Letter

Thermal response of the Nieh-Yan term

Carlos Hoyos,[1,*] Niko Jokela,[2,†] and José Manuel Penín[3,4,‡]

[1]*Departamento de Física and Instituto de Ciencias y Tecnologías Espaciales de Asturias (ICTEA), Universidad de Oviedo, c/Leopoldo Calvo Sotelo 18, ES-33007, Oviedo, Spain*

[2]*Department of Physics and Helsinki Institute of Physics, P.O. Box 64, FI-00014 University of Helsinki, Finland*

[3]*INFN, Sezione di Firene, Via G. Sansone 1, I-50019 Sesto Fiorentino (Firenze), Italy*

[4]*Dipartimento di Fisica e Astronomia, Universitá di Firenze, Via G. Sansone 1, I-50019 Sesto Fiorentino (Firenze), Italy*

(Received 12 December 2024; accepted 23 April 2025; published 10 October 2025)

We reinterpret the Nieh-Yan (NY) anomaly using holography, finding that the U(1) axial symmetry remains unbroken and the axial current coupling to an external gauge field is conserved. Instead, the anomaly arises from a breakdown of Hodge duality relations between fermion bilinears due to symmetry constraints on one-form currents. We show that the axial response associated with the NY term is distinct from the chiral vortical effect and exhibits a characteristic T^2 dependence. Torsion-induced axial transport decouples spin and axial charge dynamics, thereby clarifying the physical significance of the NY anomaly and motivating further field-theoretic and holographic studies.

DOI: 10.1103/kcdm-vvzv

Introduction. Chirality is a fundamental property of relativistic fermions, prominently featured in high energy physics and the effective theories of Weyl semimetals [1] and time-reversal-breaking superfluid and superconductor phases, such as ^{3}He-A [2,3]. A notable consequence is the emergence of unusual charge and thermal transport phenomena in response to external sources. This is often associated with anomalies in the axial current, which measure the difference between currents with opposite chiralities [4]. Anomalous transport may leave observable imprints on the charge distribution of particles produced in heavy-ion collisions [5–7], an effect currently under investigation [8–10]. Axial anomalies have also been studied in Weyl semimetals [11–16] and have been invoked to explain the large negative magnetoresistance measured in experiments [17–24].

An intriguing extension of these phenomena involves torsion-induced transport. Although torsion appears to be absent in the fabric of spacetime, it emerges effectively in condensed matter systems as lattice defects in solids [25–27] or textures of the order parameter in superfluids or superconductors [28]. The axial response to torsion in condensed matter systems has become a topic of significant interest, yet it remains highly debated and not fully understood.

Although the anomalous axial response to external gauge fields and background geometry is well established, the response to torsion remains puzzling. The putative torsional anomaly of the axial current, $\mathcal{J}^\mu_{1,5} = \bar{\psi}\gamma^\mu\gamma_5\psi$, is characterized by a topological term introduced by Nieh and Yan (NY) [29],

$$\partial_\mu \langle \mathcal{J}^\mu_{1,5} \rangle = -ic_{\mathrm{NY}} \epsilon^{\mu\nu\lambda\rho} \left(\eta_{\alpha\beta} T^\alpha_{\mu\nu} T^\beta_{\lambda\rho} - \frac{1}{2} R_{\mu\nu\lambda\rho} \right), \quad (1)$$

where $T^\alpha_{\mu\nu}$ and $R_{\mu\nu\lambda\rho}$ denote the torsion and Riemann curvature tensors, respectively. Unlike other anomalies, the coefficient c_{NY} is dimensionful. Due to the absence of any natural scale, its presence in high energy physics is dubious [30–34]. However, in condensed matter systems c_{NY} can acquire a natural interpretation [27], being proportional to a cutoff of the effective theory, to temperature T, or to various chemical potentials [35–41]. Furthermore, a universal temperature-dependent contribution, $c_{\mathrm{NY}} \propto T^2$, has been proposed [42–49]. The nature of this response remains disputed [4,50–53], with some questioning whether it genuinely reflects an axial-torsional effect or merely disguises the chiral vortical effect [54,55].

In this Letter, we demonstrate that the thermal axial response arising from (1) is indeed distinct from the chiral vortical effect, although it does not constitute an anomaly in

*Contact author: hoyoscarlos@uniovi.es
†Contact author: niko.jokela@helsinki.fi
‡Contact author: jmanpen@gmail.com

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

the conventional sense. The key insight is that previous arguments correctly address the response in the consistent axial current, but the outright dismissal of the torsional response depends on an oversimplified identification of the bilinear coupling of the fermion to torsion, $\mathcal{J}_3^{\mu\nu\lambda} = \bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]}\psi$, with the Hodge dual of the axial current. However, in the absence of anomalies, the quantum expectation value $\langle \mathcal{J}_{1,5}^{\mu} \rangle$ represents a conserved current with only transverse components, while $\langle \mathcal{J}_3^{\mu\nu\lambda} \rangle$ may have both transverse and longitudinal components. In the presence of torsion $\langle \mathcal{J}_3^{\mu\nu\lambda} \rangle$ can develop a transverse component, such that its Hodge dual no longer coincides with the conserved axial current but instead corresponds to an axial vector operator with a nonconservation law compatible with the NY anomaly. Intriguingly, this decouples spin and axial charge dynamics.

