

Four-point correlator of planar supersymmetric Yang-Mills theory at twelve loops

Jacob L. Bourjaily^{1,*}, Song He (何颂)^{1,†}, Canxin Shi (施灿欣)^{2,‡} and Yichao Tang (唐一朝)^{2,4,§}

¹*Institute for Gravitation and the Cosmos, Department of Physics, Pennsylvania State University, University Park, Pennsylvania 16802, USA*

²*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

³*School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, Hangzhou, Zhejiang 310024, China*

⁴*School of Physical Sciences, University of Chinese Academy of Sciences, No. 19A Yuquan Road, Beijing 100049, China*



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We determine the four-point correlation function and amplitude in planar, maximally supersymmetric Yang-Mills theory to 12 loops. We find that the recently introduced “double-triangle” rule in fact implies the previously described square and pentagon rules; and when applied to 12 loops, it fully determines the 11-loop correlator and fixes all but 3 of the (619,981,403) 12-loop coefficients; these remaining coefficients can be subsequently fixed using the “(single-)triangle” rule. Not only do we confirm the Catalan conjecture for antiprism graphs, but we discover evidence for a greatly *generalized Catalan conjecture* for the coefficients of all *polygon-framed fishnet* graphs. We provide all contributions through 12 loops as Supplemental Material to this work.

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I. INTRODUCTION

Much of the recent progress in our understanding of perturbative quantum field theory has resulted from the discovery of remarkable new structures within the “theoretical data” resulting from hard computations; such discoveries have led to many deep insights and fueled the development of powerful new tools for computation—extending our theoretical reach to further discovery.

The four-point amplitude in the planar limit of maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory (sYM) has long served as an important benchmark (among many) in our perturbative reach. This amplitude was first determined at the integrand level via generalized unitarity through six loops [1–6], to eight loops using the “soft-collinear bootstrap” in [7,8]. In [9] a set of graphical rules (exploiting the correspondence between this amplitude and correlation functions) was described and used to determine

the amplitude through ten loops; more recently, a new graphical rule was described by [10] which brought this benchmark to 11 loops.

Although the only “observable” associated with this amplitude is the cusp anomalous dimension (well known to high orders and at strong coupling via integrability [11,12]), studying this object at the *integrand* level has led directly to developments of generalized unitarity [13–19], on-shell recursion relations [20–22], the discovery of dual conformal invariance [23–27], and Yangian invariance [28] of sYM, along with much more. Moreover, by taking higher-point lightlike limits, it is known that this one particular function captures *complete* perturbative information about *all* n -point scattering amplitudes in sYM [29–38] (see also [39–43]).

In this work, we argue that the “double-triangle” rule of [10] in fact implies the “square” and “pentagon” rules described in [9], and we use this together with the “triangle” rule of [9] to determine the integrand of correlator/amplitude to 12 loops. This new graphical rule relates ℓ -loop contributions to those at $(\ell - 1)$ loops, fixing all contributions at $(\ell - 1)$ loops completely as we confirm at $\ell = 12$. In particular, our result shows concretely that the triangle rule used in [10] to fix the 11-loop ansatz was in fact unnecessary. A by-product is an independent confirmation of the so-called Catalan conjecture for the *antiprism* graphs which have the largest coefficients, -42 in the 12-loop case. Moreover, our results provide evidence for a

*Contact author: bourjaily@psu.edu

†Contact author: songhe@itp.ac.cn

‡Contact author: shicanxin@itp.ac.cn

§Contact author: tangyichao@itp.ac.cn

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much more general conjecture, which predicts coefficients of certain infinite families of graphs as common generalizations of antiprism graphs and f -graphs for the so-called *fishnet* integrals [44]. Without such high-loop empirical “data,” it is hard to imagine such structure being anticipated or discovered.

