

Holographic dual of defect conformal field theory with corner contributions

Xinyu Sun¹ and Shao-Kai Jian^{2,*}

¹*Institute for Advanced Study, Tsinghua University, Beijing 100084, China*

²*Department of Physics and Engineering Physics, Tulane University, New Orleans, Louisiana 70118, USA*



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We study defect conformal field theory (CFT) within the framework of holographic duality, emphasizing the impact of corner contributions. We model distinct conformal defects using interface branes that differ in tensions and are connected by a corner. Employing the relationship between CFT scaling dimensions and Euclidean gravity actions, we outline a general procedure for calculating the anomalous dimensions of defect changing operators at nontrivial cusps. Several analytical results are obtained, including the cusp anomalous dimensions at big and small angles. While $1/\phi$ universal divergence appears for small cusp angles due to the fusion of two defects, more interestingly, we uncover a bubble phase rendered by a near zero angle cusp, in which the divergence is absent.

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Introduction. Defect conformal field theory (CFT) appears across various disciplines in physics [1,2], ranging from Kondo effect [3,4] in solid-state physics to the description of D-branes in the string theory [5]. Fathoming defect CFT is not only of great importance but also has broad applications. Consider a general defect CFT on a circle S^1 , hosting two conformal defects, as illustrated in Fig. 1(a). Two CFT Hamiltonian, H_1 and H_2 , are defined on lattice sites with green and purple colors, respectively. The two defects of type a and b separating the bulk CFT are denoted by the red and blue bonds, respectively. The full Hamiltonian reads

$$H_{ab} = \sum_{r=1}^{L_1} h_1(r, r+1) + \sum_{r=L_1+1}^{L_1+L_2} h_2(r, r+1) + \kappa_a h_a(L_1 + L_2, 1) + \kappa_b h_b(L_1, L_1 + 1). \quad (1)$$

Here, h_1 and h_2 denote local terms in H_1 and H_2 defined in $r = 1, \dots, L_1$ and $r = L_1 + 1, \dots, L_1 + L_2$. h_a and h_b with the strength κ_a and κ_b are the local terms of defect type a and b , respectively. The total number of sites is $L = L_1 + L_2$, and $r + L = r$ due to the periodic boundary condition. The angle spanned by these two defects is $\phi = 2\pi L_1/L$.

In the low-energy limit, this lattice model under Renormalization Group (RG) will flow to a defect CFT.

*Contact author: sjian@tulane.edu

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The bulk CFT Hamiltonian H_1 and H_2 correspond to two CFTs separated by different conformal defects in the most general case. The ground state energy has the following form [6]:

$$E_{\text{GS}} = \epsilon_{\text{bulk}} L + \epsilon_{\text{defect}} + \frac{\Delta_{ab}^{12}(\phi)}{L} + \mathcal{O}(L^{-2}), \quad (2)$$

in which ϵ_{bulk} and ϵ_{defect} denote the bulk and defect energy density. The third term that is proportional to $1/L$ is universal, and it gives the scaling dimension, $\Delta_{ab}^{12}(\phi)$, of the defect changing operator with an additional cusp ϕ . While a partial list of known examples includes a few minimal models [7–19], the $O(N)$ model [20–26], and supersymmetric Yang-Mills models [27–33], the solution to this defect problem is relatively limited, even in 2D. For instance, folding of two minimal models is no longer a minimal model, and the conformal boundary condition [34,35] is not known in general.

In this paper, we propose a bottom-up holographic dual [36–39] to gain a general understanding. More explicitly,

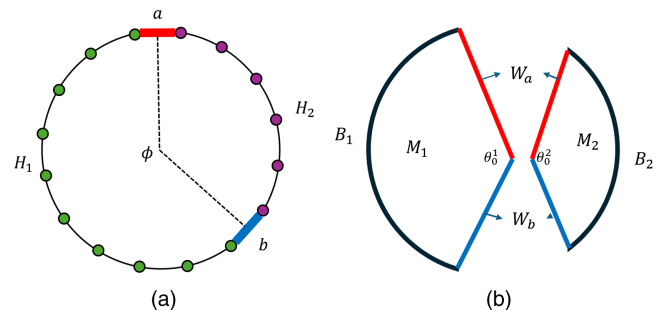


FIG. 1. Illustration of (a) the defect CFT model and (b) the holographic model. See the main text for detailed explanations.

