

Near-horizon behavior of nonequatorial accelerated particle motion and high energy particle collisions

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(Received 16 May 2025; accepted 12 September 2025; published 2 October 2025)

We consider the motion of a particle in the background of a stationary axially symmetric generic black hole. A particle experiences the action of a force of unspecified nature. We require the force to remain finite in a comoving frame. The result is expressed in terms of several integers characterizing the Taylor expansion of the metric coefficients near the horizon. We show that the polar component of the four-velocity remains finite. As a result, the scenarios of high energy particle collisions, found previously in the context of the BSW effect for equatorial motion, do not change qualitatively in the nonequatorial case. The fact that the polar component is finite, also enables us to fill some gaps in the description of the BSW effect in previous works. This includes (i) classification of particle trajectories for nonequatorial motion, (ii) more coherent derivation of kinematics of the BSW effect, and (iii) discussion of bands in which the BSW effect takes place. Our results have a quite general character and can be used not only in description of high energy particle collisions but also in diverse astrophysical problems for which motion is not constrained by the equatorial plane.

DOI: [10.1103/kjk7-pyvj](https://doi.org/10.1103/kjk7-pyvj)

I. INTRODUCTION

Motion of particles in the background of black holes is a general and very vast subject. Its properties are important both in astrophysics and theoretical physics, especially for quite generic motion when particles are not constrained to the equatorial plane. In doing so, there is an important question: which general constraints arise due to the presence of a force when a particle moves near the horizon? The answer should take into account diverse metric behavior for different types of the horizons and requirement of the finiteness of such a force there. Among all possible applications of this there is the one that is being discussed in literature intensively during the last decade. This is the so-called Bañados-Silk-West (BSW) effect [1]. The main motivation of the present article stems just from this subject. However, we would like to stress that the results have a more general area of applicability since we

scrutinize the interplay between the force, kind of a trajectory and black hole metric.

According to the findings of [1], if two particles move toward a black hole and collide near it, the energy $E_{c.m.}$ in the center-of-mass frame can grow without bound, provided one of the particles has fine-tuned energy and angular momentum. Originally, this result was obtained for (i) the Kerr black hole, (ii) an extremal horizon, (iii) free particles, and (iv) the motion within the equatorial plane. Further, each of the first three conditions was relaxed, showing the BSW effect exists for equatorial motion around different types of horizons, including nonextremal black holes [2] and generic “dirty” black holes [3]. As far as the latter condition is concerned, a scenario where motion is nonequatorial was analyzed in [4] for the Kerr black hole. In [5] it was extended to generic black holes surrounded by matter (so-called dirty ones).

The goal of the present paper is to consider the most general type of scenario when (i) a black hole is a generic stationary axially symmetric one, (ii) the horizon is quite generic, (iii) particles experience the influence of some finite force of unspecified nature, and (iv) move along nonequatorial trajectories. A similar analysis was made in [6,7] where points (i)–(iii) were taken into account.

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Now, we take the next step and consider also point (iv). In doing so, we concentrate on the properties of a trajectory, mainly on the behavior of the component of particle velocity due to the change of a polar angle. In principle, one could expect the potential appearance of new qualitative features of collision scenarios just to the behavior of the polar angle. However, previous studies showed that for the Kerr metric this is not the case. Below, we argue that also, in general, the account for nonequatorial motion does not change the properties of the BSW effect qualitatively. To elucidate, whether or not nonequatorial motion brings such features in the set of possible scenarios and affects the manifestation of the BSW effect, the analysis suggested in our paper is necessary. Processes with particles moving arbitrarily near a black hole are relevant in astrophysics (see for example the recent analysis of what a near-horizon orbiter can see on the sky [8]). And, what seems to be even more important, including nonequatorial motion enables us to make a description of the BSW effect as general as possible.

The paper is organized as follows. In Sec. II we list the main formulas describing the metric and the choice of observers acceleration in the comoving frame. In Sec. III we list the near-horizon expansion for the metric coefficients needed in what follows. In Sec. IV we elucidate the conditions when a force acting on a particle remains finite in the horizon limit for different kinds of trajectory. In Sec. V we show how the finiteness of the polar component of the four-velocity enables us to fill some gaps in previous works on the BSW effect. In Sec. VI we discuss the relation between different frames relevant for our consideration. General discussion of these results is given in Sec. VII. In Appendix A we explain how the comoving frame can be obtained from the OZAMO frame. In Appendix B we suggest a description of the frame transformation from the group viewpoint in terms of rotation angles and parameters of a local Lorentz transformation.

II. GENERAL SETUP AND DESCRIPTION OF NONGEODESIC PARTICLES

A. Metric and particle motion

We consider axially symmetric spacetimes

$$ds^2 = -N^2 dt^2 + g_{\varphi\varphi}(d\varphi - \omega dt)^2 + \frac{dr^2}{A} + g_{\theta\theta}d\theta^2. \quad (1)$$

Our convention in the choice of coordinates is $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$. In what follows, we will also use, for shortness, notations $g_\varphi = g_{\varphi\varphi}$ and $g_\theta = g_{\theta\theta}$.

Let us consider the motion of a particle in this background. If it moves freely, it follows from the equations of motion that its four-velocity is equal to

$$u^\mu = \left(\frac{X}{N^2}, \sigma \frac{\sqrt{A}}{N} P, u^\theta, \frac{L}{g_\varphi} + \omega \frac{X}{N^2} \right), \quad (2)$$

where u^θ is an azimuthal velocity, $\sigma = \pm 1$, $X = E - \omega L$, where E and L are the energy and angular momentum respectively, and the function P is defined from the condition $u_\mu u^\mu = -1$ that gives us

$$P^2 = X^2 - N^2 \left(1 + \frac{L^2}{g_\varphi} + g_\theta (u^\theta)^2 \right). \quad (3)$$

Here, the quantities E and L have, accordingly, the meaning of energy and angular momentum. For free fall, they are conserved since the metric does not depend on t and φ . If the force is present, E and L become coordinate-dependent functions.

B. Frames and tetrads

Acceleration of a particle is given by

$$a^\mu = u^\nu \nabla_\nu u^\mu = u^\nu \partial_\nu u^\mu + \Gamma_{\nu\rho}^\mu u^\nu u^\rho. \quad (4)$$

For our future computations it is sufficient to know the tetrad components of acceleration $a_C^{(a)} = a^\mu \tilde{e}_\mu^{(a)}$ in the comoving (CO) frame $\tilde{e}_\mu^{(a)}$. However, this cannot be done from the very beginning. First of all, we start from the so-called zero-angular momentum observer (ZAMO) frame [9] and make transformations to the CO frame, as described in Appendix A. The ZAMO frame is a direct generalization of the static one in the case when the metric itself is also static. It leads to simplification in calculations and is a powerful tool in relativistic astrophysics [9]. [10]. There are two different kinds of such a system. The first one is the orbital ZAMO frame, when a reference particle moves along the trajectory with $r = \text{const}$. In general, it is not geodesics. Such a frame can be realized by observers that carry a tetrad

$$e_{(0)}^\mu = \frac{1}{N}(1, 0, 0, \omega), \quad e_{(1)}^\mu = \sqrt{A}(0, 1, 0, 0), \quad (5)$$

$$e_{(2)}^\mu = \frac{1}{\sqrt{g_{\theta\theta}}}(0, 0, 1, 0), \quad e_{(3)}^\mu = \frac{1}{\sqrt{g_{\varphi\varphi}}}(0, 0, 0, 1). \quad (6)$$

We use letter ‘‘O’’ to indicate that such an observer orbits a black hole.

In this tetrad frame, the four-velocity reads

$$u^{(a)} = \left(\frac{X}{N}, \sigma \frac{P}{N}, \sqrt{g_\theta} u^\theta, \frac{L}{\sqrt{g_\varphi}} \right). \quad (7)$$

The second kind is the so-called FZAMO frame realized by free-falling particles with $L = 0$. But we will not analyze this frame until Sec. VI.

After transformations from the OZAMO frame to the CO one, we have, according to the results of Appendix A, components of acceleration in the CO frame:

$$a_C^{(0)} = 0, \quad (8)$$

$$a_C^{(1)} = P \frac{\sqrt{A} \partial_r X + L \partial_r \omega}{N \sqrt{X^2 - N^2}} - u^\theta \frac{\partial_\theta X + L \partial_\theta \omega}{\sqrt{X^2 - N^2}}, \quad (9)$$

$$a_C^{(2)} = -\frac{1}{2} \frac{P'}{P} \frac{A}{\sqrt{g_\theta}} \partial_\theta \left(\frac{P^2}{AN^2} \right) + \frac{1}{\sqrt{g_\theta}} \frac{P}{P'} \left(\frac{X}{N^2} (\partial_\theta X + L \partial_\theta \omega) - \frac{L \partial_\theta L}{g_\phi} \right) \\ + \sqrt{g_\theta} u^\theta \frac{\sqrt{A} N}{P'} \left(\frac{X}{N^2} (\partial_r X + L \partial_r \omega) - \frac{L \partial_r L}{g_\phi} \right) - P' \frac{\sqrt{A} \partial_r (g_\theta u^\theta)}{N \sqrt{g_\theta}}, \quad (10)$$

$$a_C^{(3)} = \frac{1}{\sqrt{X^2 - N^2}} \left\{ \frac{u^\theta}{\sqrt{g_\theta} P'} [(X^2 - N^2) \partial_\theta L - LX(\partial_\theta X + L \partial_\theta \omega)] - \frac{P}{\sqrt{g_\theta} P'} \frac{\sqrt{A}}{N} [(X^2 - N^2) \partial_r L - LX(\partial_r X + L \partial_r \omega)] \right\}, \quad (11)$$

where P' is defined in (A11). Here, a new axis 1 is directed along a trajectory of a moving particle.

Our strategy consists in examining the behavior of each component of acceleration near the horizon. In all cases, we do calculations in CO. In doing so, we require each term in the components of acceleration to be regular separately. In principle, one cannot exclude in advance the situations when some terms diverge, but these divergences mutually cancel. However, we do not consider such very special situations. Our approach includes also free motion when acceleration is not only regular but equals zero exactly. We also have to note that one may be interested if one can use other frames to investigate forces with respect to them. This question is analyzed in Sec. VI.

Although for our purposes we mainly need the CO frame, from time to time we mention the ZAMO one as well. In particular, the OZAMO frame is very convenient for the kinematic analysis in terms of the three-dimensional velocities that generalize the results of [11] (see below).

For the Kerr black hole, instructive analysis of relations between different frames was done in [12]. In the current work, we generalize it to a generic dirty black hole.

III. NEAR-HORIZON EXPANSIONS

Now we are interested in near-horizon properties of the metric and particle trajectories. Let us assume that the horizon is placed at $r = r_h$. For this hypersurface to be a horizon it is required that $A(r_h) = N^2(r_h) = 0$. This means that near the horizon we can write the Taylor expansions of the functions A and N^2 in the form

$$N^2 = \kappa_p v^p + o(v^p), \quad A = A_q v^q + o(v^q), \quad (12)$$

where $v = r - r_h$, p , and q are integers, and κ_p , A_q are coefficients that may depend on θ . Also, we assume that near the horizon holds

$$\omega = \hat{\omega}_H + \hat{\omega}_k v^k + \dots + \hat{\omega}_{l-1} v^{l-1} + \omega_l v^l + o(v^l), \quad (13)$$

$$g_a = g_{aH} + g_{am} v^m + o(v^m), \quad (14)$$

where we use “hat” notation for coefficients that do not depend on θ . Here, l , k , m are integers. As was shown in [13], the regularity of the Ricci scalar (and other scalar invariants) implies that integers l and k have to satisfy

$$k \geq \left\lceil \frac{p - q + 3}{2} \right\rceil, \quad l \geq \left\lceil \frac{p + 1}{2} \right\rceil. \quad (15)$$

If these requirements are satisfied, then the corresponding hypersurface r_h is indeed a horizon (and not a singularity).