To illustrate our proposal in a controlled setting, we compute the relevant expectation values in a holographic model where axial anomalies are turned off, definitely removing possible contributions from a chiral vortical effect. A proper implementation of torsional response necessitates a meticulous implementation of Hodge duality relations among fermion bilinears in the holographic dual, which we systematically verify. Our results align with the proposed interpretation of the NY anomaly, and we advocate for a direct verification through field-theoretic calculations. Such an analysis could involve explicit computations of fermion bilinear expectation values in a background torsion field using standard quantum field theory tools or a careful reexamination of prior works [35–49] to precisely identify torsional sources and consistent currents.

Furthermore, our holographic model offers a novel perspective on spin dynamics in strongly coupled systems, an area of active research [56–62]. This is particularly relevant in the context of hadron polarization observed in heavy-ion collisions [63–66], as our approach enables a clear distinction between spin polarization and effects due to chiral imbalance in a rotating plasma.

Our findings reveal a distinct and physically meaningful axial response to torsion in condensed matter systems, offering new insights into the interplay between torsion, axial anomalies, and spin dynamics. Future investigations, both in field theory and holography, could further elucidate the significance of the NY anomaly and its experimental manifestations.

Duality relations for fermion bilinears. Due to the properties of the Clifford algebra in four dimensions, not all tensor operators constructed as fermion bilinears are independent, but are related by a Hodge duality. For a Dirac fermion ψ we can construct completely antisymmetric tensor operators

$$\mathcal{J}_p^{\mu_1\cdots\mu_p} = \bar{\psi}\gamma^{[\mu_1}\cdots\gamma^{\mu_p]}\psi, \tag{2a}$$

$$\mathcal{J}_{p,5}^{\mu_1\cdots\mu_p} = \bar{\psi}\gamma^{[\mu_1}\cdots\gamma^{\mu_p]}\gamma_5\psi, \tag{2b}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and we have grouped the operators transforming either vectorially or axially. Here, $p = 0$ corresponds to scalar and pseudoscalar operators. These operators can be identified as components of p-forms

$$\mathcal{J}_{p(,5)} = \frac{1}{p!}\mathcal{J}_{p(,5)\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}. \tag{3}$$

It is convenient to introduce complex combinations

$$\mathcal{J}_p^{\pm} = \frac{1}{2}\left(\mathcal{J}_p \pm i\mathcal{J}_{p,5}\right). \tag{4}$$

Denoting by $\star_4$ the Hodge dual operation in four dimensions, the relations between operators derived from the Clifford algebra are

$$\mathcal{J}_4^{\pm} = \pm\star_4\mathcal{J}_0^{\mp}, \quad \mathcal{J}_3^{\pm} = \pm\star_4\mathcal{J}_1^{\mp}, \quad \mathcal{J}_2^{\pm} = \mp\star_4\mathcal{J}_2^{\mp}. \tag{5}$$

One can introduce external p-forms $c_p^{\pm}$ acting as sources for each of these operators

$$I_p = \int c_p^{\pm}\wedge\star_4\,\mathcal{J}_p^{\pm} = -\int \mathrm{d}^4x\frac{1}{p!}c_{\mu_1\cdots\mu_p}^{\pm}\mathcal{J}_p^{\pm\mu_1\cdots\mu_p}. \tag{6}$$

The relations (5) lead to duality relations for the sources

$$c_4^{\pm} = \mp\star_4 c_0^{\mp}, \quad c_3^{\pm} = \pm\star_4 c_1^{\mp}, \quad c_2^{\pm} = \pm\star_4 c_2^{\mp}. \tag{7}$$

When fermions are coupled to a curved background it is convenient to use the Cartan formalism with a vielbein and spin connection. Assuming flat spacetime, a nontrivial spin connection implies a nonzero torsion, which is minimally coupled to the fermions through the term [34,50,67]

$$-\frac{1}{16}\omega_{\mu\alpha\beta}\bar{\psi}\{\gamma^{\mu},[\gamma^{\alpha},\gamma^{\beta}]\}\psi. \tag{8}$$

Using the identity $\{\gamma^{\mu},[\gamma^{\alpha},\gamma^{\beta}]\} = 4\gamma^{[\mu}\gamma^{\alpha}\gamma^{\beta]}$, this contribution is equivalent to introducing three-form sources into the action

$$c_{3\mu\nu\lambda}^{+} = c_{3\mu\nu\lambda}^{-} = \frac{3}{2}\omega_{[\mu\nu\lambda]}. \tag{9}$$

Since torsion and three-form sources are equivalent, the three-form operator is proportional to the completely antisymmetric part of the spin current $\langle\mathcal{J}_3\rangle = \frac{2}{3}\langle\mathcal{S}_3\rangle$.