the CFT is dual to a bulk gravity, and each defect is dual to an interface brane [40–43], as illustrated in Fig. 1(b). $B_{1,2}$ ($M_{1,2}$) denotes the asymptotic boundary (the bulk region) of the anti-de Sitter (AdS) gravity dual to the CFTs. Two branes $W_{a,b}$ are dual to the defects a, b , and they intersect at a corner with internal angles $\theta_0^{1,2}$ in region $M_{1,2}$. It should be noted that the corner angle $\theta_0^{1,2}$ is distinct from the cusp angle ϕ . Holographic models for defect CFT have been widely explored [44–55]. While similar setups either in a single CFT [51] or without a corner contribution [50] were considered, we obtain new results as summarized in the following. We outline a general procedure of solving the brane solution and getting $\Delta_{ab}^{12}(\phi)$. This gives a universal methodology to obtain arbitrary $\Delta_{ab}^{12}(\phi)$ numerically in holography. For some special cases, we can make simplifications to get analytical results. Specifically, we focus on (a) the defect changing operator between different conformal defects, $\Delta_{ab}^{11}(\pi)$, without a cusp, see Eq. (11), and (b) the cusp anomalous dimension for the same kind of defect between the distinct CFTs $\Delta_{aa}^{12}(\phi)$, see Eq. (12) [Eq. (15)] for a big (small) cusp angle, or the same CFTs with a nontrivial corner angle $\Delta_{ab}^{11}(\phi)$, see Eq. (16). More interestingly, we uncover a bubble-solution phase at near zero cusp angle, $\phi \approx 0$, where the fusion between defects is unconventional.

Holographic duality and general solutions. We consider a dual model of Eq. (1) described by 3D Euclidean gravity action,

$$I_{\text{tot}} = I_{\text{EH}} + I_T + I_{\text{surface}} + I_{\text{corner}}. \quad (3)$$

Here, $I_{\text{EH}} = -\frac{1}{2} \sum_{i=1,2} \int_{M_i} \sqrt{g_i} (R_i + \frac{2}{l_i^2})$ is the bulk Einstein-Hilbert action for the two AdS regions M_i with g_i, R_i , and l_i being the determinant of metric, the Ricci scalar, and the AdS radius, respectively. The boundaries are $\partial M_i = S_a^i + S_b^i + B_i$, where $S_{a,b}^i$ are the interface branes and B_i is an asymptotic boundary. $I_T = \sum_{a=a,b} \int_{W_a} \sqrt{\hat{g}_a} T_a$, and $I_{\text{surface}} = -\sum_{i=1,2} \int_{\partial M_i} \sqrt{\hat{g}_i} K_i$ describes the interface brane $W_{a,b}$, $W_{a,b} = S_{a,b}^1 = S_{a,b}^2$, with $\hat{g}$, K , and T denoting the induced metric, the external curvature, and the corresponding tension, respectively. The corner contribution joining two interface branes is described by

$$I_{\text{corner}} = -\sum_{i=1,2} \int_{S_a^i \cap S_b^i} \sqrt{\hat{g}} (\theta_0^i - \theta^i), \quad (4)$$

with θ_0^i being the angle spanning between the tangents of the two branes at the corner. Note that we omit the corner at the asymptotic boundary and a counterterm $I_{\text{c.t.}}$, the detail of which can be found in Supplemental Material [56].

Consider a gravity dual for CFT in a long strip of width L_i , the AdS bulk solution is

$$ds_i^2 = (r^2 - M_i l_i^2) d\tau^2 + \frac{l_i^2 dr^2}{r^2 - M_i l_i^2} + r^2 dx^2, \quad (5)$$

where $l_{1,2}$ is the radius for $\text{AdS}_{1,2}$ related to the central charge $c_{1,2} = \frac{3l_{1,2}}{2G_N}$, where G_N is the Newton constant, and M_i is the parameter, which is related to the width L_i . The Euclidean time is periodic, $\tau = \tau + T_{\text{DCFT}}^{-1}$, $T_{\text{DCFT}}^{-1} \gg L$. It is worth explaining the physical meaning of M_i . For a pure AdS geometry, $M_i < 0$ determines the perimeter in the compactified coordinate, $x = x + 2\pi/\sqrt{-M_i}$. Nevertheless, in our geometry shown in Fig. 1(b), two AdS geometries are joined via interface branes, so that the compactified coordinate is not fixed by solely M_1 or M_2 , but both of them. We will need to determine them according to the length $L_{1,2}$ in the dual CFTs.