Also we assume that near the horizon expansions hold

$$X = \hat{X}_0 + \hat{X}_{s_1} v^{s_1} + \dots + \hat{X}_{s_2-1} v^{s_2-1} + X_{s_2} v^{s_2} + o(v^{s_2}), \quad (16)$$

$$L = \hat{L}_H + \hat{L}_{b_1} v^{b_1} + \dots + \hat{L}_{b_2-1} v^{b_2-1} + L_{b_2} v^{b_2} + o(v^{b_2}), \quad (17)$$

$$u^\theta = \hat{u}_{c_1}^\theta u^{c_1} + \dots + \hat{u}_{c_2-1}^\theta v^{c_2-1} + u_{c_2}^\theta v^{c_2} + o(v^{c_2}), \quad (18)$$

where s_1 , s_2 , b_1 , b_2 , c_1 , and c_2 are real numbers.

One reservation is in order. In principle, one can consider nonintegers p , q , k , l . However, this would pose a question about analytical continuation of the metric across the horizon, properties of regularity of the metric, etc. Even in the simplest case of a static metric such issues are quite nontrivial and require separate discussion (see [14] and references therein). In the current work, we put this issue aside and make a simple assumption that the aforementioned numbers are integers.

Below, we examine the properties of different kinds of particles. Our strategy is the following. We begin with kinematic consideration, then we consider restrictions following the requirement of the finiteness of a force. In doing so, we consider separately the cases of finite and infinite proper time needed to reach the horizon.

IV. CONDITIONS ON NEAR-HORIZON PARTICLES

A. Usual particles

In what follows we examine the conditions when the tetrad components of acceleration are finite. Let us at first consider usual particles.

1. Kinematic considerations

According to the definition accepted in literature, for them $X \neq 0$ on the horizon and, hence, $\hat{X}_0 \neq 0$. Now let us consider quantity P in this limit:

$$P = \sqrt{X^2 - N^2 \left(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2 \right)}. \quad (19)$$

This quantity has to be real that requires $X^2 \geq N^2(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2)$. If u^θ is finite (or even tends to zero) near the horizon, this condition is automatically satisfied near the horizon because $N^2 \rightarrow 0$, and thus the whole expression $N^2(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2) \rightarrow 0$, while $X^2 \rightarrow \hat{X}_0^2 \geq 0$. However, if u^θ diverges near the horizon, then this condition becomes more complicated. Then, among three terms in $1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2$ the term $g_\theta(u^\theta)^2$ is dominant which means that near the horizon

$$N^2 \left(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2 \right) \sim v^{p+2c_1}. \quad (20)$$

This quantity has to be finite (or tend to zero) for the quantity P to be real. This gives kinematic restriction on coefficient c_1

$$c_1 \geq -\frac{p}{2}. \quad (21)$$

If this is so, then in the near-horizon limit one has

$$P = \begin{cases} \hat{X}_0 & \text{if } c_1 > -\frac{p}{2} \\ \sqrt{\hat{X}_0^2 - \kappa_p g_{\theta H}((u^\theta)_{-p/2})^2} & \text{if } c_1 = -\frac{p}{2} \end{cases}, \quad (22)$$

while in all cases $P' = \hat{X}_0$ in the horizon limit.

Before we proceed with the analysis of the forces, we have to investigate the proper time of usual particles. As the radial component of the four-velocity $u^r = \frac{dr}{d\tau}$, the proper time is given by

$$\tau = \int \frac{dr}{u^r} = \sigma \int \frac{N}{\sqrt{A} P} dr. \quad (23)$$

As near the horizon for usual particles $P = O(1)$, we obtain

$$\tau \sim \int v^{\frac{p-q}{2}} dr \sim \begin{cases} v^{\frac{p-q+2}{2}} & \text{if } q \neq p+2 \\ \ln|v| & \text{if } q = p+2 \end{cases}. \quad (24)$$

The proper time is regular if $q < p+2$, and it diverges if $q \geq p+2$. In the first case such a particle can cross the horizon; in the second one it is unable to do it.

2. Dynamics considerations: Regular proper time

Let us start with the case when a usual particle can cross the horizon. As was mentioned in the previous section, this happens if $q < p+2$. We decided to consider this case at first as it corresponds to the cases most relevant in physics and astrophysics. For example, for the extremal Kerr spacetime (where the BSW effect was originally found) $q = p$, and the condition $q < p+2$ is automatically satisfied.

We remind the reader that we analyze the forces in the comoving frame. Let us start with analysis of $a_C^{(1)}$ (9). The first term will be finite if $\partial_r X \sim \frac{N}{\sqrt{A}}$ and $\partial_r \omega \sim \frac{N}{\sqrt{A}}$. Integrating we find that these terms will be regular if

$$s_1 \geq \frac{p-q}{2} + 1, \quad k \geq \frac{p-q}{2} + 1. \quad (25)$$

As in this subsection we assume $q < p+2$, both corresponding terms tend to zero for any non-negative s_1 and k .

The second term in $a_C^{(1)}$ (9) will be regular if $u^\theta \partial_\theta X$ and $u^\theta \partial_\theta \omega$ are regular (we postpone analysis of these conditions).

Now let us move to $a_C^{(2)}$ (10). We start with the analysis of the second term that will be finite if

$$\frac{X}{N^2} (\partial_\theta X + L \partial_\theta \omega) - \frac{L \partial_\theta L}{g_\varphi} = O(1). \quad (26)$$

This will be true if $\partial_\theta X \sim N^2$, $\partial_\theta \omega \sim N^2$ and $\partial_\theta L = O(1)$. These terms are regular if

$$s_2 \geq p, \quad l \geq p, \quad b_2 \geq 0. \quad (27)$$

This allows us to analyze the second term in $a_C^{(1)}$ (9) that will be regular if $u^\theta \partial_\theta X$ and $u^\theta \partial_\theta \omega$ are regular. Let us start with $u^\theta \partial_\theta X$ that near the horizon behaves like

$$u^\theta \partial_\theta X \sim v^{c_1+s_2}. \quad (28)$$

However, as follows from (27) and (21), the corresponding degree $c_1 + s_2 \geq \frac{p}{2}$. This quantity is positive and thus $u^\theta \partial_\theta X$ tends to zero. The same argument holds for $u^\theta \partial_\theta \omega$. Thus we have found that $a_C^{(1)}$ (9) will be regular if both (21) and (27) hold.

Now we are ready to analyze the first term in $a_C^{(2)}$ (10). It will be finite if

$$A\partial_\theta\left(\frac{P^2}{AN^2}\right) = A\partial_\theta\left(\frac{X^2}{AN^2} - \frac{1}{A}\left(1 + \frac{L^2}{g_\theta} + g_\theta(u^\theta)^2\right)\right) = O(1). \quad (29)$$

In principle, this expression can be rewritten as

$$\frac{\partial_\theta X^2}{N^2} - X^2 \frac{\partial_\theta(AN^2)}{AN^4} + \frac{\partial_\theta A}{A} \left(1 + \frac{L^2}{g_\theta} + g_\theta(u^\theta)^2\right) - \partial_\theta\left(\frac{L^2}{g_\theta} + g_\theta(u^\theta)^2\right) = O(1). \quad (30)$$

In (30) the first term is regular because of condition $s_2 \geq p$ [see (27)]. The second term in (30) will be finite if

$$\partial_\theta(AN^2) \sim AN^4. \quad (31)$$

Equivalently, this condition may be written as

$$\frac{\partial_\theta A}{A} + \frac{\partial_\theta N^2}{N^2} = O(N^2). \quad (32)$$

It was analyzed and integrated in Appendix C in [13]. In terms of expansion coefficients, this condition can be written through such a relation (Eqs. (C5) and (C18) in [13]):

$$A_q \kappa_p = C_p, \quad (33)$$

$$\frac{A_{q+s}}{A_q} + \frac{\kappa_{p+s}}{\kappa_p} = \sum_{n=2}^s \frac{(-1)^n}{n} \sum_{k_j} \frac{n!}{k_1! \dots k_n!} \left[\prod_{j=1}^n \left(\frac{A_{q+j}}{A_q}\right)^{k_j} + \prod_{j=1}^n \left(\frac{\kappa_{p+j}}{\kappa_p}\right)^{k_j} \right] + C_{p+s}. \quad (34)$$

Summation over k_j is conducted over all combinations of integers k_j such that

$$\sum_{j=1}^n j k_j = s. \quad (35)$$

The coefficients $\{C_p, C_{p+1}, \dots, C_{2p}\}$ are constants, $s \in \{1, \dots, p\}$.

The fourth term in (30) is finite if $\partial_\theta L = O(1)$ and $u^\theta \partial_\theta u^\theta = O(1)$. The first condition is already satisfied by (27), while the second one is new and it gives us

$$c_1 + c_2 \geq 0. \quad (36)$$

The third term in (30) is more complicated. To analyze this term we have to note that $1 + \frac{L^2}{g_\theta}$ is finite near the horizon, while $g_\theta(u^\theta)^2 \sim v^{2c_1}$. Thus the second term gives $\partial_\theta A \sim v^{-2c_1+q}$. We will investigate this condition later.

Now let us analyze the fourth term in $a_C^{(2)}$ (10). It will be finite if

$$\frac{\sqrt{A}}{N} \frac{\partial_r(g_\theta u^\theta)}{\sqrt{g_\theta}} = O(1), \quad (37)$$

which requires

$$\partial_r(g_\theta u^\theta) = \partial_r(g_\theta)u^\theta + g_\theta \partial_r(u^\theta) = O\left(\frac{N}{\sqrt{A}}\right). \quad (38)$$

The second term in (38) is of order v^{c_1-1} . This term will be regular if $c_1 \geq \frac{p-q}{2} + 1$. The first term in (38) is of order

v^{c_1+m-1} . It will be regular if $c_1 \geq \frac{p-q}{2} + 1 - m$ where m was introduced in (14). As m is considered to be non-negative [see (14)], the condition coming from the finiteness of the second term is stronger. Thus, the fourth term in $a_C^{(2)}$ (10) will be regular if

$$c_1 \geq \frac{p-q}{2} + 1. \quad (39)$$

As in this subsection we consider the case of finite proper time (that requires $q < p+2$), this automatically means $c_1 > 0$.

Also note that for c_1 there is a condition (21). This condition is less strict than $c_1 \geq \frac{p-q}{2} + 1$ for the case of a finite proper time. Thus, we can combine both conditions and write:

$$\text{If the proper time is finite } (q < p+2), \quad c_1 \geq \frac{p-q}{2} + 1. \quad (40)$$

The third term in $a_C^{(2)}$ (10) is regular if

$$u^\theta \frac{\sqrt{A}}{N} \partial_r X, \quad u^\theta \frac{\sqrt{A}}{N} \partial_r \omega, \quad u^\theta N \sqrt{A} \partial_r L, \quad (41)$$

are regular.

Substituting expansions for A , N^2 , and u^θ , one finds that near the horizon these expressions have order

$$v^{c_1 - \frac{p-q}{2} + s_1 - 1}, \quad v^{c_1 - \frac{p-q}{2} + k - 1}, \quad v^{c_1 + \frac{p+q}{2} + b_1 - 1}, \quad (42)$$

respectively.

However, as c_1 satisfies $c_1 \geq \frac{p-q}{2} + 1$ [see (40)], it is obvious that these terms are regular for any non-negative s_1, k, b_1 .

Now we are ready to find when the third term in (30) is regular. As was already mentioned, regularity of the third term in (30) will be achieved if $\frac{\partial_\theta A}{A} g_\theta (u^\theta)^2$ is regular. However, as we showed in (40), c_1 is positive, and thus this term is automatically regular. Thus, we see that the only conditions that have to hold for the functions A and N^2 are (34).

This finishes the analysis of $a_C^{(2)}$ (10). For regularity of this term, conditions (27), (36), (40), and (34) have to hold.

