \bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1)) - 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2))^2} \\ &\quad - \frac{2(k^2 + m^2)(e^{2L\sqrt{k^2+m^2}}(\bar{b}_0^2 g^2\sqrt{k^2+m^2}(e^{2L\sqrt{k^2+m^2}} + 1) - 2\bar{b}_0 g(k^2 + m^2)(2e^{2L\sqrt{k^2+m^2}} + 1) + 4(k^2 + m^2)^{3/2}e^{2L\sqrt{k^2+m^2}}))}{(b_0 g(2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}} - \bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1)) - 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2))^2} \\ \mathcal{P}_{-+} &= \frac{2e^{L\sqrt{k^2+m^2}}(b_0 g(\bar{b}_0 g\sqrt{k^2+m^2} - k^2 - m^2) + (k^2 + m^2)(\sqrt{k^2+m^2} - \bar{b}_0 g))}{b_0 g(2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}} - \bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1)) - 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2)} \\ \mathcal{P}_{+-} &= \frac{2(k^2 + m^2)^{3/2}e^{L\sqrt{k^2+m^2}}}{b_0 g(2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}} - \bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1)) - 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2)} \\ \mathcal{P}_{++} &= -\frac{(k^2 + m^2)(2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}} - b_0 g(e^{2L\sqrt{k^2+m^2}} + 1))}{b_0 g(\bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1) - 2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}}) + 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2)}. \end{aligned} \quad (27)$$

As such, we get, again for a fixed $\vec{k}$,

$$\begin{aligned} \det \mathcal{P} &= \frac{(k^2 + m^2)^2 (e^{2L\sqrt{k^2+m^2}} - 1)}{\left(b_0 g(\bar{b}_0 g(e^{2L\sqrt{k^2+m^2}} - 1) - 2\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}}) + 2e^{2L\sqrt{k^2+m^2}}(-\bar{b}_0 g\sqrt{k^2+m^2} + 2k^2 + 2m^2)\right)^3} \\ &\quad \times \left(b_0^2 g^2 (\bar{b}_0^2 g^2 (e^{2L\sqrt{k^2+m^2}} - 1)^2 - 4\bar{b}_0 g\sqrt{k^2+m^2}e^{2L\sqrt{k^2+m^2}}(e^{2L\sqrt{k^2+m^2}} - 1) + 4(k^2 + m^2)e^{4L\sqrt{k^2+m^2}})\right. \\ &\quad \left.- 4b_0 g e^{2L\sqrt{k^2+m^2}}(\bar{b}_0^2 g^2\sqrt{k^2+m^2}(e^{2L\sqrt{k^2+m^2}} - 1) - 2\bar{b}_0 g(k^2 + m^2)(2e^{2L\sqrt{k^2+m^2}} - 1) + 4(k^2 + m^2)^{3/2}e^{2L\sqrt{k^2+m^2}})\right. \\ &\quad \left.+ 4(k^2 + m^2)e^{4L\sqrt{k^2+m^2}}(\bar{b}_0 g(\bar{b}_0 g - 4\sqrt{k^2+m^2}) + 4(k^2 + m^2))\right). \end{aligned} \quad (28)$$

We pause here to draw attention to the power of the auxiliary fields method to impose the boundary conditions. Although *a priori* these were merely multipliers, they do couple to the bulk dynamics and by integrating out the latter, one ends up with a quantum corrected boundary field sector, including nontrivial propagation and induced interactions purely living on the boundaries. We restrict ourselves to the leading order corrections here (Gaussian approximation) but in principle the $1PI$ effective action $\Gamma(\beta, \bar{\beta})$ can be computed to higher orders.

To keep track of the dimensionalities, let us repeat here the relevant formulas, see also Ref. [24]. Let us denote the (infinite) 4D spacetime integration volume with $V_4 = \ell_t \ell_x \ell_y \ell_z$, with $\ell_{t,x,y,z} \rightarrow +\infty$. Then we have the path integral identity

$$Z = \int [\mathcal{D}\Phi] e^{-S[\Phi]} = e^{-\ell_t E},$$

with E the vacuum energy of the theory, and Φ a shorthand for all the fields. In a fully translationally invariant theory, one could then find the energy density $\mathcal{E}$ from $E = \ell_x \ell_y \ell_z \mathcal{E}$.