The two interface branes dual to the defects are given by $[x(\sigma), r(\sigma)]$ at a constant time slice, with σ being a parameter. Two matching conditions are [50]

$$ds^2|_{S_a^1} = ds^2|_{S_a^2}, \quad (K^1 + K^2)_{\mu\nu} = T_a h_{\mu\nu}, \quad (6)$$

where $\alpha = a, b$. The first equation states that the metric on the brane is continuous. The second equation relates the tension of the brane T_a to the extrinsic curvatures $K^{1,2}$ from the two regions, with $h_{\mu\nu}$ being the induced metric. The second equation gives a constraint on the tension $T_{\min} \leq T \leq T_{\max}$ with $T_{\min} = |l_1^{-1} - l_2^{-1}|$, $T_{\max} = l_1^{-1} + l_2^{-1}$. With the corner term, the two interface branes shall not only satisfy the matching condition (6), but they will join at a tangent angle $\theta_0^{1,2}$. Although there are two corners in two AdS spacetime, we can prove that two angles are not independent, and in the following, we require that the angle in AdS_1 is $\theta_0 \equiv \theta_0^1$. The general procedure to proceed is to solve the brane trajectory with given $(l_{1,2}, T_{a,b}, M_{1,2}, \theta_0)$ and then match the interval $L_{1,2}$ for the two CFTs, which will then lead to an equation for $M_{1,2}$ and $L_{1,2}$.

Without loss of generality, we assume $l_1 < l_2$. After solving the equations of motion with matching conditions (6), with a proper coordinate $\sigma = r^2 - Ml^2$, we have the differential equations $\dot{x}_i^\alpha$ for branes trajectories with the tension T at each AdS spacetime, where the explicit form is given in Supplemental Material [56]. Here, $i = 1, 2$ label the brane in $\text{AdS}_{1,2}$, and $T_\alpha = T_{a,b}$ for different branes. With the condition that the angle at the corner of two branes in AdS_1 is θ_0 , we can solve the coordinate of the crossing point of two branes with $\sigma = \sigma_0$ [57]. Then an equation can be constructed to match $L_{1,2}$. To this end, it is convenient to define two dimensionless variables $\gamma = \frac{L_1}{L_2} = \frac{\phi}{2\pi - \phi}$ and $\mu = \frac{M_2}{M_1}$, then, we have

$$\gamma = \frac{I'_1 + \pi(1 - \text{sgn}(I'_1))}{I'_2 + \frac{\pi}{\sqrt{\mu}}(1 - \text{sgn}(I'_2))}, \quad (7)$$

where $I'_{1,2}$ correspond to two integrals that are related to the brane trajectory, whose expression can be found in Supplemental Material [56]. In the following, we will use γ and ϕ interchangeably. In general, Eq. (7) is a complicated integral function about μ and cannot be solved analytically. However, in some special cases, we can solve it to get $\mu(\gamma)$.

Now, let us discuss the effect of the corner term Eq. (4). In AdS/BCFT, it is known that a corner term is necessary when tensions are different $T_a \neq T_b$ and plays the role of boundary condition changing operator. In our model, a corner term plays the role of defect changing operator. Specifically, a nontrivial defect changing operator exists for $\theta_0 \neq \pi$. To extract the scaling dimension of defect changing operator, we evaluate the on shell action,

$$I_{\text{tot}} = \frac{M_1 l_1 L_1 + M_2 l_2 L_2}{2T_{\text{DCFT}}}. \quad (8)$$