Now let us move to the $a_C^{(3)}$ (11). We start with the second square bracket of $a_C^{(3)}$ (11) that will be finite if

$$\frac{\sqrt{A}}{N} \partial_r L = O(1), \quad \frac{\sqrt{A}}{N} \partial_r X = O(1), \quad \frac{\sqrt{A}}{N} \partial_r \omega = O(1). \quad (43)$$

However, employing (25), we see that the second and third conditions are already satisfied, while the first one will be satisfied if

$$b_1 \geq \frac{p-q}{2} + 1. \quad (44)$$

The first term in $a_C^{(3)}$ (11) will be finite if

$$u^\theta \partial_\theta L = O(1), \quad u^\theta \partial_\theta X = O(1), \quad u^\theta \partial_\theta \omega = O(1). \quad (45)$$

These conditions will be satisfied if

$$b_2 \geq -c_1, \quad s_2 \geq -c_1, \quad l \geq -c_1. \quad (46)$$

However, according to (40) and assuming b_2, s_2 , and l to be positive, conditions (46) will be satisfied if

$$b_2, s_2, l \geq \frac{q-p-2}{2}. \quad (47)$$

But as in this subsection we consider only the case of finite proper time ($q < p+2$), these conditions are automatically satisfied by any non-negative b_2, s_2 , and l .

This completes the analysis of $a_C^{(3)}$ (11). For this component to be finite, conditions (44) and (47) have to hold.

Now let us summarize all the conditions we obtained for the case of a regular proper time ($q < p+2$). First of all, we have found that $s_1 \geq \frac{p-q}{2} + 1$ [see (25)].

The quantity s_2 is constrained by (27) and (47) and requirement $s_2 > 0$. But for a regular proper time ($q < p+2$) the number $\frac{q-p-2}{2}$ in the condition (47) is negative, and the condition (27) (namely, $s_2 \geq p$) appears to be the strongest among these three.

The next integer is k . It is constrained only by (25).

The coefficient l is constrained by (27) and (47) and requirement $l > 0$. The situation with l is the same as with s_2 and gives $l \geq p$.

The coefficient b_1 is constrained only by (44). Coefficient b_2 is constrained by (27), (47) and the condition $b_2 > 0$. As we have already explained, the quantity $\frac{q-p-2}{2}$ is negative for regular proper time. Thus the condition $b_2 > 0$ is more strict.

The last conditions constrain c_1 and c_2 . The coefficient c_1 is constrained only by (40). As was noted, for a regular proper time (when $q < p+2$) this quantity is positive. As follows from (18), $c_2 \geq c_1$. As c_1 is positive, we directly see that $c_2 > 0$.

The final list of conditions is

If proper time is regular ($q < p+2$),

$$s_1, k, b_1, c_1 \geq \frac{p-q}{2} + 1, \quad s_2, l \geq p, b_2, c_2 > 0, \quad \text{and (34)}. \quad (48)$$

3. Dynamics considerations: Diverging proper time

As was noted above, a usual particle will have diverging proper time if $q \geq p+2$. First of all, we note that, as we assume p to be positive, this automatically entails that $q > 2$. We remind the reader that we analyze the forces in the comoving frame. At first we investigate the expression for $a_C^{(1)}$ (9). The first term is of order $v^{\frac{q-p}{2}+s_1-1}$. However, as for diverging proper time $q \geq p+2$, we see that this term is regular for any positive s_1 . The same holds for the second term that is of order $v^{\frac{q-p}{2}+k-1}$. It will be regular for any positive k . The last two terms in $a_C^{(1)}$ (9) we will analyze further.

Now let us analyze the $a_C^{(2)}$ (10). We start with the second term. Corresponding expressions are independent on the proper time and have been already analyzed in the previous subsection. Corresponding conditions of regularity of these expressions are given by (27). As was also noted in the previous section, conditions (27) ensure that the last two terms in $a_C^{(1)}$ (9) are also regular. Finiteness of the first term in $a_C^{(2)}$ (10), as was argued in the previous subsection, can be rewritten as (30). However, in this section we slightly change the order in which we are analyzing corresponding terms.

The first term in (30) is regular if $s_2 \geq p$ what is true even for diverging proper time as the condition (27) has to hold. Second term in (30) is finite if conditions (33)–(34) hold. Now let us analyze the fourth term in (30), namely $\partial_\theta \left(\frac{L^2}{g_\varphi} + g_\theta (u^\theta)^2 \right)$. Expanding the derivative, we obtain

$$\begin{aligned} \partial_\theta \left(\frac{L^2}{g_\varphi} + g_\theta (u^\theta)^2 \right) &= \frac{\partial_\theta L^2}{g_\varphi} - \frac{L^2}{(g_\varphi)^2} \partial_\theta g_\varphi \\ &\quad + (\partial_\theta g_\theta) (u^\theta)^2 + g_\theta \partial_\theta (u^\theta)^2. \end{aligned} \quad (49)$$

Here all four terms have to be regular. But as the metric functions g_θ , g_φ (and all their derivatives) are assumed to be finite, it directly follows that both L and u^θ (and their derivatives) have to be regular. In terms of expansion coefficients (17) and (18), this means

$$b_{1,2} \geq 0, \quad c_{1,2} \geq 0. \quad (50)$$

The third term in (30), namely $\frac{\partial_\theta A}{A} (1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2)$, is then quite easy to analyze as $\frac{\partial_\theta A}{A}$ is regular as follows from the expansion for A [see (12)], while $(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2)$ is regular because of (50). This finishes the analysis of Eq. (30), and thus the first term in $a_C^{(2)}$ (10).

Now we are left with the third and fourth terms in $a_C^{(2)}$ (10). Let us start with the fourth one, namely $P' \frac{\sqrt{A}}{N} \frac{\partial_r(g_\theta u^\theta)}{\sqrt{g_\theta}}$. This expression, in fact, contains two terms $P' \frac{\sqrt{A}}{N} \frac{g_\theta \partial_r(u^\theta)}{\sqrt{g_\theta}}$ and $P' \frac{\sqrt{A}}{N} \frac{u^\theta \partial_r(g_\theta)}{\sqrt{g_\theta}}$, the first is of order $v^{\frac{q-p}{2}+c_1-1}$, while the second is $v^{\frac{q-p}{2}+c_1+m-1}$. However, as we are considering the case of diverging proper time ($q \geq p+2$) and as the condition $c_1 \geq 0$ has to hold (50), both of these terms are regular (note that by definition $m \geq 1$).

The third term in $a_C^{(2)}$ (10) contains three terms that are of order $v^{\frac{q-p}{2}+s_1-1}$, $v^{\frac{q-p}{2}+k-1}$, and $v^{\frac{q-p}{2}+b_1-1}$. However, as we are considering the case of diverging proper time ($q \geq p+2$) and as, by definition s_1 and k are positive (along with b_1), we obtain that the corresponding terms are regular.

Thus we are only left with $a_C^{(3)}$ (11). The first term, namely $\frac{u^\theta}{\sqrt{X^2-N^2}\sqrt{g_\theta P'}} [(X^2-N^2)\partial_\theta L - LX(\partial_\theta X + L\partial_\theta \omega)]$, is, obviously, regular as all the quantities L , X , and ω are regular near the horizon. The second term, namely $\frac{P}{\sqrt{X^2-N^2}\sqrt{g_\theta P'}} \frac{\sqrt{A}}{N} [(X^2-N^2)\partial_r L - LX(\partial_r X + L\partial_r \omega)]$, has three terms in it that are of order $v^{\frac{q-p}{2}+b_1-1}$, $v^{\frac{q-p}{2}+s_1-1}$, $v^{\frac{q-p}{2}+k-1}$. All these three terms are regular for the diverging proper time ($q \geq p+2$) and positive b_1 , s_1 , and k .

Let us now summarize all the conditions we have obtained so far for regular forces. The coefficient s_1 is unconstrained and can take any non-negative value; s_2 is constrained by (27). The coefficients c_1 , c_2 , b_1 , and b_2 are given by (50) and can take any non-negative value. The quantity k is constrained only by the regularity condition (15); l is restricted by $l \geq p$ (27). The last requirement that has to hold is the set of conditions (33) and (34). This can be summarized in such a set of conditions

$$s_2 \geq p, \quad l \geq p, \quad s_1, c_{1,2}, b_{1,2} \geq 0 \quad \text{and (33-34)}. \quad (51)$$

The situation when $\tau \rightarrow \infty$ for usual particles refers to the so-called “remote horizons.” For the spherically symmetric

metric the examples of such a kind were discussed in [15]. (In [14] such cases were classified as “null infinity” but we find such terminology in a given context inaccurate. The lapse function $N \rightarrow 0$, so the redshift grows unbounded in this limit which is typical just of horizons.)

B. Subcritical particles

Now we are able to start an analysis of the subcritical particles. But before we do this, we have to define precisely what it is meant by subcritical particles. According to classification given in [6], where authors considered the case of equatorial motion, a subcritical particle is such that near the horizon the quantity X behaves according to

$$X = \hat{X}_{s_1} v^{s_1} + o(v^{s_1}), \quad (52)$$

where $0 < s_1 < \frac{p}{2}$. Note that as in [6] only equatorial motion was investigated, corresponding expansion coefficients $\hat{X}_{s_1}$ were inevitably independent on θ . In the case of nonequatorial motion, generally it is not so. Thus, we have to assume an expansion for the quantity X in the form

$$X = \hat{X}_{s_1} v^{s_1} + \dots + \hat{X}_{s_2-1} v^{s_2-1} + X_{s_2} v^{s_2} + o(v^{s_2}), \quad (53)$$

where $0 < s_1 < \frac{p}{2}$ and no additional constraint is imposed on $s_2 \geq s_1$. Expansion of other parameters of the particle and metric functions is assumed, as in the previous case, to be in the form

$$N^2 = \kappa_p v^p + o(v^p), \quad A = A_q v^q + o(v^q), \quad (54)$$

$$L = \hat{L}_H + \hat{L}_{b_1} v^{b_1} + \dots + \hat{L}_{b_2-1} v^{b_2-1} + L_{b_2} v^{b_2} + o(v^{b_2}), \quad (55)$$

$$u^\theta = \hat{u}_{c_1}^\theta v^{c_1} + \dots + \hat{u}_{c_2-1}^\theta v^{c_2-1} + u_{c_2}^\theta v^{c_2} + o(v^{c_2}), \quad (56)$$

$$\omega = \hat{\omega}_H + \hat{\omega}_k v^k + \dots + \hat{\omega}_{l-1} v^{l-1} + \omega_l v^l + o(v^l) \quad (57)$$

$$g_a = g_{aH} + g_{am} v^m + o(v^m). \quad (58)$$

1. Kinematical considerations

Before we proceed with an analysis of acceleration components, we have to note that several physical limitations come already from kinematics. Indeed, the radial component of the four-velocity has to be real. This requires one to have $P^2 > 0$:

$$P^2 = X^2 - N^2 \left(1 + \frac{L^2}{g_\varphi} + g_\theta(u^\theta)^2 \right) > 0. \quad (59)$$

The first term, namely X^2 , is positive but tends to zero as $(\hat{X}_{s_1})^2 v^{2s_1}$. The second and the third terms, namely $-N^2(1 + \frac{L^2}{g_\varphi})$, are negative but they tend to zero with the

rate v^p . As for the subcritical particles we assumed that $s_1 < p/2$; these terms are of higher order than X^2 and do not play a crucial role in the near-horizon behavior of P^2 .

However, the fourth term $-N^2 g_\theta(u^\theta)^2$ may have an influence on the sign of P^2 . Indeed, this term is negative and the near horizon behaves as $-\kappa_p g_{\theta H}(\hat{u}_{c_1}^\theta)^2 v^{2c_1+p}$. Thus in the function P^2 there are two potentially dominant terms: $(\hat{X}_{s_1})^2 v^{2s_1}$ and $-\kappa_p g_{\theta H}(\hat{u}_{c_1}^\theta)^2 v^{2c_1+p}$. Depending on what term is dominant, we obtain such various cases:

- (i) Case $2s_1 = 2c_1 + p$. Then, both these terms are of the same order, and P^2 near the horizon

$$P^2 \approx [(\hat{X}_{s_1})^2 - \kappa_p g_{\theta H}(\hat{u}_{c_1}^\theta)^2] v^{2s_1}. \quad (60)$$

The corresponding coefficient will be positive or at least non-negative if $(\hat{X}_{s_1})^2 \geq \kappa_p g_{\theta H}(\hat{u}_{c_1}^\theta)^2$. Otherwise the subcritical particle is unable to reach the horizon.