Holographic dual. According to the holographic dictionary, each bilinear fermion operator corresponds to a complex form field in the five-dimensional dual geometry

$$\mathcal{J}_p^{\pm} \leftrightarrow C_p^{\pm}. \tag{10}$$

We will assume that the dual geometry is asymptotically AdS_5. The fermion bilinears are all of scaling dimension

$\Delta = 3$, this fixes the masses of the p-forms in units of the AdS_5 radius L of the holographic dual

$$M_p^2 L^2 = (3-p)(p-1). \tag{11}$$

Action for even forms: We will start with the even forms and study the more complicated case of the odd forms separately. The five-dimensional action for the even forms is $[\mathrm{D}C_p^{\pm} = \mathrm{d}C_p^{\pm} - i(C_1^+ - C_1^-) \wedge C_p^{\pm}]$

$$S = \sum_{p=0,2} \frac{\lambda_p}{2} \int \epsilon_{AB} (C_p^A \wedge \mathrm{D}C_{4-p}^B - \mathrm{D}C_p^A \wedge C_{4-p}^B) - \frac{\delta_{AB}}{1+\delta_{p,2}} (m_p C_p^A \wedge\star\, C_p^B + m_{4-p} C_{4-p}^A \wedge\star\, C_{4-p}^B), \tag{12}$$

where $A, B = \pm$, $\epsilon_{+-} = 1$ and we have introduced the Hodge star operator in five dimensions $\star$ ($\epsilon^{01234} = 1$). The masses in the action have the values

$$m_4 L = 1, \qquad m_0 L = -3, \qquad m_2 L = 2. \tag{13}$$

A non-Abelian version of the action for two-forms was originally introduced in [68].

Action for odd forms: Omitting the gravitational sector, the action for the odd forms is similar to the one for the even forms

$$S_1 = \lambda_1 \int \epsilon_{AB} \mathrm{d}C_1^A \wedge C_3^B - \frac{m_3}{2} \delta_{AB} C_3^A \wedge\star\, C_3^B. \tag{14}$$

The action is invariant under a zero-form gauge symmetry

$$\delta C_1^{\pm} = \mathrm{d}\Lambda_0^{\pm}. \tag{15}$$

This corresponds to the $\mathrm{U}(1) \times \mathrm{U}(1)$ global flavor symmetry in the dual field theory.

We now consider the coupling to the spin connection. In the Cartan formalism, the gravitational and torsional degrees of freedom are captured by the vielbeins e^a and spin connection ω_b^a one-forms. The indices $\{a, b\} = \{0, 1, 2, 3, z\}$ refer to the tangent space of the five-dimensional (5D) geometry. The torsion and curvature are given by the two-forms

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \qquad R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{16}$$

where indices are raised and lowered with the flat Minkowski metric. The Einstein-Hilbert action with a cosmological constant Λ is

$$S_{\mathrm{grav}} = \frac{1}{2\kappa^2} \int \left(\frac{1}{3!} \mathcal{R} - 2\Lambda \star 1 \right), \tag{17}$$

where

$$\mathcal{R} = \epsilon_{abcde} e^a \wedge e^b \wedge e^c \wedge R^{de} \tag{18}$$

and where κ^2 is the 5D gravitational constant.

The coupling to the odd forms will be through the Lorentz-invariant three-form

$$\Omega_3 = \eta_{ab} e^a \wedge T^b. \tag{19}$$

It should be noted that in $3+1$ dimensions the NY term (1) equals $\star_4\, \mathrm{d}\Omega_3$. A minimal term preserving the zero-form gauge symmetry (15) is

$$S_{\mathrm{tor}} = \frac{\lambda_t}{4\kappa^2} \int \Omega_3 \wedge [m_3 \star (C_3^+ + C_3^-) - \mathrm{d}(C_1^+ - C_1^-)]. \tag{20}$$

The last term of this action coincides with the one obtained in [52] using the descent formalism.

To fully determine our theory, we need to introduce additional terms in the action that describe the dynamics of torsion via the spin connection. The simplest action that can be constructed is as follows:

$$S_\omega = \frac{3L^2}{2\kappa^2 g_\omega^2} \int R^a{}_b \wedge\star\, R_a^b + \frac{1}{2} \epsilon_{abcde} e^a \wedge R^{bc} \wedge R^{de}. \tag{21}$$

The values of the couplings that lead to a consistent system of equations are

$$g_w^2 = 6, \quad \Lambda L^2 = -4, \quad m_3 \kappa^2 \lambda_1 = 4, \quad \lambda_t = \kappa^2 \lambda_1. \tag{22}$$

With this choice, the action contains the terms of the Lovelock-Chern-Simons action [58,60,69].