However, care must be taken as we have both 4D “bulk” $\Phi_{\text{bulk}} = \{\varphi\}$ and 3D “plate” $\Phi_{\text{plate}} = \{\beta, \bar{\beta}\}$ fields. Accordingly, the vacuum energy E will contain contributions from bulk and plate fields, the latter of which will correspond to the Casimir energy. Schematically, in the current quadratic approximation, with plate fields acting as sources for the bulk fields, it can be diagonalized by a shift of the bulk fields (which leaves the measure invariant $\mathcal{D}\Phi_{\text{bulk}} = \mathcal{D}\Phi'_{\text{bulk}}$), as follows:

$$\begin{aligned} Z &= \int [\mathcal{D}\Phi_{\text{bulk}}] [\mathcal{D}\Phi_{\text{plate}}] e^{-S[\Phi_{\text{bulk}}, \Phi_{\text{plate}}]} \\ &= \int [\mathcal{D}\Phi'_{\text{bulk}}] e^{-S_{\text{bulk}}[\Phi'_{\text{bulk}}]} \int \mathcal{D}[\Phi_{\text{plate}}] e^{-S_{\text{plate}}[\Phi_{\text{plate}}]} \\ &= e^{-\ell_t E_{\text{bulk}}} e^{-\ell_t E_{\text{Cas}}}, \end{aligned}$$

where the bulk action S_{bulk} does not depend on the plate configuration. The plate action only contains the plate fields, hence describing a 3D theory, and the Casimir energy density follows as $E_{\text{Cas}} = \ell_x \ell_y \mathcal{E}_{\text{Cas}}$.

Returning to the case at hand, and putting everything back together, we obtain via Gaussian integration in Fourier space and after exponentiation,

$$Z = \exp - \left(\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \det \mathcal{M} + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \det(-\mathcal{P}) - \frac{V_3}{2} \lim_{\vec{k} \rightarrow 0} (b_0 \quad \bar{b}_0) \cdot \mathcal{P} \cdot \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix} \right). \quad (29)$$

The first two contributions, arising from Eqs. (23) and (28), can be nicely combined and simplified to

$$\begin{aligned} & \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \det \mathcal{M} + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \det(-\mathcal{P}) \\ &= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \left[\left(1 - e^{-2L\sqrt{k^2+m^2}} \right) (k^2 + m^2)^2 \right]. \end{aligned} \quad (30)$$

Thus,

$$\begin{aligned} Z &= \exp \left(\frac{V_3}{2} \lim_{\vec{k} \rightarrow 0} (b_0 \quad \bar{b}_0) \cdot \mathcal{P} \cdot \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix} \right) \exp \left[-\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \right. \\ &\quad \left. \times \ln \left(\left(1 - e^{-2L\sqrt{k^2+m^2}} \right) (k^2 + m^2)^2 \right) \right] \\ &= \exp(-V_3 E_{\text{Cas},N}), \end{aligned} \quad (31)$$

from which we can read off the Casimir energy in the Neumann case,

$$E_{\text{Cas},N}^{(1)} = - \frac{gm^2 b_0 \bar{b}_0 (b_0 + \bar{b}_0) - m^2 e^{2Lm} (-gb_0 \bar{b}_0 (b_0 + \bar{b}_0) + 2m(b_0^2 + \bar{b}_0^2)) - 2mb_0 \bar{b}_0 e^{Lm} (g^2 b_0 \bar{b}_0 - gm(b_0 + \bar{b}_0) + 2m^2)}{2(e^{2Lm}(gb_0 - 2m)(g\bar{b}_0 - 2m) - g^2 b_0 \bar{b}_0)}. \quad (34)$$

Closed expressions exist for the various (complex) extrema of $E_{\text{Cas},N}^{(1)}$ in terms of $(b_0, \bar{b}_0)$, but these are lengthy algebraic expressions that are not very illuminating. We will therefore present results for the nonzero mass case in a purely numerical fashion.