- (ii) Case $2s_1 < 2c_1 + p$. Then, the term $(\hat{X}_{s_1})^2 v^{2s_1}$ is dominant and we are able to write

$$P^2 \approx (\hat{X}_{s_1})^2 v^{2s_1}. \quad (61)$$

- (iii) Case $2s_1 > 2c_1 + p$ is impossible because in this case P^2 is negative near the horizon.

Thus we have obtained that for subcritical particles kinematics requires

$$0 < s_1 < \frac{p}{2}, \quad c_1 \geq s_1 - \frac{p}{2}. \quad (62)$$

These conditions will be used further.

Now we are able to move to the analysis of forces acting on such particles. But before we do this, let us investigate the proper time required for such particle to cross the horizon. The radial component of four-velocity in the leading order reads

$$u^r = \sigma \frac{\sqrt{A}}{N} P \sim v^{\frac{q-p}{2}+s_1}. \quad (63)$$

The proper time, required to cross the horizon, in the main order is given by

$$\tau = \int \frac{dr}{u^r} \sim \begin{cases} v^{\frac{p-q+2}{2}-s_1} & \text{if } s_1 \neq \frac{p-q+2}{2} \\ \ln|v| & \text{if } s_1 = \frac{p-q+2}{2} \end{cases}. \quad (64)$$

This quantity may either diverge (if $s_1 \geq \frac{p-q+2}{2}$) or to be regular (if $s_1 < \frac{p-q+2}{2}$). These cases differ physically because if the proper time diverges ($s_1 \geq \frac{p-q+2}{2}$), this means that the corresponding particle cannot cross the horizon. Meanwhile, if the proper time is regular (this happens if $s_1 < \frac{p-q+2}{2}$), such a particle can cross the horizon.

2. Dynamics considerations: Diverging proper time

In this section, we start with the case of diverging proper time. Combining the defining condition for subcritical particles (62) and the condition when the proper time (64) diverges, one obtains that in this case

$$\begin{cases} 0 < s_1 < \frac{p}{2} & \text{if } q \geq p+2 \\ \frac{p-q+2}{2} \leq s_1 < \frac{p}{2} & \text{if } 2 < q < p+2 \\ \text{Impossible} & \text{if } q \leq 2 \end{cases}. \quad (65)$$

As explained above, we have to investigate components of acceleration in the CO frame (8)–(11). We start our analysis with the $a_C^{(1)}$ (9) component.

$$a_C^{(1)} = P \frac{\sqrt{A}}{N} \frac{\partial_r X + L \partial_r \omega}{\sqrt{X^2 - N^2}} - u^\theta \frac{\partial_\theta X + L \partial_\theta \omega}{\sqrt{X^2 - N^2}}. \quad (66)$$

Let us, as previously, consider each term separately. First of all, from previous analysis of kinematics [Eqs. (60) and (61)] it follows that the function P in this case is of order $\sim u^{s_1}$. Thus the first term in $a_C^{(1)}$ (9) is of order

$$P \frac{\sqrt{A}}{N} \frac{\partial_r X}{\sqrt{X^2 - N^2}} \sim v^{s_1 + \frac{q-p}{2}-1}. \quad (67)$$

This term will be regular if $s_1 + \frac{q-p}{2} - 1 \geq 0$, or, equivalently, $s_1 \geq \frac{p-q+2}{2}$. Note that this condition has to be compatible with (65). But looking at this condition can notice that (65) is stronger. Thus, the first term in (9) will be regular if (65) holds.

Now let us investigate the second term in $a_C^{(1)}$ (9). This term near the horizon behaves like

$$P \frac{\sqrt{A}}{N} \frac{L \partial_r \omega}{\sqrt{X^2 - N^2}} \sim v^{\frac{q-p}{2}+k-1}. \quad (68)$$

This term is regular if the regularity conditions hold (15).

The third term in $a_C^{(1)}$ (9) is $u^\theta \frac{\partial_\theta X}{\sqrt{X^2 - N^2}} \sim v^{c_1+s_2-s_1}$. This term is regular if

$$c_1 \geq s_1 - s_2. \quad (69)$$

The last term in $a_C^{(1)}$ (9) is $u^\theta \frac{L \partial_\theta \omega}{\sqrt{X^2 - N^2}} \sim v^{c_1+l-s_1}$. This term will be regular if

$$c_1 \geq s_1 - l. \quad (70)$$

Combining (70) with (69) one obtains

$$c_1 \geq s_1 - \min(l, s_2). \quad (71)$$

Now we are able to investigate the $a_C^{(2)}$ (10). The first term is

$$-\frac{1}{2} \frac{P'}{P} \frac{A}{\sqrt{g_\theta}} \partial_\theta \left(\frac{P^2}{AN^2} \right). \quad (72)$$

Note that as we are considering the case of subcritical particles, both quantities P and P' tend to zero near horizon with the same rate $\sim v^{s_1}$. Thus, expanding the expression for P^2 one would obtain that finiteness of this term will give exactly the condition (30), namely

$$\begin{aligned} & \frac{\partial_\theta X^2}{N^2} - X^2 \frac{\partial_\theta (AN^2)}{AN^4} + \frac{\partial_\theta A}{A} \left(1 + \frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right) \\ & - \partial_\theta \left(\frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right) = O(1). \end{aligned} \quad (73)$$

The first term in this expression is $\frac{\partial_\theta X^2}{N^2} = \frac{2X\partial_\theta X}{N^2} \sim v^{s_1+s_2-p}$. It will be regular if

$$s_2 \geq p - s_1. \quad (74)$$

The last term $\partial_\theta \left(\frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right)$ has been already analyzed in the previous section. It was shown that for the regularity of this term quantities L and u^θ have to be regular and conditions (50) have to hold. This automatically leads to regularity of the third term. However, the second term is not so easy. Finiteness of this term requires us to have $\frac{\partial_\theta (AN^2)}{AN^2} \sim v^{p-2s_1}$. This condition will be satisfied if the conditions (33)–(34) hold for all $s \in \{0, \dots, p - 2s_1\}$.

Now let us analyze the second term in $a_C^{(2)}$ (10), namely $\frac{1}{\sqrt{g_\theta}} \frac{P}{P'} \left(\frac{X}{N^2} (\partial_\theta X + L\partial_\theta \omega) - \frac{L\partial_\theta L}{g_\phi} \right)$. As for subcritical particles near the horizon $P \sim P' \sim v^{s_1}$, this term will be regular if $\frac{X}{N^2} (\partial_\theta X + L\partial_\theta \omega) - \frac{L\partial_\theta L}{g_\phi}$ is regular. The first term in this expression is of order $v^{s_1+s_2-p}$ and will be regular if (74) holds. The second term is of order v^{s_1+l-p} and it will be regular if

$$l \geq p - s_1. \quad (75)$$

The last term is regular for non-negative b_2 (50).

Now let us move to the third term in $a_C^{(2)}$ (10), namely $u^\theta \frac{\sqrt{AN}}{P'} \left(\frac{X}{N^2} (\partial_r X + L\partial_r \omega) - \frac{L\partial_r L}{g_\phi} \right)$. The first term in this expression is of order $v^{c_1+\frac{q-p}{2}+s_1-1}$. But it can be easily seen that if the conditions (65) and (50) hold, this term is regular. The second term here is of order $v^{c_1+\frac{q-p}{2}+k-1}$ and will be regular if (15) and (50) hold. The last term in this expression is of order $v^{c_1+\frac{q-p}{2}-s_1+b_1-1}$. This term is also regular because $c_1, b_1 \geq 0$ according to (50) and as $0 < s_1 < p/2$ for subcritical particles and $q > 2$ for diverging proper time.

The last term in $a_C^{(2)}$ (10) is $P' \frac{\sqrt{A}}{N} \frac{\partial_r (g_\theta u^\theta)}{\sqrt{g_\theta}} = P' \frac{\sqrt{A}}{N} \frac{\partial_r (g_\theta u^\theta + g_\theta \partial_r u^\theta)}{\sqrt{g_\theta}}$. As for subcritical particles near the horizon holds $P' \approx X \sim v^{s_1}$, the first term in this expression is of order $v^{s_1+\frac{q-p}{2}+c_1}$, while the second one is $v^{s_1+\frac{q-p}{2}+c_1-1}$. However, both these terms are regular because $c_1 \geq 0$ according to (50) and $\frac{p-q+2}{2} \leq s_1$ according to (65).

Now let us investigate the last component $a_C^{(3)}$ (11). Note that for the analysis of this component it is useful to remind the reader that $P \approx P' \approx \sqrt{X^2 - N^2} \approx X \sim v^{s_1}$. The first term in $a_C^{(3)}$ (11) is of order $v^{c_1+b_2}$. As $c_1 \geq 0$ (50), we see that this term is regular if $b_2 \geq 0$. The second term in $a_C^{(3)}$ (11) is of order $v^{c_1+s_2-s_1}$. This term is regular because $c_1 \geq 0$ (50) and as $s_2 \geq s_1$ by the definition. The third term is of order $v^{c_1-s_1+l}$. This term is regular because $c_1 \geq 0$ (50) and as $l \geq \lceil \frac{p+1}{2} \rceil$ because of (15) and as $0 < s_1 < p/2$ by the definition. The fourth term in $a_C^{(3)}$ (11) is of order $v^{s_1+\frac{q-p}{2}+b_1-1}$. But it is regular because of conditions (65) and (50). The fifth term in $a_C^{(3)}$ (11) is of order $v^{\frac{q-p}{2}+s_1-1}$ and it is regular because of (65). The last term in $a_C^{(3)}$ (11) is of order $v^{\frac{q-p}{2}+k-1}$ and it is regular because of regularity conditions (15).

Summarizing all the conditions, we see that s_1 is constrained only by conditions (65), while s_2 is constrained by (74). The coefficients $c_{1,2}, b_{1,2}$ are constrained only by (50). Coefficient k is constrained only by regularity conditions (15), while l is constrained by (75). But in addition to these conditions (33) and (34) have to hold for all $s \in \{0, \dots, p - 2s_1\}$. Thus for subcritical particles with diverging proper time we require only

$$c_{1,2} \geq 0, b_{1,2} \geq 0, \begin{cases} 0 < s_1 < \frac{p}{2} & \text{if } q \geq p+2 \\ \frac{p-q+2}{2} \leq s_1 < \frac{p}{2} & \text{if } 2 < q < p+2, \\ \text{Impossible} & \text{if } q \leq 2 \end{cases} \quad s_2 \geq p - s_1, (15), l \geq p - s_1$$

along with conditions (33–34) for $s \in \{0, \dots, p - 2s_1\}$. (76)

3. Dynamics considerations: Regular proper time

If proper time is regular for subcritical particles [what happens if $s_1 < \frac{p-q+2}{2}$, see (64)], such a particle can easily cross the horizon. As for subcritical particles the condition $0 < s_1 < p/2$ has to hold, combining this with the condition $s_1 < \frac{p-q+2}{2}$ one obtains that subcritical particles will have regular proper time if

$$\begin{cases} \text{Impossible} & \text{if } q \geq p+2 \\ 0 < s_1 < \frac{p-q+2}{2} & \text{if } 2 < q < p+2 \\ 0 < s_1 < \frac{p}{2} & \text{if } q \leq 2 \end{cases} \quad (77)$$

The first equation we are going to investigate is (9). The first term in $a_C^{(1)}$ is of order

$$P \frac{\sqrt{A}}{N} \frac{\partial_r X}{\sqrt{X^2 - N^2}} \sim u^{\frac{q-p}{2} + s_1 - 1}. \quad (78)$$

This term will be regular if $s_1 \geq \frac{p-q+2}{2}$. However, regularity of the proper time requires one to have $s_1 < \frac{p-q+2}{2}$, see (64), thus without any further analysis we can say that it is impossible to have subcritical particles with regular proper time.