Although I_{tot} seems to have a simple form, the complexity is hidden in $M_{1,2}$ as we need to express them as $(l_{1,2}, T_{a,b}, L_{1,2}, \theta_0)$ by solving Eq. (7). As the defect CFT is dual to the gravitational model with interface branes joined by a corner, this partition function reveals the information of $\Delta_{ab}^{12}(\phi)$. In particular,

$$I_{\text{tot}} = \frac{E_{ab}^{12}(\gamma)}{T_{\text{DCFT}} L}, \quad \Delta_{ab}^{12}(\phi) = E_{ab}^{12}(\gamma) - \frac{1}{2}(E_{aa}^{12} + E_{bb}^{12}), \quad (9)$$

in which $E_{aa}^{12} \equiv E_{aa}^{12}(1)$ is the eigenvalue at $\gamma = 1$. With the procedure outlined above, a general $\Delta_{ab}^{12}(\phi)$ can be solved numerically. Next, we will discuss several special cases with simpler analytical expressions.

Defect changing operator. Consider the defect changing operator between different conformal defects without a cusp in the same CFT: $\gamma = 1$ and $l_1 = l_2 = l$, the geometry has a $\mathbb{Z}_2$ symmetry, so we know $\mu = 1$ without solving Eq. (7). Plugging it into Eq. (8), we have $I_{\text{tot}} = -\frac{2l}{T_{\text{DCFT}} L} [\pi + \text{sgn}(\theta_0 - \pi)(\pi - \arccos \beta)]^2$ with

$$\beta = \frac{\cos \theta_0 + (lT_a/2)(lT_b/2)}{\sqrt{(1 - (lT_a/2)^2)(1 - (lT_b/2)^2)}}, \quad (10)$$

and consequently, $E_{ab}^{12}(\gamma = 1) = I_{\text{tot}} T_{\text{DCFT}} L$. After subtracting $E_{aa} = -2\pi^2 l$ for $T_a = T_b$ and $\theta_0 = \pi$, we arrive at

$$\Delta_{ab}^{11}(\pi) = 2l(\pi^2 - \arccos^2 \beta), \quad (11)$$

in which we require $\theta_0 \leq \pi$ to satisfy $\Delta_{ab} > \Delta_{aa} = 0$. Here, we briefly discuss the relation between our AdS results and the correspondence for the CFT. As mentioned above, the CFT central charge and the AdS radius are related by $l_{1,2} = \frac{2G_N}{3} c_{1,2}$. For the tension of brane, we can relate it to the boundary entropy of CFTs. In the simplest case,

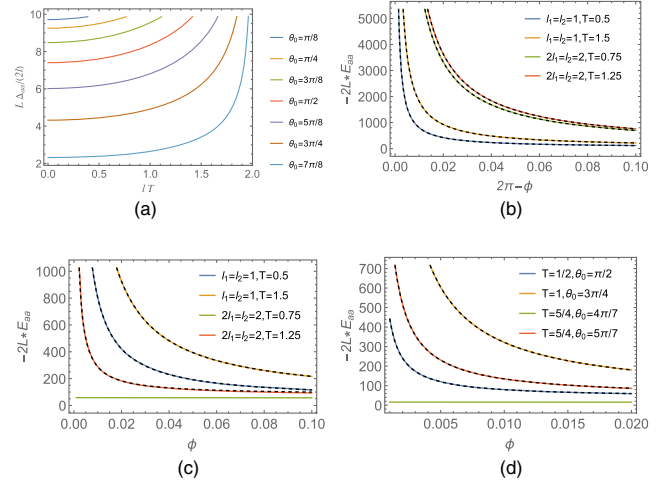


FIG. 2. (a) Scaling dimension of the defect changing operator with $l_1 = l_2 = l$ and $T_a = T_b = T$ with different θ_0 . (b),(c) On shell action for (b) $\phi \rightarrow 2\pi$ limit and (c) $\phi \rightarrow 0$ limit without corners. (d) On shell action for $\phi \rightarrow 0$ limit with a corner. The solid (dash) curves are numerical (analytical) results.

$l_1 = l_2 = l$, we have $S_{\text{bdy}} = \frac{c}{3} \tanh^{-1} \frac{lT}{2}$ [58], which gives $lT = 2 \tanh(3S_{\text{bdy}}/c) \in [0, 2]$. The expression of Eqs. (10) and (11) with CFT data S_{bdy} and c is shown in Supplemental Material [56].