II. SUMMARY OF THE DOUBLE-TRIANGLE RULE

Consider the connected four-point correlation function $\mathcal{G} := \langle \mathcal{O}(x_1) \bar{\mathcal{O}}(x_2) \mathcal{O}(x_3) \bar{\mathcal{O}}(x_4) \rangle$ of the lightest half-BPS operators $\mathcal{O}(x) := \text{tr}(\varphi(x)^2)$ in sYM. Perturbatively, loop corrections of $\mathcal{G}$ can be computed using Lagrangian insertions [35]: the ℓ -loop *integrand* is given by the *Born-level* correlator with ℓ chiral Lagrangian insertions $x_{i=5, \dots, 4+\ell}$. Normalized with an overall ℓ -independent factor, the resulting integrand is a rational function $\mathcal{F}^{(\ell)}(x_1, \dots, x_{4+\ell})$ with uniform conformal weight -4 in all $(4 + \ell)$ points enjoying complete $\mathfrak{S}_{4+\ell}$ permutation symmetry among its arguments [36]. Permutation invariance suggests that we describe this function in terms of unlabeled graphs and expand $\mathcal{F}^{(\ell)}$ into a basis of ℓ -loop “ f -graphs” constructed as rational products of edge factors. The space of inequivalent f -graphs can be easily constructed, allowing us to express $\mathcal{F}^{(\ell)} =: \sum_i c_i^\ell f_i^{(\ell)}$.

Each f -graph corresponds to a permutation-invariant rational function constructed from a graph $\Gamma(f)$ involving $(4 + \ell)$ vertices, with uniform conformal weight -4 in each. Letting $\Gamma_{\text{n,d}}(f)$ denote the subgraphs associated with the numerator and denominator, respectively, we define

$$f = \frac{\prod_{i \bullet \bullet j \in \Gamma_{\text{n}}(f)} x_{ij}^2}{\prod_{i \bullet \bullet j \in \Gamma_{\text{d}}(f)} x_{ij}^2} + \left(\frac{\text{permutations } \mathfrak{S}_{4+\ell}}{\text{mod Aut}(\Gamma(f))} \right). \quad (1)$$

That is, we may consider f -graph corresponds to an unlabeled graph $\Gamma(f)$ with $(4 + \ell)$ vertices, and solid (respectively, dashed) edges representing denominators $\Gamma_{\text{d}}(f)$ [respectively, numerators $\Gamma_{\text{n}}(f)$]. For example, the only planar (referring to the denominator’s subgraph) f -graph at $\ell = 3$ is

$$\begin{aligned} & \text{Diagram of a planar } f\text{-graph at } \ell=3 \\ & \Leftrightarrow \underbrace{\frac{x_{12}^2}{\prod_{i=3}^7 x_{1i}^2 x_{2i}^2 x_{i,i+1}^2}}_{252 \text{ distinct terms}} + \left(\frac{\text{inequivalent}}{\text{perms.}} \right), \end{aligned} \quad (2)$$

which has $|\text{Aut}(\Gamma(f))| = 20$. We are interested in the planar limit of sYM, where only planar f -graphs contribute.

Universal divergences of the correlator under physical limits impose constraints which relate $\mathcal{F}^{(\ell)}$ and $\mathcal{F}^{(\ell-1)}$; they translate nicely into graphical rules which we use to bootstrap the f -graph coefficients. The most familiar example is the *triangle rule* [9] from the log x_{12}^2 divergence in the OPE limit $x_2 \rightarrow x_1$ [37]. A more powerful constraint was discovered in [10] arising from the Sudakov log $x_{12}^2 \log x_{23}^2$ divergence in the double lightlike limit $x_{12}^2, x_{23}^2 \rightarrow 0$, which leads to the *double-triangle rule*, $\mathcal{P}(\mathcal{F}^{(\ell)}) = \mathcal{F}^{(\ell-1)}$ with the pinching operation $\mathcal{P}$ acting on all double-triangle structures:

$$\mathcal{P}: \left(\text{Diagram of a double-triangle structure} \right) \Rightarrow \left(\text{Diagram of a single-triangle structure} \right). \quad (3)$$

A. Square and pentagon rules are implied by the double-triangle rule

The “square rule” of [9] generalizes the “rung rule” of [1] and can be derived from the consistency of the term $\mathcal{A}_1 \mathcal{A}_{\ell-1} \subset \mathcal{A}^2_\ell$ arising from the perturbative expansion of the square of the amplitude (as determined from the correlator). As described in [9], it dictates equality between the coefficients of f -graphs that are related via

$$\left(\text{Diagram of a square structure} \right) \Leftrightarrow \left(\text{Diagram of a pentagon structure} \right). \quad (4)$$

It is easy to see this as a special case of the double-triangle rule, as the left-hand side is the unique preimage of the resulting lower-loop graph under $\mathcal{P}$ (3), resulting in the necessary equality between the two coefficients.