C. Critical particles

The next step is to consider an analog of the so-called critical particles. In [6], where only equatorial motion was investigated, critical particles were defined in such a way that expansion for the quantity X was given in the form

$$X = X_{p/2} v^{p/2} + o(v^{p/2}). \quad (79)$$

In our case we generalize this definition and assume that for critical particles the expansion holds,

$$X = \hat{X}_{p/2} v^{p/2} + \dots + \hat{X}_{s_2-1} v^{s_2-1} + X_{s_2} v^{s_2} + o(v^{s_2}), \quad (80)$$

where $s_2 \geq p/2$. Here, as previously, that means that the corresponding quantity is independent of θ . Expansions for all other quantities, as in the previous sections, are given by

$$N^2 = \kappa_p v^p + o(v^p), \quad A = A_q v^q + o(v^q), \quad (81)$$

$$L = \hat{L}_H + \hat{L}_{b_1} v^{b_1} + \dots + \hat{L}_{b_2-1} v^{b_2-1} + L_{b_2} v^{b_2} + o(v^{b_2}), \quad (82)$$

$$u^\theta = \hat{u}_{c_1}^\theta v^{c_1} + \dots + \hat{u}_{c_2-1}^\theta v^{c_2-1} + u_{c_2}^\theta v^{c_2} + o(v^{c_2}), \quad (83)$$

$$\omega = \hat{\omega}_H + \hat{\omega}_k v^k + \dots + \hat{\omega}_{l-1} v^{l-1} + \omega_l v^l + o(v^l), \quad (84)$$

$$g_a = g_{aH} + g_{am} v^m + o(v^m). \quad (85)$$

1. Kinematical considerations

We start with the analysis of the kinematics of such particles. Critical particles may exist if the radial component of the four-velocity is real near the horizon. This is equivalent to the requirement that P^2 is non-negative. Namely, we require

$$P^2 = X^2 - N^2 \left(1 + \frac{L^2}{g_\varphi} + g_\theta (u^\theta)^2 \right) \geq 0. \quad (86)$$

Expanding this near $v = 0$, one obtains

$$P^2 \approx \left[\hat{X}_{p/2}^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}} \right) \right] v^p - \kappa_p g_{\theta H} (\hat{u}_{c_1}^\theta)^2 v^{p+2c_1}. \quad (87)$$

There are two terms that may potentially be divergent, and we have to investigate corresponding cases distinctly.

(i) Case $c_1 < 0$. In this case the second term is dominant and

$$P^2 = -\kappa_p g_{\theta H} (\hat{u}_{c_1}^\theta)^2 v^{p+2c_1} + o(v^{p+2c_1}). \quad (88)$$

However, P^2 is negative and thus this case is prohibited. This means that *for critical particles, it is impossible to have divergent polar velocity.*

(ii) Case $c_1 = 0$. In this case both terms in (87) are of the same order, and

$$P^2 = \left[\hat{X}_{p/2}^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}} + g_{\theta H} (\hat{u}_0^\theta)^2 \right) \right] v^p + o(v^p). \quad (89)$$

This quantity will be non-negative if

$$\hat{X}_{p/2}^2 \geq \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}} + g_{\theta H} (\hat{u}_0^\theta)^2 \right). \quad (90)$$

(iii) Case $c_1 > 0$. In this case the first term in (87) is of the dominant order, and one can write

$$P^2 = \left[\hat{X}_{p/2}^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}} \right) \right] v^p + o(v^p). \quad (91)$$

Such particles may reach the horizon if

$$\hat{X}_{p/2}^2 \geq \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}} \right). \quad (92)$$

To summarize, we have to say that critical particles are possible only in the case when $c_1 \geq 0$.

Before we analyze forces acting on critical particles, let us investigate the behavior of the proper time. The radial component of the four-velocity is given by

$$u^r = \sigma \frac{\sqrt{A}}{N} P \sim v^{\frac{q}{2}}. \quad (93)$$

The proper time required to cross the horizon, in the leading order, is given by

$$\tau = \int \frac{dr}{u^r} \sim \begin{cases} v^{\frac{2-q}{2}} & \text{if } q \neq 2 \\ \ln|v| & \text{if } q = 2 \end{cases}. \quad (94)$$

The proper time diverges if $q \geq 2$ (extremal and ultra-extremal horizons). Otherwise ($q < 2$) it is regular (non-extremal horizons). We will investigate these cases distinctly.

2. Dynamics considerations: Diverging proper time ($q \geq 2$)

In this case we also use the expressions for acceleration in CO (8)–(11). Let us start with the radial component (9)

$$a_C^{(1)} = P \frac{\sqrt{A}}{N} \frac{\partial_r X + L \partial_r \omega}{\sqrt{X^2 - N^2}} - u^\theta \frac{\partial_\theta X + L \partial_\theta \omega}{\sqrt{X^2 - N^2}}. \quad (95)$$

The first term is of order (here we used that for critical particles $s = p/2$)

$$P \frac{\sqrt{A}}{N} \frac{\partial_r X}{\sqrt{X^2 - N^2}} \sim v^{\frac{q}{2}-1}. \quad (96)$$

As in this subsection we consider the case of diverging proper time ($q \geq 2$), we immediately see that this term is regular.

The second term in (9) is of order

$$P \frac{\sqrt{A}}{N} \frac{L \partial_r \omega}{\sqrt{X^2 - N^2}} \sim v^{\frac{q-p}{2}+k-1}. \quad (97)$$

As the regularity conditions (15) have to hold, this term is inevitably regular.

The third term in (9) is of order

$$\frac{u^\theta}{\sqrt{X^2 - N^2}} \partial_\theta X \sim v^{c_1+s_2-p/2}. \quad (98)$$

This term will be regular if

$$s_2 \geq p/2 - c_1.$$

Meanwhile, for critical particles we assumed that $s_2 \geq p/2$ [see the text after (80)], so this condition is automatically satisfied.

The last term in (9) is of order

$$\frac{u^\theta}{\sqrt{X^2 - N^2}} L \partial_\theta \omega \sim v^{c_1+l-p/2}. \quad (99)$$

This term will be regular if

$$c_1 \geq p/2 - l. \quad (100)$$

As from regularity conditions (15) follows that $l \geq p/2$, this condition is satisfied for any $c_1 \geq 0$,

$$c_1 \geq p/2 - \min(l, p/2). \quad (101)$$

The next component is (10),

$$\begin{aligned} a_C^{(2)} = & -\frac{1}{2} \frac{P'}{P} \frac{A}{\sqrt{g_\theta}} \partial_\theta \left(\frac{P^2}{AN^2} \right) \\ & + \frac{1}{\sqrt{g_\theta}} \frac{P}{P'} \left(\frac{X}{N^2} (\partial_\theta X + L \partial_\theta \omega) - \frac{L \partial_\theta L}{g_\phi} \right) \\ & + \sqrt{g_\theta} u^\theta \frac{\sqrt{AN}}{P'} \left(\frac{X}{N^2} (\partial_r X + L \partial_r \omega) - \frac{L \partial_r L}{g_\phi} \right) \\ & - P' \frac{\sqrt{A}}{N} \frac{\partial_r (g_\theta u^\theta)}{\sqrt{g_\theta}}. \end{aligned} \quad (102)$$

Let us start with the terms in the third line. The first two terms there (namely $\sqrt{g_\theta} u^\theta \frac{\sqrt{A}}{P'} \frac{X}{N} \partial_r X$ and $\sqrt{g_\theta} u^\theta \frac{\sqrt{A}}{P'} \frac{X}{N} L \partial_r \omega$) are of the same order as the first two terms in (9), and thus regular. The third term in the third line is of order (here we used that for critical particles $P' \sim N$)

$$\frac{\sqrt{AN}}{P'} \frac{L \partial_r L}{g_\phi} \sim v^{b_1+\frac{q}{2}-1}. \quad (103)$$

This term is regular if $b_1 \geq 1 - \frac{q}{2}$. However, now we investigate only particles with diverging proper time, meaning that $q \geq 2$. Thus the right-hand side of the condition $b_1 \geq 1 - \frac{q}{2}$ is always negative and thus the third line in (102) will be regular for any non-negative value of b_1 .

The term in the fourth line in (102) is given by

$$P' \frac{\sqrt{A}}{N} \frac{\partial_r (g_\theta u^\theta)}{\sqrt{g_\theta}} = P' \frac{\sqrt{A}}{N \sqrt{g_\theta}} (\partial_r (g_\theta) u^\theta + g_\theta \partial_r (u^\theta)). \quad (104)$$

The second term here is dominant, because the first one is of order $\sim v^{\frac{q}{2}+m-1+c_1}$ [here m is positive, see (85)], while the second is $\sim v^{\frac{q}{2}+c_1-1}$. As for diverging proper time we require $q \geq 2$ and as $c_1 \geq 0$ (comes from kinematics), this term is nondivergent. Thus, the whole second line in (10) is regular.

Now let us analyze the first line in (102). The first line therein is given by

$$-\frac{1}{2} \frac{P'}{P} \frac{A}{\sqrt{g_\theta}} \partial_\theta \left(\frac{P^2}{AN^2} \right). \quad (105)$$

Using that for critical particles both $P \sim N$ and $P' \sim N$, and expanding the expression for P^2 [see (19)], we obtain that the condition of finiteness of this term is exactly Eq. (30),

$$\begin{aligned} \frac{\partial_\theta X^2}{N^2} - X^2 \frac{\partial_\theta (AN^2)}{AN^4} + \frac{\partial_\theta A}{A} \left(1 + \frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right) \\ - \partial_\theta \left(\frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right) = O(1). \end{aligned} \quad (106)$$

Analysis of this expression is somehow analogous to the case of diverging proper time for subcritical particles. The first term here is automatically nondivergent as it is $\sim v^{s_2-p/2}$. As $s_2 \geq p/2$ (80), this term is regular. The second term in (30) is automatically finite as for critical particles $s_1 = p/2$. The third and fourth terms in (30) are regular because u^θ is regular (as the condition $c_1 \geq 0$ follows from kinematics), thus both $(\frac{L^2}{g_\phi} + g_\theta (u^\theta)^2)$ and $(1 + \frac{L^2}{g_\phi} + g_\theta (u^\theta)^2)$ are regular.

We are left with the $\frac{1}{\sqrt{g_\theta}} \frac{P'}{P'} \left(\frac{X}{N^2} (\partial_\theta X + L \partial_\theta \omega) - \frac{L \partial_\theta L}{g_\phi} \right)$ term in the second line of (102). However, as for critical particles $X \sim N$, $P' \sim P$ this term is regular for all $c_2, b_2, l \geq 0$.

Now let us move to the last component, (11):

$$\begin{aligned} a_C^{(3)} = \frac{1}{\sqrt{X^2 - N^2}} \left\{ \frac{u^\theta}{\sqrt{g_\phi} P'} [(X^2 - N^2) \partial_\theta L - LX (\partial_\theta X + L \partial_\theta \omega)] \right. \\ \left. - \frac{P}{\sqrt{g_\phi} P'} \frac{\sqrt{A}}{N} [(X^2 - N^2) \partial_r L - LX (\partial_r X + L \partial_r \omega)] \right\}. \end{aligned}$$

The first term $\frac{u^\theta}{\sqrt{g_\phi} P'} \sqrt{X^2 - N^2} \partial_\theta L \sim v^{b_2+c_1}$ and is regular for any $b_2 \geq 0$ (as $c_1 \geq 0$). The second term $\frac{u^\theta}{\sqrt{g_\phi} P'} \frac{LX}{\sqrt{X^2 - N^2}} \partial_\theta X \sim v^{s_2-p/2+c_1}$ and it is regular as $s_2 \geq p/2$ (80). The third term is of order of $\frac{u^\theta}{\sqrt{g_\phi} P'} \frac{L^2 X}{\sqrt{X^2 - N^2}} \partial_\theta \omega \sim v^{l-p/2+c_1}$ and is regular because of the regularity condition (15). The fourth term is of order $\frac{P}{\sqrt{g_\phi} P'} \frac{\sqrt{A}}{N} \sqrt{X^2 - N^2} \partial_r L \sim v^{\frac{q}{2}-1+b_1}$ and it is regular because for diverging proper time $q \geq 2$ and as $b_1 \geq 0$. The fifth term is of order $\frac{1}{\sqrt{X^2 - N^2}} \frac{P}{\sqrt{g_\phi} P'} \frac{\sqrt{A}}{N} LX \partial_r X \sim v^{\frac{q}{2}-1}$ and it is regular for diverging proper time $q \geq 2$. The sixth term is of order $\frac{1}{\sqrt{X^2 - N^2}} \frac{P}{\sqrt{g_\phi} P'} \frac{\sqrt{A}}{N} L^2 X \partial_r \omega \sim v^{\frac{q}{2}-\frac{p}{2}+k-1}$ and is regular because of the regularity condition (15).