To better understand the effect of the corner contribution, we set $T_a = T_b = T$ which leads to $\Delta_{ab}|_{T_a=T} = 2l[\pi^2 - \arccos^2(\frac{4\cos\theta_0 + (lT)^2}{4 - (lT)^2})]$ with the constraint $0 < T < T_c = 2 \sin(\theta_0/2)$. Note that, although $T_a = T_b$, it still corresponds to two different defects due to a nontrivial θ_0 . The scaling dimension of the defect changing operator is plotted in Fig. 2(a). A greater tension T and a larger deviation from $\theta_0 = \pi$ will lead to a greater scaling dimension of defect changing operators. Specially, we have $\Delta_{ab}|_{T_a=T_c} = 2l\pi^2$ and $\Delta_{ab}|_{T_a=0} = 2l(\pi^2 - \theta_0^2)$.

Cusp anomalous dimension. Apart from the defect changing operator given above, interesting conformal data exist even for the same defect: the cusp anomalous dimension. The same types of defects means $T_a = T_b = T$ and $\theta_0 = \pi$. Then we look at a big angle $\phi \sim \pi$ and small angle $\phi \sim 0, 2\pi$, respectively. Consider $\phi \sim \pi$, $\gamma = 1 + \delta_\gamma$, and $\delta_\gamma \ll 1$. We can check $\mu = 1$ is a solution for $\gamma = 1$. Therefore, we can expand $\mu = 1 + \delta_\mu$ with $\delta_\mu \ll 1$, to get $\delta_\mu = \frac{2l_1 l_2 T}{l_1 + l_2 + l_1 l_2 T} \delta_\gamma + \mathcal{O}(\delta_\gamma^2)$, where the expression of the second order $\mathcal{O}(\delta_\gamma^2)$ is given in Supplemental Material [56]. With this solution and Eq. (8), the cusp anomalous dimension near $\phi = \pi$ reads

$$\Delta_{aa}^{12}(\phi) = -l_1 \left[\pi \left(\frac{l_2}{l_1} - 1 \right) \delta_\phi + \frac{2(T - T_{\min})}{T + T_{\max}} \delta_\phi^2 \right], \quad (12)$$

where $\delta_\phi = \phi - \pi$. When $l_1 = l_2 = l$ with a $\mathbb{Z}_2$ symmetry, the linear term in Eq. (12) vanishes, and the quadratic term is negative, consistent with reflection positivity [25,59,60].

Next, consider the cusp anomalous dimension near $\phi \rightarrow 0$ or $\phi \rightarrow 2\pi$ for the same defect: $T_a = T_b = T$ and $\theta_0 = \pi$. Note that these two limits are different when the two CFTs are distinct $l_1 \neq l_2$. As $\phi \rightarrow 0$ implies $\gamma \rightarrow 0$, we simplify Eq. (7) as $\sqrt{\mu} = \frac{2\pi}{B(T, l_1, l_2)}\gamma + \mathcal{O}(\gamma^2)$, where $B(T, l_1, l_2) = 2\pi\Theta(l_2 T - 1) - \Xi_1(T, l_1, l_2)$ with

$$\Xi_{u_0}(T, l_1, l_2) = \sqrt{\frac{4v}{-u}} \left[K(u_0, v) + \frac{l_1^2 l_2^2 T^2 - l_1^2}{l_2^2} \Pi(u_0, u, v) \right], \quad (13)$$

and $u_0 = 1$. We define $v = \frac{(T-T_{\max})(T+T_{\min})}{(T+T_{\max})(T-T_{\min})}$, $u = l_1^2(T_{\max} - T)(T + T_{\min})$, $\Theta(\cdot)$ is the step function, and $K(\cdot, \cdot)$ ($\Pi(\cdot, \cdot, \cdot)$) is the incomplete elliptic integral of the first (third) kind. With this solution and Eq. (8), we can obtain the on shell action explicitly,

$$I_{\text{tot}} = -\frac{1}{2T_{\text{DCFT}}L} \left(\frac{l \cdot B(T, l_1, l_2)^2}{\gamma} + \mathcal{O}(1) \right). \quad (14)$$