However, let us first focus on the $m = 0$ case, as this case comes closest to the YM one. Everything can be handled fully analytically then. Indeed, as

$$E_{\text{Cas},N}^{(1)} = \frac{gb_0^2 \bar{b}_0^2}{2gLb_0 \bar{b}_0 - 2(b_0 + \bar{b}_0)}, \quad (35)$$

we find a nontrivial minimum at $b_0 = \bar{b}_0 = \frac{3}{gL}$ for all L , with Hessian eigenvalues $\{\frac{3}{2L}, \frac{9}{2L}\}$. There is also the trivial minimum at $b_0 = \bar{b}_0 = 0$, with lower energy. However, both are genuine ground states as one cannot penetrate (tunnel through) the infinite energy barriers separating both sectors, see Fig. 1. Likewise, even for large fields, there is no way to tunnel to the regions where the potential drops to $-\infty$ because of the energy walls.

$$\begin{aligned} E_{\text{Cas},N} &= -\frac{1}{2} \lim_{\vec{k} \rightarrow 0} (b_0 \quad \bar{b}_0) \cdot \mathcal{P} \cdot \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix} + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \\ &\quad \times \ln \left[\left(1 - e^{-2L\sqrt{k^2+m^2}} \right) (k^2 + m^2)^2 \right] \\ &\equiv E_{\text{Cas},N}^{(1)} + E_{\text{Cas},N}^{(2)}. \end{aligned} \quad (32)$$

The first part is evidently only relevant in case of non-vanishing vacuum expectation values (VEVs), so let us first focus on the second part, being

$$\begin{aligned} E_{\text{Cas},N}^{(2)} &= \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \left[\left(1 - e^{-2L\sqrt{k^2+m^2}} \right) (k^2 + m^2)^2 \right] \\ &= \int_0^\infty \frac{dk}{4\pi^2} k^2 \ln \left(1 - e^{-2L\sqrt{k^2+m^2}} \right) \\ &\quad + \int \frac{d^3 k}{(2\pi)^3} \ln(k^2 + m^2) = -\frac{m^2}{4\pi^2 L} \sum_{n=1}^{+\infty} \frac{K_2(2nmL)}{n^2} \\ &\quad + \text{divergent } L\text{-independent term.} \end{aligned} \quad (33)$$

Renormalizing here the Casimir energy in the usual way by subtracting its $L \rightarrow \infty$ counterpart, it is reassuring that we recover the well-known fact that the standard Neumann Casimir energy equals that of the Dirichlet case; see, e.g., Ref. [31]. This expression (33) has a nonsingular limit in the massless case, $\lim_{m \rightarrow 0} E_{\text{Cas},N}^{(1)} = \frac{-\pi^2}{1440L^3}$.

More interestingly, let us look at the new piece,

In the nontrivial ground state, in total we end up with

$$\begin{aligned} E_{\text{Cas},N} &= E_{\text{Cas},N}^{(1)} + E_{\text{Cas},N}^{(2)} = \frac{27}{2g^2 L^3} - \frac{\pi^2}{1440L^3} \Rightarrow F_{\text{Cas},N} \\ &= -\partial_L E_{\text{Cas},N} = \frac{81}{2g^2 L^4} - \frac{\pi^2}{480L^4}. \end{aligned} \quad (36)$$

Interestingly, for small values of g , the Casimir force is repulsive, while there is a critical $g_c = \frac{36\sqrt{15}}{\pi} \approx 44.4$ at which the Casimir force changes sign.

For the massive case, we will again choose $g = \frac{1}{2}$ and work in units $m = 1$, the behavior as reported below is generic for other parameter choices. We will also only focus on the real minima, as there are several other (complex) saddle points or local maxima of no interest to us. For $L < 1.6$, the trivial solution remains an absolute minimum, while (again shielded with an infinite energy barrier from the trivial minimum or unstable regions) another local minimum appears at $b_0 = \bar{b}_0 \approx 8.3$.