Equations of motion: The equations of motion of the forms are obtained from the variation of (12), (14), and (20),

$$\mathrm{d}\left(C_p^{\pm} - \frac{1}{4}\delta_{p,3}\Omega_3 \right) \pm (-1)^p \frac{m_{4-p}}{1+\delta_{p,2}} \star \left(C_{4-p}^{\mp} - \frac{1}{4}\delta_{p,1}\Omega_3 \right) = 0. \tag{23}$$

Combining the equations of motion we see that the forms satisfy the second-order equations

$$\star \mathrm{d} \star \mathrm{d} C_p^{\pm} + M_p^2 C_p^{\pm} = \frac{1}{4}\delta_{p,3} \star \mathrm{d} \star \mathrm{d}\Omega_3. \tag{24}$$

The equations for the spin connection are derived from the variations of (17), (20), and (21). We decompose the spin connection in the Levi-Civita connection $\mathring{\omega}$ plus a contorsion term which we treat as a fluctuation

$$\omega_b^a = \mathring{\omega}^a{}_b(e) + K_b^a. \tag{25}$$

The linearized equations around the background AdS_5 geometry produce

$$\mathring{D} \star \mathring{D}K^a{}_b + \frac{1}{2L^2}\epsilon^a{}_{bcde}K^c{}_f \wedge e^f \wedge e^d \wedge e^e = 0, \quad (26)$$

where $\mathring{D}$ is the covariant derivative constructed with the background spin connection $\mathring{\omega}$.

Asymptotic expansions: We will consider an asymptotically AdS_5 black brane background

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f(z)} - f(z)dt^2 + (dx^i)^2\right), \quad f(z) = 1 - \frac{z^4}{z_h^4} \quad (27)$$

which, for vanishing background values of the forms, is a solution to the equations of motion. Here z_h is the position of the horizon which is related to the temperature $T = (\pi z_h)^{-1}$. The asymptotic boundary is at $z = 0$.

The asymptotic form of the solutions to the second-order equations (24) is

$$C_p^{\pm} \sim c_p^{\pm}\left(\frac{z}{L}\right)^{1-p} + k_p^{\pm}\left(\frac{z}{L}\right)^{3-p}\log\frac{z}{L} + j_p^{\pm}\left(\frac{z}{L}\right)^{3-p}. \quad (28)$$

The factors $c_p^{\pm}$, $k_p^{\pm}$, and $j_p^{\pm}$ are p-forms along the field theory directions depending on the x^μ coordinates. $c_p^{\pm}$ and $j_p^{\pm}$ are respectively the coefficients of the "leading" and "subleading" solutions, while $k_p^{\pm} \sim \Box c_p^{\pm}$. We identify the components of $c_p^{\pm}$ along the field theory directions as sources for the corresponding dual bilinear operators, while the components of $j_p^{\pm}$ determine their vacuum expectation values.

We can further simplify the equations of the spin connection (26). At $T = 0$ we can work in the gauge $K^a{}_{bz} = 0$ and expand the contorsion in the vielbein basis

$$K^a{}_b = \Omega^a{}_{bc}e^c, \qquad \Omega_3 = \Omega_{[abc]}e^a \wedge e^b \wedge e^c. \quad (29)$$

The nonzero components of the solutions are[1]

$$\Omega_{abc} = -\epsilon_{abcd}\eta^{d\mu}\partial_\mu\Omega_0, \qquad a, b, c, \mu \neq z. \quad (30)$$

Here Ω_0 is a scalar satisfying the same second-order equations as C_0 in the anti–de Sitter (AdS) background. Therefore, the boundary expansion of the scalar coincides with (28) for $p = 0$, taking $c_0 = \omega_0^{(0)}$, $j_0 = S_0$. Then, the leading terms in the boundary expansion of the Ω_3 components coincide with (28) for $p = 3$, identifying

$$\omega_3^{(0)} = \star_4 d\omega_0^{(0)}, \qquad S_3 = \star_4 dS_0. \quad (31)$$

This is the same kind of expansion we have for a massless three-form. Note that in the gauge we have chosen the coefficients of the asymptotic expansion are transverse,

$$\partial^\mu\omega^{(0)}_{\mu\nu\lambda} = 0 = \partial^\mu S_{\mu\nu\lambda}. \quad (32)$$

From the first-order equations (23) we derive additional conditions relating the coefficients of the asymptotic expansions that reproduce (7) for the coefficients of the leading terms, except for $c_3^{\pm}$ which agrees with (9). There are similar relations for the subleading coefficients (see Supplemental Material [74] for the explicit expressions), in particular for orthogonal components

$$j_3^{\pm} = \frac{3}{2}S_3^{\perp} \pm \frac{1}{m_3 L}\star_4\left(2j_1^{\perp\mp} - \frac{L^2}{2}\Box c_1^{\perp\mp}\right). \quad (33)$$

The longitudinal components vanish $j_1^{\parallel} = S_3^{\parallel} = 0$.

Expectation values from holography. According to the holographic dictionary, the generating functional of the quantum field theory equals the on-shell action of the gravity dual [70,71]. Correlation functions of fermion bilinears are then obtained from the variation of the on-shell action with respect to the leading coefficients in the expansion of the forms (28). It is important to note that the currents obtained in this way are always *consistent* currents.

The action is usually divergent and needs to be regularized and the divergences removed from physical quantities. This is done following the procedure of holographic renormalization [72,73], by which we introduce a cutoff in the holographic radial direction and add a counterterm action to remove the divergences.