While powerful, there are many graphs not susceptible to the square rule. Taking inspiration from the square rule, but taking a five-point lightlike limit resulted in the somewhat peculiar “pentagon” rule described in [9] (to which we refer the reader for details). Essentially, the pentagon rule dictated that the sum of coefficients of a collection of f -graphs sharing a peculiar subtopology must vanish. In light of the double-triangle rule, it is easy to now see that the pentagon rule also follows as a special case of the double-triangle rule:

$$\left\{ \left(\text{Diagram 1} \right), \left(\text{Diagram 2} \right) \right\} \Rightarrow \left(\text{Diagram 3} \right). \quad (5)$$

The collection of terms appearing on the left-hand side of (5) are precisely those appearing as preimages under $\mathcal{P}$ (3) of the nonplanar graph on the right-hand-side. Because the

image is nonplanar, the double-triangle rule dictates that the sum of coefficients must vanish.

III. DETERMINING THE 12-LOOP CORRELATOR/AMPLITUDE

The computation of the 12-loop amplitude and correlator via the graphical bootstrap proceeded in four steps: (1) generating all 12-loop f -graphs; (2) imposing the double-triangle rule to obtain bootstrap equations; (3) solving the bootstrap equations to obtain all visible 12-loop coefficients; and (4) using the triangle rule to fix the very few (three) remaining coefficients.

To generate all f -graphs at 12 loops, we started with all possible denominators, or planar graphs with 16 vertices and minimal valency 4, which was generated using PLANTRI [45] ignoring different embeddings. From these, it is straightforward to find all possible numerator “decorations” which give a valid f -graph [one of conformal weight (-4) in every vertex]. Some planar graphs admit many numerators: at 12 loops, there is one admitting 213,082 graphically distinct numerators.

In order to impose the double-triangle rule relating the ℓ -loop and $(\ell - 1)$ -loop coefficients, we first identify all possible ways of highlighting a double-triangle subgraph in an ℓ -loop f -graph $f_i^{(\ell)}$. The pinching operation takes the double-triangle-highlighted f -graph $f_i^{(\ell)\triangleleft\triangleright}$ to a cusp-highlighted f -graph with one fewer vertex, $f^{(\ell-1)\vee}$. Including symmetry factors, the sum of preimages for each cusp-highlighted, pinched graph must equal the coefficient of the lower-loop graph (zero if the pinched graph is not a planar or has double poles). It is worth noting that applying these rules to each graph is easily parallelized: the double-triangle rule can be applied to one graph at a time. This was done using the high-performance computing cluster of ITP-CAS, requiring approximately 22,200 core hours to complete, resulting in 10,315,532,348 bootstrap equations (with many duplications). Although the number of equations is large, they are extremely sparse and easy to solve sequentially. The detailed method is described in an earlier article [10]. It required 3 days to solve the resulting equations (albeit at a cost of 512 gigabytes of local memory).

The resulting equations suffice to completely determine the 11-loop correlator/amplitude, and leave merely 3 undetermined 12-loop coefficients among the 619,981,403 12-loop f -graphs; these remaining coefficients were determined by application of the triangle rule. Specifically, in [10] it was observed that the $\ell \rightarrow (\ell - 1)$ -loop bootstrap equations appeared to fully determine both all $(\ell - 1)$ -loop coefficients as well as all *visible* ℓ -loop coefficients—those with double-triangle subtopologies among their denominators. Only one 11-loop graph is invisible, and only 2 at 12 loops; these are shown in Fig. 1. Although we confirm that the $12 \rightarrow 11$ -loop bootstrap