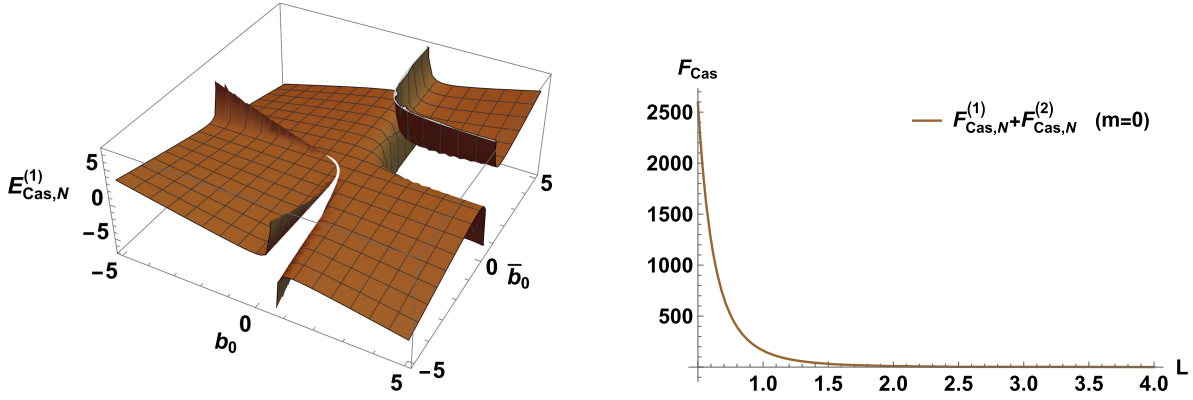


FIG. 1. Left: $E_{\text{Cas},N}^{(1)}$ from Eq. (35) for $g = \frac{1}{2}$, $L = 1$. Right: corresponding nonlinear Neumann Casimir force.

For larger L , $L > 1.6$, $(0, 0)$ remains a global minimum, while the other minimum evolves into a saddle point or local maximum if L grows further. Overall, $(0, 0)$ is thus a global minimum in a region that is shielded with infinite energy barriers from unstable areas.

III. NONLINEAR DIRICHLET BOUNDARY CONDITIONS

Let us also briefly discuss the nonlinear Dirichlet case. The analysis is largely similar to the Neumann case, by simply redefining $J(k, z)$ as follows:

$$J(\vec{k}, z) = (b_0(2\pi)^3 \delta^{(3)}(\vec{k}) + \beta(\vec{k}))\delta_+ + (\bar{b}_0(2\pi)^3 \delta^{(3)}(\vec{k}) + \bar{\beta}(\vec{k}))\delta_-, \quad (37)$$

which leads to a redefinition of the matrix $\mathcal{P}$, as follows:

$$\mathcal{P} = \begin{pmatrix} G(-L/2, -L/2) & G(-L/2, +L/2) \\ G(+L/2, -L/2) & G(+L/2, +L/2) \end{pmatrix}. \quad (38)$$

Eventually, we find

$$E_{\text{Cas},D}^{(1)} = -\frac{e^{2Lm}(2m(b_0^2 + \bar{b}_0^2) - b_0\bar{b}_0g(b_0 + \bar{b}_0)) + b_0\bar{b}_0g(b_0 + \bar{b}_0) + 4b_0\bar{b}_0me^{Lm}}{2(e^{2Lm}(gb_0 - 2m)(g\bar{b}_0 - 2m) - g^2b_0\bar{b}_0)}. \quad (39)$$

For $m \neq 0$, there is a nontrivial extremum at

$$b_0 = \bar{b}_0 = \frac{4me^{Lm}}{g(1 + e^{Lm})}. \quad (40)$$

The Hessian eigenvalues associated to Eq. (40) are given by $\{\frac{1+e^{-Lm}}{2m}, \frac{\sinh(Lm)}{m(-3+e^{Lm})}\}$. As such, Eq. (40) corresponds to a nonperturbative local minimum iff

$$L > \frac{\ln 3}{m}, \quad (41)$$

and a saddle point otherwise.

As such, in the Dirichlet case, new dynamics emerge only in the massive case, and for sufficiently large plate separation. Said otherwise, we have found a nontrivial stable absolute minimum, shielded from tunneling to other regions because of infinite energy walls. We note here that expression (39), just as its Neumann counterpart (34), does not explicitly depend on the spacetime dimensionality,

which can be easily seen from the expression (25) for what concerns the $\lim_{\vec{k} \rightarrow 0} (b_0\bar{b}_0) \cdot \mathcal{P} \cdot \begin{pmatrix} b_0 \\ \bar{b}_0 \end{pmatrix}$ contribution.