Thus we see that if the proper time diverges (that happens in the case $q \geq 2$), the regularity of the forces

acting on such particles does not require anything new as compared to kinematic restrictions [namely $c_1 \geq 0$, (90) and (92)] except for conditions of regularity of horizons (15).

3. Dynamics considerations: Regular proper time ($q < 2$)

We start with the (9) component,

$$a_C^{(1)} = P \frac{\sqrt{A}}{N} \frac{\partial_r X + L \partial_r \omega}{\sqrt{X^2 - N^2}} - u^\theta \frac{\partial_\theta X + L \partial_\theta \omega}{\sqrt{X^2 - N^2}}. \quad (107)$$

The first term is of order

$$P \frac{\sqrt{A}}{N} \frac{\partial_r X}{\sqrt{X^2 - N^2}} \sim v^{\frac{q}{2}-1}, \quad (108)$$

where we took into account that for critical particles $X \sim N \sim P$.

However, in this subsection we assume that the proper time is regular (so that $q < 2$), and thus this term diverges. This means that such a situation is impossible.

To conclude this section devoted to critical particles, we have to note that they can reach the horizon only if $c_1 \geq 0$ and conditions (90) and (92) have to hold. Forces acting on such particles are regular only if $q \geq 2$. However, this leads to the diverging proper time [namely, $c_1 \geq 0$, (90) and (92)].

D. Ultracritical particles

The last type of particles we are going to investigate are ultracritical ones. In [6] this type of particles was defined in such a way that for them an expansion holds:

$$X = X_{p/2} v^{p/2} + o(v^{p/2}). \quad (109)$$

1. Kinematic considerations

We require that for ultracritical particles the expansion coefficients are such that several dominant terms in the expression for P cancel each other and one obtains that the radial component of the four-velocity behaves as $u^r \sim v^i$, where $i > \frac{q}{2}$ (the latter value was obtained for critical particles). To find what conditions have to hold in this case, let us consider the function P :

$$P^2 = X^2 - N^2 \left(1 + \frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right). \quad (110)$$

As was noted in the previous section, P^2 will be non-negative if $c_1 \geq 0$. Then, P becomes

$$P^2 = \left[X_{p/2}^2 - \kappa_p \left(1 + \frac{L_{\phi H}^2}{g_{\phi H}} + g_{\theta H} (u_{\theta H}^\theta)^2 \right) \right] v^p + o(v^p). \quad (111)$$

The radial component of the four-velocity is given by $u^r = \sigma \frac{\sqrt{A}}{N} P$. This quantity will be of order v^i if we choose coefficients X_s in such a way that

$$X^2 - N^2 \left(1 + \frac{L^2}{g_\phi} + g_\theta (u^\theta)^2 \right) \sim v^{2i+p-q}. \quad (112)$$

We will not write an explicit expression for X_s because it is quite complicated and it is unnecessary.

2. Dynamics considerations

Detailed analysis of forces is not required because this case differs from the case of critical particles only in the behavior of the function P with P decreasing to zero faster for ultracritical particles than for critical particles. To see this, we have to note that in (8)–(11) the function P appears only in the numerator. In turn, this implies that the conditions of the regularity of forces for critical particles lead to the fact that for ultracritical particles these conditions also ensure the regularity of forces.

This completes consideration of different types of particles. Meanwhile, we must clarify one important feature. In general, expression (86) depends on the angle θ . This means that fulfilment of this condition depends not only on parameters of a particle but also on angle θ . The most representative in this sense is the situation near the poles of black holes. For regular black holes (without any conical singularities) it holds typically that near the poles $g_{\phi H} \sim \theta^2$ or $g_{\phi H} \sim (\pi - \theta)^2$. Thus the right-hand side of (90) behaves as $\sim \theta^{-2}$ as one approaches pole $\theta = 0$, and the inequality (86) cannot be satisfied everywhere. This creates the “belts” near the equator where a particle can move and where different scenarios of collisions may occur. (For the details of these belts in the Kerr metric and their role for properties of critical particles and, hence, the BSW effect, see Sec. IV of [4]. However, we would like to stress that the existence of belts is a generic property of nonequatorial motion, inherent not only to critical particles.)

Concluding all the previous sections, we present Table I which summarizes the kinematics and regularity of forces constraints. It can be seen that the regularity of forces requires that usual particles can have either regular or diverging proper time, while fine-tuned particles can have only diverging proper time.

V. CONSEQUENCES OF FINITENESS OF u^θ

In previous sections, we have shown that regularity of force implies that negative c_1 is incompatible with the regularity of acceleration and, hence, u^θ and the third component of the velocity in the OZAMO frame (7) near the horizon remain finite (if $c_1 = 0$) or even tends to zero (if $c_1 > 0$). This has important consequences. First of all, it means that a general classification of possible scenarios of

the BSW effect found in the equatorial case remains valid here as well. Namely, the BSW effect occurs for collisions of such particles: usual and subcritical, two subcritical, and usual or subcritical collides with critical or ultracritical (see Table II in [6] for details).

Also, the fact that $c_1 \geq 0$ enables us to fill some gaps in previous consideration. A general explanation of the BSW effect was suggested earlier [16] in which the following expansion of the four-velocity was used:

$$u^\mu = \frac{l^\mu}{2\alpha} + \beta N^\mu + s^\mu, \quad (113)$$

where l^μ and N^μ are null vectors and s^μ is the spacelike one that is orthogonal to l^μ and N^μ . The same expansion was used in [17] to elucidate the role of the bifurcation surface as a potential particle accelerator. In both cases, the assumption was made according to which $s_\mu s^\mu$ remained finite. Now this is proved since

$$s_\mu s^\mu = g_\phi (u^\phi)^2 + g_\theta (u^\theta)^2, \quad (114)$$

where both terms are finite.

The same assumption of the finiteness of u^θ was made in Appendix B of [18] to show that in the OZAMO frame a usual particle hits the horizon perpendicularly. Now this assumption is proven.

VI. THREE-VELOCITIES AND RELEVANT FRAMES

Kinematically, unbounded growth of the energy of collisions $E_{c.m.}$ in the center-of-mass frame (the BSW effect) can be understood in terms of three-dimensional velocities. Such an entity can be defined in the terad formalism according to

$$V^{(i)} = - \frac{e_\mu^{(i)} u^\mu}{e_\mu^{(0)} u^\mu}, \quad (115)$$

where $e_\mu^{(a)}$ ($a = 0, 1, 2, 3$) is a tetrad attached to an observer.

The BSW effect occurs if the relative velocity of both particles w approaches in this case that of light, so boost between them becomes divergent. One of the easiest ways to show this is to use the so-called OZAMO frame for evaluation of components of $V^{(i)}$. This was shown in [11], where usual and critical particles were considered. Now, we can generalize the corresponding results to scenarios with other types of particles. We can look at the properties of w in two ways. Either we (i) choose an OZAMO frame similarly to [11] and calculate w or (ii) carry out calculations of velocity of particle 2 directly in the frame CO comoving with particle 1.

Let us consider both approaches separately.

A. Evaluation of relative velocity in OZAMO frame

At first, we discuss case (i). Then, the form of tetrads is given in Appendix A. This approach has an additional motivation since the ZAMO frame (in both versions, OZAMO and FZAMO) is a powerful standard tool for description of particle motion near a black hole. It follows from (A3) that in the horizon limit the ratio $V^{(2)}/V^{(1)} \rightarrow 0$ as well as $V^{(3)}/V^{(1)}$ and $V^{(1)} \rightarrow 1$ for usual and subcritical particles. Therefore, such kinds of particles hit the horizon perpendicularly, so any two of them move in the same direction. Then, we follow Sec. III of [11]. Using the formulas of special relativity, we have for the relative velocity of two particles

$$w = \frac{V_1 - V_2}{1 - V_1 V_2}. \quad (116)$$

Let near the horizon $V = 1 - \varepsilon$, where $\varepsilon \ll 1$. Then, in the limit under consideration

$$w_H = \frac{|\varepsilon_1 - \varepsilon_2|}{\varepsilon_1 + \varepsilon_2}. \quad (117)$$

The key point here is the rate with which $V \rightarrow 1$. For a usual particles, it follows from (A4) that $\varepsilon_1 \sim \varepsilon_2 \sim N^2$, so $w_H < 1$ and the boost is finite. If particle 1 is usual and particle 2 is subcritical, $\varepsilon_1 \sim N^2 \sim v^p$ and $\varepsilon_2 \sim v^{p-2s_1}$, so $\varepsilon_1/\varepsilon_2 \rightarrow 0$. Then, $w_H = 1$ and the boost is infinite. If both particles are subcritical, $\varepsilon_1 \sim v^{p-2s_1} \sim \varepsilon_2$, $w_H < 1$, so the boost is finite.

If particle 1 is critical or ultracritical, its limiting velocity $V_{1H} < 1$ and it hits the horizon under some angle, not perpendicularly. Then, if particle 2 is usual the relative velocity $w_H \rightarrow 1$ and the boost is infinite [11]. The same is true if particle 2 is subcritical. If both particles are critical or ultracritical, $w_H < 1$ and the boost is finite.

We obtained the set of possibilities represented in Table I. As, in general, there are three relevant frames (OZAMO, FZAMO, and CO), we present the extended version of Table II in which we included characteristics in corresponding frames in addition to OZAMO. In the last two columns it is implied that the CO is realized by a particle indicated in the first column. In all cases FZAMO is

realized by usual particles, while OZAMO is realized by critical and ultracritical ones. In addition, the boost between FZAMO and OZAMO is always infinite. This is in correspondence with row 4 in Table II in [6] that shows that the relative gamma factor diverges if one of the particles is usual (equivalent to FZAMO) and the second particle is critical or ultracritical (equivalent to OZAMO).

B. Evaluation of relative velocity in CO frame

Now, we turn to approach (ii). In doing so, we attach a tetrad to particle 1 and look at the velocity of particle 2 in this frame. We can obtain even more detailed information concerning the motion of the particle than it was found above. This is because now we are able to analyze not only the absolute value of relative velocity but also its different components. To this end, we use the results listed in Appendix A. We attach the tetrad $\tilde{e}_\mu^{(i)}$ to some particle 1. Then, for any other particle 2 with four-velocity $\hat{u}^\mu$ we have using (2):

$$\hat{V}^i = \frac{\tilde{e}_\mu^{(i)} \hat{u}^\mu}{\gamma}, \quad (118)$$

where we use hat for quantities related to this particle, and $\gamma = -\tilde{e}_\mu^{(0)} \hat{u}^\mu$ is the Lorentz gamma factors of relative motion of both particles. In other words, we measure the velocity of particle 2 from the point of view of an observer comoving with particle 1.