Variation of the action: The variation of the five-dimensional on-shell action with respect to each form is a total derivative term

$$\delta S_{\text{on-shell}} = -\sum_{p=0}^{4}\sigma_p\int d[\delta_{AB}\delta C_p^A \wedge\star\, dC_p^B], \quad (34)$$

where

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\frac{\lambda_0}{2m_4}, \frac{\lambda_1}{m_3}, \frac{2\lambda_2}{m_2}, 0, \frac{\lambda_0}{2m_0}\right). \quad (35)$$

The form of the variation (34) is the same as for the action of forms with ordinary kinetic terms. Since the three-form variation vanishes, we should add a total derivative term giving a nonzero variation. A candidate term appears if one integrates by parts the action

[1]At nonzero T the *ansatz* becomes more complicated but the asymptotic behavior is the same, see Supplemental Material [74].

$$S_{B1} = \alpha_1 \int \mathrm{d}\left[\epsilon_{AB} C_1^A \wedge C_3^B - \frac{\lambda_t}{4\kappa^2 \lambda_1}(C_1^+ - C_1^-) \wedge \Omega_3\right]. \quad (36)$$

This action is not invariant off shell under the zero-form symmetry (15), however, one can confirm that it is invariant on shell. The variation of the on-shell action with the new terms for the one-form is still of the form (34) with $\sigma_1' = \frac{\lambda_1+\alpha_1}{m_3}$. The variation for the three-form is

$$\delta S_{\text{on shell}} = \alpha_1 \int \mathrm{d}[C_1^+ \wedge \delta C_3^- - C_1^- \wedge \delta C_3^+]. \quad (37)$$

Another candidate is to add the total derivative term

$$S_{B2} = -\frac{\sigma_3'}{2} \int \mathrm{d}[\delta_{AB} C_3^A \wedge \star \mathrm{d} C_3^B]. \quad (38)$$

In this case the variation is the same as for the other forms (34). The right choice for the coefficients turns out to be

$$\sigma_3{}' = -\frac{4}{3}\frac{L^2}{2\kappa^2 g_\omega^2}, \ \alpha_1 = -\frac{12}{7(12+g_\omega^2)}\lambda_1 \quad (39)$$

We relegate the details of the holographic renormalization to Supplemental Material [74]. The counterterm action takes the expected form of a sum of mass and kinetic terms of the fields evaluated at the cutoff, except for a term $\alpha_1 \int_{\partial \mathrm{AdS}_5} \epsilon_{AB} c_1^A \wedge (C_3^B - \frac{3}{2}\Omega_3)$. This term depends on the source $c_\mu^\pm$ rather than on the pullback of the one-form $C_\mu^\pm$ and makes the whole action gauge-invariant off shell.

Expectation values: The expectation value of the bilinear operators is determined by the variation $\langle \delta I_p \rangle = \delta S_{\text{on shell}} + \delta S_{B1} + \delta S_{B2} + \delta S_{ct}$. We obtain[2]

$$\langle \mathcal{J}_p^\pm \rangle = \frac{\sigma_p'}{L}\left(2 j_p^\pm - \frac{L^2}{2}\Box \hat{c}_p^\pm\right) + \delta_{p,3}\langle \mathcal{J}_3^\pm \rangle_{\alpha_1}, \quad (40)$$

where $\hat{c}_p^\pm = c_p^\pm$ for p even, $\hat{c}_1^\pm = c_1^{\perp\pm}$ and $\hat{c}_3^\pm = 0$. The additional term in the three-form is

$$\langle \mathcal{J}_3^\pm \rangle_{\alpha_1} = \mp \alpha_1 \star_4 \left(j_1^{\perp\mp} - \frac{L^2}{4}\Box c_1^{\perp\mp}\right). \quad (41)$$

[2]The contact terms can be modified by finite counterterms, see Supplemental Material [74].

On the other hand, the spin current obtained from the variation with respect to the spin connection is

$$\langle \mathcal{S}_3 \rangle = -\frac{6L}{\kappa^2 g_\omega^2} S_3 - \frac{3}{2}(\langle \mathcal{J}_3^+ \rangle_{\alpha_1} + \langle \mathcal{J}_3^- \rangle_{\alpha_1}). \quad (42)$$

Using (33), one can check that the conditions $\langle \mathcal{J}_3 \rangle = \frac{2}{3}\langle \mathcal{S}_3 \rangle$ and $\langle \mathcal{J}_3^{\pm\parallel} \rangle = \pm \star_4 \langle \mathcal{J}_1^{\mp} \rangle$ are satisfied.