For completeness, we mention there is also a trivial maximum at

$$b_0 = \bar{b}_0 = 0. \quad (42)$$

The Hessian eigenvalues at $(0, 0)$ are given by $\frac{-1 \pm e^{-Lm}}{2m}$, so the trivial vacuum is unstable now. This is quite distinctive from the Neumann findings. Furthermore, there are also two saddle points at $(b_0, \bar{b}_0) = (\bar{b}_0, b_0) = (\frac{4me^{2Lm}}{g(-1+e^{Lm})}, \frac{4me^{Lm}}{g(-1+e^{Lm})})$. It can be verified that the Hessian eigenvalues always have opposite signs in both cases.

For $L < \frac{\ln 3}{m}$, there is nowhere a minimum to be found, only the local maximum and three (complex) saddle points. Higher order corrections to the Casimir energy in the coupling g are necessary to decide whether a minimum reappears. It would be most interesting to study this possible phase transition in terms of L in a truly dynamical setting by considering a (slowly) moving plate, but the dynamical Casimir effect, see, e.g., Ref. [2] and references therein, is another story and beyond the scope of this paper.

Returning to the novel minimum (40), under the condition (41), we eventually end up with a Casimir energy given by

$$E_{\text{Cas,D}} = \frac{e^{Lm}}{1 + e^{Lm}} \frac{m}{8g^2} - \frac{m^2}{4\pi^2 L} \sum_{n=1}^{+\infty} \frac{K_2(2nmL)}{n^2}, \quad (43)$$

where we note that for large separations L , the nonperturbative first piece will dominate.⁵ The Casimir force itself remains negative and thus repulsive in the Dirichlet case, even upon including the nonperturbative boundary mass effect, in the region $L < \frac{\ln 3}{m}$, we cannot make definite statements.

IV. CONCLUSIONS

We implemented a set of nonlinear boundary conditions for an otherwise free scalar field theory by means of auxiliary fields restricted to a lower-dimensional interface, in our case two infinite parallel plates. This problem is extremely relevant in itself since nonlinear boundary conditions arise in many contexts (such as chemical diffusion [32], nonlinear optics [33], vibration problems in nuclear reactor dynamics [34], etc.). As such, interactions are included on these interfaces, and we investigated the possibility that these affect the Casimir energy in a nonperturbative manner. We have given evidence, at least at leading order (mean field), that such is the case, both in the nonlinear Dirichlet and Neumann models. Although the computations are quite similar, the results are rather different. The most interesting finding is that in the Neumann case with massless scalars, two ground states are found with different energies (a trivial and a nonperturbative one), but separated from one another via infinite energy walls. In the nonperturbative ground state, the Casimir force can either be repulsive or attractive depending on the strength of the coupling. In the Dirichlet case, only for massive scalars, and for sufficiently large enough plate separation, a new nonperturbative ground state is found, still leading to an attractive Casimir force.

On the other hand, besides the intrinsic interest in analyzing the effect of nonlinear boundary conditions on

the Casimir energy, the most important motivation stems from the fact that, in the Yang-Mills case, the field strength is quadratic in the gauge potential. Since the boundary conditions for the Yang-Mills Casimir effect are linear in the field strength, they are thus quadratic in the gauge field itself. Therefore, our findings strongly motivate the investigation of whether similar nonperturbative boundary effects can emanate from the gauge-invariant YM boundary conditions, which are automatically nonlinear when written in terms of the field strength tensor. Hence, when extended to YM, our results might shed novel light on the recent series of lattice YM Casimir works [6–9], which suggest the existence of a new light mass scale. Possibly, an interesting interplay with the dynamical bulk gluon mass generation could arise in YM, where the bulk gluon mass can provide the (fitted) exponential decay of the Casimir energy in terms of the plate separation; see Refs. [6–9] and also Refs. [12,13] where such bulk mass effects were discussed from a continuum viewpoint.

In particular, the case of nonlinear Neumann boundary conditions analyzed in the present manuscript suggests the possibility of coexisting vacua (one trivial and the other nonperturbative), which can significantly affect the Yang-Mills Casimir energy. We will revisit this interesting issue in future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

⁵Strictly speaking, if we adopt the convention to define the Casimir energy subtraction so that $E_{\text{Cas,D}}(\infty) = 0$, we should subtract $\frac{m}{8g^2}$ from Eq. (43). This is anyhow irrelevant for the Casimir force itself.

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