The other limit, $\phi \rightarrow 2\pi$, implies instead $\gamma \rightarrow \infty$, which can be solved similarly. With Eq. (9), the cusp anomalous dimension is given by

$$\Delta_{aa}^{12}(\phi) = \begin{cases} -\frac{l_2 B(T, l_2, l_1)^2}{2} \frac{2\pi}{2\pi - \phi} & \phi \rightarrow 2\pi \\ -\frac{l_1 B(T, l_1, l_2)^2}{2} \frac{2\pi}{\phi} & \phi \rightarrow 0 \end{cases}. \quad (15)$$

The cusp anomalous dimension diverges as $1/\phi$ or $1/(2\pi - \phi)$, under the limit [25,61]. On the CFT side, the two defects fuse into a trivial defect. The divergent behavior can be interpreted the Casimir energy, and the prefactor is the fusion coefficient. It is positive or negative for repulsive and attractive interaction. We plot the numerical results obtained by our general procedure and analytical results in Figs. 2(b) and 2(c), showing the limit $\phi \rightarrow 2\pi, 0$, respectively. For $\phi \rightarrow 2\pi$, the two results, i.e., numerical (analytical) results shown by solid (dashed) curves, agree quite well in Fig. 2(b). However, for $\phi \rightarrow 0$, some results agree, but some do not, for instance, the green solid curve with $2l_1 = l_2 = 2$, $T = 0.75$ does not diverge, unlike the analytical solution. The reason is that there can emerge a bubble solution in the limit $\phi \rightarrow 0$, as is discussed next.

Bubble phase and exotic fusion. As mentioned above, for $\phi \rightarrow 0$, two defects fuse into one, and lead to a divergent behavior. However, it is not always the case. In the following, we consider more general solutions and also, discuss the role of the corner contribution with $\theta_0 \neq \pi$.

First, we consider the same defect, $T_a = T_b$, $l_1 \neq l_2$, and $\theta_0 = \pi$. Given $B(T, l_1, l_2) > 0$, the solution

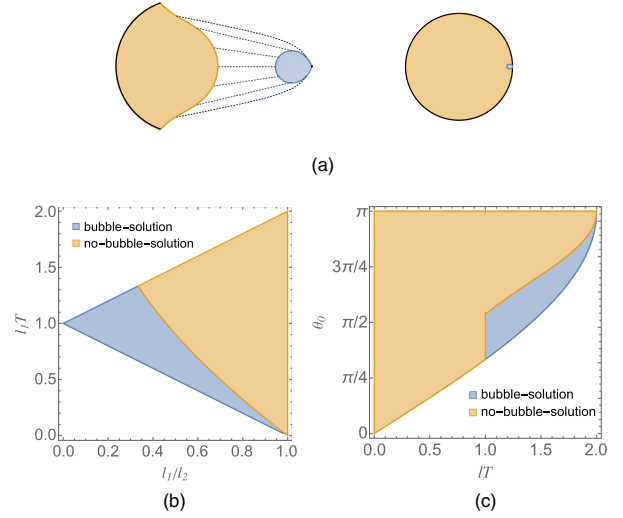


FIG. 3. (a) Configurations of the bubble-solution phase on the left panel and no-bubble-solution phase on the right panel, where the blue (orange) region illustrates $\text{AdS}_{1(2)}$. (b) Phase diagram from small cusp angles $\phi \sim 0$ for $\theta_0 = \pi$. (c) Phase diagram from small cusp angles $\phi \sim 0$ with a corner.

$\sqrt{\mu} = \frac{2\pi}{B(T, l_1, l_2)}\gamma$ becomes $2\pi L_1/L_2 = B(T, l_1, l_2)\sqrt{M_2/M_1}$. It shows that when the length of one CFT shrinks to zero, $L_1 \ll L_2$, its dual AdS also shrinks to zero $M_1 \gg M_2$. A schematic plot of the solution is given in Fig. 3(a) right panel. However, if $B(T, l_1, l_2) \rightarrow 0$, that the length of one CFT shrinks to zero, $L_1 \ll L_2$, does not imply its dual AdS shrinks to zero. Instead, we find that at $B(T, l_1, l_2) = 0$, the solution of Eq. (7) at $\gamma \rightarrow 0$ is given by a finite $M_2/M_1 \rightarrow \mu_0 > 0$. A schematic plot of the two geometries given in Fig. 3(a), showing a qualitative difference between the two solutions. The one with $M_2/M_1 \rightarrow 0$ is termed as the no-bubble-solution phase, and the one with $M_2/M_1 \rightarrow \mu_0 > 0$ is termed as the bubble-solution phase. Their phase boundary is precisely given by $B(T, l_1, l_2) = 0$. We can show that $B(T, l_1, l_2) = 0$ is possible only when $l_1 < l_2$. It means that the dual of CFTs with a greater central charge can support a bubble of CFTs with a smaller central charge, but the opposite cannot occur. With independent parameters $(l_1/l_2, T)$, the phase diagram is plotted in Fig. 3(b).