Nieh-Yan term in holography. Our analysis reveals that the Hodge duality relations (5) for the odd forms are not satisfied in general. Defining $\langle \mathcal{K}_1^\pm \rangle = \pm \star_4 \langle \mathcal{J}_3^\mp \rangle$, the corresponding axial and vector combinations are

$$\langle \mathcal{K}_1 \rangle = \langle \mathcal{J}_1 \rangle, \qquad \langle \mathcal{K}_{1,5} \rangle = \langle \mathcal{J}_{1,5} \rangle - \frac{2i}{3} \star_4 \langle \mathcal{S}_3 \rangle. \quad (43)$$

Thus, while the Hodge dual vector current is conserved, the axial current acquires a longitudinal component

$$\mathrm{d} \star_4 \langle \mathcal{K}_1 \rangle = 0, \qquad \mathrm{d} \star_4 \langle \mathcal{K}_{15} \rangle = \frac{2i}{3} \mathrm{d}\langle \mathcal{S}_3 \rangle. \quad (44)$$

This introduces a temperature-dependent NY anomaly term. At nonzero temperature the dual geometry can be taken to be the AdS_5 black brane (27). The regular solution at low frequency and momentum is such that the components of S_3 transverse to the spatial momentum are nonzero. Details of the solution are given in the Supplemental Material [74]

$$S_3 = -\frac{g_w^2 \kappa^2}{4L^3} c_{\mathrm{NY}} \omega_3^{(0)},$$
$$c_{\mathrm{NY}} \approx 4.275 \frac{L^3}{\kappa^2} T^2. \quad (45)$$

Acknowledgments. We thank Karl Landsteiner and Teemu Ojanen for useful discussions. C. H. is partially supported by the Agencia Estatal de Investigación and the Ministerio de Ciencia, Innovación y Universidades through the Spanish Grant No. PID2021-123021NB-I00. N. J. has been supported in part by Research Council of Finland Grant No. 354533.

Data availability. No data were created or analyzed in this study.

[1] B. Yan and C. Felser, Topological materials: Weyl semimetals, Annu. Rev. Condens. Matter Phys. **8**, 337 (2017).

[2] G. E. Volovik, *The Universe in a helium droplet*, Vol. 117 (Oxford University Press, United Kingdom, 2006).

[3] T. Meng and L. Balents, Weyl superconductors, Phys. Rev. B **86**, 054504 (2012).

[4] M. N. Chernodub, Y. Ferreiros, A. G. Grushin, K. Landsteiner, and M. A. H. Vozmediano, Thermal transport, geometry, and anomalies, Phys. Rep. **977**, 1 (2022).

[5] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, The chiral magnetic effect, Phys. Rev. D **78**, 074033 (2008).

[6] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Real-time dynamics of the chiral magnetic effect, Phys. Rev. Lett. **104**, 212001 (2010).

[7] D. E. Kharzeev, The chiral magnetic effect and anomaly-induced transport, Prog. Part. Nucl. Phys. **75**, 133 (2014).

[8] B. I. Abelev *et al.* (STAR Collaboration), Azimuthal charged-particle correlations and possible local strong parity violation, Phys. Rev. Lett. **103**, 251601 (2009).

[9] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Observation of the chiral magnetic effect in ZrTe5, Nat. Phys. **12**, 550 (2016).

[10] D. E. Kharzeev, J. Liao, and P. Tribedy, Chiral magnetic effect in heavy ion collisions: The present and future, Int. J. Mod. Phys. E **33**, 2430007 (2024).

[11] H. B. Nielsen and M. Ninomiya, Adler-Bell-Jackiw anomaly and Weyl fermions in crystal, Phys. Lett. **130B**, 389 (1983).

[12] D. T. Son and B. Z. Spivak, Chiral anomaly and classical negative magnetoresistance of Weyl metals, Phys. Rev. B **88**, 104412 (2013).

[13] R. Lundgren, P. Laurell, and G. A. Fiete, Thermoelectric properties of Weyl and Dirac semimetals, Phys. Rev. B **90**, 165115 (2014).

[14] A. A. Burkov, Negative longitudinal magnetoresistance in Dirac and Weyl metals, Phys. Rev. B **91**, 245157 (2015).

[15] B. Z. Spivak and A. V. Andreev, Magnetotransport phenomena related to the chiral anomaly in Weyl semimetals, Phys. Rev. B **93**, 085107 (2016).

[16] A. Lucas, R. A. Davison, and S. Sachdev, Hydrodynamic theory of thermoelectric transport and negative magnetoresistance in Weyl semimetals, Proc. Natl. Acad. Sci. U.S.A. **113**, 9463 (2016).

[17] H.-J. Kim, K.-S. Kim, J.-F. Wang, M. Sasaki, N. Satoh, A. Ohnishi, M. Kitaura, M. Yang, and L. Li, Dirac versus Weyl fermions in topological insulators: Adler-Bell-Jackiw anomaly in transport phenomena, Phys. Rev. Lett. **111**, 246603 (2013).

[18] J. Xiong, S. K. Kushwaha, T. Liang, J. W. Krizan, M. Hirschberger, W. Wang, R. J. Cava, and N. P. Ong, Evidence for the chiral anomaly in the Dirac semimetal Na_3Bi, Science **350**, 413 (2015).

[19] X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang, X. Dai, and G. Chen, Observation of the chiral-anomaly-induced negative magnetoresistance in 3d Weyl semimetal taas, Phys. Rev. X **5**, 031023 (2015).