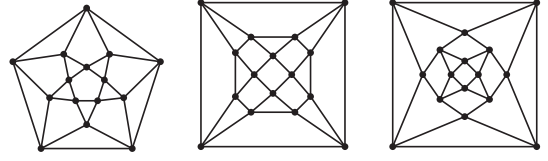


FIG. 1. All 11-, 12-loop f -graphs without double-triangles—and hence “invisible” to double-triangle rule at the corresponding loop order. Interestingly, the 11-loop graph’s coefficient is determined by the $12 \rightarrow 11$ -loop double-triangle rule.

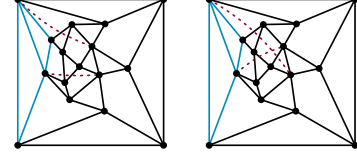


FIG. 2. 12-loop f -graphs whose coefficients are related but *not* determined by the $12 \rightarrow 11$ -loop double-triangle rule.

equations suffice to fully determine all 11-loop coefficients (including the “invisible” graph’s), these equations did *not* fully determine all “visible” 12-loop graphs’ coefficients. The one exception is a pair of “next-to-invisible” graphs whose coefficients are related (but not fixed) by the double-triangle rule. These are shown in Fig. 2. These 3 coefficients were subsequently determined using the (single-) triangle rule.

It is interesting to compare these results to other possible methods. Using on-shell recursion [22,46–48], the 12-loop amplitude would require $> 6 \times 10^{15}$ individual expressions [49]; and using unitarity with a basis of (dihedrally symmetrized,) dual-conformally invariant master integrals, there would be 19,117,648,284 coefficients to determine. These are summarized in Table I.

We have included in Supplemental Material [50] all contributions to the amplitude and correlation function through $\ell = 12$ loops [51].

A. Statistics of coefficients

We have given the statistics of f -graph coefficients up to $\ell = 12$ in Table II. These coefficients are in line with previous conjectures and observations. For example, half-integer coefficients appear for $\ell \geq 8$, multiples of $\frac{1}{4}$ at $\ell \geq 10$, and multiples of $\frac{1}{8}$ for $\ell \geq 12$; and the coefficients of antiprism graphs are given by (signed) Catalan numbers at even loop-orders.

Several comments are in order. First, the number of nonvanishing coefficients decreases rapidly with ℓ : for $\ell = 8, \dots, 12$, only about 39%, 24%, 15%, 9.3%, and 5.7% of coefficients are nonzero. Moreover, most coefficients are concentrated within a very narrow ranges: e.g. for $\ell = 10, 11, 12$ only 0.1%, 0.04%, and 0.02% lie beyond the range $[-1, 1]$; at 12 loops, only 4×10^{-6} fall outside $[-2, 2]$ and a mere 2×10^{-8} outside $[-5, 5]$. From the data, it is clear that

TABLE I. Numbers of terms required to represent the ℓ -loop amplitude via on-shell recursion, in terms of (dihedrally symmetrized) dual-conformally invariant (DCI) master integrals, or f -graphs—and how many have nonvanishing coefficients.

	$\ell = 1$	2	3	4	5	6	7	8	9	10	11	12
Recursed cells:	1	10	146	2,684	59,914	1,329,324	33,291,164	878,836,728	24,175,924,094	687,444,432,396	20,086,271,785,340	600,384,612,445,304
DCI integrals:	1	1	2	8	34	278	3,125	49,935	981,984	23,045,474	623,496,933	19,117,648,284
(Contributing:)	1	1	2	8	34	224	1,818	19,198	236,823	3,412,129	56,145,999	1,049,691,130
f -graphs:	1	1	1	3	7	36	220	2,707	42,979	898,353	22,024,902	619,981,403
(Contributing:)	1	1	1	3	7	26	127	1,060	10,525	136,433	2,048,262	35,503,735

TABLE II. Numbers of f -graphs contributing with each distinct coefficient for $\ell \leq 12$. The vertical line is not to scale, but correctly ordered so as to highlight how new coefficients arise between gaps separating lower-loop coefficients.