We discuss the implication for the cusp anomalous dimension. For $B(T, l_1, l_2) > 0$, namely, in the no-bubble-solution phase, the cusp anomalous dimension given in Eq. (15) is valid. However, in the bubble-solution phase, the solution with $\mu = \mu_0$ renders a finite on shell action without the $1/\phi$ divergence, in contrast to Eq. (14). Therefore, we expect the existence of the bubble-solution phase corresponds to an exotic fusion of two defects. We are ready to explain Figs. 2(b) and 2(c). For parameters located in the no-bubble-solution phase, the numerical and analytical results agree, whereas for parameters located in the bubble-solution phase, specially, for $l_2 = 2l_1 = 2$, $T = 0.75$, the cusp anomalous dimension converges at $\phi \rightarrow 0$ shown by the solid curve.

Next, we consider a more general case with a corner contribution $\theta_0 \neq \pi$ that indicates a nontrivial defect changing operator. We further assume $l_1 = l_2 = l$ and $T_a = T_b = T$. Then, similar to the previous discussion, In the limit $\phi \rightarrow 0$, or equivalently $\gamma \rightarrow 0$, we obtain $\sqrt{\mu} = \frac{2\pi}{\tilde{B}(T, l, \theta_0)}\gamma + \mathcal{O}(\gamma^2)$, where $\tilde{B}(T, l, \theta_0)$ is defined in Supplemental Material [56]. For $\theta_0 < \pi$ and $T > 1$, it can be simplified to $\tilde{B}(T, l, \theta_0) = 2\pi - 2\Xi_1(T, l, l) + \Xi_{u_0}(T, l, l)$, where $u_0 = \left[\frac{l^2 T^2 + 2 \cos \theta_0 - 2}{(lT - 2)(\sqrt{(l^2 T^2 + l^2 T^2 \cos \theta_0 - 2)(\cos \theta_0 - 1)} - lT \cos \theta_0)} \right]^{1/2}$. Again, $\tilde{B}(T, l, \theta_0) = 0$ defines the phase transition between the no-bubble-solution phase and the bubble-solution phase. With $lT \in [0, 2]$ and $\cos \theta_0 \in [-1, 1 - (lT)^2/2]$ which is required to have a solution, the phase diagram is shown in Fig. 3(c). From the phase diagram, we can find that there is no bubble-solution phase for $lT < 1$.

The asymptotic behavior of the cusp anomalous dimension for $\phi \rightarrow 0$ in the no-bubble-solution phase is

$$\Delta_{ab}(\phi) = -\frac{l\tilde{B}(T, l, \theta_0)^2 2\pi}{2\phi}, \quad (16)$$

which has a $1/\phi$ singularity from the Casimir energy. Finally, we compare the numerical and analytical results of the anomalous dimension and plot them in Fig. 2(d).

It is illustrative to consider $lT = 1$. In this case, $\tilde{B}(1/l, l, \theta_0) = \theta_0 - \theta_0^c$ with $\theta_0^c = \frac{2}{\sqrt{3}} \cdot K(-\frac{1}{3})$. Hence, the bubble-solution phase and the no-bubble-solution phase exist for $\theta_0 \in (\frac{\pi}{3}, \theta_0^c)$ and $\theta_0 \in (\theta_0^c, \frac{5\pi}{3})$, respectively.