Straightforward calculation gives us

$$\hat{V}^{(1)} = \frac{1}{\gamma V} \left[-\sigma(X^2 - N^2) \frac{\hat{X}}{N^2 X} + \left(\sigma L \frac{\hat{L}}{g_\phi} + \sigma g_\theta u^\theta \hat{u}^\theta \right) + \frac{\hat{\sigma} P \hat{P}}{N^2} \right], \quad (119)$$

$$\hat{V}^{(2)} = \frac{\sqrt{g_\theta}}{P' \gamma} (-\sigma \hat{\sigma} \hat{P} u^\theta + P \hat{u}^\theta), \quad (120)$$

$$\hat{V}^{(3)} = \frac{1}{P' X V \sqrt{g_\phi} \gamma} [\hat{L} (P')^2 - g_\theta u^\theta \hat{u}^\theta L N^2 - \sigma \hat{\sigma} P \hat{P} L], \quad (121)$$

TABLE I. List of conditions that have to hold for different types of particles. The second column represents conditions coming from kinematic conditions. The conditions in the last two columns are the ones that have to hold in addition to the kinematic restrictions to have regular forces. “Same as for k.r.” means that the kinematic restrictions (denoted for shortness as “k.r.”) are sufficient for the regularity of the force.

Type of particle	Kinematic restrictions	Regular proper time	Diverging proper time
Usual ($s_1 = 0$)	None	(48)	(51)
Subcritical ($0 < s_1 < p/2$)	(62)	Impossible	(76)
Critical ($s_1 = p/2$)	(90) if $c_1 = 0$ and (92) if $c_1 > 0$	Impossible	Same as for k.r.
Ultracritical ($s_1 > p/2$)	(112) and $c_1 \geq 0$	Impossible	Same as for k.r.

TABLE II. Table showing different boost types between CO-OZAMO and CO-FZAMO frames for different types of particles.

Particle	Boost CO-OZAMO	Boost CO-FZAMO
Usual	Infinite	Finite
Subcritical	Infinite	Infinite
Critical and ultracritical	Finite	Infinite

where

$$\gamma = -\tilde{e}_\mu^{(0)} \hat{u}^\mu = \frac{X\hat{X} - \sigma\hat{\sigma}P\hat{P} - N^2 g_\theta u^\theta \hat{u}^\theta}{N^2} - \frac{L\hat{L}}{g_\phi}. \quad (122)$$

If both particles coincide, one can check that (A19) holds, so $\hat{V}^{(i)} = 0$ as it should be.

Now, in contrast to the OZAMO frame, a usual particle can hit the horizon under an arbitrary angle and with an arbitrary value of velocity. For a simplified case of motion of particle in the background of a spherically symmetric static black hole this issue is discussed in Sec. 7 of [19]. In this case, OZAMO turns into a static frame. It was stressed in [19] that the boost between the static frame and CO is diverging near the horizon. Then, if the angular component of velocity of particle 2 is very small in the static frame, it becomes of order 1 in the frame comoving with particle 1, provided particle 1 moves freely or under the action of finite force. Formulas (119)–(121) can be of use in any analysis of particle motion near the horizon in a physical (nonsingular frame) analysis of particle motion, say, in flows around a generic rotating black hole.

VII. CONCLUSIONS

In this work we have investigated motion of particles moving near the horizon of a generic axially symmetric black hole. In doing so, we did not constrain consideration by motion in the equatorial plane. We also discussed different types of horizons characterized by a set of integers p, q, k, l . Moreover, we took into account the action of an arbitrary (regular) force on such particles.

During the analysis of restrictions of the four-velocity caused by regularity of forces in the properly chosen frame, we have found that the absolute value of the projection of the four-velocity on 2-sphere of constant radius (namely, the quantity s^μ , introduced in Sec. V) remains regular as one approaches the horizon. Moreover, this statement is true for any type of particle and is independent on whether the proper time for achieving the horizon is regular or not. This statement fills the gap in previous investigations where regularity of the polar component of the four-velocity was assumed but not proven. Also, we have to emphasize that this property does not follow from kinematics and is obtained solely from the dynamics of considered particles under the action of a regular force.

Concerning other restrictions, kinematics shows that the only new phenomenon due to nonequatorial motion is the appearance of the so-called “belts” where the motion of particles may occur. These belts encircle the equatorial plane and never reach poles $\theta = 0, \pi$. In particular, this phenomenon reveals itself in the context of the BSW effect (see discussion in [4] for the particular case of the Kerr metric). Other conditions come mainly from the dynamics, specifying the range in which powers in the Taylor expansion of the metric functions near the horizon may vary. The final list is presented in Tables I and II. It is seen from them that a regular proper time is possible only for usual particles (according to the classification, introduced in [6]), while for subcritical, critical, and ultracritical ones the proper time diverges, provided the force is regular and the horizon is regular.

As far as the properties of high energy collisions are concerned, classification of scenarios of the BSW effect found earlier for equatorial motion is reproduced for the nonequatorial one as well. Meanwhile, the results obtained in the present work have quite a general nature. They are not restricted by high energy collisions and can apply to any physical or astrophysical problem in which nonequatorial motion is essential. In particular, this can be of use for the description of particle motion and collision in the accretion disc, properties of chaotic motion, and so on.

ACKNOWLEDGMENTS

O. Z. was supported in part by Grants No. 2024/22940-0 and No. 2021/10128-0 of São Paulo Research Foundation (FAPESP).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ESTABLISHING THE COMOVING FRAME

In this appendix we describe how one can establish a comoving frame from the OZAMO frame [9]. We rely on relations (5), (6):

$$\begin{aligned} e_{(0)}^\mu &= \frac{1}{N}(1, 0, 0, \omega), & e_{(1)}^\mu &= \sqrt{A}(0, 1, 0, 0), \\ e_{(2)}^\mu &= \frac{1}{\sqrt{g_{\theta\theta}}}(0, 0, 1, 0), & e_{(3)}^\mu &= \frac{1}{\sqrt{g_{\phi\phi}}}(0, 0, 0, 1). \end{aligned} \quad (A1)$$

In this tetrad frame, the four-velocity reads

$$u^{(a)} = \left(\frac{X}{N}, \sigma \frac{P}{N}, \sqrt{g_\theta} u^\theta, \frac{L}{\sqrt{g_\phi}} \right). \quad (A2)$$

To establish transformations between OZAMO and CO frames, we have to find the three-velocity in the OZAMO

frame. Direct computation gives us

$$V^{(i)} = -\frac{e_\mu^{(i)} u^\mu}{e_\mu^{(0)} u^\mu} = \left(\frac{\sigma P}{X}, \frac{\sqrt{g_\theta} u^\theta N}{X}, \frac{LN}{\sqrt{g_\varphi} X} \right). \quad (\text{A3})$$

The absolute value of the three-velocity is

$$|V| = \sqrt{1 - \frac{N^2}{X^2}}. \quad (\text{A4})$$

To transform from the OZAMO frame to the comoving frame, one has to perform the following operations:

(i) Rotation in the $r\theta$ plane

$$e'_{(1)} = e_{(1)} \cos \delta + e_{(2)} \sin \delta, \quad e'_{(0)} = e_{(0)}, \quad (\text{A5})$$

$$e'_{(2)} = e_{(2)} \cos \delta - e_{(1)} \sin \delta, \quad e'_{(3)} = e_{(3)}, \quad (\text{A6})$$

with

$$\tan \delta = \sigma \sqrt{g_\theta} \frac{u^\theta N}{P}. \quad (\text{A7})$$

(ii) Rotation in the $r\varphi$ plane

$$e''_{(1)} = e'_{(1)} \cos \psi + e'_{(3)} \sin \psi, \quad e''_{(0)} = e'_{(0)}, \quad (\text{A8})$$

$$e''_{(3)} = e'_{(1)} \cos \psi - e'_{(3)} \sin \psi, \quad e''_{(2)} = e'_{(2)}, \quad (\text{A9})$$

with

$$\tan \psi = \sigma \frac{LN}{\sqrt{g_\varphi} P'}, \quad (\text{A10})$$

where

$$P' = \sqrt{X^2 - N^2} \left(1 + \frac{L^2}{g_\varphi} \right). \quad (\text{A11})$$

(iii) Perform a boost

$$\tilde{e}_{(0)} = \gamma(e''_{(0)} + \sigma |V| e''_{(1)}), \quad \tilde{e}_{(2)} = e''_{(2)}, \quad (\text{A12})$$

$$\tilde{e}_{(1)} = \gamma(e''_{(1)} + \sigma |V| e''_{(0)}), \quad \tilde{e}_{(3)} = e''_{(3)}, \quad (\text{A13})$$

where $\gamma = \frac{1}{\sqrt{1-V^2}} = \frac{X}{N}$.

Combining all these transformations, one obtains that comoving tetrad is given by

$$\begin{pmatrix} \tilde{e}_{(0)} \\ \tilde{e}_{(1)} \\ \tilde{e}_{(2)} \\ \tilde{e}_{(3)} \end{pmatrix} = \begin{pmatrix} \frac{X}{N} & \sigma \frac{P}{N} & \sqrt{g_\theta} u^\theta & \frac{L}{\sqrt{g_\varphi}} \\ \sigma \frac{X}{N} |V| & \frac{P}{N|V|} & \sigma \sqrt{g_\theta} \frac{u^\theta}{|V|} & \sigma \frac{L}{\sqrt{g_\varphi} |V|} \\ 0 & -\sigma \sqrt{g_\theta} \frac{u^\theta N}{P} & \frac{P}{P'} & 0 \\ 0 & -\sigma \frac{LN}{\sqrt{g_\varphi} X |V|} \frac{P}{P'} & -\sqrt{\frac{g_\theta}{g_\varphi}} \frac{L u^\theta N^2}{P' X |V|} & \frac{P'}{X |V|} \end{pmatrix} \begin{pmatrix} e_{(0)} \\ e_{(1)} \\ e_{(2)} \\ e_{(3)} \end{pmatrix}, \quad (\text{A14})$$

or, explicitly,

$$\tilde{e}_\mu^{(0)} = \left(X + L\omega, -\sigma \frac{P}{\sqrt{AN}}, -g_\theta u^\theta, -L \right), \quad (\text{A15})$$

$$\tilde{e}_\mu^{(1)} = \frac{1}{|V|} \left(-\sigma \left(X + L\omega - \frac{N^2}{X} \right), \frac{P}{\sqrt{AN}}, \sigma g_\theta u^\theta, \sigma L \right), \quad (\text{A16})$$

$$\tilde{e}_\mu^{(2)} = \left(0, -\sigma \frac{\sqrt{g_\theta} N u^\theta}{\sqrt{AP'}}, \sqrt{g_\theta} \frac{P}{P'}, 0 \right), \quad (\text{A17})$$

$$\tilde{e}_\mu^{(3)} = \frac{1}{|V|} \left(-\sqrt{g_\varphi} \frac{P' \omega}{X}, -\sigma \frac{LN}{\sqrt{Ag_\varphi} X} \frac{P}{P'}, -\frac{g_\theta}{\sqrt{g_\varphi}} \frac{u^\theta LN^2}{P' X}, \sqrt{g_\varphi} \frac{P'}{X} \right). \quad (\text{A18})$$

One can check that in this frame holds

$$\tilde{V}^i = -\frac{\tilde{e}_\mu^{(i)} u^\mu}{\tilde{e}_\mu^{(0)} u^\mu} = (0, 0, 0), \quad (\text{A19})$$

as it should be, and the four-velocity

$$\tilde{u}^{(a)} = \tilde{e}_\mu^{(a)} u^\mu = (1, 0, 0, 0). \quad (\text{A20})$$

Calculation of acceleration in this frame $a_C^{(a)} = \tilde{e}_\mu^{(a)} a^\mu$ gives (8)–(11).