coeff	$\ell=1$	2	3	4	5	6	7	8	9	10	11	12
+57/8										1	1	9
+33/8										1	15	194
+17/4												2
+4										1	8	88
+15/4												2
+7/2												3
+27/8												1
+13/4												5
+25/8												3
+3											3	33
+23/8												5
+11/4											1	30
+21/8												6
+5/2										3	24	338
+19/8										1	13	280
+9/4												6
+17/8						1	1	8	40	306	2,631	26,524
+15/8										5	75	1,355
+7/4												70
+13/8									1	61	1,368	23,703
+3/2										8	240	5,796
+11/8												314
+5/4												2,985
+9/8	1	1	1	2	5	15	70	472	4,013	39,649	422,353	4,715,081
+7/8										278	9,830	247,553
+3/4												6,618
+5/8								78	1,289	25,603	448,236	7,496,410
+1/2										2,717	141,343	4,766,337
+3/8												428,464
+1/4												428,118
0						10	93	1,647	32,454	761,920	19,976,640	584,477,668
-1/8										2,714	141,393	4,769,077
-1/4												41,252
-3/8								63	1,240	25,030	444,432	7,467,462
-1/2										281	9,834	247,461
-5/8												2,938
-3/4												285
-7/8				1	2	10	56	434	3,906	39,300	421,022	4,709,533
-9/8										10	240	5,852
-5/4												70
-11/8								3	21	268	2,896	34,040
-3/2											15	657
-13/8												20
-7/4												29
-15/8								1	14	184	2,181	25,182
-17/8										3	41	544
-9/4												8
-19/8										1	21	359
-5/2												9
-21/8											1	41
-11/4												52
-13/4											3	2
-27/8												5
-7/2												2
-15/4												8
-4												8
-17/4												3
-9/2												1
-5								1	1	8	42	328
-10												1
-14												1
-42												1
No. contributions:	1	1	1	3	7	26	127	1,060	10,525	136,433	2,048,262	35,503,735
No. f -graphs:	1	1	1	3	7	36	220	2,707	42,979	898,353	22,024,902	619,981,403

these ranges are dictated by Catalan numbers. Finally, we note the absence of any new coefficients between two consecutive Catalan numbers: for $\ell = 8, 9$ no coefficient lies within $(-2, -5)$, for $\ell = 10, 11$ none between $(5, 14)$, and for $\ell = 12$ none within $(-14, -42)$.

IV. THE GENERALIZED CATALAN CONJECTURE

As we accumulate more and more data, patterns start to emerge that are wholly invisible at low loop orders. The most striking example is the Catalan conjecture proposed in [9]: at $2m$ points [i.e. $(2m - 4)$ loops], the largest coefficient (in magnitude) is

$$A_m := (-1)^{m-1} C_{m-3} = (-1)^{m-1} \frac{1}{m-2} \binom{2(m-3)}{m-3} \quad (6)$$

where C_n denotes the n th Catalan number. For example, $A_m = 1, -1, 2, -5, 14, -42$ for $m = 3, \dots, 8$ (that is, for $\ell = 2, 4, 6, 8, 10, 12$). The corresponding f -graph with coefficient A_m is the $2m$ -point “antiprism” illustrated in Fig. 3. The conjecture based on $m \leq 7$ data was later confirmed at $m = 8$ in [10] using a *local system*. Our full calculation at 12 loops confirms this completely.

Now, with the complete data for $\ell \leq 12$ at hand, we observe that this pattern generalizes to a broader class of f -graphs we denote as “polygon-framed fishnets.” These can be obtained as follows: start with a $2m$ -point antiprism, and repeatedly “thread” along diagonal directions [52]. A “thread” is a sequence of denominators along a diagonal direction of the m -gon, together with a numerator connecting the end points of the diagonal [53]. For example, start from the $m = 6$ antiprism of Fig. 3, sequentially threading along diagonals leads to the hexagon-framed fishnets such as the first two of Fig. 4. Note that the third graph of Fig. 4 is *not* a hexagon-framed fishnet due to the wrong placement of the numerators (although it is a valid

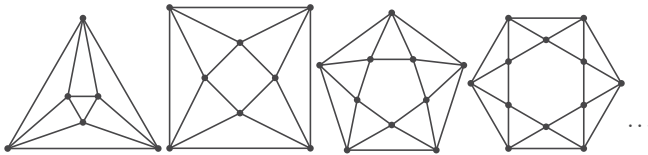


FIG. 3. Antiprism graphs for $m \in \{3, 4, 5, 6\}$.