Here, the exotic fusion means the emergence of the bubble phase in the limit of $\phi \rightarrow 0, 2\pi$ [50]. This leads to the absence of the divergent Casimir energy when $\phi \rightarrow 0, 2\pi$. There is a trivial example of bubble-solution phase in Ref. [25]. Consider a $d = 4$ free scalar field with two pinning field defects $\vec{h}_1$ and $\vec{h}_2$. When the pinning fields, $\vec{h}_1$ and $\vec{h}_2$, are perpendicular to each other, the cusp anomalous dimension will not diverge for $\phi \rightarrow 0$. Also, notice that if $\vec{h}_1 = \vec{h}_2$, the cusp anomalous dimension will always diverge. This is consistent with the phase diagram: for $l_1 = l_2$ without a corner term $\theta_0 = \pi$, there is no exotic fusion. To realize the exotic fusion with $l_1 = l_2$, the corner term is necessary, and two defects must be different, even if they lead to the same tension of branes, as shown in Figs. 3(b) and 3(c).

Concluding remarks. To illustrate the connection between our results and concrete CFT models, we consider the defect Ising model $H = -\frac{1}{2} \sum_r [\sigma^z(r) + \sigma^x(r)\sigma^x(r+1)] - \frac{1}{2} \kappa_a \sigma^x(L_1 + L_2) \sigma^x(1) - \frac{1}{2} \kappa_b \sigma^x(L_1) \sigma^x(L_1 + 1)$, where $\sigma^\mu(r)$ denotes the Pauli matrix at site r and the first summation omits the terms at the two defect bonds. We calculate the defect changing operator at $\phi = \pi$ (see Supplemental Material [56]),

$$\Delta_{ab}^{11}(\pi) = \frac{1}{2\pi^2} (\arctan \kappa_a - (-1)^Q \arctan \kappa_b)^2, \quad (17)$$

where $Q = 0$ (1) denotes the symmetric (antisymmetric) sector. We observe that in the $Q = 1$ sector, when $\kappa_a = \kappa_b = \kappa$, $\Delta_{ab}^{11}(\pi) = \frac{4}{\pi} \arctan^2 \kappa$ still nontrivially depends on κ . Note that this happens in the holographic model when $\theta_0 \neq \pi$. Hence, the effect of a nontrivial corner term is similar to the different sector $Q = 0, 1$. Further investigation on the corner angle θ_0 on the CFT side is important.

We also consider the cusp anomalous dimension of the same types of defects in a weak defect limit (see Supplemental Material [56]),

$$\Delta_{aa}(\phi) = -\frac{1}{2} \frac{1}{2\pi^2} \left(\frac{\pi - \phi}{\sin \phi} - 1 \right) \delta \kappa^2, \quad \kappa = 1 - \delta \kappa. \quad (18)$$

It is quadratic near $\phi \sim \pi$ and $1/\phi$ divergent near $\phi \sim 0$ both with a negative prefactor, consistent with our result in the no-bubble-solution phase.

In the exotic fusion, if two defects are the same with $\theta_0 = \pi$, the bubble region will have a smaller AdS radius, and act as a true vacuum [50]. It is exotic because interface and anti-interface do not fuse to a trivial identity. More interestingly, we find that a nontrivial corner term ($\theta_0 \neq \pi$) can also support a bubble solution even with $l_1 = l_2$ and $T_a = T_b$. The exotic fusion on the CFT side is also an interesting open question.

Another future direction is to investigate the role of black holes in this context. If a black hole is present in the bulk, the situation becomes more intricate. As discussed in Ref. [50], there are no-go theorems that prohibit certain pairings between space-time slices, but it may be evaded in the presence of corner terms. For example, with a corner term, the pairing [H1, H1] may become viable because the brane can avoid the horizon. Furthermore, in the presence of corner terms, the classification of space-time phases with or without a horizon becomes more subtle. For instance, a corner term can connect two different branes, blurring the distinctions between phases previously classified in Ref. [50].

We can also generalize this framework to other settings. For $\text{AdS}_3/\text{CFT}_2$ without the time-translation symmetry—for example, quenched systems or CFTs with measurements—our method and discussion may still apply. Here, the defect is aligned along a spatial direction on a fixed time slice. Additionally, for higher dimensions, there are more varieties of defect changing operators, and whether a “bubble phase” or “exotic fusion” occurs in these cases is an open question. The fusion of two higher-dimensional defects, such as two line operators, is a particularly interesting direction for future research. Furthermore, we can consider adding degrees of freedom on branes

[36,62–64] or defects [65]. This may affect, for example, the cusp anomalous dimension in dramatic ways.

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