[20] C.-Z. Li, L.-X. Wang, H. Liu, J. Wang, Z.-M. Liao, and D.-P. Yu, Giant negative magnetoresistance induced by the chiral anomaly in individual Cd3As2 nanowires, Nat. Commun. **6**, 10137 (2015).

[21] H. Li, H. He, H.-Z. Lu, H. Zhang, H. Liu, R. Ma, Z. Fan, S.-Q. Shen, and J. Wang, Negative magnetoresistance in Dirac semimetal Cd3As2, Nat. Commun. **7**, 10301 (2016).

[22] C.-L. Zhang *et al.*, Signatures of the Adler–Bell–Jackiw chiral anomaly in a Weyl fermion semimetal, Nat. Commun. **7**, 10735 (2016).

[23] M. Hirschberger, S. Kushwaha, Z. Wang, Q. Gibson, S. Liang, C. Belvin, B. Bernevig, R. Cava, and N. Ong, The chiral anomaly and thermopower of Weyl fermions in the Half-Heusler GdPtBi, Nat. Mater. **15**, 1161 (2016).

[24] J. Gooth *et al.*, Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP, Nature (London) **547**, 324 (2017).

[25] F. de Juan, A. Cortijo, and M. A. H. Vozmediano, Dislocations and torsion in graphene and related systems, Nucl. Phys. **B828**, 625 (2010).

[26] T. L. Hughes, R. G. Leigh, and O. Parrikar, Torsional anomalies, Hall viscosity, and bulk-boundary correspondence in topological states, Phys. Rev. D **88**, 025040 (2013).

[27] O. Parrikar, T. L. Hughes, and R. G. Leigh, Torsion, parity-odd response and anomalies in topological states, Phys. Rev. D **90**, 105004 (2014).

[28] Y. Ishihara, T. Mizushima, A. Tsuruta, and S. Fujimoto, Torsional chiral magnetic effect due to skyrmion textures in a Weyl superfluid, Phys. Rev. B **99**, 024513 (2019).

[29] H. T. Nieh and M. L. Yan, An identity in Riemann-Cartan geometry, J. Math. Phys. (N.Y.) **23**, 373 (1982).

[30] O. Chandia and J. Zanelli, Topological invariants, instantons and chiral anomaly on spaces with torsion, Phys. Rev. D **55**, 7580 (1997).

[31] Y. N. Obukhov, E. W. Mielke, J. Budczies, and F. W. Hehl, On the chiral anomaly in nonRiemannian space-times, Found. Phys. **27**, 1221 (1997).

[32] D. Kreimer and E. W. Mielke, Comment on: Topological invariants, instantons, and the chiral anomaly on spaces with torsion, Phys. Rev. D **63**, 048501 (2001).

[33] O. Chandia and J. Zanelli, Reply to the comment by D. Kreimer and E. Mielke, Phys. Rev. D **63**, 048502 (2001).

[34] J. Erdmenger, I. Matthaiakakis, R. Meyer, and D. Vassilevich, The chiral torsional anomaly and the Nieh-Yan invariant with and without boundaries, J. High Energy Phys. 12 (2024) 149.

[35] H. Sumiyoshi and S. Fujimoto, Torsional chiral magnetic effect in a Weyl semimetal with a topological defect, Phys. Rev. Lett. **116**, 166601 (2016).

[36] Y. Ferreiros, Y. Kedem, E. J. Bergholtz, and J. H. Bardarson, Mixed axial-torsional anomaly in Weyl semimetals, Phys. Rev. Lett. **122**, 056601 (2019).

[37] Z.-M. Huang, L. Li, J. Zhou, and H.-H. Zhang, Torsional response and Liouville anomaly in Weyl semimetals with dislocations, Phys. Rev. B **99**, 155152 (2019).

[38] Z.-M. Huang and B. Han, Torsional anomalies and bulk-dislocation correspondence in Weyl systems, arXiv:2003.04853.

[39] Z.-M. Huang, B. Han, and M. Stone, Hamiltonian approach to the torsional anomalies and its dimensional ladder, Phys. Rev. B **101**, 165201 (2020).