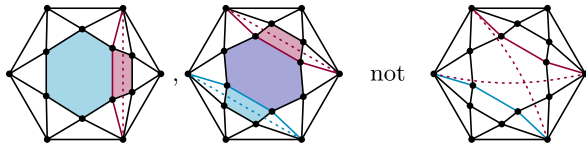


FIG. 4. Two examples of valid, hexagon-framed fishnets, and one with an inconsistent numerator prescription; the coefficient of the last graph is zero.

f -graph). The polygon-framed fishnets are so named because the square-framed fishnets ($m = 4$) (see e.g. the first three figures of Fig. 5) are the f -graph versions (after dividing by ξ_4) of the more familiar rectangular fishnet integrals contributing to $\mathcal{G}$ [44].

The interesting observation is that, at least up to 12 loops, the coefficient of any polygon-framed fishnet is given by the product of A_p ’s for each of its “faces” (each p -gon tile) excluding the outer polygon frame. For example, the $2m$ -point antiprism has a bunch of triangle tiles and an m -gon tile:

$$A_m \times \prod A_3 = A_m. \quad (7)$$

Any rectangular fishnet graph will have coefficient ± 1 , as the product consists of $A_3 = 1$ and $A_4 = -1$ tiles; for a fishnet with $\ell = a \times b$ loops, its f -graph contains $(a-1)(b-1)$ squares, thus the coefficient is $(-1)^{(a-1)(b-1)}$ [54,55].

Polygon-framed fishnets are common generalizations of these two infinite families. See Fig. 5 for several nontrivial examples up to 12 loops (with $m = 5, 6, 7, 8$). In particular, the final graph of Fig. 5 is the 12-loop antiprism with coefficient $A_8 = -42$ and the penultimate example has coefficient $A_5 A_6 = 2 \times (-5) = -10$. We conjecture this to hold to all loops: e.g. Fig. 6 shows predictions at 14 loops, while Fig. 7 shows a prediction at 37 loops.

While a proof of the original Catalan conjecture remains elusive, this evidence for its broader generalization strengthens our confidence that it will hold to arbitrary loop orders; promisingly, as we describe in the Appendix, the coefficients appearing in the bootstrap equation involving the antiprism also follow a nice pattern that resonates with a highly nontrivial identity of Catalan numbers.

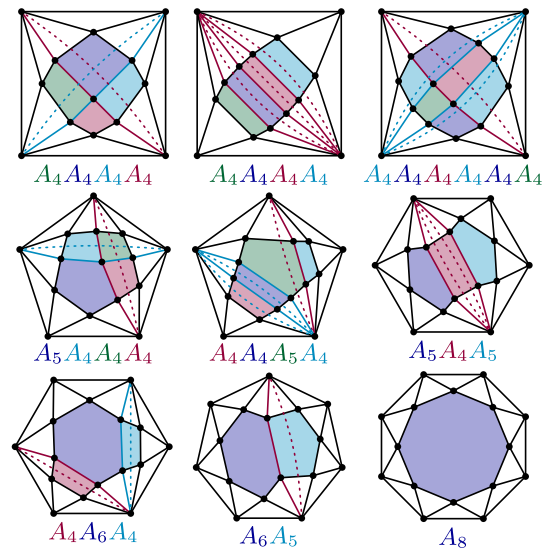


FIG. 5. Several nontrivial examples of graphs with coefficients given by the generalized Catalan conjecture.