For completeness and the readers' convenience, we give below also the expressions of the acceleration in the OZAMO frame $a_O^{(0)} = a^\mu e_{\mu(a)}$, although we do not use them directly:

$$a_O^{(0)} = \sigma P \frac{\sqrt{A}}{N^2} (\partial_r X + L \partial_r \omega) + \frac{u^\theta}{N} (\partial_\theta X + L \partial_\theta \omega), \quad (\text{A21})$$

$$a_O^{(1)} = X \frac{\sqrt{A}}{N^2} \left(\partial_r X + L \partial_r \omega - \frac{N^2}{X} \frac{L \partial_r L}{g_\phi} \right) + u^\theta \sqrt{A} \left(\sigma \partial_\theta \left(\frac{P}{N \sqrt{A}} \right) - \partial_r (g_\theta u^\theta) \right), \quad (\text{A22})$$

$$a_O^{(2)} = \sigma \frac{\sqrt{A}}{N} \frac{P}{\sqrt{g_\theta}} \partial_r (g_\theta u^\theta) + u^\theta \partial_\theta (\sqrt{g_\theta} u^\theta) + \frac{P^2 \partial_\theta A}{2 A N^2 \sqrt{g_\theta}} + \frac{X^2 \partial_\theta N^2}{2 N^4 \sqrt{g_\theta}} + \frac{L X \partial_\theta \omega}{\sqrt{g_\theta} N^2} - \frac{L^2 \partial_\theta g_\phi}{2 g_\phi^2 \sqrt{g_\theta}}, \quad (\text{A23})$$

$$a_O^{(3)} = \sigma \frac{\sqrt{A}}{N} \frac{P}{\sqrt{g_\phi}} \partial_r L + \frac{u^\theta \partial_\theta L}{\sqrt{g_\phi}}. \quad (\text{A24})$$

APPENDIX B: RELATION TO LORENTZ GROUP TRANSFORMATIONS

Above, we built the matrix that transforms a stationary frame to a frame that is comoving with a particle. It includes dynamic characteristics of a particle—such as energy and components of momentum. In this appendix we give an alternative approach that relies on the group transformations within the Lorentz group.

It follows from Eq. (A14) that

$$\tilde{e}_{(0)} = \frac{X}{N^2} \partial_t + \sigma P \frac{\sqrt{A}}{N} \partial_r + u^\theta \partial_\theta + \left(\frac{L}{g_\phi} + \frac{X \omega}{N^2} \right) \partial_\phi, \quad (\text{B1})$$

$$\tilde{e}_{(1)} = \sigma \frac{X}{N^2} \partial_t + \frac{P}{|V|} \frac{\sqrt{A}}{N} \partial_r + \sigma \frac{u^\theta}{|V|} \partial_\theta + \sigma \left(\frac{L}{g_\phi |V|} + \frac{X \omega}{N^2} \right) \partial_\phi, \quad (\text{B2})$$

$$\tilde{e}_{(2)} = \frac{-\sigma \sqrt{g_\theta} u^\theta N}{P'} \sqrt{A} \partial_r + \frac{P}{P'} \frac{1}{\sqrt{g_\theta}} \partial_\theta, \quad (\text{B3})$$

$$\tilde{e}_{(3)} = -\sigma \frac{LN}{\sqrt{g_\phi} X V} \frac{P}{P'} \sqrt{A} \partial_r - \frac{L u^\theta N^2}{\sqrt{g_\phi} P' X V} \partial_\theta + \frac{P'}{V X \sqrt{g_\phi}} \partial_\phi. \quad (\text{B4})$$

Let us introduce a new parameter q and use also angles δ and ψ , satisfying:

$$\cosh q = \frac{X}{N}, \quad \sinh q = \sigma \frac{X|V|}{N}, \quad (\text{B5})$$

$$\cos \delta = \frac{P'}{X|V|}, \quad \sin \delta = \sigma \frac{LN}{\sqrt{g_\phi} X|V|}, \quad (\text{B6})$$

$$\cos \psi = \frac{P}{P'}, \quad \sin \psi = \sigma \sqrt{g_\theta} \frac{u^\theta N}{P'}. \quad (\text{B7})$$

Then,

$$\begin{aligned} \sigma \frac{P}{N} &= \sinh q \cos \delta \cos \psi, \\ \frac{L}{\sqrt{g_\phi}} &= \sin \delta \sinh q, \\ \sqrt{g_\theta} u^\theta &= \sinh q \cos \delta \sin \psi, \end{aligned} \quad (\text{B8})$$

and

$$\begin{aligned} \tilde{e}_{(0)} &= \frac{\cosh q}{N} \partial_t + \sinh q \cos \delta \cos \psi \sqrt{A} \partial_r \\ &\quad + \sinh q \cos \delta \sin \psi \frac{1}{\sqrt{g_\theta}} \partial_\theta \\ &\quad + \left(\frac{\sinh q \sin \delta}{\sqrt{g_\phi}} + \frac{\omega \cosh q}{N} \right) \partial_\phi, \end{aligned} \quad (\text{B9})$$

$$\begin{aligned} \tilde{e}_{(1)} &= \frac{1}{N} \sinh q \partial_t + \cosh q \cos \delta \cos \psi \sqrt{A} \partial_r \\ &\quad + \frac{\cosh q \cos \delta \sin \psi}{\sqrt{g_\theta}} \partial_\theta \\ &\quad + \left(\frac{\cosh q \sin \delta}{\sqrt{g_\phi}} + \frac{\omega \sinh q}{N} \right) \partial_\phi, \end{aligned} \quad (\text{B10})$$

$$\tilde{e}_{(2)} = -\sin \psi \sqrt{A} \partial_r + \cos \psi \partial_\theta, \quad (\text{B11})$$

$$\tilde{e}_{(3)} = -\sin \delta \cos \psi \partial_r - \sin \psi \sin \delta \partial_\theta + \cos \delta \partial_\phi. \quad (\text{B12})$$

The table that enters Eq. (A14) can be rewritten in the form

$$\Lambda = \begin{pmatrix} \cosh q & \sinh q \cos \delta \cos \psi & \sinh q \cos \delta \sin \psi & \sinh q \sin \delta \\ \sinh q & \cosh q \cos \delta \cos \psi & \cosh q \cos \delta \sin \psi & \cosh q \sin \delta \\ 0 & -\sin \psi & \cos \psi & 0 \\ 0 & -\sin \delta \cos \psi & -\sin \psi \sin \delta & \cos \delta \end{pmatrix}. \quad (\text{B13})$$

One can check that in the equatorial case ($\theta = \frac{\pi}{2}$) this parametrization reduces to that in [20] and Eq. (42) of [21].

1. Null tetrad

We also use the null tetrad bearing in mind possible application in the framework of the Newman-Penrose formalism. Let us introduce the null vectors $k, l, m, \bar{m}$ according to

$$e_{(0)} = \frac{k+l}{\sqrt{2}}, \quad e_{(1)} = \frac{k-l}{\sqrt{2}}, \quad (\text{B14})$$

$$e_{(2)} = \frac{m+\bar{m}}{\sqrt{2}}, \quad e_{(3)} = \frac{m-\bar{m}}{\sqrt{2}i}. \quad (\text{B15})$$

Then, one checks that the transformations (A5)–(A13) with $v = \tanh q$ can be rewritten in the form

$$\tilde{k} = e^q \left[k \left(\frac{1 + \cos \delta \cos \psi}{2} \right) + l \left(\frac{1 - \cos \delta \cos \psi}{2} \right) + m \left(\frac{\sin \psi \cos \delta}{2} + \frac{\sin \delta}{2i} \right) + \bar{m} \left(\frac{\sin \psi \cos \delta}{2} - \frac{\sin \delta}{2i} \right) \right], \quad (\text{B16})$$

$$\tilde{l} = e^{-q} \left[l \left(\frac{1 + \cos \delta \cos \psi}{2} \right) + k \left(\frac{1 - \cos \delta \cos \psi}{2} \right) - m \left(\frac{\sin \psi \cos \delta}{2} + \frac{\sin \delta}{2i} \right) - \bar{m} \left(\frac{\sin \psi \cos \delta}{2} - \frac{\sin \delta}{2i} \right) \right], \quad (\text{B17})$$

$$\tilde{m} = m \left(\frac{\cos \psi + \cos \delta - i \sin \psi \sin \delta}{2} \right) + \bar{m} \left(\frac{\cos \psi - \cos \delta - i \sin \psi \sin \delta}{2} \right) - (k-l) \frac{\sin \psi + i \sin \delta \cos \psi}{2}, \quad (\text{B18})$$

$$\tilde{\bar{m}} = m \left(\frac{\cos \psi - \cos \delta + i \sin \psi \sin \delta}{2} \right) + \bar{m} \left(\frac{\cos \psi + \cos \delta + i \sin \psi \sin \delta}{2} \right) - (k-l) \frac{\sin \psi - i \sin \delta \cos \psi}{2}. \quad (\text{B19})$$

If one uses the vectors $\tilde{e}$ constructed from a tilted null tetrad similar to (B14) and (B15), the result will coincide with ((B10)–(B12)).

The above formulas establish relationship between the Lorentz transformations in a locally flat space-time and particle dynamics in the curved background.

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- [1] M. Bañados, J. Silk Pand, and S. M. West, Kerr black holes as particle accelerators to arbitrarily high energy, *Phys. Rev. Lett.* **103**, 111102 (2009).
 - [2] A. A. Grib and Yu. V. Pavlov, On particles collisions in the vicinity of rotating black holes, *Pis'ma Zh. Eksp. Teor. Fiz.* **92**, 147 (2010) [*JETP Lett.* **92**, 125 (2010)].
 - [3] O. B. Zaslavskii, Acceleration of particles as universal property of rotating black holes, *Phys. Rev. D* **82**, 083004 (2010).
 - [4] T. Harada and M. Kimura, Collision of two general geodesic particles around a Kerr black hole, *Phys. Rev. D* **83**, 084041 (2011).
 - [5] O. B. Zaslavskii, Ultra-high energy collisions of nonequatorial geodesic particles near dirty black holes, *J. High Energy Phys.* **12** (2012) 032.
 - [6] H. V. Ovcharenko and O. B. Zaslavskii, Bañados-Silk-West effect with finite forces near different types of horizons: General classification of scenarios, *Phys. Rev. D* **108**, 064029 (2023).
 - [7] H. V. Ovcharenko and O. B. Zaslavskii, Bañados-Silk-West phenomenon for near-fine-tuned particles with an external force: General classification of scenarios, *Phys. Rev. D* **109**, 124041 (2024).
 - [8] D. E. A. Gates and S. Hadar, Signatures of particle collisions near extreme black holes, *Phys. Rev. D* **109**, 024052 (2024).

- [9] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation, *Astrophys. J.* **178**, 347 (1972).
- [10] *Black Holes: The Membrane Paradigm*, edited by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale University Press, London, 1986).
- [11] O. B. Zaslavskii, Acceleration of particles by black holes: Kinematic explanation, *Phys. Rev. D* **84**, 024007 (2011).
- [12] T. Piran and J. Shaham, Upper bounds on collisional Penrose processes near rotating black hole horizons, *Phys. Rev. D* **16**, 1615 (1977).
- [13] H. V. Ovcharenko and O. B. Zaslavskii, Axially symmetric rotating black holes, Boyer–Lindquist coordinates, and regularity conditions on horizons, *Gravit. Cosmol.* **29**, 269 (2023).
- [14] H. Maeda and C. Martinez, Existence and absence of Killing horizons in static solutions with symmetries, *Classical Quantum Gravity* **41**, 245013 (2024).
- [15] K. A. Bronnikov, E. Elizalde, S. D. Odintsov, and O. B. Zaslavskii, Horizons vs. singularities in spherically-symmetric space-times, *Phys. Rev. D* **78**, 064049 (2008).
- [16] O. B. Zaslavskii, Acceleration of particles by black holes—A general explanation. *Classical Quantum Gravity* **28**, 105010 (2011).
- [17] O. B. Zaslavskii, High energy particle collisions near the bifurcation surface, *Int. J. Mod. Phys. D* **22**, 1350044 (2013).
- [18] H. V. Ovcharenko and O. B. Zaslavskii, Naked and truly naked rotating black holes, *J. Math. Phys. (N.Y.)* **66**, 042502 (2025).
- [19] A. V. Toporensky and O. B. Zaslavskii, Flow and peculiar velocities for generic motion in spherically symmetric black holes, *Gravitation Cosmol.* **27**, 126 (2021).
- [20] P. Gusin, B. Kusnierz, and A. Radosz, Observers in spacetimes with spherical and axial symmetries, [arXiv:1507.01617](https://arxiv.org/abs/1507.01617).
- [21] I. V. Tanatarov and O. B. Zaslavskii, Collisional super-Penrose process and Wald inequalities, *Gen. Relativ. Gravit.* **49**, 119 (2017).