[40] J. Nissinen, Emergent spacetime and gravitational Nieh-Yan anomaly in chiral $p+ip$ Weyl superfluids and superconductors, Phys. Rev. Lett. **124**, 117002 (2020).
[41] S. Laurila and J. Nissinen, Torsional Landau levels and geometric anomalies in condensed matter Weyl systems, Phys. Rev. B **102**, 235163 (2020).
[42] J. Nissinen and G. E. Volovik, On thermal Nieh–Yan anomaly in topological Weyl materials, Pis'ma Zh. Eksp. Teor. Fiz. **110**, 797 (2019).
[43] J. Nissinen and G. E. Volovik, Thermal Nieh-Yan anomaly in Weyl superfluids, Phys. Rev. Res. **2**, 033269 (2020).
[44] Z.-M. Huang, B. Han, and M. Stone, Nieh-yan anomaly: Torsional landau levels, central charge, and anomalous thermal Hall effect, Phys. Rev. B **101**, 125201 (2020).
[45] L. Liang and T. Ojanen, Topological magnetotorsional effect in Weyl semimetals, Phys. Rev. Res. **2**, 022016 (2020).
[46] Z. V. Khaidukov and M. A. Zubkov, Chiral torsional effect, JETP Lett. **108**, 670 (2018).
[47] S. Imaki and A. Yamamoto, Lattice field theory with torsion, Phys. Rev. D **100**, 054509 (2019).
[48] S. Imaki and Z. Qiu, Chiral torsional effect with finite temperature, density and curvature, Phys. Rev. D **102**, 016001 (2020).
[49] J. Nissinen and G. E. Volovik, Anomalous chiral transport with vorticity and torsion: Cancellation of two mixed gravitational anomaly currents in rotating chiral p + ip Weyl condensates, Phys. Rev. D **106**, 045022 (2022).
[50] Y. Ferreiros and K. Landsteiner, On chiral responses to geometric torsion, Phys. Lett. B **819**, 136419 (2021).
[51] T. Amitani and Y. Nishida, Torsion-induced chiral magnetic current in equilibrium, Ann. Phys. (Amsterdam) **448**, 169181 (2023).
[52] M. Valle and M. A. Vazquez-Mozo, On Nieh-Yan transport, J. High Energy Phys. 03 (2022) 177.
[53] M. Valle and M. A. Vazquez-Mozo, Torsional constitutive relations at finite temperature, J. High Energy Phys. 02 (2024) 185.
[54] K. Landsteiner, E. Megias, L. Melgar, and F. Pena-Benitez, Holographic gravitational anomaly and chiral vortical effect, J. High Energy Phys. 09 (2011) 121.
[55] V. Braguta, M. N. Chernodub, K. Landsteiner, M. I. Polikarpov, and M. V. Ulybyshev, Numerical evidence of the axial magnetic effect, Phys. Rev. D **88**, 071501 (2013).
[56] R. G. Leigh, N. N. Hoang, and A. C. Petkou, Torsion and the gravity dual of parity symmetry breaking in AdS(4)/CFT(3) holography, J. High Energy Phys. 03 (2009) 033.
[57] A. C. Petkou, Torsional degrees of freedom in AdS4/CFT3, arXiv:1004.1640.
[58] A. D. Gallegos and U. Gürsoy, Holographic spin liquids and Lovelock Chern-Simons gravity, J. High Energy Phys. 11 (2020) 151.
[59] A. D. Gallegos, U. Gursoy, and A. Yarom, Hydrodynamics, spin currents and torsion, J. High Energy Phys. 05 (2023) 139.
[60] J. Erdmenger, B. Heß, I. Matthaiakakis, and R. Meyer, Universal Gibbons-Hawking-York term for theories with curvature, torsion and non-metricity, SciPost Phys. **14**, 099 (2023).
[61] J. Erdmenger, B. Heß, R. Meyer, and I. Matthaiakakis, Gibbons-Hawking-York boundary terms and the generalized geometrical trinity of gravity, Phys. Rev. D **110**, 066002 (2024).
[62] C. Cartwright, D. Gallegos, U. Gürsoy, R. Klein, and A. Yarom, A supersymmetric spin current, arXiv:2408.04399.
[63] L. Adamczyk *et al.* (STAR Collaboration), Global Λ hyperon polarization in nuclear collisions: Evidence for the most vortical fluid, Nature (London) **548**, 62 (2017).
[64] J. Adam *et al.* (STAR Collaboration), Polarization of Λ ($\bar{\Lambda}$) hyperons along the beam direction in Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV, Phys. Rev. Lett. **123**, 132301 (2019).
[65] S. Acharya *et al.* (ALICE Collaboration), Evidence of spin-orbital angular momentum interactions in relativistic heavy-ion collisions, Phys. Rev. Lett. **125**, 012301 (2020).
[66] J. Adam *et al.* (STAR Collaboration), Global polarization of Ξ and Ω hyperons in Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV, Phys. Rev. Lett. **126**, 162301 (2021); **131**, 089901(E) (2023).
[67] J. L. Mañes, M. Valle, and M. A. Vázquez-Mozo, Chiral torsional effects in anomalous fluids in thermal equilibrium, J. High Energy Phys. 05 (2021) 209.
[68] R. Alvares, C. Hoyos, and A. Karch, An improved model of vector mesons in holographic QCD, Phys. Rev. D **84**, 095020 (2011).
[69] M. Banados, O. Miskovic, and S. Theisen, Holographic currents in first order gravity and finite Fefferman-Graham expansions, J. High Energy Phys. 06 (2006) 025.
[70] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[71] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[72] S. de Haro, S. N. Solodukhin, and K. Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. **217**, 595 (2001).
[73] M. Bianchi, D. Z. Freedman, and K. Skenderis, Holographic renormalization, Nucl. Phys. **B631**, 159 (2002).
[74] See Supplemental Material at http://link.aps.org/supplemental/10.1103/kcdm-vvzv for helpful intermediate steps and detailed conventions that will assist in reproducing the results presented in the main text.