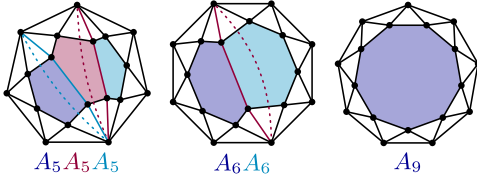


FIG. 6. We predict the coefficients of these contributions to the 14-loop correlator to be $A_5 A_5 A_5 = 8$, $A_6 A_6 = 25$, and $A_9 = 132$, respectively.

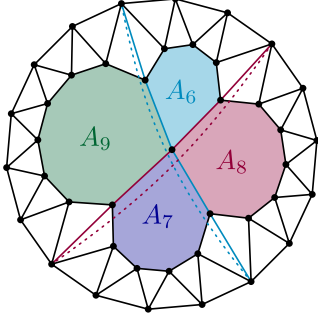


FIG. 7. We predict the coefficient of this 37-loop graph to be $A_6 A_8 A_7 A_9 = 388, 080$.

The appearance of Catalan numbers (which count trivalent graphs) associated to m -gon tiles suggests that they are related to leading singularities (trivalent graphs in momentum space) through the duality to amplitudes.

V. OUTLOOK

In this paper, we bootstrapped the 12-loop integrand of the four-point half-BPS correlator in planar sYM, using constraints from the leading divergent behavior of cusp and OPE limits. Computational growth aside, the way we fix the ansatz straightforwardly through 12 loops suggests that we have not exhausted the constraining power of our graphical rules. With enough computational resources, we are quite optimistic that such a bootstrap program could pursue several loop orders higher. Meanwhile, a few more interesting questions naturally emerge.

First, it would be nice to explore whether these limiting behaviors completely characterize the four-point correlator, at least perturbatively, by proving whether these constraints (or the double-triangle rule by itself) suffice to determine the integrand to all loop orders.

Second, the integrand of the correlator provides valuable data for scattering amplitudes [56,57] and IR-safe weighted cross sections [58–63] in sYM. Hopefully, the duality to amplitudes will provide a different perspective for the Catalan conjecture.

Third, it would be very interesting to understand the relation between the manifestly symmetric and local f -graph representation of the correlator and the more geometrical twistor representation [64–66].

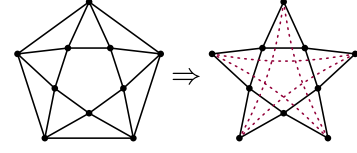


FIG. 8. A five-point integral seen as a pentagon-framed fishnet.

Another important direction is to study the higher-point Feynman integrals that these f -graphs provide (see e.g. Fig. 8). It is likely that the polygon-framed fishnets, like the rectangular fishnets [44,67–69], could be studied using integrability techniques [70–74].

Last but not least, the cusp limit could be applied more generally to nonplanar corrections [75], higher-point correlators [76,77], or heavier half-BPS correlators [54,78–80]. In particular, since the generating function of heavier half-BPS correlators is represented by f -graphs lifted to 10 dimensions [54,80] and thus also satisfies the graphical rules, a natural question is to explore the significance of these graphical rules in the $10d$ context. Could there be a cusped “ $10d$ Wilson-loop” that participates in a correlator generating function/“ $10d$ Wilson-loop”/Coulomb-branch amplitude triality?

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DATA AVAILABILITY

The data that support the findings of this article are openly available [50].

APPENDIX: LOCAL SYSTEM FOR ANTIPRISM GRAPHS

Approaching the Catalan conjecture from a different perspective, we examine the unique bootstrap equation involving the $2n$ -point antiprism (corresponding to pinching the unique double-triangle subgraph on the “belt”). The f -graphs involved in this equation (Fig. 9) are characterized as follows. Suppose the highlighted double-triangle is $(\Delta aba')(\Delta abb')$. For some $1 \leq m \leq n-2$, shoot $(m-1)$ raylike diagonals from a on the inner n -gon and another $(m-1)$ raylike diagonals from b on the outer

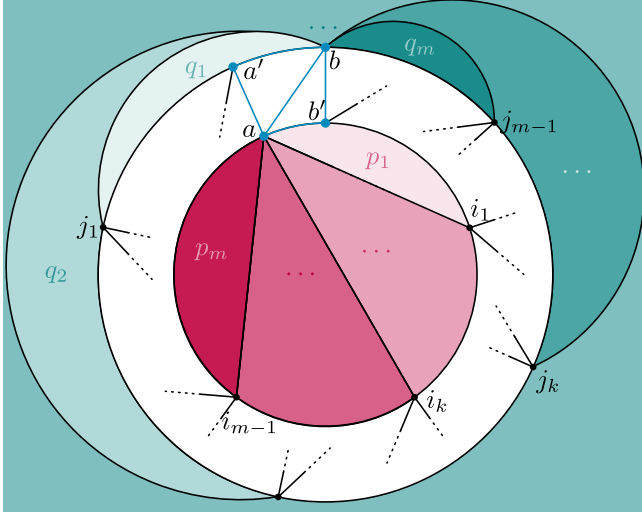


FIG. 9. Typical diagram contributing to the bootstrap equation involving the $2n$ -point antiprism. Numerators $\prod_{k=1}^{m-1} x_{a,j_k}^2 x_{b,i_k}^2$ are omitted for clarity. p_k/q_k denote the sizes of faces colored purple/teal, respectively.

n -gon, dividing the two n -gon faces into $2m$ faces of size p_k/q_k . Label the end points $i_1, \dots, i_{m-1}$ of the inner diagonals *clockwise* and the end points $j_1, \dots, j_{m-1}$ of the outer diagonals *counterclockwise*. These $2(m-1)$ denominators $(\prod_{k=1}^{m-1} x_{a,i_k} x_{b,j_k})^{-1}$ are compensated by a numerator $\prod_{k=1}^{m-1} x_{a,j_k} x_{b,i_k}$ to restore the correct conformal weight. These f -graphs pinch to the same cusp-highlighted f -graph as the $2n$ -point antiprism because the extra denominators and numerators precisely cancel.

We shall represent these f -graphs by the sequence $(p_1, \dots, p_m; q_1, \dots, q_m)$. It is easy to count that there are $\binom{2(n-3)}{n-3}$ allowed sequences by counting ways to shoot out the diagonals, but among these, the pair $(p_1, \dots, p_m; q_1, \dots, q_m)$ and $(q_1, \dots, q_m; p_1, \dots, p_m)$ lead to isomorphic f -graphs, since flipping one inside out yields the other. Surprisingly, we find that the coefficients of these f -graphs are all described by the formula:

$$\begin{aligned} c(p_1, p_2, \dots, p_m; q_1, q_2, \dots, q_m) \\ = (-1)^{m + \sum_{k=1}^m p_k} \\ \times \prod_{k=1}^m C(p_{1,k} - q_{m-k+2,m} - 3) C(q_{m-k+1,m} - p_{1,k}), \end{aligned} \quad (\text{A1})$$

where $C(p)$ is the p th Catalan number for $p \geq 0$ and $C(p) := 0$ for $p < 0$. We use the shorthand notation $p_{k,k'} := p_k + p_{k+1} + \dots + p_{k'}$ and similar for $q_{k,k'}$. For example, the antiprism coefficient is $c(n; n) = (-1)^{n+1} C(n-3)$, which agrees with the Catalan conjecture. Other examples include $c(3, 3, \dots, 3; 3, 3, \dots, 3) = +1$, which can be verified by recursively using the “square rule” to reduce to the unique planar six-point f -graph. Another nontrivial check is that this formula is indeed invariant under exchanging $(p_1, \dots, p_m) \leftrightarrow (q_1, \dots, q_m)$. This formula successfully reproduces the correct coefficients for all such graphs up to 16 points. Furthermore, we checked that up to $2n \leq 28$ points, the $\binom{2(n-3)}{n-3}$ coefficients always sum up to 0, as they should according to the bootstrap equation.

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- [51] The reader may also download the data files from https://huggingface.co/datasets/shicanxin/correlator_12loops/tree/main.

- [52] More precisely, we require any two intersecting threads to intersect normally (not tangentially) at a valency-4 vertex, and all threads along the same diagonal direction must lie in parallel.
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